(3+1)d boundaries with gravitational anomaly of (4+1)d invertible topological order for branch-independent bosonic systems

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We study bosonic systems on a space-time lattice (with imaginary time) defined by path integrals of commuting fields. We introduce a concept of branch-independent bosonic (BIB) systems, whose path integral is independent of the branch structure of the space-time simplicial complex, even for a space-time with boundaries. In contrast, a generic lattice bosonic (GLB) system's path integral may depend on the branch structure. We find the invertible topological order characterized by the Stiefel-Whitney cocycle [such as (4+1)d w_2w_3] to be nontrivial for branch-independent bosonic systems, but this topological order and a trivial gapped tensor product state belong to the same phase (via a smooth deformation without any phase transition) for generic lattice bosonic systems. This implies that the invertible topological orders in generic lattice bosonic systems on a space-time lattice are not classified by the oriented cobordism. The branch dependence on a lattice may be related to the orthonormal frame of smooth manifolds and the framing anomaly of continuum field theories. In general, the branch structure on a discretized lattice may be related to a frame structure on a smooth manifold that trivializes any Stiefel-Whitney classes. We construct branch-independent bosonic systems to realize the w_2w_3 topological order, and its (3+1)d gapped or gapless boundaries. One of the gapped boundaries is a (3+1)d $\mathbb{Z}_2$ gauge theory with (1) fermionic $\mathbb{Z}_2$ gauge charge particle which trivializes w_2 and (2) "fermionic" $\mathbb{Z}_2$ gauge flux line trivializes w_3. In particular, if the flux loop's world sheet is unorientable, then the orientation-reversal one-dimensional world line must correspond to a fermion world line that *does not carry the* $\mathbb{Z}_2$ *gauge charge*. We also explain why Spin and Spin^c structures trivialize the w_2w_3 nonperturbative global pure gravitational anomaly to zero [which helps to construct the anomalous (3+1)d gapped $\mathbb{Z}_2$ and gapless all-fermion U(1) gauge theories], but the Spin^h and Spin$\times_{\mathbb{Z}_2}$Spin($n \geqslant 3$) structures modify the w_2w_3 into a nonperturbative global mixed gauge-gravitational anomaly, which helps to constrain grand unifications (e.g., $n = 10, 18$) or construct new models.

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I. INTRODUCTION

Gapped quantum states of matter (or, more precisely, gapped quantum liquids [1,2]) with no symmetry can be divided into two classes [3]: (1) states with no topological order (all those states belong to the trivial phase represented by tensor product states, with no entanglement or short-range entanglement); (2) states with topological order [4,5] (i.e., gapped states with long-range entanglement [6]).

In the presence of global symmetry that is not spontaneously broken, the above two classes can be further divided into some subclasses: (1a) states with no topological order and no symmetry-protected topological (SPT) order, or synonymously, symmetry-protected trivial (SPT) order, with no entanglement or short-range entanglement; (1b) states with no topological order but with nontrivial SPT order [7–19], with short-range entanglement; (2a) states with both topological order and symmetry. Those states are said to have symmetry-enriched topological (SET) orders [20–31], with long-range entanglement.

In this work, we aim to study those topological states of matter and their boundaries for bosonic systems. We realize that even bosonic systems without any symmetries can have many different types, such as bosonic systems on a space-time lattice with imaginary time, bosonic systems in continuum space-time with real or imaginary time, and bosonic systems defined via lattice Hamiltonian with real continuous time. Those different bosonic theories require different mathematical descriptions. In this work, we will only study bosonic systems on a space-time lattice with imaginary time, that satisfy the reflection positivity. In fact, we will study a simpler problem, the so-called invertible topological states of matter in the bulk, and their boundaries, with or without symmetry.

Stacking two topological states $\mathscr{C}_1$ and $\mathscr{C}_2$ gives us a third topological state $\mathscr{C}_3 = \mathscr{C}_1 \boxtimes \mathscr{C}_2$. The stacking operation $\boxtimes$ makes the set of topological states into a monoid. (A monoid is like a group except its elements may not have inverse.) If a topological state $\mathscr{C}$ has an inverse under the stacking operation $\boxtimes$, i.e., there exists a topological state $\mathscr{D}$ such that $\mathscr{C} \boxtimes \mathscr{D}$ is a

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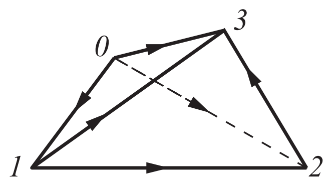

FIG. 1. A branch structure of a simplicial complex is given by assigning arrows to the links which do not form loops around every triangle. The branch structure gives rise to a local order of vertices for every simplex. Stiefel-Whitney cocycle a simplicial complex can be constructed after assigning a branch structure (see Appendix F, Figs. 5 and 6).

trivial product state, then $\mathscr{C}$ will be called an invertible topological state. It turns out that all SPT states are invertible. Only a small fraction of topologically ordered states and SET states are invertible. For example, a fermionic integer quantum Hall state is an invertible topologically ordered state [a fermionic invertible SET with U(1) symmetry]. An antiferromagnetic spin-1 Haldane chain is a SPT state protected by spin rotation SO(3) or time-reversal $\mathbb{Z}_2^T$ symmetry, which is always invertible. (In contrast, an antiferromagnetic spin-2 chain, another Haldane phase, has trivial SPT order.)

Invertible topological states of bosonic systems are characterized by a simple class of invertible topological invariants [16,32–38]. In this paper, we will derive the possible boundary theory from those topological invariants, especially the (4+1)-dimensional [(4+1)d] invertible topological order characterized by the Stiefel-Whitney $\mathrm{w}_2\mathrm{w}_3$ topological invariant in five dimensions (5d). There are some earlier works in this direction [34,39–49] which construct boundary theories of the $\mathrm{w}_2\mathrm{w}_3$ invertible topological order. In this work, we will present a more complete and systematic derivation.

Branch-independent bosonic system vs generic lattice bosonic system. Even bosonic systems on a space-time lattice can have different types. We find that in order to discuss invertible topological order, we need to introduce a concept of branch-independent bosonic system (the L-type system studied in [50] happens to be a branch-independent bosonic system, L for Lagrangian formulated on the space-time lattice), whose partition function computed from path integral is independent of the branch structures (or branching structures) of space-time lattice (i.e., space-time simplicial complex), even for space-time with boundaries. Here a branch structure is a local ordering of the vertices for each simplex (see Fig. 1 and Appendix A). Generic lattice bosonic systems do not have this requirement, and their path integrals may depend on the branch structures of space-time complex.

Branch independence is a property of lattice bosonic systems. It appears that such a lattice property is related to frame independence, a property of continuum field theory. An orthonormal frame of a d-dimensional manifold is a set of d vector fields, which is orthonormal at every point with respect to the metrics tensor of the manifold. We will abbreviate orthonormal frame as frame (which is also called the vielbein). After assigning a frame to a manifold, we can define an SO(d) connection to describe the curvature. Continuum field theory explicitly depends on the SO(d) connection and the frame. Thus, the partition function of the continuum field theory may depend on the frame of the space-time manifold. If the partition function does depend on the frame, we say that the theory has a *framing anomaly* [4]. Otherwise, the theory is *free of framing anomaly*.

In [51], (2+1)-dimensional [(2+1)d] generic lattice bosonic systems are constructed to realize topological orders with any integral chiral central charge and the corresponding gravitational Chern-Simons term. Since the central charge is not 0 mod 24, those models contain framing anomaly and are not frame independent. This implies that the partition function of generic lattice bosonic systems may depend on the frames of the space-time manifold in the continuum limit. This example and the above discussions suggest a close relation between the branch structure on a lattice and the frame structure on a continuum manifold.[1]

We conjecture that the independence of the branch structure of space-time complex for lattice models implies the independence of frame structure of space-time manifold in the continuum limit. Under such a conjecture, the branch-independent bosonic systems on lattice only give rise to continuum effective field theories that are free of framing anomaly. As a result, a (2+1)d branch-independent bosonic system can only realize topological orders with chiral central charge $c = 0$ mod 24, where the (2+1)d invertible topological orders are generated by three copies of E_8 quantum Hall states (say E_8^3 topological order). Indeed, using SO(∞) nonlinear σ model, Ref. [50] constructed a branch-independent bosonic model that realizes the E_8^3 topological order with $c = 24$. For more details, see Appendix I.

In this work, we find that the (4+1)d invertible topological order characterized by the Stiefel-Whitney class $\mathrm{w}_2\mathrm{w}_3$ is nontrivial for branch-independent bosonic systems. On the other hand, the $\mathrm{w}_2\mathrm{w}_3$ topological order and trivial tensor product states belong to the same phase for generic lattice bosonic systems. This implies that we cannot use the oriented cobordism (i.e., manifolds with special orthogonal group SO structures) [16,34–36] to classify invertible topological orders for the generic lattice bosonic systems (where the lattice means the space-time lattice).

Can we use oriented cobordism to classify invertible topological orders for branch-independent bosonic systems? The

[1]Branch structure on a discretized lattice vs frame structure on a smooth manifold that trivializes any Stiefel-Whitney classes: A framing of a d-dimensional manifold of M is a choice of trivialization of its tangent bundle, hence, a choice of a section of the corresponding frame bundle. If there exists a framing on the tangent bundle TM, then there exist d linearly independent sections of the tangent bundle. If the ith Stiefel-Whitney class $w_i(TM) \neq 0$, then there can not exist $d - i + 1$ linearly independent sections of the tangent bundle. So $w_i(TM) = 0$ for $1 \leqslant i \leqslant d$. Thus, note the following:
(1) The tangential frame structure on a smooth manifold trivializes any Stiefel-Whitney classes.
(2) We will later suggest that a branch structure on a discretized lattice or on a simplicial complex also trivializes Stiefel-Whitney classes.
(3) Thus, our work suggests a possible relation: a branch structure on a discretized lattice may be related to the frame structure on a smooth manifold that trivializes any Stiefel-Whitney classes.

oriented cobordism suggests a $\mathbb{Z}$ class of (2+1)d invertible bosonic topological orders, generated by the E_8 topological order.[2] However, currently, we do not have a realization of E_8 topological order using branch-independent bosonic systems. We only know a branch-dependent bosonic model that realizes the E_8 topological order [51], and a branch-independent bosonic model that realizes the E_8^3 topological order [50]. Thus, it is not clear if oriented cobordism classifies invertible topological orders for the branch-independent bosonic systems or not. In this paper, we will concentrate on branch-independent lattice bosonic systems.

The boundary theories of bulk invertible topological orders with no symmetry belong to a special class: they are theories with gravitational anomalies. In fact, the gravitational anomalies in field theories are classified by invertible topological orders in one higher dimension [35,56]. From this point of view, we study various anomalous theories with a given gravitational anomaly on the (3+1)d boundary of (4+1)d w_2w_3 topological order. We shall call this gravitational anomaly as the w_2w_3 anomaly, which is a *nonperturbative global gravitational anomaly*.[3]

For example, it is known that the (4+1)d invertible topological order has a gapped (3+1)d boundary described by a $\mathbb{Z}_2$ gauge theory with the w_2w_3 anomaly, where the $\mathbb{Z}_2$ gauge charge is fermionic. However, the fermionic $\mathbb{Z}_2$ gauge charge does not fully characterize the gravitational anomaly. In particular, there is also an anomaly-free (3+1)d $\mathbb{Z}_2$ gauge theory with a fermionic $\mathbb{Z}_2$ gauge charge, i.e., there is a (3+1)d lattice bosonic system that can realize a $\mathbb{Z}_2$ gauge theory with a fermionic $\mathbb{Z}_2$ gauge charge [59]. In this work, we show that the 2d world sheet of the $\mathbb{Z}_2$ gauge flux in the space-time must carry a noncontractible 1d fermionic orientation-reversal world line if the 2d world sheet is unorientable. This 1d fermionic orientation-reversal world line with neutral gauge charge is, however, distinct from the fermionic world line of $\mathbb{Z}_2$ gauge charge. This crucial property, together with the fermionic $\mathbb{Z}_2$ gauge charge, characterizes the gravitational anomaly.

The above result about the fermion world line on unorientable world sheet of $\mathbb{Z}_2$ flux line was first obtained in Sec. 3.3 of [44]. In this paper we give a different derivation of the result using a path-integral formulation on a space-time lattice. This result was also obtained recently in Ref. [48] using Hamiltonian formulation on spatial lattice. The "fermionic" nature of the $\mathbb{Z}_2$ flux line can also be characterized by the statistical hopping algebra for strings [49], a generalization of the statistical hopping algebra for particles [59].

A. Notations and conventions

We denote the (n')d for the space-time dimensions to be $n' = (n+1)$ with an n-dimensional space and a one-dimensional time. Typically, in this paper, the dimension always refers to the space-time dimension altogether. We may simply call the 0d $\mathbb{Z}_2$ gauge charge as $\mathbb{Z}_2$ charge, whose space-time trajectory is a 1d world line. We may simply call the 1d $\mathbb{Z}_2$ gauge flux loop as $\mathbb{Z}_2$ flux, which can be a 1d loop which bounds a 2d surface enclosing gauge flux, whose space-time trajectory is a 2d world sheet.

In this work, we use a lot of formalisms of chain and cochain, as well as the associated derivative cup product, Steenrod square, etc. A brief coverage of those topics can found in Appendix A. The $\mathbb{Z}_n$ values are chosen to be $\{0, 1, \dots, n-1\}$. In this paper, we always use this set to extend $\mathbb{Z}_n$ values to $\mathbb{Z}$ values, and treat $\mathbb{Z}_n$-valued quantities as $\mathbb{Z}$-valued quantities. To help to express $\mathbb{Z}_n$-valued relation using $\mathbb{Z}$-valued quantities, we denote $\overset{n}{=}$ to mean equal up to a multiple of n (thus, it is a mod n relation: two sides of the equality are equal mod n), and use $\overset{d}{=}$ to mean equal up to a coboundary df (i.e., with the coboundary operator d).

We denote the Lorentz group as SO (for bosonic systems) and Spin (for fermionic systems graded by the fermion parity $\mathbb{Z}_2^f$). In $(n+1)$d space-time, the SO really means the $\mathrm{SO}(n+1)$ for the Euclidean rotational symmetry group and the $\mathrm{SO}(n,1)$ for the Lorentz rotational + boost symmetry group; the Spin really means the $\mathrm{Spin}(n+1)$ for the Euclidean rotational symmetry group and the Spin(n, 1) for the Lorentz rotational + boost symmetry group. We denote $\overset{n,d}{=}$ to mean equal up to a mod n relation and also equal up to a coboundary df. We will use the group $N \rtimes_{e_2,\alpha} G$ to describe the extension of a group G by an Abelian group N via

$$1 \to N \to N \rtimes_{e_2,\alpha} G \to G \to 1,$$

which is characterized by $e_2 \in H^2(G;N)$ of the second cohomology class, where N is an Abelian group with a G action via $\alpha : G \to \mathrm{Aut}[N]$. We will also use $G_1 \times_N G_2 \equiv \frac{G_1 \times G_2}{N}$ to define as the product group of G_1 and G_2 modding out their common normal subgroup N. Other mathematical conventions and definitions (such as Stiefel-Whitney class) can be found in Appendix B. We provide many Appendixes on the toolkits of cochain, cocycle, cohomology, characteristic class, and cobordism.

Dynamical gauge fields are associated with the gauge connections of gauge bundles that are summed over in the path integral (or partition function). *Background gauge fields*

[2]The (2+1)d Abelian bosonic topological orders are classified (in a many-to-one fashion) by symmetric integral matrices with even diagonals [52], which are called K matrices. Those topological orders are realized by K-matrix quantum Hall wave functions $\Psi(z_i^I) = \prod_{i<j;I,J}(z_i^I - z_j^J)^{K_{IJ}} e^{-\frac{1}{4}\sum|z_i^I|^2}$ and are described by K-matrix U(1) Chern-Simons theories $\frac{K_{IJ}}{4\pi} a_I\, da_J$. The K matrices with $|\det(K)| = 1$ classify invertible topological orders. Here the E_8 topological order is an invertible topological order described by a K matrix given by the Cartan matrix of E_8, denoted as K_{E_8}. The (1+1)d boundary carries the chiral central charge $c = 8$ [53]. In contrast, the E_8^3 topological order is described by a K matrix $K = K_{E_8} \oplus K_{E_8} \oplus K_{E_8}$ and has its boundary carrying the chiral central charge $c = 24$. It was suggested that the $\mathbb{Z}$ class of the oriented cobordism is generated by the E_8 topological order (see, for example, Freed's work [54] or Freed-Hopkins [16]). See Sec. 7 of [55] for an elaborated interpretation of the related cobordism invariants.

[3]Perturbative local anomalies are detectable via infinitesimal gauge and diffeomorphism transformations continuously deformable from the identity, captured by perturbative Feynman diagrams [57]. Nonperturbative global anomalies are detectable via large gauge and diffeomorphism transformations that cannot be continuously deformed from the identity [58].

are associated with the nondynamical gauge connections of gauge bundles that are fixed, not summed over in the path integral. We will distinguish their gauge transformations: for dynamical fields as *dynamical gauge transformations*, for background fields as *background gauge transformations*, see Appendix J.

In this work, the *anomalous gauge theory* merely means its partition function alone is only *noninvariant* under background gauge transformations (but still *invariant* under dynamical gauge transformations), namely, the anomalous gauge theory with only 't Hooft anomaly of the global symmetry [60] can still be *well defined* on the boundary of one-higher dimensional invertible topological phase. The cancellation of background gauge transformations between the bulk and boundary theories are known as the anomaly inflow [61].

II. BRANCH-INDEPENDENT BOSONIC SYSTEM AND GENERIC LATTICE BOSONIC SYSTEM

In this work, we are going to use cochains on a space-time simplicial complex as bosonic fields. In order to construct the action S in the path integral, using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A local scheme to order the vertices is given by a branch structure [10,62,63]. A branch structure is a choice of orientation of each link in the complex so that there is no oriented loop on any triangle (see Fig. 1). Relative to a base branch structure, all other branch structures can be described by a $\mathbb{Z}$-valued 1-cochain s (see Appendix A 6). After assigning a branch structure to the space-time complex, we can define cup product $\overset{s}{\smile}$ of cochains that depend on the branch structure s. For the base branch structure $s=0$, we abbreviate $\overset{s}{\smile}$ by $\smile$.

We find that for two cocycles, f and g, $f \overset{s}{\smile} g - f \smile g$ is coboundary, that depends on s, f, g. Let us use $d\nu(s,f,g)$ to denote such a coboundary (for details, see Appendix A 6):

$$f \overset{s}{\smile} g + d\nu(s,f,g) = f \smile g. \tag{1}$$

Using derivative and cup product of the cochains, we can construct a local action S. So, in general, the action amplitude e^{-S} may depend on the choices of the branch structures.

A. Branch-independent bosonic (BIB) system

Now we are ready to define the branch-independent bosonic (BIB) system: A branch-independent bosonic system is defined by a path integral on a branch space-time simplicial complex, such that the value of the path integral is independent of the choices of branch structures, even when the space-time has boundaries.

Let us give an example of branch-independent bosonic system. The bosonic system has two fields: a $\mathbb{Z}_2$-valued 1-cochain field $a^{\mathbb{Z}_2}$ and a $\mathbb{Z}_2$-valued 2-cochain field $b^{\mathbb{Z}_2}$, which give rise to the following partition function:

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} (\mathrm{w}_2^s + da^{\mathbb{Z}_2}) \overset{s}{\smile} (\mathrm{w}_3^s + db^{\mathbb{Z}_2})} \times e^{i\pi \int_{\partial M^5} \nu(s, \mathrm{w}_2^s + da^{\mathbb{Z}_2}, \mathrm{w}_3^s + db^{\mathbb{Z}_2})}, \tag{2}$$

where $\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}}$ sums over all the cochain fields. The summation $\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}}$ in the path integral is known as a summation of degrees of freedom. Here, the w_n^s is the nth Stiefel-Whitney cocycle computed from a simplicial complex M^5 with a branch structure s, as described in [64] and in Appendix F. The cocycle w_n^s is a representation of the Stiefel-Whitney class $w_n(TM)$ of the tangent bundle (TM) of the space-time manifold M.[4] We omit the normalization factor here in the partition function (2).

Let w_n be the Stiefel-Whitney cocycle for the base branch structure: $\mathrm{w}_n \equiv \mathrm{w}_n^{s=0}$. In general, w_n^s depends on the branch structure s on M^5. However, $\mathrm{w}_n^s - \mathrm{w}_n$ is a coboundary:

$$\mathrm{w}_n^s = \mathrm{w}_n + d\upsilon_{n-1}(s). \tag{3}$$

We can show that (2) is independent of branch structure s. First, from the definition of ν, with a bulk M^5 but without any boundary ∂M^5, we have

$$\begin{aligned} Z &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} (\mathrm{w}_2^s + da^{\mathbb{Z}_2}) \smile (\mathrm{w}_3^s + db^{\mathbb{Z}_2})} \\ &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} (\mathrm{w}_2^s + da^{\mathbb{Z}_2})(\mathrm{w}_3^s + db^{\mathbb{Z}_2})}. \end{aligned} \tag{4}$$

In this paper, we abbreviate the cup product for the base branch structure $f \smile g$ as fg. From the relation between w_n^s and w_n, we have

$$\begin{aligned} Z &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} [\mathrm{w}_2 + d\upsilon_1(s) + da^{\mathbb{Z}_2}][\mathrm{w}_3 + d\upsilon_2(s) + db^{\mathbb{Z}_2}]} \\ &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} (\mathrm{w}_2 + da^{\mathbb{Z}_2})(\mathrm{w}_3 + db^{\mathbb{Z}_2})}. \end{aligned} \tag{5}$$

We see that the partition function (2) is indeed independent of branch structure.

The partition function for a generic bosonic system on a space-time complex M in a quantum liquid phase has a form

$$Z(M) = e^{-S^{\text{eff}}} Z^{\text{top}}(M) \tag{6}$$

in a thermodynamic limit, where $S^{\text{eff}} = \int_M$ energy density is the nonuniversal volume part, and $Z^{\text{top}}(M)$ is the universal topological partition function (i.e., the topological invariant)

[4]Here we treat w_n as the Stiefel-Whitney cocycle. In contrast, the mathematical definition of Stiefel-Whitney class as cohomology class and characteristic class w_n is given in Appendix B. Because the cohomology class is also a cocycle, so we may also abuse the notation to write Stiefel-Whitney class w_n in terms of cocycle w_n. A cohomology class is an equivalence class that has many representatives; any elements in the equivalence class are representatives. The w_n is Stiefel-Whitney class, also it can be referred to as Stiefel-Whitney cocycle when we choose a representative w_n; the w_n is a Stiefel-Whitney number only when n is the top space-time dimension.
The Stiefel-Whitney product cocycle $\mathrm{w}_2\mathrm{w}_3$ is a representative of the cup product of two Stiefel-Whitney classes (w_2 and w_3). When the $\mathrm{w}_2\mathrm{w}_3$ paired with the fundamental class, as $\int_{M^5} \mathrm{w}_2\mathrm{w}_3$, is called the Stiefel-Whitney number.

that characterizes the topological order. (The topological partition function Z^{top} and its isolation are discussed in much more details in Refs. [35,65,66].) For a branch-independent bosonic system, the topological invariant $Z^{\text{top}}(M)$ does not depend on the branch structures on M, and possibly, nor the framing of the space-time manifold in continuum limit. This leads to the conjecture that the topological invariant $Z^{\text{top}}(M)$ is a cobordism invariant [16,34,35] for invertible topological orders in the branch-independent bosonic systems.

The above example (2) is exactly solvable when M^5 has no boundary since the partition function can be calculated exactly

$$Z = 2^{N_l+N_f}\, e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}, \tag{7}$$

where N_l is the number of links (namely, 1-simplices that can be paired with $a^{\mathbb{Z}_2}$) and N_f is the number of faces (namely, 2-simplices that can be paired with $b^{\mathbb{Z}_2}$) in M^5. Thus, the summation $\sum_{a^{\mathbb{Z}_2},b^{\mathbb{Z}_2}}$ gives a $2^{N_l+N_f}$ factor. After dropping the nonuniversal volume term $e^{-S^{\text{eff}}} = 2^{N_l+N_f}$, we see that the topological partition function is given by

$$Z^{\text{top}}(M^5) = e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}. \tag{8}$$

The nontrivial topological invariant $e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}$ suggests that the bosonic model (2) realizes a nontrivial topological order. The topological order is invertible since the topological invariant is a phase factor for any closed space-time M^5. We will refer such an invertible topological order as the $\mathrm{w}_2\mathrm{w}_3$ topological order.

B. Generic lattice bosonic (GLB) system

We also like to define a concept of generic lattice bosonic (GLB) system, as a bosonic system on a spacetime simplicial complex, whose path integral may or may not depend on the branch structures. Indeed, when we define a generic lattice bosonic system, the local branch structure of space-time complex is given, and the definition may depend on the choices of the local branch structure.

Let us give an example of a generic lattice bosonic system, that happens to have "no degrees of freedom." Here "no degrees of freedom" means that there is only one term in the path integral, and also means that there is only one state $|0\rangle_j$ per single site j. By definition, the ground state of such a system is a *trivial tensor product state* $\otimes_j|0\rangle_j$ (or, simply, a *product state*). Since the generic lattice bosonic system may have an action that depends on the branch structure on the space-time complex M^5, we can choose the action amplitude to be

$$\begin{aligned} e^{-S} &= e^{i\pi\int_{M^5} \mathrm{w}_2^s \overset{s}{\smile} \mathrm{w}_3^s}\, e^{i\pi\int_{\partial M^5} \nu(s,\mathrm{w}_2^s,\mathrm{w}_3^s)} \\ &= e^{i\pi\int_{M^5} \mathrm{w}_2^s \smile \mathrm{w}_3^s} = e^{i\pi\int_{M^5} \mathrm{w}_2^s\mathrm{w}_3^s}. \end{aligned} \tag{9}$$

This is possible since a branch simplicial complex fully determines the cup product and the cocycles w_n^s that represent Stiefel-Whitney cohomology classes [18,67]. The w_n^s depends on the branch structure and different choices of branch structures s can only change w_n^s by a coboundary. Thus, when M^5 has boundaries, the action amplitude $e^{i\pi\int_{M^5} \mathrm{w}_2^s\mathrm{w}_3^s}$ will depend on the branch structure. So, such an action amplitude is only allowed by generic lattice bosonic systems.

Since there are no degrees of freedom, the path integral is only a summation of one term and the topological partition function is given by the action amplitude

$$Z^{\text{top}}(M^5) = e^{i\pi\int_{M^5} \mathrm{w}_2^s\mathrm{w}_3^s}. \tag{10}$$

This implies the following:

In generic lattice bosonic systems (GLB), a cobordism invariant $e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}$ can be realized by a system with no degrees of freedom, and corresponds to a trivial tensor product state with no SPT nor topological order. Here "no degrees of freedom" means that there is only one term in the path integral, and also means that there is only one state $|0\rangle_j$ per single site j. So, the Hilbert space only contains a single tensor product state $\otimes_j|0\rangle_j$ for all sites.

In contrast, note the following:

In branch-independent bosonic systems (BIB), a cobordism invariant $e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}$ cannot be realized by a system with no degrees of freedom, and thus corresponds to a nontrivial invertible topological order.

In other words, we can smoothly deform the topological order characterized by $e^{i\pi\int_{M^5} \mathrm{w}_2\mathrm{w}_3}$ into a trivial product state in the parameter space (or landscape) of generic lattice bosonic systems, but such a smooth deformation path does not exist in the parameter space (or landscape) of branch-independent bosonic systems. The smooth deformation is defined as the continuous and differentiable tuning of any coupling strength of Lagrangian or Hamiltonian terms in the parameter space, that do not cause any phase transitions.

To construct the above-mentioned smooth deformation in the parameter space of generic lattice bosonic systems, let us consider the following generic lattice bosonic system:

$$Z = \sum_{a^{\mathbb{Z}_2},b^{\mathbb{Z}_2}} e^{i\pi\int_{M^5}(\mathrm{w}_2+da^{\mathbb{Z}_2})(\mathrm{w}_3+db^{\mathbb{Z}_2})}\, e^{-U\int_{M^5}|a^{\mathbb{Z}_2}|^2+|b^{\mathbb{Z}_2}|^2}, \tag{11}$$

where

$$\int_{M^5}|a^{\mathbb{Z}_2}|^2 \equiv \sum_{(ij)}\left|a_{ij}^{\mathbb{Z}_2}\right|^2, \quad \int_{M^5}|b^{\mathbb{Z}_2}|^2 \equiv \sum_{(ijk)}\left|b_{ijk}^{\mathbb{Z}_2}\right|^2. \tag{12}$$

Changing U leads to the smooth deformation. When $U = 0$, the path integral (11) is (2). In the limit $U \to \infty$, the path integral (11) becomes (10). The path integral (11) is still exactly solvable when M^5 has no boundaries

$$\begin{aligned} Z &= \sum_{a^{\mathbb{Z}_2},b^{\mathbb{Z}_2}} e^{i\pi\int_{M^5}(\mathrm{w}_2+da^{\mathbb{Z}_2})(\mathrm{w}_3+db^{\mathbb{Z}_2})}\, e^{-U\int_{M^5}|a^{\mathbb{Z}_2}|^2+|b^{\mathbb{Z}_2}|^2} \\ &= (1+e^{-U})^{N_l+N_f}\, e^{i\pi\int_{M^5}\mathrm{w}_2\mathrm{w}_3}. \end{aligned} \tag{13}$$

The system is gapped, namely, with a short-range correlation for any values of U. So the deformation from $U = 0$ to $U = +\infty$ is a smooth deformation that does *not* cause any phase transition. This is why a $\mathrm{w}_2\mathrm{w}_3$ topological order and a product state belong to the same phase for generic lattice bosonic systems. Crucially, here our discussion applies to the Stiefel-Whitney class *only* because the Stiefel-Whitney class has a local expression in terms of a branch structure (see Appendix F). Other topological invariants (such as fermionic eta invariants) may not have local expressions, and may not be written in this form (13), thus cannot have a smooth deformation to a trivial state. Namely, other topological invariants may not be trivialized by a branch structure.

From the above discussion, it is clear that invertible topological orders have different classifications for branch-independent bosonic systems and for generic lattice bosonic systems. However, the two kinds of invertible topological orders are still related. Let TP denote a classification of "topological phases" following the notation of Freed-Hopkins [16]. Let $\text{TP}_d(\text{BIB})$ be the Abelian group that classifies the topological phases invertible topological orders for branch-independent bosonic (BIB) systems. Let $\text{TP}_d(\text{GLB})$ be the Abelian group that classifies invertible topological orders for generic lattice bosonic (GLB) systems. Since invertible topological orders for branch-independent bosonic systems can also be viewed as invertible topological orders for generic lattice bosonic systems, we have a group homomorphism

$$\text{TP}_d(\text{BIB}) \xrightarrow{p_d} \text{TP}_d(\text{GLB}). \tag{14}$$

Since invertible topological order characterized by Stiefel-Whitney classes all become trivial for generic lattice bosonic systems, the map p_d sends all the Stiefel-Whitney class to zero in the topological invariant. The map p_d is not injective. The map p_d may also not be surjective. So it may only tell us a subset of invertible topological orders for generic lattice bosonic systems. In the rest of this paper, we will mainly concentrate on branch-independent bosonic systems and its w_2w_3 invertible topological order.

C. Lattice bosonic systems with time-reversal symmetry

The above discussion can be easily generalized to bosonic systems with time-reversal symmetry, defined via path integrals on a space-time lattice with a real action amplitude e^{-S}. In order words, we restrict ourselves to consider only the real action amplitudes e^{-S}, as a way to implement time-reversal symmetry,

The simplest bosonic invertible topological order with time-reversal symmetry (or time-reversal SPT order) appears in 2d and is characterized by the topological invariant $(-)^{\int_{M^2} \text{w}_1^2}$ on a closed space-time M^2. A branch-independent bosonic model that realizes the SPT order is given by

$$Z = \sum_{g^{\mathbb{Z}_2}} (-)^{\int_{M^2} (\text{w}_1 + dg^{\mathbb{Z}_2})^2}, \tag{15}$$

where $\sum_{g^{\mathbb{Z}_2}}$ sums over all $\mathbb{Z}_2$-valued 0-cochains $g^{\mathbb{Z}_2}$ and the space-time M^2 can have boundaries. The branch independence is ensured by the invariance of the action amplitude $(-)^{\int_{M^2} (\text{w}_1 + dg^{\mathbb{Z}_2})^2}$ under the following transformation:

$$\text{w}_1 \to \text{w}_1 + dv_0, \quad g^{\mathbb{Z}_2} \to g^{\mathbb{Z}_2} + v_0, \tag{16}$$

even when M^2 has boundaries.

We also have a generic lattice bosonic model given by

$$Z = \sum_{g^{\mathbb{Z}_2}} (-)^{\int_{M^2} (\text{w}_1 + dg^{\mathbb{Z}_2})^2} e^{-U \int_{M^2} |g^{\mathbb{Z}_2}|^2}. \tag{17}$$

The model has the time-reversal symmetry since the action amplitude is real. The model is exactly solvable when M^2 has no boundaries, and has the following partition function:

$$Z = (1 + e^{-U})^{N_v} (-)^{\int_{M^2} \text{w}_1^2}, \tag{18}$$

where N_v is the number of the vertices in the space-time complex M^2. Since the partition function only depends on the area of space-time M^2, but does not depend on the shape of space-time M^2, the model (17) is gapped (i.e., has short-range correlations) for any values of U. There is no phase transition as we change U.

When $U = 0$, the model (17) becomes (15) and realizes the time-reversal SPT order characterized by topological invariant $(-)^{\int_{M^2} \text{w}_1^2}$. When $U = \infty$, the model (17) becomes a model with no degrees of freedom which must correspond to a trivial tensor product state. We see that, for generic lattice bosonic models with time-reversal symmetry, the time-reversal SPT order characterized by topological invariant $(-)^{\int_{M^2} \text{w}_1^2}$ and the trivial tensor product state belong to the same phase. This result suggests that the bosonic invertible topological orders with time-reversal symmetry (i.e., with real action amplitudes) for generic lattice bosonic systems are not classified by unoriented cobordism (i.e., manifolds with orthogonal group O structures).

III. BOUNDARIES OF w_2w_3 INVERTIBLE TOPOLOGICAL ORDER

A. Branch independence and background gauge invariance

In the last section, we studied a 5d branch-independent bosonic model on space-time complex M^5 defined by the path integral (2). That path integral can be simplified by using the base branch structure to define the cup product:

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \left(\text{w}_2^s + da^{\mathbb{Z}_2}\right)(\text{w}_3^s + db^{\mathbb{Z}_2})}. \tag{19}$$

When M^5 is a closed manifold with no boundary, the above path integral gives rise to a topological invariant (also known as a cobordism invariant) $e^{i\pi \int_{M^5} \text{w}_2\text{w}_3}$. In this section, we consider the case when M^5 has a boundary. We shall obtain the possible boundary theories for the w_2w_3 topological order.

One might guess that, when M^5 has a boundary, the partition function is still given by $Z(M^5) = e^{i\pi \int_{M^5} \text{w}_2^s\text{w}_3^s}$. If this was true, we could conclude that the boundary is gapped since the partition function $Z(M^5) = e^{i\pi \int_{M^5} \text{w}_2^s\text{w}_3^s}$ does not depend on the metrics on the boundary $B^4 \equiv \partial M^5$. [A gapless system must have a partition function that depends on the metrics (i.e., the shape and size) of the space-time.]

Such a gapped boundary is possible for generic lattice bosonic systems, but it is impossible for the branch-independent bosonic model (19). This is because the partition function in (19) is independent of the branch structure on M^5 even when M^5 has boundaries, while our guess $Z(M^5) = e^{i\pi \int_{M^5} \text{w}_2^s\text{w}_3^s}$ depends on the branch structure since

$$\text{w}_2^s = \text{w}_2 + dv_1(s), \quad \text{w}_3^s = \text{w}_3 + dv_2(s). \tag{20}$$

Thus, $e^{i\pi \int_{M^5} \text{w}_2^s\text{w}_3^s}$ cannot be the partition function of (19) which must be independent of the branch structure.

Due to (20), we see that we can encode the branch-structure independence, via the invariance under the following transformation:

$$\mathrm{w}_2 \to \mathrm{w}_2 + dv_1, \quad \mathrm{w}_3 \to \mathrm{w}_3 + dv_2. \tag{21}$$

Thus, the branch independence of (19) can be rephrased as the invariance of the path integral

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} (\mathrm{w}_2 + da^{\mathbb{Z}_2})(\mathrm{w}_3 + db^{\mathbb{Z}_2})} \tag{22}$$

under the above transformation (21), even for space-time M^5 with boundaries. We refer to (21) as a "background gauge transformation," which is a change in the parameters in the Lagrangian rather than a change in the dynamical fields in the Lagrangian. See the comparison between "background gauge transformation" and "dynamical gauge transformation" in Appendix J.

In the following, we will use the background gauge invariance under (21) to ensure the independence of branch structures. For the branch-independent bosonic model (22), the boundary must be nontrivial. We may assume the boundary to be described by

$$Z(M^5) = \sum_{\phi} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3 - \int_{\partial M^5} \mathcal{L}_{\text{bndry}}(\phi, \mathrm{w}_2, \mathrm{w}_3)}. \tag{23}$$

The boundary Lagrangian $\mathcal{L}_{\text{bndry}}$ is not invariant under the background gauge transformation (21), which cancels the noninvariance of $\mathrm{w}_2\mathrm{w}_3$ in $e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3}$ when M^5 has boundaries. This cancellation of noninvariances of the bulk and boundary theories is actually the idea of anomaly inflow [61].

B. 4d $\mathbb{Z}_2$ gauge boundary of the $\mathrm{w}_2\mathrm{w}_3$ topological order

In this section, we are going to explore the possibility that the boundary Lagrangian $\mathcal{L}_{\text{bndry}}$ describes a $\mathbb{Z}_2$ gauge theory, more precisely the dynamical Spin structure summed over in the path integral.

1. Effective boundary theory

A 4d $\mathbb{Z}_2$ gauge theory can be described by $\mathbb{Z}_2$-valued 1-cochain $a^{\mathbb{Z}_2}$ and 2-cochain $b^{\mathbb{Z}_2}$ fields (for example, see [65,68,69]) with the following path integral:

$$\begin{aligned} Z &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} \overset{s}{\smile} db^{\mathbb{Z}_2}} e^{i\pi \int_{\partial B^4} \nu(s, a^{\mathbb{Z}_2}, db^{\mathbb{Z}_2})} \\ &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} \smile db^{\mathbb{Z}_2}} = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2}}, \end{aligned} \tag{24}$$

where $\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}}$ is a summation over $\mathbb{Z}_2$-valued 1-cochains $a^{\mathbb{Z}_2}$ and 2-cochains $b^{\mathbb{Z}_2}$ on B^4. But, such a theory is invariant under the background gauge transformation (21), and cannot cancel the noninvariant of $e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3}$.

We can add coupling to w_2 and w_3 to fix this problem and obtain the following boundary theory (with the bulk topological invariant included):

$$\begin{aligned} &Z(M^5, B^4 = \partial M^5) \\ &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2} \text{ on } B^4} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + \mathrm{w}_2 b^{\mathbb{Z}_2}}. \end{aligned} \tag{25}$$

Indeed, such a partition function is independent of the branch structure since it is invariant under the background gauge transformation (21). To see this point, we note that the change in w_2 and w_3 can be absorbed by $a^{\mathbb{Z}_2}$ and $b^{\mathbb{Z}_2}$. In other words, the action amplitude $e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{\partial M^5} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + \mathrm{w}_2 b^{\mathbb{Z}_2}}$ is invariant under the following generalized transformation:

$$\begin{aligned} a^{\mathbb{Z}_2} &\to a^{\mathbb{Z}_2} + v_1, \quad b^{\mathbb{Z}_2} \to b^{\mathbb{Z}_2} + v_2, \\ \mathrm{w}_2 &\to \mathrm{w}_2 + dv_1, \quad \mathrm{w}_3 \to \mathrm{w}_3 + dv_2. \end{aligned} \tag{26}$$

So (25) is a boundary theory of the branch-independent bosonic theory (22).

In fact, we can obtain (25) directly from (22) by assuming M^5 to have boundaries and not adding anything on the boundaries:

$$\begin{aligned} Z &= \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2} \text{ on } M^5} e^{i\pi \int_{M^5} (\mathrm{w}_2 + da^{\mathbb{Z}_2})(\mathrm{w}_3 + db^{\mathbb{Z}_2})} \\ &= 2^{N_l^b + N_f^b} \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2} \text{ on } B^4} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + \mathrm{w}_2 b^{\mathbb{Z}_2}}, \end{aligned} \tag{27}$$

where N_2^b and N_f^b are the number of links and faces in M^5 that are not on the boundary ∂M^5.

Both (24) and (25) describe some kinds of $\mathbb{Z}_2$ gauge theories. However, the two $\mathbb{Z}_2$ gauge theories are very different. Equation (24) is anomaly free and can be realized by a branch-independent bosonic model in 4d. In fact, (24) itself is a branch-independent bosonic model in 4d that realizes the anomaly-free $\mathbb{Z}_2$ gauge theory. On the other hand, (25) has an invertible gravitational anomaly. It can only be realized as a boundary of 5d invertible topological order. In our example, (22) is a branch-independent bosonic model that realizes the 5d invertible topological order, and (25) is a boundary theory of of the 5d model (22).

Due to different gravitational anomalies, the two $\mathbb{Z}_2$ gauge theories (24) and (25) have different properties. In the $\mathbb{Z}_2$ gauge theory (24), the $\mathbb{Z}_2$ gauge charge is a bosonic particle and the $\mathbb{Z}_2$ gauge flux line behaves like a bosonic string. On the other hand, in the $\mathbb{Z}_2$ gauge theory (25), the $\mathbb{Z}_2$ gauge charge is a fermionic particle and the $\mathbb{Z}_2$ gauge flux line has certain fermionic nature.

To see the above result, we include the world line of $\mathbb{Z}_2$ gauge charge and world sheet of $\mathbb{Z}_2$ gauge flux into the boundary theory (25):

$$\begin{aligned} &\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + \mathrm{w}_2 b^{\mathbb{Z}_2}} \\ &\quad \times e^{i\pi \int_{B^4} l_3 a^{\mathbb{Z}_2} + s_2 b^{\mathbb{Z}_2}}, \end{aligned} \tag{28}$$

where l_3 and s_2 are $\mathbb{Z}_2$-valued 3- and 2-cocycles which correspond to the Poincaré dual of the world line and the world sheet. But the above action with the world lines and world sheets is not invariant under the transformation (26), and thus is not a correct boundary theory for branch-independent bosonic systems. We may fix this problem by considering the following modified boundary theory:

$$\begin{aligned} &\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + b^{\mathbb{Z}_2}\mathrm{w}_2} \\ &\quad \times e^{i\pi \int_{B^4} l_3 a^{\mathbb{Z}_2} + s_2 b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} l_3 \mathrm{w}_2 + s_2 \mathrm{w}_3}. \end{aligned} \tag{29}$$

In the above, we have assumed that l_3 and s_2 on the boundary B^4 can be extended to the bulk M^5 as cocycles. The invariance of the above path integral under transformation (26) can be checked directly.

But the above expression also has a problem: It depends on how we extend l_3 and s_2 on the boundary B^4 to the bulk M^5. To fix this problem, we propose the path integral

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \mathrm{w}_2\mathrm{w}_3} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2}\mathrm{w}_3 + \mathrm{w}_2 b^{\mathbb{Z}_2}} \times e^{i\pi \int_{B^4} l_3 a^{\mathbb{Z}_2} + s_2 b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2 + \mathrm{Sq}^2\beta_2 s_2 + s_2\mathrm{w}_3}, \quad (30)$$

that is fully invariant under the transformations in (26). Here Sq is the Steenrod square (A20) and β_2 is the generalized Bockstein homomorphism (A9) that acts on cocycles (see Appendix A for details).

Let us first show that the term $e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2 + \mathrm{Sq}^2\beta_2 s_2 + s_2\mathrm{w}_3}$ only depends on the fields on the boundary $B^4 = \partial M^5$, so that the above path integral is well defined. To do so, we note that, according to Wu relation for a $\mathbb{Z}_2$-valued cocycle x_{d-n} in the top d dimension on the complex M^d,

$$\mathrm{Sq}^n x_{d-n} \overset{2,d}{=} u_n x_{d-n}, \quad (31)$$

where u_n is Wu class:

$$\begin{aligned} &u_0 \overset{2,d}{=} 1, \quad u_1 \overset{2,d}{=} \mathrm{w}_1, \quad u_2 \overset{2,d}{=} \mathrm{w}_1^2 + \mathrm{w}_2, \\ &u_3 \overset{2,d}{=} \mathrm{w}_1\mathrm{w}_2, \quad u_4 \overset{2,d}{=} \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4, \\ &u_5 \overset{2,d}{=} \mathrm{w}_1^3\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_4, \\ &u_6 \overset{2,d}{=} \mathrm{w}_1^2\mathrm{w}_2^2 + \mathrm{w}_1^3\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_4 + \mathrm{w}_2\mathrm{w}_4, \\ &u_7 \overset{2,d}{=} \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_3^2 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_4, \\ &u_8 \overset{2,d}{=} \mathrm{w}_1^8 + \mathrm{w}_2^4 + \mathrm{w}_1^2\mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_4 + \mathrm{w}_1\mathrm{w}_3\mathrm{w}_4 + \mathrm{w}_4^2 \\ &\quad + \mathrm{w}_1^3\mathrm{w}_5 + \mathrm{w}_3\mathrm{w}_5 + \mathrm{w}_1^2\mathrm{w}_6 + \mathrm{w}_2\mathrm{w}_6 + \mathrm{w}_1\mathrm{w}_7 + \mathrm{w}_8. \end{aligned} \quad (32)$$

From (31) and (32), we can show that $\mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2$ is a coboundary on oriented M^5 with $\mathrm{w}_1 \overset{2}{=} 0$. Thus, $e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2}$ only depends on l_3 on the boundary $B^4 = \partial M^5$.

The term $e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2}$ makes l_3 on the boundary to be a fermion world line via a higher-dimensional bosonization [17,65,70–72].[5] (For details, see Appendix C.) In other words, the anomalous $\mathbb{Z}_2$ gauge theory on the boundary has a special property that the $\mathbb{Z}_2$ gauge charge is a fermion.

There is another way to understand why the $\mathbb{Z}_2$ gauge charge is a fermion. The $\mathrm{w}_2\mathrm{w}_3$ topological order, as a bosonic state, can live on any five-dimensional orientable manifold with a SO(5) connection for the tangent bundle. One way to obtain a gapped boundary of $\mathrm{w}_2\mathrm{w}_3$ topological order is to trivialize w_2 for the SO(5) connection on the boundary. Such a trivialization can be viewed as a group extension $\mathrm{Spin}(5) = \mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(5)$ via

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(5) \to \mathrm{SO}(5) \to 1. \quad (33)$$

This is consistent with the fact that the Spin manifold is necessary and sufficient condition for its second Stiefel-Whitney class $\mathrm{w}_2 = 0$ for its tangent bundle. Trivializing w_2 on the boundary can be thought as promoting the SO(5) connection in the bulk to a Spin(5) connection on the boundary, where Spin(5) is a $\mathbb{Z}_2$ central extension of SO(5). This implies that the boundary is described by a twisted $\mathbb{Z}_2$ gauge theory, where the $\mathbb{Z}_2$ connection 1-cochain $a^{\mathbb{Z}_2}$ satisfies

$$da^{\mathbb{Z}_2} \overset{2}{=} \mathrm{w}_2, \quad (34)$$

instead of $da^{\mathbb{Z}_2} \overset{2}{=} 0$. The above relation can be obtained from (25) by integrating out $b^{\mathbb{Z}_2}$ first. In this case, the $\mathbb{Z}_2$ charge couples to the Spin(5) connection on the boundary, instead of $\mathbb{Z}_2 \times \mathrm{SO}(5)$ connection. So the $\mathbb{Z}_2$ charge carries a half-integer spin representation of the space-time Spin group if we interpret the extended normal $\mathbb{Z}_2$ as the fermion parity $\mathbb{Z}_2^f$ in (33) as

$$1 \to \mathbb{Z}_2^f \to \mathrm{Spin}(5) \to \mathrm{SO}(5) \to 1. \quad (35)$$

The odd $\mathbb{Z}_2$ gauge charge in (33) is also the half-integer spin representation of Spin group, which is also odd under the fermion parity $\mathbb{Z}_2^f$ in (35). Then, according to the usual lattice belief in terms of the space-time–spin statistics relation, the half-integer spin representation of this $\mathbb{Z}_2$ gauge charge is also a fermion. The Spin structure, which contains the emergent fermion parity $\mathbb{Z}_2^f$ on the boundary, is also called the emergent dynamical Spin structure [44].

However, a 4d $\mathbb{Z}_2$ gauge theory with a fermionic $\mathbb{Z}_2$ gauge charge may not have gravitational anomaly since such theory can be realized by a 4d bosonic model

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{B^4} b^{\mathbb{Z}_2}(\mathrm{w}_2 + da^{\mathbb{Z}_2})}. \quad (36)$$

The action amplitude is invariant under the following transformation, even when B^4 has boundaries:

$$a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + v_1, \quad b^{\mathbb{Z}_2} \to b^{\mathbb{Z}_2}, \quad \mathrm{w}_2 \to \mathrm{w}_2 + dv_1. \quad (37)$$

After including the world line of $\mathbb{Z}_2$ charge and the world sheet of $\mathbb{Z}_2$ flux, the above becomes

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2} \text{ on } B^4} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + b^{\mathbb{Z}_2}\mathrm{w}_2} \times e^{i\pi \int_{B^4} l_3 a^{\mathbb{Z}_2} + s_2 b^{\mathbb{Z}_2}} e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2}, \quad (38)$$

which is the path-integral description of the anomaly-free $\mathbb{Z}_2$ gauge theory with fermionic $\mathbb{Z}_2$ charge [59].

Therefore, the $\mathbb{Z}_2$ flux line must also have some special properties as a realization of the anomaly in the $\mathbb{Z}_2$ gauge theory (25). Let us first show that $\mathrm{Sq}^2\beta_2 s_2 + s_2\mathrm{w}_3$ is also a coboundary on M^5. Using

$$\mathrm{Sq}^1 x \overset{2}{=} \beta_2 x, \quad \mathrm{Sq}^1(\mathrm{w}_j) \overset{2,d}{=} \mathrm{w}_1\mathrm{w}_j + (j-1)\mathrm{w}_{j+1}, \quad (39)$$

[5]Higher-dimensional bosonization here especially in [71] means to use the purely bosonic commuting fields (i.e., cochains) with only Steenrod algebras (but without using Grassmann variables) to represent the fermionic systems with the fermionic parity symmetry $\mathbb{Z}_2^f$. The fermionic system here may be regarded as a system with emergent fermions living on the boundary of bosonic topological order. See Appendix K for a summary.

we find that

$$\beta_2 w_2 \stackrel{2}{=} \mathrm{Sq}^1 w_2 \stackrel{2,d}{=} w_1 w_2 + w_3. \tag{40}$$

Noticing $w_1 \stackrel{2}{=} 0$, we find that

$$s_2 w_3 \stackrel{2,d}{=} s_2 \beta_2 w_2 \stackrel{2}{=} w_2 \beta_2 s_2 + \beta_2(w_2 s_2) \stackrel{2,d}{=} w_2 \beta_2 s_2. \tag{41}$$

The first equality uses (40), and the second equality uses the chain rule. The third equality uses $\beta_2(w_2 s_2) \stackrel{2}{=} \mathrm{Sq}^1(w_2 s_2) \stackrel{2,d}{=} w_1 w_2 s_2 \stackrel{2}{=} 0$, where we have also used the Wu relation (31) and $w_1 \stackrel{2}{=} 0$. Now, according to (41), we have $w_2\beta_2 s_2 + s_2 w_3 \stackrel{2,d}{=} 0$. Then, using (31) and $w_1 \stackrel{2}{=} 0$, we have $\mathrm{Sq}^2\beta_2 s_2 + s_2 w_3 \stackrel{2,d}{=} 0$, and $\mathrm{Sq}^2\beta_2 s_2 + w_2\beta_2 s_2 \stackrel{2,d}{=} 0$. Thus, $e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + s_2 w_3}$ and $e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + w_2\beta_2 s_2}$ only differ by a surface term on $B^4 = \partial M^5$. Also, $e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + w_2\beta_2 s_2}$ itself is a surface term on $B^4 = \partial M^5$.

The term $e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + w_2\beta_2 s_2}$ makes $\beta_2 s_2$ on the boundary to be the Poincaré dual of a fermion world line via a higher-dimensional bosonization [17,65,71]. We note that if the 2d world sheet for the $\mathbb{Z}_2$ flux loop is *orientable*, then its Poincaré dual s_2 is a $\mathbb{Z}$-valued 2-cocycle, thus, $s_2 \in Z^2(M^5;\mathbb{Z})$. In this case $\beta_2 s_2 = \frac{1}{2} ds_2 = 0$ because the cocycle condition imposes $ds_2 = 0$.

Therefore, a nontrivial $\beta_2 s_2$ comes from an *unorientable* 2d world sheet.[6] On an unorientable world sheet, we have a world line that marks the reversal of the orientation, whose Poincaré dual is $\beta_2 s_2$. So, such an orientation-reversal world line corresponds to a fermion world line.

In other words, the anomalous $\mathbb{Z}_2$ gauge theory (25) on the boundary has a special property that an unorientable world sheet of the $\mathbb{Z}_2$ flux carries a noncontractible fermionic world line. Such a fermionic world line corresponds to the orientation-reversal loop on the unorientable world sheet.

2. Trivialization picture

We have used the trivialization picture to understand the half-integer spin and the Fermi statistic of the $\mathbb{Z}_2$ charge. Can we use the similar trivialization picture to understand the above "fermionic" properties of $\mathbb{Z}_2$ flux lines?

In the above we have associated the Fermi statistic of the $\mathbb{Z}_2$ charge (described by l_3 in space-time B^4) with the topological invariant

$$e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + i\pi\int_{M^5}\mathrm{Sq}^2 l_3 + l_3 w_2}, \quad B^4 = \partial M^5, \tag{45}$$

expressed in one higher dimension M^5. Similarly, we can associate the "Fermi statistic" of the $\mathbb{Z}_2$ flux line (described by s_2 in space-time B^4) with the topological invariant

$$e^{i\pi\int_{B^4} s_2 b^{\mathbb{Z}_2} + i\pi\int_{M^5}\mathrm{Sq}^2\mathrm{Sq}^1 s_2 + s_2 w_3}, \quad B^4 = \partial M^5. \tag{46}$$

To gain a better understanding of the "Fermi" statistics of $\mathbb{Z}_2$ charged particle and $\mathbb{Z}_2$ flux line, we like to express topological invariants in the same dimension rather than one higher dimension.

We start with the path integral (25) describing the $\mathbb{Z}_2$ boundary of the $w_2 w_3$ invertible bosonic topological order. Then, we add the world line for $\mathbb{Z}_2$ charge and worksheet for $\mathbb{Z}_2$ flux line. But here we assume the world line and world sheet are boundaries. Thus, their Poincaré dual's l_3 and s_2 are coboundaries

$$l_3 = d\tilde{l}_2, \quad s_2 = d\tilde{s}_1. \tag{47}$$

Adding the world line for $\mathbb{Z}_2$ charge and worksheet for $\mathbb{Z}_2$ flux line to the boundary space-time B^4, we obtain the following path integral:

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi\int_{M^5} w_2 w_3} e^{i\pi\int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2} w_3 + b^{\mathbb{Z}_2} w_2} \times e^{i\pi\int_{B^4}(d\tilde{l}_2)a^{\mathbb{Z}_2} + (d\tilde{s}_1)b^{\mathbb{Z}_2}}. \tag{48}$$

But, the new term breaks the invariance under (26), which to ensure the branch independence. To fix this, we consider

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi\int_{M^5} w_2 w_3} e^{i\pi\int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2} w_3 + b^{\mathbb{Z}_2} w_2} \times e^{i\pi\int_{B^4}(d\tilde{l}_2)a^{\mathbb{Z}_2} + \tilde{l}_2 w_2 + (d\tilde{s}_1)b^{\mathbb{Z}_2} + \tilde{s}_1 w_3}. \tag{49}$$

However, the fix causes another problem: $\tilde{l}_2$ and $\tilde{l}_2 + \bar{l}_2$ described the same world line if $\bar{l}_2$ is a $\mathbb{Z}_2$-valued cocycle; $\tilde{s}_1$ and $\tilde{s}_1 + \bar{s}_1$ described the same world sheet if $\bar{s}_1$ is a $\mathbb{Z}_2$-valued cocycle. Therefore, the path integral must be invariant under the following transformations

$$\begin{aligned} \tilde{l}_2 &\to \tilde{l}_2 + \bar{l}_2, \quad d\bar{l}_2 \stackrel{2}{=} 0;\\ \tilde{s}_1 &\to \tilde{s}_1 + \bar{s}_1, \quad d\bar{s}_1 \stackrel{2}{=} 0. \end{aligned} \tag{50}$$

To have such an invariance, we consider

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi\int_{M^5} w_2 w_3} e^{i\pi\int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2} + a^{\mathbb{Z}_2} w_3 + b^{\mathbb{Z}_2} w_2} \times e^{i\pi\int_{B^4}(d\tilde{l}_2)a^{\mathbb{Z}_2} + \tilde{l}_2 w_2 + \mathbb{Sq}^2\tilde{l}_2 + (d\tilde{s}_1)b^{\mathbb{Z}_2} + \tilde{s}_1 w_3 + \mathbb{Sq}^2\mathbb{Sq}^1\tilde{s}_1}, \tag{51}$$

where $\mathbb{Sq}^n$ is the generalized Steenrod square introduced in [71], which acts on cochains and is defined in (A21). $\mathbb{Sq}^n$ is equal to Pontryagin square mod 2 when acting on n-cochains. Thus, the generalized Steenrod square $\mathbb{Sq}^n$ is in general *not* the same as the convention Pontryagin square studied in [73,74].

Using (A29), we find

$$\begin{aligned} (\tilde{l}_2 + \bar{l}_2)w_2 + \mathbb{Sq}^2(\tilde{l}_2 + \bar{l}_2) &\stackrel{2,d}{=} (\tilde{l}_2 + \bar{l}_2)w_2 + \mathbb{Sq}^2\tilde{l}_2 + \mathbb{Sq}^2\bar{l}_2\\ &\stackrel{2,d}{=} (\tilde{l}_2 + \bar{l}_2)w_2 + \mathbb{Sq}^2\tilde{l}_2 + w_2\bar{l}_2 \stackrel{2,d}{=} \tilde{l}_2 w_2 + \mathbb{Sq}^2\tilde{l}_2, \end{aligned} \tag{52}$$

[6] Although we require the *oriented* and *orientable* special orthogonal SO symmetry for this 4d boundary and 5d bulk theory, we do allow *unorientable* world sheets on 2d submanifolds. Earlier we wrote for *oriented* world sheet $s_2 \in Z^2(M^5;\mathbb{Z})$ with the topological term

$$e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + w_2\beta_2 s_2} = e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + w_3 s_2}. \tag{42}$$

However, for *unorientable* world sheets $s_2 \in Z^2(M^5;\mathbb{Z}_2)$, we can use Steenrod square Sq^1 to rewrite the above equation as

$$e^{i\pi\int_{M^5}\mathrm{Sq}^2\mathrm{Sq}^1 s_2 + w_2\mathrm{Sq}^1 s_2} = e^{i\pi\int_{M^5}\mathrm{Sq}^2\mathrm{Sq}^1 s_2 + w_3 s_2}. \tag{43}$$

In Appendix H, we prove a generalized Wu relation

$$\mathrm{Sq}^2\mathrm{Sq}^1 x_{d-3} = (w_1^3 + w_3)x_{d-3} \tag{44}$$

on a manifold M^d where w_i is the Stiefel-Whitney class of M^d.

where we have used $\mathbb{Sq}^2\bar{l}_2 \stackrel{2}{=} \mathrm{Sq}^2\bar{l}_2 \stackrel{2,d}{=} (\mathrm{w}_2+\mathrm{w}_1^2)\bar{l}_2$ and $\mathrm{w}_1 \stackrel{2,d}{=} 0$ for B^4. This allows us to show the invariance of the path integral (51) under $\tilde{l}_2 \to \tilde{l}_2 + \bar{l}_2$.

Similarly, using (A29) and (A29), we find

$$\begin{aligned}(\tilde{s}_1+\bar{s}_1)\mathrm{w}_3 + \mathbb{Sq}^2\mathbb{Sq}^1(\tilde{s}_1+\bar{s}_1) &\stackrel{2,d}{=} (\tilde{s}_1+\bar{s}_1)\mathrm{w}_3 + \mathbb{Sq}^2\mathbb{Sq}^1\tilde{s}_1 + \mathbb{Sq}^2\mathbb{Sq}^1\bar{s}_1 \\ &\stackrel{2,d}{=} \tilde{s}_1\mathrm{w}_3 + \bar{s}_1\mathbb{Sq}^1\mathrm{w}_2 + \mathbb{Sq}^2\mathbb{Sq}^1\tilde{s}_1 + \mathrm{w}_2\mathbb{Sq}^1\bar{s}_1, \end{aligned} \quad (53)$$

where we have used $\mathbb{Sq}^1\mathrm{w}_2 \stackrel{2,d}{=} \mathrm{w}_3$ when $\mathrm{w}_1 \stackrel{2,d}{=} 0$. Using (A33) and (A9), we have

$$\begin{aligned}\bar{s}_1\mathbb{Sq}^1\mathrm{w}_2 + \mathrm{w}_2\mathbb{Sq}^1\bar{s}_1 &\stackrel{2}{=} \bar{s}_1\beta_2\mathrm{w}_2 + \mathrm{w}_2\beta_2\bar{s}_1 \\ &\stackrel{2,d}{=} (\beta_2\mathrm{w}_2)\bar{s}_1 + \mathrm{w}_2\beta_2\bar{s}_1 \stackrel{2,d}{=} \beta_2(\mathrm{w}_2\bar{s}_1) \stackrel{d}{=} 0. \end{aligned} \quad (54)$$

Therefore,

$$(\tilde{s}_1+\bar{s}_1)\mathrm{w}_3 + \mathbb{Sq}^2\mathbb{Sq}^1(\tilde{s}_1+\bar{s}_1) \stackrel{2,d}{=} \tilde{s}_1\mathrm{w}_3 + \mathbb{Sq}^2\mathbb{Sq}^1\tilde{s}_1. \quad (55)$$

This allows us to show the invariance of the path integral (51) under $\tilde{s}_1 \to \tilde{s}_1 + \bar{s}_1$. Thus, (51) is a correct boundary theory for $\mathrm{w}_2\mathrm{w}_3$ topological order.

Let us examine the $\tilde{l}_2$ terms in the theory. The term $e^{i\pi\int_{B^4}\mathbb{Sq}^2\tilde{l}_2}$, quadratic in $\tilde{l}_2$, gives the $\mathbb{Z}_2$ charge (described by $l_3 = d\tilde{l}_2$) a Fermi statistics. The accompanying linear term $e^{i\pi\int_{B^4}\mathrm{w}_2\tilde{l}_2}$ gives the $\mathbb{Z}_2$ charge a half-integer spin, which is associated with the statement that fermion is related to the trivialization of w_2 (for details, see Appendix C).

Now let us examine the $\tilde{s}_1$ terms. Due to the very similar structure, we can say that the term $e^{i\pi\int_{B^4}\mathbb{Sq}^2\mathbb{Sq}^1\tilde{s}_1}$ gives the $\mathbb{Z}_2$ flux line (described by $s_2 = d\tilde{s}_1$) an unusual statistics. We may also say that the accompanying linear term $e^{i\pi\int_{B^4}\mathrm{w}_3\tilde{s}_1}$ is associated with the trivialization of w_3. So, the unusual statistics of the $\mathbb{Z}_2$ flux line is associated with the trivialization of w_3, just like the Fermi statistics of a particle is associated with the trivialization of w_2.

One way to characterize the unusual statistics of the $\mathbb{Z}_2$ flux line is to notice that $d(\mathbb{Sq}^1\tilde{s}_1) = \mathbb{Sq}^1 d\tilde{s}_1 = \mathbb{Sq}^1 s_2 \stackrel{2}{=} \beta_2 s_2$ describes the Poincaré dual of the orientation-reversal line of the world sheet of the $\mathbb{Z}_2$ flux line. So the orientation-reversal line of the world sheet behaves like a fermion world line due to the term $\mathbb{Sq}^2(\mathbb{Sq}^1\tilde{s}_1)$.

From the relation $\beta_2\mathrm{w}_2 \stackrel{2,d}{=} \mathrm{w}_3$, we see that the trivialization of w_2 on the boundary also implies a trivialization of w_3. The world sheet s_2 of the $\mathbb{Z}_2$ flux line couples to a 2-cochain $b^{\mathbb{Z}_2}$ via $e^{i\pi\int_{B^4}s_2 b^{\mathbb{Z}_2}}$ [see Eq. (30)]. For our anomalous $\mathbb{Z}_2$ gauge theory, the w_3 is trivialized as a coboundary via the split into a lower-dimensional cochain $b^{\mathbb{Z}_2}$, namely,

$$db^{\mathbb{Z}_2} \stackrel{2}{=} \mathrm{w}_3. \quad (56)$$

Such a relation can be obtained from (25) by integrating out $a^{\mathbb{Z}_2}$ first. But, (56) is not independent from (34) because $\mathrm{w}_3 \stackrel{d,2}{=} \mathrm{Sq}^1\mathrm{w}_2$.

To summarize, (1) the (4+1)d invertible topological order has a boundary described by a dynamical $\mathbb{Z}_2$ gauge theory with gravitational anomaly; (2) in the continuum limit, the gauge charge transforms as $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty) = \mathrm{Spin}(\infty)$ under the $\mathbb{Z}_2$ gauge transformation and space-time rotation; (3) such a $\mathbb{Z}_2$ gauge charge is a fermion in the spin statistics; and (4) the *orientation-reversal* world line on the *unorientable* world sheet of $\mathbb{Z}_2$ flux loop is a fermion world line, but, such a fermion world line does not carry the $\mathbb{Z}_2$ gauge charge.

3. A physical consequence of the fermion-carrying $\mathbb{Z}_2$ flux world sheet

In a usual anomaly-free $\mathbb{Z}_2$ gauge theory, if we proliferate the $\mathbb{Z}_2$ flux world sheets in space-time, we will get a new gapped state, which corresponds to a confined phase $\mathbb{Z}_2$ gauge theory with no topological order. Why can proliferating the $\mathbb{Z}_2$ flux world sheet give rise to a gapped state? This is because the path-integral amplitude for the $\mathbb{Z}_2$ flux world sheets is positive. If the surface tension of the world sheet is zero, the equal-weight superposition of the world sheet leads to a short-range correlated state.

For the anomalous $\mathbb{Z}_2$ gauge theory on the boundary of $\mathrm{w}_2\mathrm{w}_3$ topological order, the path-integral amplitude for the $\mathbb{Z}_2$ flux world sheets is no longer positive. It contains a $\pm$ sign from the braiding of the fermionic world lines carried by the unorientable world sheet. In this case, we are not sure that the proliferation the $\mathbb{Z}_2$ flux worldsheets can give rise to a gapped $\mathbb{Z}_2$ confined state. This result is consistent with our expectation that the anomalous $\mathbb{Z}_2$ gauge theory at the boundary of the $\mathrm{w}_2\mathrm{w}_3$ topological order cannot have a trivial confined phase.

If we only allow orientable world sheets of the $\mathbb{Z}_2$ flux lines, then the path-integral amplitude for those orientable world sheets can be all positive. We can have a phase where the orientable world sheet proliferates. But, the proliferation of orientable world sheet gives rise to a U(1) gauge theory, instead of $\mathbb{Z}_2$ confined phase. Such a $\mathrm{U}(1)^f$ gauge boundary of the $\mathrm{w}_2\mathrm{w}_3$ topological order will be discussed in the next subsection.

C. 4d $\mathrm{U}(1)^f$ gauge boundary of the $\mathrm{w}_2\mathrm{w}_3$ topological order

The anomalous $\mathbb{Z}_2$ gauge theory is only one possible boundary of the $\mathrm{w}_2\mathrm{w}_3$ topological order. In this section, we will discuss another boundary: a U(1) gauge theory with gravitational anomaly. To obtain such an anomalous boundary U(1) gauge theory, we first rewrite the topological invariant $\mathrm{w}_2\mathrm{w}_3 \stackrel{2}{=} \mathrm{w}_2\mathrm{Sq}^1\mathrm{w}_2$ as

$$\begin{aligned}Z^{\mathrm{top}}(M^5) &= e^{i\pi\int_{M^5}\mathrm{w}_2\mathrm{Sq}^1\mathrm{w}_2} \\ &= e^{i\pi\int_{M^5}\mathrm{w}_2\beta_2\mathrm{w}_2} = e^{i\pi\int_{M^5}\mathrm{w}_2\frac{1}{2}d\mathrm{w}_2}, \end{aligned} \quad (57)$$

where we have used $\mathrm{Sq}^1\mathrm{w}_2 \stackrel{2,d}{=} \beta_2\mathrm{w}_2$ [see (A33)] and $\beta_2\mathrm{w}_2 = \frac{1}{2}d\mathrm{w}_2$ [see (A9)]. In (57), both $(\mathrm{w}_2)(\beta_2\mathrm{w}_2)$ and $(\mathrm{w}_2)(\frac{1}{2}d\mathrm{w}_2)$ pair between the $\mathbb{Z}_2$ valued w_2 and the $\mathbb{Z}$ valued $(\beta_2\mathrm{w}_2)$ or $(\frac{1}{2}d\mathrm{w}_2)$, which altogether can be well defined in the $\mathbb{Z}_2$ value.

We find that if M^5 has a Spin^c structure, then $\mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1(c_1 \mod 2) \stackrel{2,d}{=} 0$, thus $\beta_2\mathrm{w}_2 \stackrel{d}{=} 0$ and $Z^{\mathrm{top}}(M^5) = 1$ (see a proof in Appendix D 1's Remark 3). Thus, the $\mathrm{w}_2\mathrm{w}_3$ is *not* a cobordism invariant for the Spin^c structure. In this case, the $\mathrm{SO}(\infty)$ connection on M^5 can be lifted into a $\mathrm{U}(1)^f \rtimes_{\frac{1}{2}\mathrm{w}_2} \mathrm{SO}(\infty)$ connection. The $\mathrm{U}(1)^f$ implies that

$$\mathrm{U}(1)^f \supset \mathbb{Z}_2^f$$

contains the fermion parity as a normal subgroup. Here $\mathrm{U}(1)^f \rtimes_{\frac{1}{2}\mathrm{w}_2} \mathrm{SO}(\infty)$ is the $\mathrm{U}(1)^f$ extension of $\mathrm{SO}(\infty)$ characterized by $\frac{1}{2}\mathrm{w}_2 \in H^2[\mathrm{BSO}(\infty);\mathbb{R}/\mathbb{Z}]$.

Such a group extension to a total group $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{U}(1)^f \equiv \frac{\mathrm{Spin}\times\mathrm{U}(1)^f}{\mathbb{Z}_2^f}$ via the short exact sequence

$$1 \to \mathrm{U}(1)^f \to \mathrm{Spin}^c(5) \to \mathrm{SO}(5) \to 1 \tag{58}$$

implies the $\mathrm{w}_2\mathrm{w}_3$ in SO is trivialized in Spin^c. The Spin^c structure, which contains the emergent $\mathrm{U}(1)^f \supset \mathbb{Z}_2^f$ on the boundary, is also called the emergent dynamical Spin^c structure [44].

The above discussions also imply that the $\mathrm{w}_2\mathrm{w}_3$ topological order has another boundary described by a $\mathrm{U}(1)^f$ gauge theory with gravitational anomaly. To write such a $\mathrm{U}(1)^f$ gauge theory, we start with a 5d branch-independent bosonic model that realizes the $\mathrm{w}_2\mathrm{w}_3$ invertible topological order:

$$Z = \sum_{a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}} e^{i\pi \int_{M^5} (\mathrm{w}_2 + 2 d a^{\mathbb{R}/\mathbb{Z}^f})(\beta_2 \mathrm{w}_2 + d b^{\mathbb{Z}})}. \tag{59}$$

The brunching independence is ensured by the invariance of the above path integral under the following dynamical gauge transformations (α_0) and background gauge transformations (υ_1 and υ_2):

$$\begin{aligned} &a^{\mathbb{R}/\mathbb{Z}^f} \to a^{\mathbb{R}/\mathbb{Z}^f} + d\alpha_0 + \tfrac{1}{2}\upsilon_1, \quad b^{\mathbb{Z}} \to b^{\mathbb{Z}} + \upsilon_2,\\ &\quad \mathrm{w}_2 \to \mathrm{w}_2 - d\upsilon_1 - 2\upsilon_2,\\ &\quad \beta_2\mathrm{w}_2 \to \beta_2\mathrm{w}_2 - \beta_2 d\upsilon_1 - d\upsilon_2 = \beta_2\mathrm{w}_2 - d\upsilon_2. \end{aligned} \tag{60}$$

The $\upsilon_1 \in C^1(M^5;\mathbb{Z})$ and $\upsilon_2 \in C^2(M^5;\mathbb{Z})$ are $\mathbb{Z}$-valued 1- and 2-cochains and $\alpha_0 \in C^0(B^4;\mathbb{R}/\mathbb{Z})$ is a $\mathbb{R}/\mathbb{Z}$-valued function (i.e., 0-cochain). Note that $\beta_2 d\upsilon_1 = 0$ because $\beta_2 = \frac{1}{2}d$, while $\frac{1}{2}\upsilon_1 \in C^1(M^5,\mathbb{R}/\mathbb{Z})$, and $dd = d^2 = 0$ on $C^1(M^5,\mathbb{R}/\mathbb{Z})$. Here, the gauge transformation of $a^{\mathbb{R}/\mathbb{Z}^f}$ is related to that of $\frac{1}{2}a^{\mathbb{Z}_2}$ where the gauge transformation of $a^{\mathbb{Z}_2}$ is in (26).

When M^5 has boundaries and after integrating out $a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}$ in the bulk, we obtain the following boundary theory:

$$\begin{aligned} Z = &\sum_{a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}} e^{i\pi \int_{M^5} \mathrm{w}_2\beta_2\mathrm{w}_2} e^{i2\pi \int_{B^4} a^{\mathbb{R}/\mathbb{Z}^f} db^{\mathbb{Z}} + a^{\mathbb{R}/\mathbb{Z}^f}\beta_2\mathrm{w}_2 + \frac{1}{2}\mathrm{w}_2 b^{\mathbb{Z}}}\\ &\times e^{-\frac{1}{2}g\int_{B^4}|b + da^{\mathbb{R}/\mathbb{Z}^f} + \frac{1}{2}\mathrm{w}_2|^2}, \end{aligned} \tag{61}$$

where $a^{\mathbb{R}/\mathbb{Z}^f} \in C^1(B^4;\mathbb{R}/\mathbb{Z})$ is a $\mathbb{R}/\mathbb{Z}$-valued 1-cochain[7] and $b^{\mathbb{Z}} \in C^2(B^4;\mathbb{Z})$ is a $\mathbb{Z}$-valued 2-cochain.[8] We add a gauge-invariant term $e^{-\frac{1}{2}g\int_{B^4}|b + da^{\mathbb{R}/\mathbb{Z}^f} + \frac{1}{2}\mathrm{w}_2|^2}$, which will produce the Maxwell term after we integrating out the $b^{\mathbb{Z}}$ field. We find that a small g leads to a semiclassical $\mathrm{U}(1)^f$ gauge theory. Thus, the above describes a $\mathrm{U}(1)^f$ gauge theory,[9] with a gravitational anomaly of $\mathrm{w}_2\mathrm{w}_3$.

In the presence of the 1d world line of $\mathrm{U}(1)^f$ electric charge (i.e., Wilson line) and the world line of $\mathrm{U}(1)^f$ magnetic

[7]Note that the isomorphism $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$, however, we use $\mathbb{R}/\mathbb{Z}$ to emphasize the group operation is addition as in $\mathbb{R}/\mathbb{Z}$, instead of multiplication in U(1). Also, we denote $\mathbb{R}/\mathbb{Z}^f = \mathrm{U}(1)^f \supset \mathbb{Z}_2^f$ to include the fermion parity normal subgroup.

[8]First let us clarify why $b^{\mathbb{Z}} da^{\mathbb{R}/\mathbb{Z}^f} + (\beta_2\mathrm{w}_2)a^{\mathbb{R}/\mathbb{Z}^f} + b^{\mathbb{Z}}(\frac{1}{2}\mathrm{w}_2)$ is well defined in $\mathbb{R}/\mathbb{Z}$ valued, for the cup product between a $\mathbb{Z}$-valued cohomology class (e.g., here $b^{\mathbb{Z}}$, $\beta_2\mathrm{w}_2$, etc.) and a $\mathbb{R}/\mathbb{Z}$-valued cohomology class (e.g., here $da^{\mathbb{R}/\mathbb{Z}^f}$, $a^{\mathbb{R}/\mathbb{Z}^f}$, $\frac{1}{2}\mathrm{w}_2$, etc.). Generally, for $z \in \mathbb{Z}$ and the equivalence class $[x] \in \mathbb{R}/\mathbb{Z}$ where the representative $x \in \mathbb{R}$, we have that the definition $z[x] \equiv [zx] \in \mathbb{R}/\mathbb{Z}$ is well defined since if $[x] = [y] \in \mathbb{R}/\mathbb{Z}$, then $x - y \in \mathbb{Z}$ and $z(x-y) \in \mathbb{Z}$. So $[zx] = [zy] \in \mathbb{R}/\mathbb{Z}$, thus, we prove that $z[x] \in \mathbb{R}/\mathbb{Z}$ is well defined. Thus, we prove that $b^{\mathbb{Z}} da^{\mathbb{R}/\mathbb{Z}^f} + (\beta_2\mathrm{w}_2)a^{\mathbb{R}/\mathbb{Z}^f} + b^{\mathbb{Z}}(\frac{1}{2}\mathrm{w}_2)$ is well defined in $\mathbb{R}/\mathbb{Z}$ valued.

[9]We can motivate better about the 4d action $\int_{B^4} b^{\mathbb{Z}} da^{\mathbb{R}/\mathbb{Z}^f}$. Schematically, without any other source or operator insertions in the path integral, we have the equation of motion $db^{\mathbb{Z}} = 0$. Although the cocycle (closed) is not necessarily coboundary (exact), locally we can write $b^{\mathbb{Z}} = d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f}$ since the $b^{\mathbb{Z}} \in C^2(B^4;\mathbb{Z})$ has the integer-quantized electric flux, then $\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f} \in C^1(B^4;\mathbb{R}/\mathbb{Z}^f)$ is the dual gauge field (the 't Hooft magnetic gauge field). Since the pure Abelian U(1) gauge theory has the action $da^{\mathbb{R}/\mathbb{Z}^f} \wedge \star da^{\mathbb{R}/\mathbb{Z}^f} = \star d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f} \wedge d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f}$ with the U(1) gauge coupling suppressed, we can also regard $b^{\mathbb{Z}}$ as the integer-quantized electric flux $\oiint d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f} = \oiint \star da^{\mathbb{R}/\mathbb{Z}^f} \in \mathbb{Z}$ on a closed 2-cycle, while $\star b^{\mathbb{Z}}$ as the integer-quantized magnetic flux $\oiint \star d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f} = \oiint da^{\mathbb{R}/\mathbb{Z}^f} \in \mathbb{Z}$ on a closed 2-cycle. Thus, in this special case, we may also treat $\int_{B^4} b^{\mathbb{Z}} da^{\mathbb{R}/\mathbb{Z}^f}$ as $\int_{B^4} da^{\mathbb{R}/\mathbb{Z}^f} \wedge \star da^{\mathbb{R}/\mathbb{Z}^f} = \int_{B^4} \star d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f} \wedge d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f}$. By the Maxwell equations, $d da^{\mathbb{R}/\mathbb{Z}^f} = d \star da^{\mathbb{R}/\mathbb{Z}^f} = 0$, so da is a harmonic form (closed and coclosed) and the Hodge dual $\star da^{\mathbb{R}/\mathbb{Z}^f} = d\mathrm{v}^{\mathbb{R}/\mathbb{Z}^f}$ is also a harmonic form. By the Hodge theorem, each cohomology class has a unique harmonic representative. So, Hodge star is an isomorphism from the harmonic representatives of $H^k_{DR}(M)$ to the harmonic representatives of $H^{n-k}_{DR}(M)$, here we can take $M = B^4$. Poincaré duality says that $\langle\alpha,\beta\rangle = \int_M \alpha \wedge \beta$ is a perfect pairing between $H^k_{DR}(M)$ and $H^{n-k}_{DR}(M)$. So $\int_M \alpha \wedge \star\alpha > 0$ for any nonzero α and we can define $\int_M |\alpha|^2$ as $\int_M \alpha \wedge \star\alpha$. The sum of $|\alpha|^2$ on a 2-simplex over the space-time simplex is also the action $\int_M |\alpha|^2$. For a cohomology class α, the integral $\int_M |\alpha|^2$ is defined to be $\int_M |\alpha|^2$ for any cocycle representative α of the cohomology class α. In particular, we can choose the harmonic representative. For a harmonic form α, we have $\int_M |\alpha|^2 = \int_M \alpha \wedge \star\alpha$ since $\alpha \wedge \star\alpha = |\alpha|^2$.

monopole (i.e., 't Hooft line) on the boundary, the boundary theory becomes

$$Z = \sum_{a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}} e^{i\pi \int_{M^5} w_2\beta_2 w_2} e^{i2\pi \int_{B^4} a^{\mathbb{R}/\mathbb{Z}^f} db^{\mathbb{Z}} + a^{\mathbb{R}/\mathbb{Z}^f}\beta_2 w_2 + \frac{1}{2} w_2 b^{\mathbb{Z}}} \times e^{-\frac{1}{2}g\int_{B^4} |b + da^{\mathbb{R}/\mathbb{Z}^f} + \frac{1}{2}w_2|^2} \times e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} b^{\mathbb{Z}}} \times e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3^{e\,\mathbb{Z}} + l_3^{e\,\mathbb{Z}} w_2 + \mathrm{Sq}^2 d\eta_2^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} dw_2}, \tag{62}$$

where $l_3^{e\,\mathbb{Z}}$ is a $\mathbb{Z}$-valued 3-cocycle, the Poincaré dual of the Wilson line $e^{i\oint_{S^1} a^{\mathbb{R}/\mathbb{Z}^f}}$ [the world line of the $U(1)^f$ electric charge], and $\eta_2^{\mathbb{R}/\mathbb{Z}^f}$ is a $\mathbb{R}/\mathbb{Z}$-valued 2-cocycle, i.e., it satisfies

$$d\eta_2^{\mathbb{R}/\mathbb{Z}^f} = l_3^{m\,\mathbb{Z}}. \tag{63}$$

Here, l_3^m is a $\mathbb{Z}$-valued cocycle, which is the Poincaré dual of the 't Hooft line $e^{i\oint_{S^1} v^{\mathbb{R}/\mathbb{Z}^f}}$ [the world line of $U(1)^f$ magnetic monopole] written in terms of the dual gauge field.[10] We note that $\eta_2^{\mathbb{R}/\mathbb{Z}^f}$ [of $\mathbb{R}/\mathbb{Z}$ or U(1) value] in (62) is related to $\frac{1}{2}s_2$ where s_2 (of $\mathbb{Z}_2$ or $\mathbb{Z}$ value) appears in (30).[11]

The theory (62) is invariant under the gauge transformation (60). Also, the term

$$e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3^{e\,\mathbb{Z}} + l_3^{e\,\mathbb{Z}} w_2 + \mathrm{Sq}^2 d\eta_2^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} dw_2}$$

only depends on the fields on the boundary $B^4 = \partial M^5$. Based on the relation (63), this term can be also read schematically as

$$e^{i\pi \int_{M^5} (\mathrm{Sq}^2 + w_2)(l_3^{e\,\mathbb{Z}} + l_3^{m\,\mathbb{Z}})}. \tag{66}$$

This term makes the $U(1)^f$ electric charge (Wilson line $e^{i\oint_{S^1} a^{\mathbb{R}/\mathbb{Z}^f}}$ as Poincaré dual of $l_3^{e\,\mathbb{Z}}$) and the $U(1)^f$ magnetic monopole ('t Hooft line $e^{i\oint_{S^1} v^{\mathbb{R}/\mathbb{Z}^f}}$ as Poincaré dual of $l_3^{m\,\mathbb{Z}}$) to be fermions, via a higher-dimensional bosonization [17,65,71]. Thus, we show that the boundary of w_2w_3 topological order $U(1)^f$ electric charge has the fermionic statistics, the $U(1)^f$ magnetic monopole has the fermionic statistics, and their bound object $U(1)^f$ dyon also has the fermionic statistics. This is known as the all-fermion quantum electrodynamics (QED$_4$).

Before ending this section, we like to write two bosonic theories with no gravitational anomaly. The first one is given by the following path integral:

$$Z = \sum_{a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}} e^{i2\pi \int_{B^4} a^{\mathbb{R}/\mathbb{Z}^f} db^{\mathbb{Z}} - \frac{1}{2}g\int_{B^4} b\star b} \times e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} b^{\mathbb{Z}}} \tag{67}$$

which is invariant under the following gauge transformation:

$$a^{\mathbb{R}/\mathbb{Z}^f} \to a^{\mathbb{R}/\mathbb{Z}^f} + d\alpha_0, \quad b^{\mathbb{Z}} \to b^{\mathbb{Z}}. \tag{68}$$

When g is small, the above bosonic model realizes a U(1) gauge theory at low energies, where both electric charge described by $l_3^{e\,\mathbb{Z}}$ and magnetic charge described by $d\eta_2^{m\,\mathbb{R}/\mathbb{Z}}$ are bosons.

The second bosonic model is given by

$$Z = \sum_{a^{\mathbb{R}/\mathbb{Z}^f}, b^{\mathbb{Z}}} e^{i2\pi \int_{B^4} a^{\mathbb{R}/\mathbb{Z}^f} db^{\mathbb{Z}} + \frac{1}{2}w_2 b^{\mathbb{Z}}} \times e^{-\frac{1}{2}g\int_{B^4} (b + da^{\mathbb{R}/\mathbb{Z}^f} + \frac{1}{2}w_2)\star(b + da^{\mathbb{R}/\mathbb{Z}^f} + \frac{1}{2}w_2)} \times e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} b^{\mathbb{Z}}} e^{i\pi \int_{M^5} \mathrm{Sq}^2 l_3^{e\,\mathbb{Z}} + l_3^{e\,\mathbb{Z}} w_2} \tag{69}$$

which is invariant under the following gauge transformation:

$$a^{\mathbb{R}/\mathbb{Z}^f} \to a^{\mathbb{R}/\mathbb{Z}^f} + d\alpha_0 + \tfrac{1}{2}v_1, \quad b^{\mathbb{Z}} \to b^{\mathbb{Z}}, \quad w_2 \to w_2 - dv_1. \tag{70}$$

When g is small, the above bosonic model realizes a U(1) gauge theory at low energies, where the electric charge described by $l_3^{e\,\mathbb{Z}}$ is a fermion, and the magnetic charge described by $d\eta_2^{m\,\mathbb{R}/\mathbb{Z}}$ is a boson.

IV. 4D BOUNDARY OF SU(2) OR OTHER SPIN(N) INTERNAL SYMMETRIC THEORIES

So far, we have formulated two kinds of 4d boundary theories of 5d w_2w_3: the 4d $\mathbb{Z}_2$ gauge theories (where the local $\mathbb{Z}_2$ gauge field is more precisely the dynamical Spin structure summed over in the path integral) and the 4d U(1) gauge theories [where the local U(1) gauge field is more precisely the dynamical U(1) gauge connection of Spin^c structure summed over in the path integral]. We also have provided these boundary gauge theory constructions as the trivialization

[10]Note that $e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} b^{\mathbb{Z}}}$ in (62) can be schematically rewritten as

$$e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} dv^{\mathbb{R}/\mathbb{Z}^f}} = e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + (d\eta_2^{\mathbb{R}/\mathbb{Z}^f}) v^{\mathbb{R}/\mathbb{Z}^f}} = e^{i2\pi \int_{B^4} l_3^{e\,\mathbb{Z}} a^{\mathbb{R}/\mathbb{Z}^f} + l_3^{m\,\mathbb{Z}} v^{\mathbb{R}/\mathbb{Z}^f}} \tag{64}$$

via $b^{\mathbb{Z}} = dv^{\mathbb{R}/\mathbb{Z}^f} = \star da^{\mathbb{R}/\mathbb{Z}^f}$. Hence, we can identify the Poincaré dual of the Wilson line ($a^{\mathbb{R}/\mathbb{Z}^f}$) as $l_3^{e\,\mathbb{Z}}$, while the Poincaré dual of the 't Hooft line ($v^{\mathbb{R}/\mathbb{Z}^f}$) as $l_3^{m\,\mathbb{Z}}$.

[11]If we map $\eta_2^{\mathbb{R}/\mathbb{Z}^f} \mapsto \frac{1}{2}s_2$, both are $\mathbb{R}/\mathbb{Z}$ valued, then we can map between the expressions in (62) and (30):

$$\mathrm{Sq}^2 d\eta_2^{\mathbb{R}/\mathbb{Z}^f} + \eta_2^{\mathbb{R}/\mathbb{Z}^f} dw_2 \mapsto \mathrm{Sq}^2 d\frac{1}{2}s_2 + \frac{1}{2}s_2 dw_2 = \mathrm{Sq}^2\mathrm{Sq}^1 s_2 + s_2\mathrm{Sq}^1 w_2 = \mathrm{Sq}^2(\beta_2 s_2) + s_2 w_3. \tag{65}$$

TABLE I. For a single space-time Weyl spinor $\mathbf{2}_L$ in 4d, it has a Witten SU(2) anomaly if the space-time Weyl spinor is also the $(\mathbf{4}r+\mathbf{2})$-dimensional representation (Rep) of SU(2) (or isospin $2r+\frac{1}{2}$), for some non-negative integer r. It has a new SU(2) anomaly if the space-time Weyl spinor is also the $(\mathbf{8}r+\mathbf{4})$-dimensional Rep of SU(2) (or isospin $4r+\frac{3}{2}$), for some non-negative integer r. The checkmark ✓ means the fermion theory has the corresponding anomaly. These SU(2) anomalies can be interpreted as either a 't Hooft anomaly of global symmetry [if the SU(2) is global symmetry not gauged] or dynamical gauge anomaly [if the SU(2) is dynamically gauged].

SU(2) isospin	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	mod 4	$2\,r+\frac{1}{2}$	$4\,r+\frac{3}{2}$	mod 4
SU(2) Rep **R** (dim)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	mod 8	**4** r +**2**	**8** r +**4**	mod 8
Witten SU(2) anomaly		✓				✓				✓		
New SU(2) anomaly				✓							✓	

of w_2w_3 of SO structure via its pullback p to the $p^*w_2w_3=0$ in Spin or Spin^c structures [namely, $1\to\mathbb{Z}_2\to \text{spin}\xrightarrow{p}\text{SO}\to 1$ and $1\to \text{U}(1)\to \text{Spin}^c\xrightarrow{p}\text{SO}\to 1$ see further details in Appendix D]. The $\text{Spin}^c=\text{Spin}\times_{\mathbb{Z}_2}\text{U}(1)$ structure constrains that the fermions carry odd U(1) charge while the bosons carry even U(1) charge. In this section, we consider the $\text{Spin}^h=\text{Spin}\times_{\mathbb{Z}_2}\text{SU}(2)$ structure such that the fermions are in even-dimensional representation (e.g., $\mathbf{2},\mathbf{4},\ldots$; or isospin $\frac{1}{2},\frac{3}{2},\ldots$) of SU(2) while the bosons are in odd-dimensional representation (e.g., $\mathbf{1},\mathbf{3},\ldots$; or isospin $0,1,\ldots$) of SU(2). More generally, we can consider the $\text{Spin}\times_{\mathbb{Z}_2}\text{Spin}(n)$ structure. For example, for $n=10$, fermions are in the spinor representation of spin(10) (e.g., $\mathbf{16},\ldots$) while bosons are in other tensor representation of Spin(10) (e.g., $\mathbf{10},\ldots$). We list the cobordism invariants from $\text{TP}_5[\text{Spin}\times_{\mathbb{Z}_2}\text{Spin}(n)]$ in Appendix D. In this section, we summarize and enumerate other 4d boundary theories of 5d w_2w_3, based on the $\text{Spin}\times_{\mathbb{Z}_2}\text{Spin}(n)$ structure construction, in particular, $n=3$ and 10.

(1) Boundary with the SU(2) and $\text{Spin}^h=\text{Spin}\times_{\mathbb{Z}_2}\text{SU}(2)=\text{Spin}\times_{\mathbb{Z}_2}\text{Spin}(3)$ structures:

(a) When the SU(2) is a global symmetry, note the following.

(i) An odd number of the fundamental two-dimensional representation (Rep) of SU(2) of the space-time Weyl spinor ψ cannot be gapped by quadratic mass term while preserving the Lorentz and SU(2) symmetries. This is due to that the only quadratic mass term $\epsilon^{\alpha\beta}\epsilon^{ij}\psi_{\alpha i}\psi_{\beta j}=0$ [where α,β are the Lorentz indices and $i,j\in\{1,2\}$ are SU(2) indices; we take both the singlet $\mathbf{1}$ out of $\mathbf{2}\otimes\mathbf{2}=\mathbf{1}\oplus\mathbf{3}$] vanish due to the Fermi statistics. This suggests a possible anomaly: another hint is that the fermion spectrum under the SU(2) gauge bundle over S^4 with an instanton number 1 background gives an odd number of fermion zero mode. More generally, an odd number of $\mathbf{4}r+\mathbf{2}$ Rep of SU(2) Weyl spinor has the same anomaly known as Witten anomaly as a 't Hooft anomaly of the Spin^h symmetry (see Table I). But, these 4d theories live on the boundary of another 5d cobordism invariant, known as $\tilde{\eta}\text{PD}[c_2(V_{\text{SU}(2)})]$.[12] These 4d theories do not live on the boundary of the 5d w_2w_3.

(ii) An odd number of the four-dimensional representation (Rep) of SU(2) of the space-time Weyl spinor Ψ also cannot be gapped by quadratic mass term while preserving the Lorentz and SU(2) symmetries. The singlet of both Lorentz and SU(2) symmetries requires any quadratic mass term vanishes: $\epsilon^{\alpha\beta}C^{IJ}\Psi_{\alpha I}\Psi_{\beta J}=0$.[13] Reference [44] shows that on a 4d nonspin manifold, the complex projective space $\mathbb{CP}^2$, with an appropriate large diffeomorphism by complex conjugation the $\mathbb{CP}^2$ coordinates $z_j\to\bar{z}_j$ and an appropriate SU(2) large gauge transformation, we can construct a mapping torus 5d Dold manifold $\mathbb{CP}^2\rtimes S^1$ such that the large gauge diffeomorphism is transformed along the fifth dimension. Moreover, together with the SU(2) bundle, the whole theory is compatible with the Spin^h structure. But, the path integral gets a (-1) sign under this large gauge-diffeomorphism transformation. This odd (-1) noninvariance shows the new SU(2) anomaly. More generally, an odd number of $\mathbf{8}r+\mathbf{4}$ Rep of SU(2) Weyl spinor has the same anomaly known as the new SU(2) anomaly as a 't Hooft anomaly of the Spin^h symmetry (see Table I).

(b) When the SU(2) is dynamically gauged, note the following.

(i) The Witten anomaly gives a dynamical gauge anomaly constraint such that an odd number of $\mathbf{4}r+\mathbf{2}$ Rep of SU(2) Weyl spinor coupled to dynamical SU(2) gauge fields are ill defined. It is not physically sensible to study its gauge dynamics.

(ii) The gauge theories with new SU(2) anomalies are still well-defined theories with well-defined gauge dynamics on Spin manifolds because $w_2=0$ means no w_2w_3 anomaly on the spin manifolds. However, their gauge dynamics become ill defined in 4d on non-Spin manifolds.

(7) Let us discuss further about the SU(2) theory with a $\mathbf{4}$ Rep of SU(2) Weyl spinor.

(i) *Explicit symmetry breaking*: If we are allowed to break this SU(2) theory with the new SU(2) anomaly, for example, by choosing the quadratic fermion mass term via the $\epsilon^{\alpha\beta}C'^{IJ}\Psi_{\alpha I}\Psi_{\beta J}$ such that the pairing $C'^{IJ}\Psi_{\alpha I}\Psi_{\beta J}$

[12]The $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator from $\text{TP}_1(\text{Spin})=\mathbb{Z}_2$ or $\Omega_1^{\text{Spin}}=\mathbb{Z}_2$. A 1d manifold generator for the cobordism invariant $\tilde{\eta}$ is a 1d S^1 for fermions with periodic boundary condition, so called the Ramond circle. A 4d manifold generator for the $c_2(V_{\text{SU}(2)})$ is the nontrivial SU(2) bundle over the S^4, such that the instanton number is 1. The PD is Poincaré dual.

[13]Here $I,J\in\{1,2,3,4\}$ are SU(2) indices which correspond to isospin-$\frac{3}{2}$ indices $\{\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\}$. We take the singlet $\mathbf{1}$ out of the tensor product of $\mathbf{4}$ of SU(2): $\mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{7}$. Based on the Clebsch-Gordan coefficients $\frac{1}{2}(|\frac{3}{2},\frac{3}{2}\rangle|\frac{3}{2},-\frac{3}{2}\rangle-|\frac{3}{2},\frac{1}{2}\rangle|\frac{3}{2},-\frac{1}{2}\rangle+|\frac{3}{2},-\frac{1}{2}\rangle|\frac{3}{2},\frac{1}{2}\rangle-|\frac{3}{2},-\frac{3}{2}\rangle|\frac{3}{2},\frac{3}{2}\rangle)=|0,0\rangle$, we have $C^{ij}\Psi_{\alpha i}\Psi_{\beta j}=\frac{1}{2}[(\Psi_{\alpha 1}\Psi_{\beta 4}-\Psi_{\alpha 4}\Psi_{\beta 1})-(\Psi_{\alpha 2}\Psi_{\beta 3}-\Psi_{\alpha 3}\Psi_{\beta 2})]$. Since $\epsilon^{\alpha\beta}$ and C^{ij} are both antisymmetric, while all Ψ are Grassmannian variables due to Fermi statistics, thus, $\epsilon^{\alpha\beta}C^{ij}\Psi_{\alpha i}\Psi_{\beta j}=0$.

selects the three-dimensional Rep of SU(2) [which is also the vector Rep of SO(3), and the isospin-1 Rep of SU(2) and SO(3)], then C'^{IJ} is symmetric under $I \leftrightarrow J$, and such a mass term does not vanish under Fermi statistics.[14] This mass term explicitly breaks the SU(2) down to U(1) symmetry.

(ii) *Spontaneous symmetry breaking*: Instead, we can consider the spontaneous symmetry breaking by introducing the Yukawa-Higgs term $\epsilon^{\alpha\beta}(\vec{C'}^{IJ}\Psi_{\alpha I}\Psi_{\beta J})\vec{\Phi}$ such that not merely $(\vec{C'}^{IJ}\Psi_{\alpha I}\Psi_{\beta J})$ is the **3** of SU(2) but also the Higgs scalar $\vec{\Phi}$ is also the **3** of SU(2), while we again select the SU(2) singlet **1** out of their tensor product pairing $\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}$. Then, the Higgs condensation $\langle\vec{\Phi}\rangle \neq 0$ only breaks the SU(2) *spontaneously* down to U(1).

(iii) *SU(2) to all-fermion U(1) gauge theories*: Reference [44] shows that the spherical rotationally symmetric 't Hooft-Polyakov monopole [up to SU(2) gauge transformation] in the presence of **4** Rep of SU(2) Weyl spinor traps fermionic zero modes in the spectrum. Reference [44] shows that the 't Hooft-Polyakov monopole becomes fermion. This means under the breaking from SU(2) to U(1), such that the isospin-$\frac{3}{2}$ Weyl fermion of SU(2) becomes the fermionic electric charge 1 of U(1) and the fermionic 't Hooft-Polyakov monopole of SU(2) becomes the fermionic magnetic monopole of U(1).

Together with the fermionic electric charge of U(1), the dyon of U(1) is also a fermion. Reference [44] suggests that a possible renormalization group (RG) flow such that the high-energy theory is an asymptotic free SU(2) gauge theory while the low-energy theory is the all-fermion U(1) gauge theory. The subtlety is that this 4d SU(2) gauge theory is ill defined on a nonspin manifold and cannot be put on the boundary of 5d w_2w_3. But, once we break SU(2) down to U(1), the 4d U(1) gauge theory can be put on a nonspin manifold on the boundary of 5d w_2w_3.

(iv) *All-fermion U(1) to $\mathbb{Z}_2$ gauge theories*: If we introduce another Higgs also the **3** of SU(2), with a different vacuum expectation value, we can further Higgs down the U(1) down to $\mathbb{Z}_2$, such that the fermionic electric charge 1 of U(1) becomes the fermionic electric charge 1 of $\mathbb{Z}_2$, and the fermionic monopole's 1d 't Hooft line of U(1) becomes the orientation-reversal 1d fermionic world line on an unorientable 2d world sheet of $\mathbb{Z}_2$ gauge theory.

(2) Boundary with the Spin(10) and Spin $\times_{\mathbb{Z}_2}$ Spin(10) structures:

(a) The standard so(10) grand unification [75,76] with Spin(10) internal symmetry group and with Weyl fermions in the **16** of Spin(10) does not have the w_2w_3 anomaly [more precisely, it is the $w_2w_3 = w'_2w'_3$ mixed gauge-gravitational anomaly, which is also a nonperturbative global anomaly, with $w_j = w_j(TM)$ and $w'_j = w_j(V_{SO(n=10)})$]. This is the only 5d cobordism invariant from TP_5[Spin $\times_{\mathbb{Z}_2}$ Spin(10)], thus the only 4d global anomaly for Spin $\times_{\mathbb{Z}_2}$ Spin(10) structure. The absence of $w_2w_3 = w'_2w'_3$ anomaly means that the standard so(10) grand unification is free from all perturbative local and nonperturbative global anomalies within Spin $\times_{\mathbb{Z}_2}$ Spin(10) structure [44,55,77].

(b) However, it is possible to construct a modified so(10) grand unification with Spin(10) internal symmetry group, also with Weyl fermions in the **16** of Spin(10), but with additional discrete torsion class of Wess-Zumino-Witten–type term written on the 4d boundary and 5d bulk coupled system [45–47]. Here we summarize the results in [45–47]:

(i) When Spin(10) internal symmetry group is treated as a global symmetry, this modified so(10) grand unification can live on the boundary of 5d $w_2w_3 = w'_2w'_3$ invertible topological order. The $w_2w_3 = w'_2w'_3$ anomaly is saturated by the discrete torsion class of Wess-Zumino-Witten–type term alone. The discrete torsion class of Wess-Zumino-Witten–type term gives rise to various possible gauge theory realizations of low-energy dynamics in 4d. The various possible gauge theory realizations are the emergent gauge theories [similar to the emergent dynamical Spin structure of the $\mathbb{Z}_2$ gauge theory and emergent dynamical Spin^c structure of the all-fermion U(1) gauge theory that we studied earlier].

(ii) When Spin(10) internal symmetry group is dynamically gauged, the Spin(10) gauge field in the 5d bulk $w_2w_3 = w'_2w'_3$ is also gauged. The 5d bulk is no longer a gapped invertible topological order; the 5d bulk becomes gapless and further long-range entangled. Thus, the Spin(10) gauge fields can live only on the 4d boundary, but also propagate into the 5d bulk.

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APPENDIX A: SPACE-TIME COMPLEX WITH BRANCH STRUCTURE, COCHAINS, HIGHER CUP PRODUCT

1. Space-time complex and branch structure

In this paper, we consider models defined on a space-time lattice. A space-time lattice is a triangulation of the d-dimensional space-time, which is denoted by M^d. We will also call the triangulation M^d as a space-time complex, which

[14]Here we take the **3** out of the tensor product of **4** of SU(2): $\mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{7}$. Based on the Clebsch-Gordan coefficients, we may choose $[\sqrt{\frac{9}{20}}(|\frac{3}{2},\frac{3}{2}\rangle|\frac{3}{2},-\frac{3}{2}\rangle+|\frac{3}{2},-\frac{3}{2}\rangle|\frac{3}{2},\frac{3}{2}\rangle)-\sqrt{\frac{1}{20}}(|\frac{3}{2},\frac{1}{2}\rangle|\frac{3}{2},-\frac{1}{2}\rangle+|\frac{3}{2},-\frac{1}{2}\rangle|\frac{3}{2},\frac{1}{2}\rangle)]=|1,0\rangle$, we have $C'^{IJ}\Psi_{\alpha I}\Psi_{\beta J}=[\sqrt{\frac{9}{20}}(\Psi_{\alpha 1}\Psi_{\beta 4}+\Psi_{\alpha 4}\Psi_{\beta 1})-\sqrt{\frac{1}{20}}(\Psi_{\alpha 2}\Psi_{\beta 3}+\Psi_{\alpha 3}\Psi_{\beta 2})]$. Since C'^{IJ} are symmetric, $\epsilon^{\alpha\beta}C'^{IJ}\Psi_{\alpha I}\Psi_{\beta J}\neq 0$.

is formed by simplices: the vertices, links, triangles, etc. We will use $i, j, \ldots$ to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled by $(i, j), (j, k), \ldots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i, j, k), (j, k, l), \ldots$.

In order to define a generic lattice theory on the space-time complex M^d using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branch structure [10,62,63]. A branch structure is a choice of orientation of each link in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 2).

The branch structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, etc. So the simplex in Fig. 2(a) has the following vertex ordering: 0,1,2,3. We always use ordered vertices to label a simplex. So, the simplex in Figs. 2(a) and 2(b) are labeled as [0,1,2,3].

The branch structure also gives the simplex (and its subsimplices) a canonical orientation. Figure 2 illustrates two 3-simplices with opposite canonical orientations compared with the three-dimensional space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

2. Chain, cochain, cycle, cocycle

Given an Abelian group $(\mathbb{M}, +)$, $\mathbb{M}$ can also be viewed a $\mathbb{Z}$ module (i.e., a vector space with integer coefficient) that also allows scaling by an integer:

$$x + y \in \mathbb{M}, \quad x * y \in \mathbb{M}, \quad mx = \underbrace{x + x + \cdots + x}_{m \text{ terms}} \in M,$$
$$x, y \in \mathbb{M}, \quad m \in \mathbb{Z}. \tag{A1}$$

The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \overset{\text{as set}}{=} \mathbb{M}_1 \times \mathbb{M}_2. \tag{A2}$$

An n-cochain α_n in M^d is a formal combination of n-simplexes in M^d with coefficients in $\mathbb{M}$:

$$\alpha_n = \sum_{[i,j,\ldots,k]} \alpha_{n;i,j,\ldots,k}[i, j, \ldots, k], \quad alpha_{n;i,j,\ldots,k} \in \mathbb{M} \tag{A3}$$

where $\sum_{[i,j,\ldots,k]}$ sums over all simplexes in M^d. The collection of all such n-chains is denoted as $C_n(M^d; \mathbb{M})$. For example, a

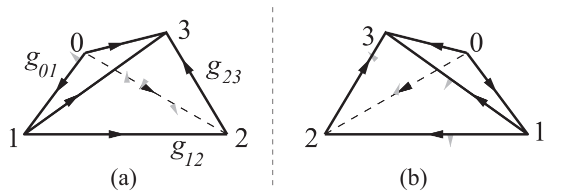

FIG. 2. Two branch simplices with opposite orientations. (a) A branch simplex with positive orientation and (b) a branch simplex with negative orientation.

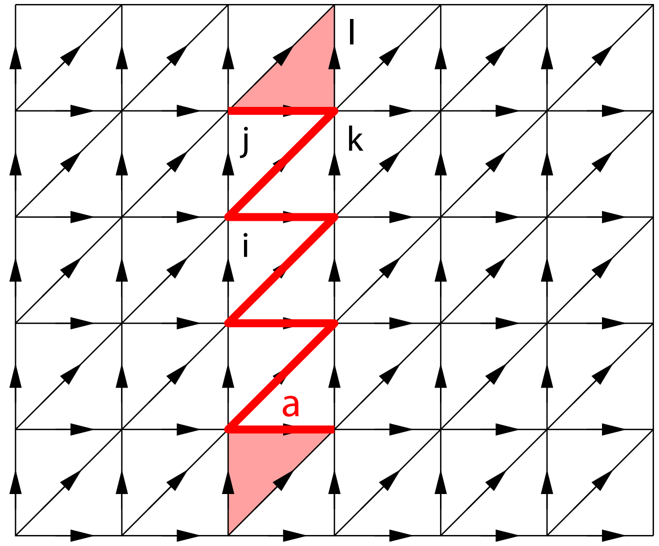

FIG. 3. A 1-cochain a has a value 1 on the red links: $a_{ik} = a_{jk} = 1$ and a value 0 on other links: $a_{ij} = a_{kl} = 0$. da is nonzero on the shaded triangles: $(da)_{jkl} = a_{jk} + a_{kl} - a_{jl}$. For such 1-cochain, we also have $a \smile a = 0$. So when viewed as a $\mathbb{Z}_2$-valued cochain, $\beta_2 a \neq a \smile a$ mod 2.

2-chain can be a 2-simplex: $[i, j, k]$, a sum of two 2-simplices: $[i, j, k] + [j, k, l]$, a more general composition of 2-simplices: $[i, j, k] - 2[j, k, l]$, etc.

An n-cochain f_n in M^d is an assignment of values in $\mathbb{M}$ to each n-simplex M^d. For example, a value $f_{n;i,j,\ldots,k} \in \mathbb{M}$ is assigned to n-simplex $(i, j, \ldots, k)$ (see Fig. 3). We also denote the value $f_{n;i,j,\ldots,k}$ as $f_n([i, j, \ldots, k])$. So *a cochain f_n can be viewed as a bosonic field, $f_n([i, j, \ldots, k])$, on the space-time lattice M^d*. To be more precise f_n is a linear map f_n : n-chain $\to \mathbb{M}$. We can denote the linear map as $f_n(n\text{-chain})$, or

$$f_n(\alpha_n) = \sum_{[i,j,\ldots,k]} \alpha_{n;i,j,\ldots,k} f_n([i, j, \ldots, k]) \in \mathbb{M}\,, \tag{A4}$$

where $\sum_{[i,j,\ldots,k]}$ sums over all simplexes in M^d.

We will use $C^n(M^d; \mathbb{M})$ to denote the set of all n-cochains on M^d. $C^n(M^d; \mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-valued fields (or paths) on M^d. Note that $C^n(M^d; \mathbb{M})$ is an Abelian group under the $+$ operation.

The total space-time lattice M^d correspond to a d-chain. We will use the same M^d to denote it:

$$M^d = \sum_{[i,j,\ldots,k]} s_{i,j,\ldots,k}[i, j, \ldots, k], \tag{A5}$$

where $s_{i,j,\ldots,k} = \pm 1$, describing the relative orientation between M^d and $[i, j, \ldots, k]$. Viewing f_d as a linear map of d-chains, we can define an "integral" over M^d:

$$\int_{M^d} f_d \equiv f_d(M^d) = \sum_{[i,j,\ldots,k]} f_{d;i,j,\ldots,k} s_{i,j,\ldots,k}$$
$$= \sum_{[i,j,\ldots,k]} f_d([i, j, \ldots, k]) s_{i,j,\ldots,k}. \tag{A6}$$

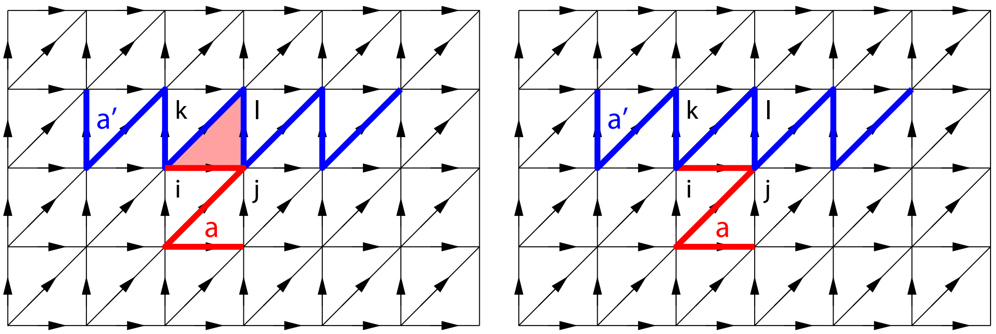

FIG. 4. A 1-cochain a has a value 1 on the red links. Another 1-cochain a' has a value 1 on the blue links. On the left, $a \smile a'$ is nonzero on the shade triangles: $(a \smile a')_{ijl} = a_{ij}a'_{jl} = 1$. On the right, $a' \smile a$ is zero on every triangle. Thus, $a \smile a' + a' \smile a$ is not a coboundary.

3. Derivative operator on cochains

We can define a derivative operator d acting on an n-cochain f_n, which gives us an $(n+1)$-cochain (see Fig. 3):

$$(df_n)([i_0 i_1 i_2 \dots i_{n+1}]) = \sum_{m=0}^{n+1} (-)^m f_n([i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}]), \quad (A7)$$

where $i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}$ is the sequence $i_0 i_1 i_2 \dots i_{n+1}$ with i_m removed, and $i_0, i_1, i_2 \dots i_{n+1}$ are the ordered vertices of the $(n+1)$-simplex $(i_0 i_1 i_2 \dots i_{n+1})$.

A cochain $f_n \in C^n(M^d;\mathbb{M})$ is called a *cocycle* if $df_n = 0$. The set of cocycles is denoted by $Z^n(M^d;\mathbb{M})$. A cochain f_n is called a *coboundary* if there exists a cochain f_{n-1} such that $df_{n-1} = f_n$. The set of coboundaries is denoted by $B^n(M^d;\mathbb{M})$. Both $Z^n(M^d;\mathbb{M})$ and $B^n(M^d;\mathbb{M})$ are Abelian groups as well. Since $d^2 = 0$, a coboundary is always a cocycle: $B^n(M^d;\mathbb{M}) \subset Z^n(M^d;\mathbb{M})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles $[f_n]$ form the so-called cohomology group denoted by

$$H^n(M^d;\mathbb{M}) = Z^n(M^d;\mathbb{M})/B^n(M^d;\mathbb{M}), \quad (A8)$$

$H^n(M^d;\mathbb{M})$, as a group quotient of $Z^n(M^d;\mathbb{M})$ by $B^n(M^d;\mathbb{M})$, is also an Abelian group.

For the $\mathbb{Z}_N$-valued cochain x_n, we lift $\mathbb{Z}_N$ to $\mathbb{Z}$, via $\{0,1,\dots,N-1\} \subset \mathbb{Z}_N$ to $\{0,1,\dots,N-1\} \subset \mathbb{Z}$, and define

$$\beta_N x_n \equiv \frac{1}{N} dx_n. \quad (A9)$$

When x_n is a cocycle, we have $dx_n \overset{N}{=} 0$. In this case, $\beta_N x_n$ is a $\mathbb{Z}$-valued cocycle, and β_N is Bockstein homomorphism.

4. Cup product and higher cup product

From two cochains f_m and h_n, we can construct a third cochain p_{m+n} via the cup product (see Fig. 4):

$$p_{m+n} = f_m \smile h_n,$$
$$p_{m+n}([0 \to m+n]) = f_m([0 \to m])h_n([m \to m+n]), \quad (A10)$$

where $i \to j$ is the consecutive sequence from i to j:

$$i \to j \equiv i, i+1, \dots, j-1, j. \quad (A11)$$

Note that the the order of vertices in a simplex $(0 \to m)$ and the notion of consecutive sequence are determined by the branch structure. Thus, the cup product (and the higher cup product below) on a simplicial complex can be defined only after we assign a branch structure to the simplicial complex. The value of the cup product depends on the branch structure.

The cup product has the following property

$$d(h_n \smile f_m) = (dh_n) \smile f_m + (-)^n h_n \smile (df_m) \quad (A12)$$

for cochains with global or local values. We see that $h_n \smile f_m$ is a cocycle if both f_m and h_n are cocycles. If both f_m and h_n are cocycles, then $f_m \smile h_n$ is a coboundary if one of f_m and h_n is a coboundary. So the cup product is also an operation on cohomology groups $\smile$: $H^m(M^D;\mathbb{M}) \times H^n(M^D;\mathbb{M}) \to H^{m+n}(M^D;\mathbb{M})$. The cup product of two *cocycles* has the following property (see Fig. 4):

$$f_m \smile h_n = (-)^{mn} h_n \smile f_m + \text{coboundary}. \quad (A13)$$

We can also define higher cup product $f_m \underset{k}{\smile} h_n$ which gives rise to a $(m+n-k)$-cochain [78]:

$$(f_m \underset{k}{\smile} h_n)([0,1,\dots,m+n-k]) = \sum_{0 \leqslant i_0 < \dots < i_k \leqslant n+m-k} (-)^p f_m([0 \to i_0, i_1 \to i_2, \dots]) \times h_n([i_0 \to i_1, i_2 \to i_3, \cdots]), \quad (A14)$$

and $f_m \underset{k}{\smile} h_n = 0$ for $k < 0$ or for $k > m$ or n. Here $i \to j$ is the sequence $i, i+1, \dots, j-1, j$, and p is the number of permutations to bring the sequence

$$0 \to i_0, i_1 \to i_2, \dots; i_0+1 \to i_1-1, i_2+1 \to i_3-1, \dots \quad (A15)$$

to the sequence

$$0 \to m+n-k. \quad (A16)$$

For example,

$$(f_m \underset{1}{\smile} h_n)([0\to m+n-1]) = \sum_{i=0}^{m-1}(-)^{(m-i)(n+1)} \times f_m([0\to i, i+n\to m+n-1])h_n([i\to i+n]). \tag{A17}$$

We can see that $\underset{0}{\smile}=\smile$. Unlike cup product at $k=0$, the higher cup product $\underset{k}{\smile}$ of two cocycles may not be a cocycle. For cochains f_m, h_n, we have

$$d(f_m \underset{k}{\smile} h_n) = df_m \underset{k}{\smile} h_n + (-)^m f_m \underset{k}{\smile} dh_n + (-)^{m+n-k} f_m \underset{k-1}{\smile} h_n + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m. \tag{A18}$$

Let f_m and h_n be cocycles and c_l be a cochain, from (A18) we can obtain

$$d(f_m \underset{k}{\smile} h_n) = (-)^{m+n-k} f_m \underset{k-1}{\smile} h_n + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m,$$
$$d(f_m \underset{k}{\smile} f_m) = [(-)^k + (-)^m] f_m \underset{k-1}{\smile} f_m,$$
$$d(c_l \underset{k-1}{\smile} c_l + c_l \underset{k}{\smile} dc_l) = dc_l \underset{k}{\smile} dc_l - [(-)^k - (-)^l](c_l \underset{k-2}{\smile} c_l + c_l \underset{k-1}{\smile} dc_l). \tag{A19}$$

5. Steenrod square and generalized Steenrod square

From (A19), we see that, for $\mathbb{Z}_2$-valued cocycles z_n,

$$\mathrm{Sq}^{n-k}(z_n) \equiv z_n \underset{k}{\smile} z_n \tag{A20}$$

is always a cocycle. Here Sq is called the Steenrod square. More generally, $h_n \underset{k}{\smile} h_n$ is a cocycle if $n+k=$ odd and h_n is a cocycle. Usually, the Steenrod square is defined only for $\mathbb{Z}_2$-valued cocycles or cohomology classes. Here, we like to define a generalized Steenrod square for $\mathbb{M}$-valued cochains c_n:

$$\mathbb{Sq}^{n-k} c_n \equiv c_n \underset{k}{\smile} c_n + c_n \underset{k+1}{\smile} dc_n. \tag{A21}$$

From (A19), we see that

$$\begin{aligned} d\mathbb{Sq}^k c_n &= d(c_n \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} dc_n) \\ &= \mathbb{Sq}^k dc_n + (-)^n \begin{cases} 0, & k = \text{odd} \\ 2\mathbb{Sq}^{k+1} c_n, & k = \text{even}. \end{cases} \end{aligned} \tag{A22}$$

In particular, when c_n is a $\mathbb{Z}_2$-valued cochain, we have

$$d\mathbb{Sq}^k c_n \overset{2}{=} \mathbb{Sq}^k dc_n. \tag{A23}$$

Next, let us consider the action of $\mathbb{Sq}^k$ on the sum of two $\mathbb{M}$-valued cochains c_n and c'_n:

$$\begin{aligned} \mathbb{Sq}^k(c_n + c'_n) &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c'_n + c_n \underset{n-k}{\smile} c'_n + c'_n \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} dc'_n + c'_n \underset{n-k+1}{\smile} dc_n \\ &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c'_n + [1+(-)^k] c_n \underset{n-k}{\smile} c'_n - (-)^{n-k}[-(-)^{n-k} c'_n \underset{n-k}{\smile} c_n + (-)^n c_n \underset{n-k}{\smile} c'_n] + c_n \underset{n-k+1}{\smile} dc'_n + c'_n \underset{n-k+1}{\smile} dc_n. \end{aligned} \tag{A24}$$

Notice that [see (A18)]

$$-(-)^{n-k} c'_n \underset{n-k}{\smile} c_n + (-)^n c_n \underset{n-k}{\smile} c'_n = d(c'_n \underset{n-k+1}{\smile} c_n) - dc'_n \underset{n-k+1}{\smile} c_n - (-)^n c'_n \underset{n-k+1}{\smile} dc_n, \tag{A25}$$

we see that

$$\begin{aligned} \mathbb{Sq}^k(c_n + c'_n) &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c'_n + [1+(-)^k] c_n \underset{n-k}{\smile} c'_n + (-)^{n-k}[dc'_n \underset{n-k+1}{\smile} c_n + (-)^n c'_n \underset{n-k+1}{\smile} dc_n] \\ &\quad - (-)^{n-k} d(c'_n \underset{n-k+1}{\smile} c_n) + c_n \underset{n-k+1}{\smile} dc'_n + c'_n \underset{n-k+1}{\smile} dc_n \\ &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c'_n + [1+(-)^k] c_n \underset{n-k}{\smile} c'_n + [1+(-)^k] c'_n \underset{n-k+1}{\smile} dc_n - (-)^{n-k} d(c'_n \underset{n-k+1}{\smile} c_n) \\ &\quad - [(-)^{n-k+1} dc'_n \underset{n-k+1}{\smile} c_n - c_n \underset{n-k+1}{\smile} dc'_n]. \end{aligned} \tag{A26}$$

Notice that [see (A18)]

$$(-)^{n-k+1} dc'_n \underset{n-k+1}{\smile} c_n - c_n \underset{n-k+1}{\smile} dc'_n = d(dc'_n \underset{n-k+2}{\smile} c_n) + (-)^n dc'_n \underset{n-k+2}{\smile} dc_n, \tag{A27}$$

and we find

$$\begin{aligned}\mathbb{Sq}^k(c_n+c_n') &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + [1+(-)^k] c_n \underset{n-k}{\smile} c_n' + [1+(-)^k] c_n' \underset{n-k+1}{\smile} dc_n - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n) \\ &\quad - d(dc_n' \underset{n-k+2}{\smile} c_n) - (-)^n dc_n' \underset{n-k+2}{\smile} dc_n \\ &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' - (-)^n dc_n' \underset{n-k+2}{\smile} dc_n + [1+(-)^k][c_n \underset{n-k}{\smile} c_n' + c_n' \underset{n-k+1}{\smile} dc_n] \\ &\quad - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n) - d(dc_n' \underset{n-k+2}{\smile} c_n). \end{aligned} \tag{A28}$$

We see that, if one of the c_n and c_n' is a cocycle,

$$\mathbb{Sq}^k(c_n+c_n') \overset{2,d}{=} \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n'. \tag{A29}$$

We also see that

$$\begin{aligned}\mathbb{Sq}^k(c_n+df_{n-1}) &= \mathbb{Sq}^k c_n + \mathbb{Sq}^k df_{n-1} + [1+(-)^k] df_{n-1} \underset{n-k}{\smile} c_n - (-)^{n-k} d(c_n \underset{n-k+1}{\smile} df_{n-1}) - d(dc_n \underset{n-k+2}{\smile} df_{n-1}) \\ &= \mathbb{Sq}^k c_n + [1+(-)^k][df_{n-1} \underset{n-k}{\smile} c_n + (-)^n \mathbb{Sq}^{k+1} f_{n-1}] + d[\mathbb{Sq}^k f_{n-1} - (-)^{n-k} c_n \underset{n-k+1}{\smile} df_{n-1} - dc_n \underset{n-k+2}{\smile} df_{n-1}]. \end{aligned} \tag{A30}$$

Using (A28), we can also obtain the following result if $dc_n =$ even:

$$\mathbb{Sq}^k(c_n+2c_n') \overset{4}{=} \mathbb{Sq}^k c_n + 2d(c_n \underset{n-k+1}{\smile} c_n') + 2dc_n \underset{n-k+1}{\smile} c_n' \overset{4}{=} \mathbb{Sq}^k c_n + 2d(c_n \underset{n-k+1}{\smile} c_n'). \tag{A31}$$

As another application, we note that, for a $\mathbb{Q}$-valued cochain m_d and using (A18),

$$\begin{aligned}\mathbb{Sq}^1(m_d) &= m_d \underset{d-1}{\smile} m_d + m_d \underset{d}{\smile} dm_d \\ &= \tfrac{1}{2}(-)^d[d(m_d \underset{d}{\smile} m_d) - dm_d \underset{d}{\smile} m_d] + \tfrac{1}{2} m_d \underset{d}{\smile} dm_d \\ &= (-)^d \beta_2(m_d \underset{d}{\smile} m_d) - (-)^d \beta_2 m_d \underset{d}{\smile} m_d + m_d \underset{d}{\smile} \beta_2 m_d \\ &= (-)^d \beta_2 \mathbb{Sq}^0 m_d - 2(-)^d \beta_2 m_d \underset{d+1}{\smile} \beta_2 m_d \\ &= (-)^d \beta_2 \mathbb{Sq}^0 m_d - 2(-)^d \mathbb{Sq}^0 \beta_2 m_d. \end{aligned} \tag{A32}$$

This way, we obtain a relation between Steenrod square and Bockstein homomorphism, when m_d is a $\mathbb{Z}_2$-valued cocycle

$$\mathbb{Sq}^1(m_d) \overset{2}{=} \beta_2 m_d, \tag{A33}$$

where we have used $\mathbb{Sq}^0 m_d = m_d$ when the value of the cochain is only 0,1.

For a k-cochain a_k, $k =$ odd, we find that

$$\begin{aligned}\mathbb{Sq}^k a_k &= a_k a_k + a_k \underset{1}{\smile} da_k \\ &= \tfrac{1}{2}[da_k \underset{1}{\smile} a_k - a_k \underset{1}{\smile} da_k - d(a_k \underset{1}{\smile} a_k)] + a_k \underset{1}{\smile} da_k \\ &= \tfrac{1}{2}[da_k \underset{2}{\smile} da_k - d(da_k \underset{2}{\smile} a_k)] - \tfrac{1}{2} d(a_k \underset{1}{\smile} a_k) \\ &= \tfrac{1}{4} d(da_k \underset{3}{\smile} da_k) - \tfrac{1}{2} d(a_k \underset{1}{\smile} a_k + da_k \underset{2}{\smile} a_k). \end{aligned} \tag{A34}$$

Thus, $\mathbb{Sq}^k a_k$ is always a $\mathbb{Q}$-valued coboundary, when k is odd.

6. Branch-structure dependence

Note that the concepts of chain and cochain do not depend on the branch structure. Although the definition of the derivative operator d formally depends on the branch structure, in fact, it is independent of the branch structure as a map between cochains.

However, the cup product and higher cup product do depend on the branch structure, as maps from two cochains to one cochain. To stress this dependence, we write a higher cup product as $\overset{B}{\underset{k}{\smile}}$, where B denotes the branch structure. In this section, we like to study this branch-structure dependence. First, we need to find a quantitative way to describe a change of branch structures.

Let us compare two branch structures B_0 and B. We can use a $\mathbb{Z}_2$-valued 1-cochain s to describe the difference between B_0 and B: $s_{ij} = 1$ if the arrow on the link (ij) is different for B_0 and B, and $s_{ij} = 0$ otherwise. However, not every 1-cochain s corresponds to the difference between two branch structures. We find that s describes the difference between two branch structures if and only if (iff) on every triangle (ijk), $i < j < k$,

s has a form

$$\begin{aligned} & s_{ij}=1, \quad s_{jk}=0, \quad s_{ik}=0; \\ \text{or} \quad & s_{ij}=0, \quad s_{jk}=1, \quad s_{ik}=0; \\ \text{or} \quad & s_{ij}=0, \quad s_{jk}=1, \quad s_{ik}=1; \\ \text{or} \quad & s_{ij}=1, \quad s_{jk}=0, \quad s_{ik}=1; \\ \text{or} \quad & s_{ij}=1, \quad s_{jk}=1, \quad s_{ik}=1, \end{aligned} \tag{A35}$$

where the order $i<j<k$ is determined by the base branch structure B_0. Thus, after we fixed a base branch structure B_0, all other branch structures can be described by s. We may write higher cup product as $\overset{s}{\underset{k}{\smile}}$. The higher cup product for the base branch structure B_0 is written as $\underset{k}{\smile}$, which corresponds to $s=0$.

We believe that, for cocycles f, g, $f \overset{s}{\smile} g - f \smile g$ is a coboundary. Thus,

$$f \overset{s}{\smile} g + d\nu(s,f,g) = f \smile g. \tag{A36}$$

If f and g are 1-cocycles, then we find that

$$\begin{aligned} & f \smile g - f \overset{s}{\smile} g \\ &= d(s \underset{1}{\smile} f \underset{1}{\smile} g) + 2(s \underset{1}{\smile} f) \smile g + 2f \smile (s \underset{1}{\smile} g) \\ &\quad - 2(s \underset{1}{\smile} f) \smile (s \underset{1}{\smile} g) - 2(s \underset{1}{\smile} g) \underset{1}{\smile} (s \smile f) \\ &\quad - 2(s \underset{1}{\smile} f) \underset{1}{\smile} (g \smile s) + 2(s \underset{1}{\smile} g) \underset{1}{\smile} [s \smile (s \underset{1}{\smile} f)] \\ &\quad + 2(s \underset{1}{\smile} f) \underset{1}{\smile} [(s \underset{1}{\smile} g) \smile s] \end{aligned} \tag{A37}$$

holds on a triangle (ijk) for all the five choices of s in (A35). We prove it as follows. The value of the right-hand side of (A37) on a triangle (ijk) with the branch structure B_0 is

$$\begin{aligned} & s_{ij} f_{ij} g_{ij} + s_{jk} f_{jk} g_{jk} - s_{ik} f_{ik} g_{ik} \\ &\quad + 2 s_{ij} f_{ij} g_{jk} + 2 f_{ij} s_{jk} g_{jk} - 2 s_{ij} f_{ij} s_{jk} g_{jk} \\ &\quad + 2 s_{ik} g_{ik} s_{ij} f_{jk} + 2 s_{ik} f_{ik} g_{ij} s_{jk} \\ &\quad - 2(g_{ik} f_{jk} + g_{ij} f_{ik}) s_{ij} s_{jk} s_{ik}, \end{aligned} \tag{A38}$$

where we have used (A17). Note the following:

(i) If $s_{ij}=1$, $s_{jk}=0$, $s_{ik}=0$, then the value of the left-hand side of (A37) on the triangle (ijk) is $f_{ij}g_{jk} - f_{ji}g_{ik}$, while the value of (A38) is $f_{ij}g_{ij} + 2f_{ij}g_{jk} = f_{ij}g_{jk} - f_{ji}g_{ik}$ since $g_{ij}+g_{jk}-g_{ik}=0$ where we have used the cocycle condition for g.

(ii) If $s_{ij}=0$, $s_{jk}=1$, $s_{ik}=0$, then the value of the left-hand side of (A37) on the triangle (ijk) is $f_{ij}g_{jk} - f_{ik}g_{kj}$, while the value of (A38) is $f_{jk}g_{jk} + 2f_{ij}g_{jk} = f_{ij}g_{jk} - f_{ik}g_{kj}$ since $f_{ij}+f_{jk}-f_{ik}=0$ where we have used the cocycle condition for f.

(iii) If $s_{ij}=0$, $s_{jk}=1$, $s_{ik}=1$, then the value of the left-hand side of (A37) on the triangle (ijk) is $f_{ij}g_{jk} - f_{ki}g_{ij}$, while the value of (A38) is

$$\begin{aligned} & f_{jk}g_{jk} - f_{ik}g_{ik} + 2f_{ij}g_{jk} + 2f_{ik}g_{ij} \\ &= f_{ij}g_{jk} + f_{ik}g_{ij} + (f_{ij}+f_{jk})g_{jk} + f_{ik}(g_{ij}-g_{ik}) \\ &= f_{ij}g_{jk} - f_{ki}g_{ij} + f_{ik}g_{jk} - f_{ik}g_{jk} \\ &= f_{ij}g_{jk} - f_{ki}g_{ij} \end{aligned} \tag{A39}$$

since $f_{ij}+f_{jk}-f_{ik}=0$ and $g_{ij}+g_{jk}-g_{ik}=0$ where we have used the cocycle condition for f and g.

(iv) If $s_{ij}=1$, $s_{jk}=0$, $s_{ik}=1$, then the value of the left-hand side of (A37) on the triangle (ijk) is $f_{ij}g_{jk} - f_{jk}g_{ki}$, while the value of (A38) is

$$\begin{aligned} & f_{ij}g_{ij} - f_{ik}g_{ik} + 2f_{ij}g_{jk} + 2f_{jk}g_{ik} \\ &= f_{ij}g_{jk} + f_{jk}g_{ik} + f_{ij}(g_{ij}+g_{jk}) + g_{ik}(f_{jk}-f_{ik}) \\ &= f_{ij}g_{jk} - f_{jk}g_{ki} + f_{ij}g_{ik} - g_{ik}f_{ij} \\ &= f_{ij}g_{jk} - f_{jk}g_{ki} \end{aligned} \tag{A40}$$

since $f_{ij}+f_{jk}-f_{ik}=0$ and $g_{ij}+g_{jk}-g_{ik}=0$ where we have used the cocycle condition for f and g.

(v) If $s_{ij}=1$, $s_{jk}=1$, $s_{ik}=1$, then the value of the left-hand side of (A37) on the triangle (ijk) is $f_{ij}g_{jk} - f_{kj}g_{ji}$, while the value of (A38) is

$$\begin{aligned} & f_{ij}g_{ij} + f_{jk}g_{jk} - f_{ik}g_{ik} + 2f_{ij}g_{jk} \\ &\quad + 2f_{jk}g_{ik} + 2f_{ik}g_{ij} - 2(g_{ik}f_{jk} + g_{ij}f_{ik}) \\ &= f_{ij}g_{jk} + f_{ij}(g_{ij}+g_{jk}) + f_{jk}g_{jk} - f_{ik}g_{ik} \\ &= f_{ij}g_{jk} + (f_{ij}-f_{ik})g_{ik} + f_{jk}g_{jk} \\ &= f_{ij}g_{jk} + f_{jk}(g_{jk}-g_{ik}) \\ &= f_{ij}g_{jk} - f_{jk}g_{ij} \\ &= f_{ij}g_{jk} - f_{kj}g_{ji} \end{aligned} \tag{A41}$$

since $f_{ij}+f_{jk}-f_{ik}=0$ and $g_{ij}+g_{jk}-g_{ik}=0$ where we have used the cocycle condition for f and g.

Thus, we have proved that (A37) holds on a triangle (ijk) for all the five choices of s in (A35). So $f \smile g - f \overset{s}{\smile} g$ is a coboundary modulo 2 if f and g are 1-cocycles.

7. Poincaré dual and pseudoinverse of Poincaré dual

The Poincaré dual of a cochain $f \in C^n(K;\mathbb{Z}_2)$ is defined to be the cap product $[K] \frown f \in C_{m-n}(K;\mathbb{Z}_2)$ where $[K]$ is the fundamental class of K (the sum modulo 2 of all m-simplices of K). The cap product $\sigma \frown f$ for an m-simplex $\sigma = [v_0, \dots, v_n, \dots, v_m]$ and $f \in C^n(K;\mathbb{Z}_2)$ is an $(m-n)$-chain, which is defined as

$$\sigma \frown f := f([v_0,\dots,v_n])[v_n,\dots,v_m]. \tag{A42}$$

So, the Poincaré dual $\mathrm{PD}(f) = [K] \frown f$ is

$$\mathrm{PD}(f) = \sum_{[v_0,\dots,v_n,\dots,v_m]} f([v_0,\dots,v_n])[v_n,\dots,v_m], \tag{A43}$$

where $\sum_{[v_0,\dots,v_n,\dots,v_m]}$ is the sum of all m-simplices of K.

Since the Poincaré dual is an isomorphism between cohomology and homology, it has a pseudoinverse (defined on cycles and cocycles and up to a boundary or coboundary). The pseudoinverse Poincaré dual of a *cycle* $\psi \in C_{m-n}(K;\mathbb{Z}_2)$ is a *cocycle* $\overline{\mathrm{PD}}(\psi) \in C^n(K;\mathbb{Z}_2)$, which is defined via its values on all the n-simplices $[v_0,\dots,v_n]$: first assume that no summand of ψ is of the form $[v,\dots]$ where v is any one of the first n vertices according to the order given by the branch structure. We can first determine the value of $\overline{\mathrm{PD}}(\psi)$ on the "minimal"

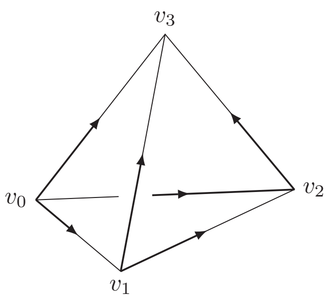

FIG. 5. The surface of a branch tetrahedron.

subset of all the n-simplices $[v_0, \dots, v_n]$:

$$\overline{\mathrm{PD}}(\psi)([v_0, \dots, v_n]) = \psi_{v_n,\dots,v_m} \quad \text{for } \psi = \sum_{[v_n,\dots,v_m]} \psi_{v_n,\dots,v_m}[v_n, \dots, v_m], \tag{A44}$$

where $\psi_{v_n,\dots,v_m} = 0, 1$. Here by minimal we mean the following: since $\overline{\mathrm{PD}}(\psi)$ has to be a cocycle, its value on any boundary is zero, if we have determined the values of $\overline{\mathrm{PD}}(\psi)$ on the n-simplices consist of vertices that are prior according to the order given by the branch structure, then we can determine the values of $\overline{\mathrm{PD}}(\psi)$ on other n-simplices $[v_0, \dots, v_n]$ and $\overline{\mathrm{PD}}(\psi)$ is defined up to a coboundary.

Note that the summand of the Poincaré dual of any cochain $f \in C^n(K; \mathbb{Z}_2)$ can not be of the form $[v, \dots]$ where v is any one of the first n vertices according to the order given by the branch structure. If $\psi \in C_{m-n}(K; \mathbb{Z}_2)$ is a cycle, we can modify ψ by a boundary such that no summand of ψ is of the form $[v, \dots]$ where v is any one of the first n vertices according to the order given by the branch structure. So, the definition of the pseudoinverse Poincaré dual is complete.

For example, let K be the surface of a tetrahedron and $|K| = S^2$. Given the branch structure on K so that the four vertices of K are ordered as v_0, v_1, v_2, and v_3 (see Fig. 5). If $\psi = [v_0, v_1] + [v_0, v_3] + [v_1, v_2] + [v_2, v_3]$, then modify ψ by a boundary $[v_0, v_1] + [v_0, v_3] + [v_1, v_3]$ such that no summand of ψ is of the form $[v_0, \dots]$, we get $\psi' = [v_1, v_2] + [v_1, v_3] + [v_2, v_3]$. By (A44), $\overline{\mathrm{PD}}(\psi')$ takes value 1 on $[v_0, v_1]$, so the sum of the values of $\overline{\mathrm{PD}}(\psi')$ on $[v_0, v_2]$ and $[v_1, v_2]$ is 1. The values of $\overline{\mathrm{PD}}(\psi')$ on other 1-simplices can also be determined and $\overline{\mathrm{PD}}(\psi')$ is determined up to a coboundary. By (A43), $\mathrm{PD}[\overline{\mathrm{PD}}(\psi')] = [v_1, v_2] + [v_1, v_3] + [v_2, v_3]$. So $\mathrm{PD}[\overline{\mathrm{PD}}(\psi')]$ and ψ' are equal.

Since $v_0, \dots, v_n, \dots, v_m$ are ordered according to the branch structure, the Poincaré dual of a cochain and the pseudoinverse Poincaré dual of a chain depends on the branch structure, i.e., the same cochain can have different Poincaré duals and the same chain can have different pseudoinverse Poincaré duals for different branch structures.

APPENDIX B: COMPARISON WITH STANDARD MATHEMATICAL CONVENTIONS AND STIEFEL-WHITNEY CLASS

In the main text of this paper, we use the Stiefel-Whitney *cocycle* w_n and the Steenrod algebra for cochains, for example, summarized in Appendix A. In this Appendix, instead, we make the comparison with the standard mathematical conventions and Stiefel-Whitney *characteristic class* w_n. Some of the math notations and conventions can be found also in the summary of [79].

Let us define mathematically carefully about Stiefel-Whitney class. The Stiefel-Whitney classes of a real vector bundle $\xi : \mathbb{R}^n \to \mathcal{E}(\xi) \to \mathcal{B}(\xi)$ [here $\mathcal{E}(\xi)$ is the total space of ξ and $\mathcal{B}(\xi)$ is the base of ξ] are the cohomology classes $w_j(\xi) \in H^j(\mathcal{B}(\xi); \mathbb{Z}_2)$ $(j = 0, 1, 2, \dots)$ satisfying the following axioms:

A1: $w_0(\xi) = 1 \in H^0(\mathcal{B}(\xi); \mathbb{Z}_2)$, $w_j(\xi) = 0$, for $\forall\ j > n$.

A2: Naturality. For $f : \mathcal{B}(\xi) \to \mathcal{B}(\eta)$ covered by a bundle map (so that $\xi = f^*\eta$), $w_j(\xi) = f^* w_j(\eta)$.

A3: Whitney sum formula. If ξ and η are vector bundles over the same base $\mathcal{B}$, then $w_k(\xi \oplus \eta) = \sum_{j=0}^{k} w_j(\xi) \smile w_{k-j}(\eta)$.

A4: For a canonical line bundle γ_1^1 over $\mathbb{RP}^1$, $w_1(\gamma_1^1) \neq 0$ (i.e., the γ_1^1 is the Möbius strip and the γ_n^1 is the canonical line bundle over $\mathbb{RP}^n$).

The Steenrod square is $\mathrm{Sq}^{n-k} c_n \equiv c_n \underset{k}{\smile} c_n$ for a $\mathbb{Z}_2$-valued n-cohomology class $c_n \in H^n(-, \mathbb{Z}_2)$. The first Steenrod square $\mathrm{Sq}^1 : H^n(-, \mathbb{Z}_2) \mapsto H^{n+1}(-, \mathbb{Z}_2)$ is the Bockstein homomorphism associated with the group extension $\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2$. The $\beta_2 : H^n(-, \mathbb{Z}_2) \mapsto H^{n+1}(-, \mathbb{Z})$ is the Bockstein homomorphism associated with the group extension $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$. Poincaré dual means $\mathrm{PD}(B) = B \frown [M]$ where $\frown$ is the cap product, the PD maps a cohomology class B to a homology class, and $[M]$ is the fundamental class of the manifold. So PD is the cap product between a cohomology class and the fundamental class of the manifold. The cup product $\smile$ is a product between a cochain and another cochain. We shall make the cup product $\smile$ implicit whenever the product is clear written between cochains.

APPENDIX C: EMERGENCE OF HALF-INTEGER SPIN AND FERMI STATISTICS

In the following, we like to explain more carefully why $\mathbb{Z}_2$ gauge charge current l_3 is a fermion current, or why the $\mathbb{Z}_2$ gauge charge is a fermion. We like to show that for a twisted $\mathbb{Z}_2$ gauge theory satisfying $da^{\mathbb{Z}_2} \overset{2}{=} \mathrm{w}_2$, the corresponding $\mathbb{Z}_2$ gauge charge is a fermion. This is because $da^{\mathbb{Z}_2} \overset{2}{=} \mathrm{w}_2$ implies that under a combined $\mathbb{Z}_2$ gauge and $\mathrm{SO}(\infty)$ space-time rotation transformation, the $\mathbb{Z}_2$ gauge charge transforms as $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$. In other words, the $\mathbb{Z}_2$ gauge charge couple to a $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$ connection in space-time. Let us use

$$\left(a_{ij}^{\mathbb{Z}_2}, \gamma_{ij}\right), \quad a_{ij}^{\mathbb{Z}_2} \in \mathbb{Z}_2, \quad \gamma_{ij} \in \mathrm{SO}(\infty) \tag{C1}$$

on link (ij) to describe a $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$. Here we use a pair

$$(a^{\mathbb{Z}_2}, \gamma), \quad a^{\mathbb{Z}_2} \in \mathbb{Z}_2, \quad \gamma \in \mathrm{SO}(\infty) \tag{C2}$$

to label an element in $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$. The group multiplication in $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$ is given by

$$\left(a_1^{\mathbb{Z}_2}, \gamma_1\right) \left(a_2^{\mathbb{Z}_2}, \gamma_2\right) = \left(a_1^{\mathbb{Z}_2} + a_2^{\mathbb{Z}_2} + \mathrm{w}_2(\gamma_1, \gamma_2), \gamma_1\gamma_2\right), \tag{C3}$$

where $\mathrm{w}_2(\gamma_1, \gamma_2) \in H^2(B\mathrm{SO}(\infty); \mathbb{Z}_2)$.

For a nearly flat connection $(a^{\mathbb{Z}_2}_{ij}, \gamma_{ij})$ on a triangle (ijk), we have

$$\left(a^{\mathbb{Z}_2}_{ij}, \gamma_{ij}\right)\left(a^{\mathbb{Z}_2}_{jk}, \gamma_{jk}\right) = \left(a^{\mathbb{Z}_2}_{ij} + a^{\mathbb{Z}_2}_{jk} + \mathrm{w}_2(\gamma_{ij}, \gamma_{jk}), \gamma_{ij}\gamma_{jk}\right) \approx \left(a^{\mathbb{Z}_2}_{ik}, \gamma_{ik}\right). \tag{C4}$$

We see that

$$\mathrm{w}_2(\gamma_{ij}, \gamma_{jk}) \overset{2}{=} a^{\mathbb{Z}_2}_{ik} - a^{\mathbb{Z}_2}_{ij} - a^{\mathbb{Z}_2}_{jk} \tag{C5}$$

which is $\mathrm{w}_2 \overset{2}{=} da^{\mathbb{Z}_2}$. This way we show that the twisted $\mathbb{Z}_2$ gauge theory has a $\mathbb{Z}_2$ gauge charge that transforms as $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty) = \mathrm{Spin}(\infty)$ simply denoted as Spin. In other words, the $\mathbb{Z}_2$ gauge charge carries a half-integer spin, and is a fermion using the spin-statistics theorem.

We may also compute the statistics of the $\mathbb{Z}_2$ gauge charge directly (which is phrased as the high-dimensional bosonization in Refs. [17,65]). Let us assume the world line of the $\mathbb{Z}_2$ gauge charge is a *boundary*. In this case, the Poincaré dual of the world line is a *coboundary*

$$l_3 \overset{2}{=} db^{\mathbb{Z}_2}_c. \tag{C6}$$

Now we can rewrite

$$\begin{aligned} &e^{i\pi\int_{M^5}\mathrm{Sq}^2 l_3 + l_3\mathrm{w}_2} \\ &= e^{i\pi\int_{M^5}\mathrm{Sq}^2 db^{\mathbb{Z}_2}_c + db^{\mathbb{Z}_2}_c\mathrm{w}_2} \\ &= e^{i\pi\int_{M^5}\mathbb{Sq}^2 db^{\mathbb{Z}_2}_c + db^{\mathbb{Z}_2}_c\mathrm{w}_2} = e^{i\pi\int_{M^5} d\mathbb{Sq}^2 b^{\mathbb{Z}_2}_c + db^{\mathbb{Z}_2}_c\mathrm{w}_2} \\ &= e^{i\pi\int_{B^4}\mathbb{Sq}^2 b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_c\mathrm{w}_2}. \end{aligned} \tag{C7}$$

Here $\mathbb{Sq}$ is a generalized Steenrod square that acts on a cochain c_n

$$\mathbb{Sq}^{n-k} c_n \equiv c_n \underset{k}{\smile} c_n + c_n \underset{k+1}{\smile} dc_n, \tag{C8}$$

where $\underset{k}{\smile}$ is the higher cup product. It has properties

$$\begin{aligned} &d\mathbb{Sq}^k c_n \overset{2}{=} \mathbb{Sq}^k dc_n, \\ &\mathbb{Sq}^k(c_n + c'_n) \overset{2}{=} \mathbb{Sq}^k c_n + \mathbb{Sq}^k c'_n + dc'_n \underset{n-k+2}{\smile} dc_n \\ &\quad + d(c'_n \underset{n-k+1}{\smile} c_n) + d(dc'_n \underset{n-k+2}{\smile} c_n). \end{aligned} \tag{C9}$$

Now the term in the path integral that contains l_3 becomes

$$e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + \mathbb{Sq}^2 b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_c \mathrm{w}_2}, \tag{C10}$$

where $b^{\mathbb{Z}_2}_c$ is given by l_3 via $db^{\mathbb{Z}_2}_c \overset{2}{=} l_3$. The above phase factor has a gauge invariance

$$\mathrm{w}_2 \to \mathrm{w}_2 + du_1, \quad a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + u_1. \tag{C11}$$

We note that $b^{\mathbb{Z}_2}_c$ is determined up to a cocycle $b^{\mathbb{Z}_2}_0$. Since

$$\begin{aligned} &e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + \mathbb{Sq}^2(b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_0) + (b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_0)\mathrm{w}_2} \\ &= e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + \mathbb{Sq}^2 b^{\mathbb{Z}_2}_c + \mathrm{Sq}^2 b^{\mathbb{Z}_2}_0 + (b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_0)\mathrm{w}_2} \\ &= e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + \mathbb{Sq}^2 b^{\mathbb{Z}_2}_c + b^{\mathbb{Z}_2}_c \mathrm{w}_2}, \end{aligned} \tag{C12}$$

therefore, the phase factor (C10) does not depend on this $b^{\mathbb{Z}_2}_0$ ambiguity. In the above, we have used $\mathrm{Sq}^2 b^{\mathbb{Z}_2}_0 + b^{\mathbb{Z}_2}_0 \mathrm{w}_2 \overset{2,d}{=} 0$ since $\mathrm{w}_1 \overset{2,d}{=} 0$.

The linear l_3 term in the phase factor

$$e^{i\pi\int_{B^4} l_3 a^{\mathbb{Z}_2} + b^{\mathbb{Z}_2}_c \mathrm{w}_2} \tag{C13}$$

describes the coupling to the background $\mathbb{Z}_2 \rtimes_{\mathrm{w}_2} \mathrm{SO}(\infty)$ connection, which indicates that the $\mathbb{Z}_2$ gauge charge carry half-integer spin. The quadratic l_3 term

$$e^{i\pi\int_{B^4}\mathbb{Sq}^2 b^{\mathbb{Z}_2}_c} = e^{i\pi\int_{B^4}(b^{\mathbb{Z}_2}_c)^2 + l_3 \underset{1}{\smile} b^{\mathbb{Z}_2}_c} \tag{C14}$$

describes the Fermi statistics of the $\mathbb{Z}_2$ gauge charge. The absence of $b^{\mathbb{Z}_2}_0$ cocycle ambiguity requires the linear l_3 term and the quadratic l_3 term to appear together as a combination (C10). Similarly, the WZW-type phase factor $e^{i\pi\int_{M^5}\mathrm{Sq}^2 l_3 + \mathrm{w}_2 l_3}$ will not depend on how we extend from B^4 to M^5 only when the linear l_3 term $\mathrm{w}_2 l_3$ and the quadratic l_3 term $\mathrm{Sq}^2 l_3$ to appear together. This corresponds to the spin statistical theorem.

To summarize, adding a phase factor $e^{i\pi\int_{M^5}\mathrm{Sq}^2 l_3 + \mathrm{w}_2 l_3}$ will make the current l_3 on the boundary $B^4 = \partial M^5$ to become a fermion current where the fermions carry a half-integer spin. Similarly, adding a phase factor $e^{i\pi\int_{M^5}\mathrm{Sq}^2\beta_2 s_2 + \mathrm{w}_2\beta_2 s_2}$ will make $\beta_2 s_2$ on the boundary to become a fermion current as well.

APPENDIX D: COBORDISM GROUP DATA AND ANOMALY CLASSIFICATION

Let us systematically enumerate the pertinent cobordism group $\mathrm{TP}_d(G)$ with some space-time internal G symmetry in Table II.

In particular, we focus on $\mathrm{TP}_5(G)$ which classifies the invertible topological phases of space-time-internal symmetry G in the five-dimensional space-time. We would like to comment why the invertible topological order characterized by $w_2 w_3$ is present or absent in the given G symmetry.

Here we denote the $\mathrm{w}_j = \mathrm{w}_j(TM) = \mathrm{w}_j(a^{\mathrm{SO}})$ as the jth Stiefel-Whitney class of the space-time tangent bundle (TM) of the space-time manifold M, while $\mathrm{w}'_j = \mathrm{w}_j(V_{\mathrm{SO}(n)})$ is the

TABLE II. The cobordism group $\mathrm{TP}_d(G)$ classifies the invertible topological phases or invertible topological field theories (including both the G-SPT state and the invertible topological order with G symmetry) of space-time internal symmetry G in the d-dimensional space-time. See the cobordism computations in [55,77,79].

d	1	2	3	4	5	6	...
$\mathrm{TP}_d(\mathrm{SO})$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	...
$\mathrm{TP}_d(\mathrm{Spin})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
$\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{U}(1)]$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	0	...
$\mathrm{TP}_d[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{U}(1)]$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	0	...
$\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{SU}(2)]$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
$\mathrm{TP}_d[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2)]$	0	0	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	...
$\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{SO}(3)]$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}^2$	0	0	0	...
$\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{Spin}(n \geqslant 7)]$ $\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{Spin}(10)]$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	0	0	...
$\mathrm{TP}_d[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(n \geqslant 7)]$ $\mathrm{TP}_d[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(10)]$	0	0	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
$\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{SO}(n \geqslant 7)]$ $\mathrm{TP}_d[\mathrm{Spin} \times \mathrm{SO}(10)]$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	0	0	...

jth Stiefel-Whitney class of the associated vector bundle of the principal gauge bundle of $\mathrm{SO}(n) = \mathrm{Spin}(n)/\mathbb{Z}_2$.

Below let us explain why the cobordism invariant $\mathrm{w}_2\mathrm{w}_3$ vanishes in some G symmetry [e.g., Spin and $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{U}(1)$], but why the $\mathrm{w}_2\mathrm{w}_3$ persists in other symmetry [e.g., SO and $\mathrm{Spin}^h = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2)$].

1. Space-time and gauge bundle constraint

(1) There is no particular constraint on w_2 or w_3 for the SO structure and SO manifold, thus the cobordism invariant $\mathrm{w}_2\mathrm{w}_3$ derived in the cobordism group of an SO structure still survives.

(2) The constraint for the Spin structure and spin manifold is $\mathrm{w}_2 = 0$, thus, $\mathrm{w}_3 = \mathrm{Sq}^1\mathrm{w}_2 = 0$ and $\mathrm{w}_2\mathrm{w}_3 = 0$. We had also used the symmetry extension to trivialize the $\mathrm{w}_2\mathrm{w}_3$ term in an SO structure via a pullback to 0 in a Spin structure under the group extension $1 \to \mathbb{Z}_2 \to \mathrm{Spin} \to \mathrm{SO} \to 1$.

(3) The constraint for the $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{U}(1)$ structure is $\mathrm{w}_2 = c_1 \mod 2$ also $\mathrm{w}_3 = 0$, so $\mathrm{w}_2\mathrm{w}_3 = 0$. To derive this, we use the $\mathrm{w}_3 = \mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1(c_1 \mod 2) = 0$ since $\mathrm{Sq}^1\rho = 0$ where ρ is the mod 2 map. The $\mathrm{Sq}^1 = \rho\beta_2$ where β_2 is the Bockstein associated with the short exact sequence $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1$, which induces the fiber sequence in their classifying spaces as $\cdots \to \mathrm{B}^2\mathbb{Z} \to \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^3\mathbb{Z} \to \ldots$. The β_2 sends the $\rho c_1 = c_1 \mod 2 \in \mathbb{Z}_2$ in $H^2(M;\mathbb{Z}_2)$ to $\beta_2\rho c_1 = \beta_2(c_1 \mod 2) \in \mathbb{Z}$ in $H^3(M;\mathbb{Z})$. Moreover, the group of homotopy classes of the maps from M to the higher classifying space B^nG is the cohomology group $H^n(M;G)$, which implies the long exact sequence of cohomology groups $\cdots \to H^2(M;\mathbb{Z}) \xrightarrow{\rho} H^2(M;\mathbb{Z}_2) \xrightarrow{\beta_2} H^3(M;\mathbb{Z}) \to \ldots$. This implies that the $\mathrm{im}\rho = \ker\beta_2$, thus, $\beta_2\rho = 0$. We derive the $\mathrm{w}_3 = \mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1(c_1 \mod 2) = \mathrm{Sq}^1(\rho c_1) = \rho\beta_2(\rho c_1) = \rho(\beta_2\rho)c_1 = 0$, thus $\mathrm{w}_2\mathrm{w}_3 = 0$.

The constraint for the $\mathrm{Spin} \times \mathrm{U}(1)$ structure still requires a spin manifold ($\mathrm{w}_2 = 0$) with a tensor product structure of space-time tangent bundle and the principal U(1) gauge bundle, thus, $\mathrm{w}_2\mathrm{w}_3 = 0$.

(4) The constraint for the $\mathrm{Spin}^h \equiv \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ structure includes $\mathrm{w}_2 = \mathrm{w}_2'$, where we denote $\mathrm{w}_j' = \mathrm{w}_j(V_{\mathrm{SO}(3)})$. Thus, $\mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1\mathrm{w}_2' \Rightarrow \mathrm{w}_3 = \mathrm{w}_3'$, so $\mathrm{w}_2\mathrm{w}_3 = \mathrm{w}_2'\mathrm{w}_3'$ can be nonzero.

The constraint for the $\mathrm{Spin} \times \mathrm{SU}(2)$ or $\mathrm{Spin} \times \mathrm{SO}(3)$ structure still requires a Spin manifold ($\mathrm{w}_2 = 0$) with a tensor product structure of space-time tangent bundle and the principal SU(2) or SO(3) gauge bundle, thus, $\mathrm{w}_2\mathrm{w}_3 = 0$.

(5) Now we discuss $\mathrm{Spin} \times \mathrm{Spin}(n \geqslant 7)$, $\mathrm{Spin}\times_{\mathbb{Z}_2}\mathrm{Spin}(n \geqslant 7)$, and $\mathrm{Spin} \times \mathrm{SO}(n \geqslant 7)$, especially when $n = 10$ or 18 suitable for grand unified theories [76,80,81].

The constraint for the $\mathrm{Spin} \times \mathrm{Spin}(n \geqslant 7)$ and $\mathrm{Spin} \times \mathrm{SO}(n \geqslant 7)$ structure still requires a Spin manifold ($\mathrm{w}_2 = 0$) with a tensor product structure of space-time tangent bundle and the principal SU(2) or SO(3) gauge bundle, thus, $\mathrm{w}_2\mathrm{w}_3 = 0$.

The constraint for the $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(n \geqslant 7)$ structure includes $\mathrm{w}_2 = \mathrm{w}_2'$, where we denote $\mathrm{w}_j' = \mathrm{w}_j(V_{\mathrm{SO}(n)})$. Thus, $\mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1\mathrm{w}_2' \Rightarrow \mathrm{w}_3 = \mathrm{w}_3'$, so $\mathrm{w}_2\mathrm{w}_3 = \mathrm{w}_2'\mathrm{w}_3'$ can be nonzero.

2. Cobordism invariants

To summarize their 5d cobordism invariants, note the following:

(1) $\mathrm{TP}_5(\mathrm{SO}) = \mathbb{Z}_2$ is generated by the cobordism invariant $\mathrm{w}_2\mathrm{w}_3$. The manifold generator is a non-Spin manifold such as a Wu manifold

$$\mathrm{SU}(3)/\mathrm{SO}(3)$$

or a Dold manifold

$$\mathbb{CP}^2 \rtimes S^1$$

[which identifies the complex conjugation of coordinates in $z \in \mathbb{CP}^2$ with the antipodal inversion of $x \in S^1$, so $(z, x) \sim (\bar{z}, -x)$].

(2) $\mathrm{TP}_5(\mathrm{Spin}) = 0$ trivializes the cobordism invariant $\mathrm{w}_2\mathrm{w}_3$ to none via a pullback from SO to Spin.

(3) $\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{U}(1)] = \mathbb{Z}^2$ classes are generated by two 5d cobordism invariants ac_1^2 and $\mu[\mathrm{PD}(c_1)]$. The 5d ac_1 corresponds to the perturbative local anomaly captured by Feynman diagram of U(1)-U(1)-U(1) fields acting on the three vertices of the triangle diagram. The 5d $\mu[\mathrm{PD}(c_1)]$ corresponds to the perturbative local anomaly captured by Feynman diagram of U(1)-gravity-gravity fields acting on the three vertices of the triangle diagram.

Here the a is the U(1) 1-form gauge connection. Here the first Chern class $c_1 = c_1(V_{\mathrm{U}(1)})$ is written as the associated vector bundle of U(1). The $\mu[\mathrm{PD}(c_1)]$ is the 3d Rokhlin invariant of $\mathrm{PD}(c_1)$, where $\mathrm{PD}(c_1)$ is the submanifold of a Spin 5-manifold which represents the Poincaré dual of c_1. In general, the Poincaré dual means $\mathrm{PD}(B) = B \frown [M]$ where $\frown$ is the cap product, PD maps a cohomology class B to a homology class, and $[M]$ is the fundamental class of the manifold.

The $\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{U}(1)] = \mathbb{Z}^2$ are also descended from the two 6d topological invariants of the bordism group $\Omega_6^{\mathrm{Spin}}(\mathrm{BU}(1))$: c_1^3 and $\frac{1}{8}[\sigma(\mathrm{PD}(c_1)) - \mathrm{F}\cdot\mathrm{F})$ from the free part of the bordism group $\Omega_6^{\mathrm{Spin}}(\mathrm{BU}(1))$. The $\mathrm{PD}(c_1)$ is the submanifold of a Spin 6-manifold which represents the Poincaré dual of $c_1 = c_1(V_{\mathrm{U}(1)})$. The $\sigma[\mathrm{PD}(c_1)]$ is the signature of the 4-manifold $\mathrm{PD}(c_1)$. The F is a 2d characteristic surface of the 4-manifold $\mathrm{PD}(c_1)$, where F represents $\mathrm{PD}(B)$ where $B \in H^2(\mathrm{PD}(c_1);\mathbb{Z})$. The $\mathrm{F}\cdot\mathrm{F}$ is the intersection form of the 4-manifold $\mathrm{PD}(c_1)$. The intersection form $\mathrm{F}\cdot\mathrm{F} = \langle B \smile B, [\mathrm{PD}(c_1)]\rangle$ is computed via the pairing between a cohomology class with a homology class, where $[\mathrm{PD}(c_1)]$ is the fundamental class of $\mathrm{PD}(c_1)$. By Rokhlin's theorem, $\sigma[\mathrm{PD}(c_1)] - \mathrm{F}\cdot\mathrm{F}$ is a multiple of 8 and $\frac{1}{8}[\sigma(\mathrm{PD}(c_1)] - \mathrm{F}\cdot\mathrm{F} = \mathrm{Arf}(\mathrm{PD}(c_1), \mathrm{F}) \mod 2$. The $\mathrm{Arf}(\mathrm{PD}(c_1), \mathrm{F})$ is the Arf invariant of a quadratic form $\tilde{q} : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$, it is $\mathbb{Z}_2$ valued, the left-hand side is $\mathbb{Z}$ valued and equals to the right-hand side modulo 2. The $\mathrm{F}\cdot h = h\cdot h \mod 2$ is true for all $h \in H_2(\mathrm{PD}(c_1);\mathbb{Z})$.

(4) $\mathrm{TP}_5[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{U}(1)] = \mathrm{TP}_5(\mathrm{Spin}^c) = \mathbb{Z}^2$ classes are generated by two 5d cobordism invariants $\frac{1}{2}ac_1^2$ and $\frac{1}{48}c_1\mathrm{CS}_3(TM)$. The 5d $\frac{1}{2}ac_1^2$ corresponds to the perturbative local anomaly captured by Feynman diagram of U(1)-U(1)-U(1) fields acting on the three vertices of the triangle diagram. The 5d $\frac{1}{48}c_1\mathrm{CS}_3(TM)$ corresponds to the perturbative local

anomaly captured by Feynman diagram of U(1)-gravity-gravity fields acting on the three vertices of the triangle diagram.

The $\mathrm{TP}_5(\mathrm{Spin}^c) = \mathbb{Z}^2$ are also descended from the two 6d topological invariants of the bordism group $\Omega_6^{\mathrm{Spin}^c}$: $\frac{1}{2}c_1^3$ and $\frac{1}{16}\sigma[\mathrm{PD}(c_1)]$ from the free part of the bordism group $\Omega_6^{\mathrm{Spin}^c}$. The $\mathrm{PD}(c_1)$ is a Spin submanifold of the Spin^c 6-manifold which represents the Poincaré dual of c_1.

(5) $\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{SU}(2)] = \mathbb{Z}_2$ class is generated by a 5d cobordism invariant $\tilde{\eta}\mathrm{PD}(c_2(V_{\mathrm{SU}(2)}))$, where the $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator from $\mathrm{TP}_1(\mathrm{Spin}) = \mathbb{Z}_2$ or $\Omega_1^{\mathrm{Spin}} = \mathbb{Z}_2$. A 1d manifold generator for the cobordism invariant $\tilde{\eta}$ is a 1d S^1 for fermions with periodic boundary condition, so called the Ramond circle. A 4d manifold generator for the $c_2(V_{\mathrm{SU}(2)})$ is the nontrivial SU(2) bundle over the S^4, such that the instanton number is 1. So the 5d manifold generator for the cobordism invariant $\tilde{\eta}\mathrm{PD}(c_2(V_{\mathrm{SU}(2)}))$ is the

$$S^1 \times S^4$$

with the fermion periodic boundary condition on S^1 and the SU(2) bundle over S^4 with an instanton number 1. The 4d boundary for a 5d $\tilde{\eta}\mathrm{PD}(c_2(V_{\mathrm{SU}(2)}))$ captures the Witten SU(2) anomaly [82].

(6) $\mathrm{TP}_5[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2)] = \mathrm{TP}_5(\mathrm{Spin}^h) = \mathbb{Z}_2^2$ classes are generated by two 5d cobordism invariants. One is the similar cobordism invariant as that of the $\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{SU}(2)] = \mathbb{Z}_2$ whose 4d boundary has the Witten SU(2) anomaly [82]. The other is the $\mathrm{w}_2\mathrm{w}_3 = \mathrm{w}_2'\mathrm{w}_3'$ with the $\mathrm{w}_j' = \mathrm{w}_j(V_{\mathrm{SO}(3)})$.

(7) $\mathrm{TP}_5[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(n)] = \mathbb{Z}_2$ has a $\mathbb{Z}_2$ class generated by $\mathrm{w}_2\mathrm{w}_3 = \mathrm{w}_2'\mathrm{w}_3'$ with the $\mathrm{w}_j' = \mathrm{w}_j(V_{\mathrm{SO}(n)})$.

Anomalies in SU(2) vs SO(3): Cobordism vs homotopy group

We note that the 4d nonperturbative global anomalies of an internal SU(2) symmetric theory on non-Spin manifolds are classified by

$$\mathrm{TP}_5[\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2)] = \mathbb{Z}_2^2,$$

whose generators are the Witten SU(2) anomaly and the new SU(2) anomaly $\mathrm{w}_2\mathrm{w}_3 = \mathrm{w}_2'\mathrm{w}_3'$; while on spin manifolds are classified by

$$\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{SU}(2)] = \mathbb{Z}_2,$$

whose generator is the Witten SU(2) anomaly. The global anomalies of an internal SO(3) symmetric theory on nonspin manifolds are classified by

$$\mathrm{TP}_5[\mathrm{SO} \times \mathrm{SO}(3)] = \mathbb{Z}_2^2,$$

whose generators are the $\mathrm{w}_2\mathrm{w}_3$ gravitational anomaly from the $\mathrm{w}_j(TM)$ of space-time tangent bundle TM and the $\mathrm{w}_2'\mathrm{w}_3'$ gauge anomaly from the $\mathrm{w}_j'(V_{\mathrm{SO}(3)})$ of internal gauge bundle; while on spin manifolds are classified by

$$\mathrm{TP}_5[\mathrm{Spin} \times \mathrm{SO}(3)] = 0.$$

We shall compare the cobordism group classification of global anomalies with the traditional homotopy group analysis [82] of global anomalies. We will find that the homotopy group analysis is insufficient, such that the homotopy group sometimes leads to incomplete or misleading results.

TABLE III. The homotopy group $\pi_d(G)$ sometimes leads to incomplete or misleading results of global anomalies. The lesson is that we should use the dth cobordism group $\mathrm{TP}_d(G)$ to classify perturbative local and nonperturbative global anomalies in the $(d-1)$d space-time, such as the cobordism group data in Table II. See the cobordism computations in [55,79].

d	1	2	3	4	5	6	...
$\pi_d[\mathrm{SU}(2)]$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	...
$\pi_d[\mathrm{SO}(3)]$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	...

For example, in Table III, we learn that the homotopy group $\pi_4[\mathrm{SU}(2)] = \mathbb{Z}_2$ only gives the Witten SU(2) anomaly [82] but misses the new SU(2) anomaly [44]. We also learn that the homotopy group $\pi_4[\mathrm{SO}(3)] = \mathbb{Z}_2$ gives a possible global anomaly but in fact the topological invariant in the homotopy theory does not correspond to any 5d cobordism invariant on Spin manifolds [with Spin × SO(3) structures]. Thus, there is no corresponding Witten SU(2) anomaly in an internal SO(3) symmetric theory.

3. Trivialization via group extension

(1) Trivialization via the pullback p^* in

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(5) \xrightarrow{p} \mathrm{SO}(5) \to 1,$$

following (33) and the symmetry extension method [43], gauging the normal subgroup $\mathbb{Z}_2$ provides a boundary $\mathbb{Z}_2$ gauge theory construction of the 5d bulk $\mathrm{w}_2\mathrm{w}_3$.

(a) The 5d cobordism invariant $\mathrm{w}_2\mathrm{w}_3$ for the 5d space-time with the SO symmetry becomes trivialized to 0 for the space-time with the Spin symmetry because the Spin structure requires $\mathrm{w}_2 = 0$ and $\mathrm{w}_3 = \mathrm{Sq}^1\mathrm{w}_2 = 0$ on the Spin manifold.

(b) The $\mathrm{w}_2\mathrm{w}_3$ of $\mathrm{TP}_5(\mathrm{SO})$ is trivialized as $p^*(\mathrm{w}_2\mathrm{w}_3) = 0$ in $\mathrm{TP}_5(\mathrm{Spin})$.

(c) So there is *no* Spin symmetric theory with some internal global $\mathbb{Z}_2^f$ fermionic parity symmetry in 4d with the boundary anomaly of 5d $\mathrm{w}_2\mathrm{w}_3$.

(d) This means the 4d boundary anomaly of 5d $\mathrm{w}_2\mathrm{w}_3$ of SO is also vanished in Spin. However, dynamically gauging the normal $\mathbb{Z}_2$ subgroup provides a boundary $\mathbb{Z}_2$ gauge theory that preserves the SO symmetry but with the 't Hooft anomaly of $\mathrm{w}_2\mathrm{w}_3$.

(6) Trivialization via the pullback p^* in

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}^c(5) \xrightarrow{p} \mathrm{SO}(5) \to 1,$$

following the symmetry extension method [43], gauging the normal subgroup U(1) provides a boundary U(1) gauge theory construction of the 5d bulk $\mathrm{w}_2\mathrm{w}_3$.

(a) The 5d cobordism invariant $\mathrm{w}_2\mathrm{w}_3$ for the 5d space-time with the SO symmetry becomes trivialized to 0 for the space-time with the Spin^c symmetry because the Spin^c structure requires $\mathrm{w}_2 = c_1 \mod 2$ and $\mathrm{w}_3 = \mathrm{Sq}^1\mathrm{w}_2 = \mathrm{Sq}^1(c_1 \mod 2) = 0$ on the Spin^c manifold.

(b) The $\mathrm{w}_2\mathrm{w}_3$ of $\mathrm{TP}_5(\mathrm{SO})$ is trivialized as $p^*(\mathrm{w}_2\mathrm{w}_3) = 0$ in $\mathrm{TP}_5(\mathrm{Spin}^c)$.

(c) So there is *no* Spin^c symmetric theory with some internal global U(1) symmetry in 4d with the boundary anomaly of 5d w_2w_3.

(d) This means the 4d boundary anomaly of 5d w_2w_3 of SO is also vanished in Spin^c. However, dynamically gauging the normal U(1) subgroup provides a boundary U(1) gauge theory that preserves the SO symmetry but with the 't Hooft anomaly of w_2w_3.

(11) The group extension

$$1 \to \mathrm{SU}(2) \to \mathrm{Spin}^h(5) \xrightarrow{p} \mathrm{SO}(5) \to 1,$$

however, does *not* provide the trivialization of w_2w_3.

(a) The 5d cobordism invariant w_2w_3 for the 5d space-time with the SO symmetry becomes $w_2w_3 = w_2'w_3'$ for the space-time with the Spin^h symmetry because the Spin^h structure requires $w_2 = w_2'$ and $w_3 = \mathrm{Sq}^1 w_2 = \mathrm{Sq}^1 w_2' = w_3'$ on the Spin^h manifold. This also means the 4d gravitational anomaly on the boundary of 5d w_2w_3 term becomes the 4d mixed gauge-gravitational anomaly on the boundary of 5d $w_2'w_3' = w_2w_3$ term.

(b) The w_2w_3 of $\mathrm{TP}_5(\mathrm{SO})$ is *not* trivialized but becomes $p^*(w_2w_3) = w_2w_3 = w_2'w_3'$ in $\mathrm{TP}_5(\mathrm{Spin}^h)$.

(c) So there indeed exists certain Spin^h symmetric theory with some internal global SU(2) symmetry in 4d with the boundary 't Hooft anomaly of 5d $w_2'w_3' = w_2w_3$. In fact, the Weyl fermion as a two-component space-time spinor and a four-component internal SU(2) spinor, in the representation of $(\mathbf{2}_L, \mathbf{4})$ of Spin $\times_{\mathbb{Z}_2}$ SU(2) $\equiv$ Spin^h has this precise so-called new SU(2) anomaly [44] of 5d $w_2'w_3' = w_2w_3$.

(d) This means the 4d boundary anomaly of 5d w_2w_3 of SO does not need to vanish in Spin^h. However, we can ask whether it is sensible to dynamically gauge the normal SU(2) subgroup in this Spin^h symmetric Weyl fermion theory with the new SU(2) anomaly [44].

(i) If we only restrict to the spin manifold with $w_2 = w_2' = 0$ thus also $w_3 = w_3' = 0$, then, yes, we can obtain a well-defined SU(2) gauge theory on a Spin manifold (such as a flat Euclidean or Minkowski space-time) and we can study its dynamics [44].

(ii) If we construct this SU(2) gauge theory on a generic non-Spin manifold with a Spin^h structure, then we have $w_2 = w_2' \neq 0$ thus also $w_3 = w_3' \neq 0$. Then, no, we obtain an ill-defined SU(2) gauge theory by summing over the SU(2) bundle with the SU(2) connections on a generic non-Spin manifold. We cannot study the dynamics of an ill-defined SU(2) gauge theory with dynamical gauge-gravitational anomaly uncanceled [44].

(18) The group extension

$$1 \to \mathrm{Spin}(n \geqslant 3) \to \mathrm{Spin}(5) \times_{\mathbb{Z}_2} \mathrm{Spin}(n \geqslant 3) \xrightarrow{p} \mathrm{SO}(5) \to 1,$$

however, also does *not* provide the trivialization of w_2w_3, but modifies the w_2w_3 to $w_2w_3 = w_2'w_3'$. The situation for $n \geqslant 3$ is similar to our previous remark on Spin(3) = SU(2).

(a) In comparison to the Spin(2) = U(1) case, there *exists* an all-fermion QED_4 as a U(1) gauge theory definable on a generic nonspin manifold with a pure 4d gravitational anomaly as a 't Hooft anomaly of the space-time diffeomorphism SO symmetry from the 5d w_2w_3.

(b) But, for Spin($n \geqslant 3$), we do *not* have a Spin($n \geqslant 3$) gauge theory (such a 4d gauge theory is *not* definable on a generic non-Spin manifold) with a pure 4d gravitational anomaly as a 't Hooft anomaly of the space-time diffeomorphism SO symmetry from the 5d w_2w_3.

(c) For Spin($n \geqslant 3$), we do have a Spin(n)-symmetric theory definable on a generic nonspin manifold with a 4d mixed gauge-gravitational anomaly as a 't Hooft anomaly of the gauge-diffeomorphism symmetry from the 5d $w_2w_3 = w_2'w_3'$.

(i) Dynamically gauging the Spin(n) in 4d alone makes sense only on a Spin manifold, which results in a 4d Spin(n) gauge theory with a well-defined dynamics on a 4d spin manifold.

(ii) Dynamically gauging the Spin(n) in 4d alone on a nonspin manifold is ill defined. But, dynamically gauging the Spin(n) on a non-Spin manifold can result in a well-defined 4d-5d coupled fully gauged system. This 4d-5d coupled system for Spin($n = 10$) is studied in [45–47].

APPENDIX E: ORIENTED BORDISM GROUPS AND MANIFOLD GENERATORS

In Thom's famous 1954 article [83], he showed that the oriented bordism ring is isomorphic to stable homotopy groups of the Thom spectrum $M\mathrm{SO}$: $\Omega_*^{\mathrm{SO}} = \pi_*(M\mathrm{SO})$. All of the homotopy groups are a direct sum $\mathbb{Z}^r \oplus \mathbb{Z}_2^s$. Bordism classes of oriented manifolds are completely determined by their Pontryagin and Stiefel-Whitney numbers. The mod 2 cohomology of $M\mathrm{SO}$ is the same as the mod 2 cohomology of BSO, a polynomial ring on the Stiefel-Whitney classes $w_2, w_3, \ldots$ whose Poincaré series is

$$\prod_{i \geqslant 2} \frac{1}{1 - t^i}.$$

Rationally, the oriented bordism ring is a polynomial algebra $\mathbb{Q}[x_4, x_8, x_{12}, \ldots]$ on generators in degrees that are a multiple of 4. This tells us the rank r of each group. The Poincaré series for the free part of Ω_*^{SO} is thus

$$p_{\mathrm{free}}(t) = \prod_{i \geqslant 1} \frac{1}{1 - t^{4i}}.$$

2-locally, the Thom spectrum $M\mathrm{SO}$ is a wedge sum of suspensions of Eilenberg-Mac Lane spectra $H\mathbb{Z}_2$ and $H\mathbb{Z}$. This allows us to write

$$H^*(M\mathrm{SO}; \mathbb{Z}_2) \cong \bigoplus_{\text{free summands}} H^*(H\mathbb{Z}; \mathbb{Z}_2) \oplus \times \bigoplus_{\text{torsion summands}} H^*(H\mathbb{Z}_2; \mathbb{Z}_2). \tag{E1}$$

Let the Poincaré series for $H^*(H\mathbb{Z}_2; \mathbb{Z}_2)$ and $H^*(H\mathbb{Z}; \mathbb{Z}_2)$ be $p_{H\mathbb{Z}_2}(t)$ and $p_{H\mathbb{Z}}(t)$, respectively, then by [84], we have

$$p_{H\mathbb{Z}_2}(t) = \prod_{k = 2^i - 1} \frac{1}{1 - t^k}$$

TABLE IV. Oriented bordism groups and manifold generators. As manifold Y^5 (respectively Y^9, Y^{11}) we may take the nonsingular hypersurface of degree (1,1) in the product $\mathbb{RP}^2 \times \mathbb{RP}^4$ (respectively $\mathbb{RP}^2 \times \mathbb{RP}^8$ or $\mathbb{RP}^4 \times \mathbb{RP}^8$) of real projective spaces. These manifolds are called real Milnor manifolds. The 5d Wu manifold is SU(3)/SO(3). The 5d Dold manifold is $S^1 \times_\tau \mathbb{CP}^2$ where the involution τ sends $(x, [y])$ to $(-x, [\bar{y}])$.

d	Ω_d^{SO}	Manifold generators
0	$\mathbb{Z}$	
1	0	
2	0	
3	0	
4	$\mathbb{Z}$	$\mathbb{CP}^2$
5	$\mathbb{Z}_2$	Y^5, Wu SU(3)/SO(3), or Dold $S^1 \times_\tau \mathbb{CP}^2$ manifolds
6	0	
7	0	
8	$\mathbb{Z}^2$	$\mathbb{CP}^4$, $\mathbb{CP}^2 \times \mathbb{CP}^2$
9	$\mathbb{Z}_2^2$	Y^9, $Y^5 \times \mathbb{CP}^2$
10	$\mathbb{Z}_2$	$Y^5 \times Y^5$
11	$\mathbb{Z}_2$	Y^{11}

and

$$p_{H\mathbb{Z}}(t) = \frac{1}{1+t} p_{H\mathbb{Z}_2}(t).$$

Since the Poincaré series $p_{\text{tors}}(t)$ of the torsion part in Ω_*^{SO} satisfies

$$p_{\text{tors}}(t) \cdot p_{H\mathbb{Z}_2}(t) + p_{\text{free}}(t) \cdot p_{H\mathbb{Z}}(t) = \prod_{k \geqslant 2} \frac{1}{1-t^k},$$

we can solve

$$p_{\text{tors}}(t) = \left[(1-t) \prod_{k \geqslant 2, k \neq 2^i - 1} \left(\frac{1}{1-t^k} \right) \right] - \left[\frac{1}{1+t} \prod_{k \geqslant 1} \left(\frac{1}{1-t^{4k}} \right) \right].$$

In particular, we have Table IV [[85], p. 203].

In 5d, the bordism group $\Omega_5^{SO} = \mathbb{Z}_2$ and the bordism invariant is $w_2 w_3$ since the only nonvanishing Stiefel-Whitney number of oriented 5-manifolds is $w_2 w_3$.

APPENDIX F: COMBINATORIAL FORMULA FOR STIEFEL-WHITNEY CLASSES

In 1940, Whitney obtained an explicit combinatorial formula for the Stiefel-Whitney classes [86]. The formula is as follows. Let K be an m-dimensional combinatorial manifold and K' the first barycentric subdivision of K. Let C_n be the sum modulo 2 of all $(m-n)$-dimensional simplices of K'. Then the chain C_n is a cycle modulo 2 and represents the homology class W_n Poincaré dual to the nth Stiefel-Whitney class of K.

In [64], the authors obtained a formula for the Stiefel-Whitney homology classes in the original triangulation without passing to the first barycentric subdivision. Their formula is as follows. A branch structure on a triangulation is an orientation of the links with no closed loops which in turn provides an order to the vertices of simplices. Given a branch structure on K so that any representation of a simplex in K is written with its vertices in increasing order. Let s be an $(m-n)$-simplex in K, say $s = [v_0, v_1, \dots, v_{m-n}]$. Let t be another simplex which has s as a face; i.e., $s \subset t$ (s may be equal to t). Let B_{-1} = set of vertices of t less than v_0, B_0 = set of vertices of t strictly between v_0 and v_1, B_k = set of vertices of t strictly between v_k and v_{k+1}, B_{m-n} = set of vertices of t greater than v_{m-n}. We say that s is regular in t, if B_k is empty for every odd k. Let $\partial_{m-n}(t)$ denote the mod 2 chain which consists of all $(m-n)$-simplices s in t so that s is regular in t. Then $C'_n = \sum_{\dim t \geqslant m-n} \partial_{m-n}(t)$ is a chain which represents the homology class W_n Poincaré dual to the nth Stiefel-Whitney class of K.

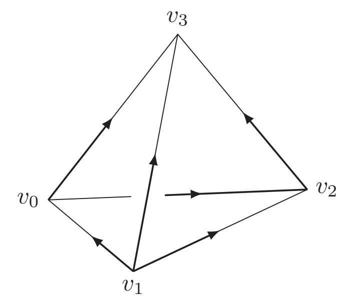

FIG. 6. The surface of a tetrahedron with another branch structure in which the order between v_0 and v_1 is reversed.

For example, let K be the surface of a tetrahedron and $|K| = S^2$. Then, $C'_1 = \sum_{\dim t \geqslant 1} \partial_1(t)$. Given the branch structure on K so that the four vertices of K are ordered as v_0, v_1, v_2, and v_3 (see Fig. 5). For $\dim t \geqslant 1$, t can be chosen as $[v_0, v_1]$, $[v_0, v_2]$, $[v_1, v_2]$, $[v_0, v_3]$, $[v_1, v_3]$, and $[v_2, v_3]$ if $\dim t = 1$ and $[v_0, v_1, v_2]$, $[v_0, v_1, v_3]$, $[v_0, v_2, v_3]$, and $[v_1, v_2, v_3]$ if $\dim t = 2$. If $\dim t = 1$ and $s \in \partial_1(t)$, then $s = t$, if $\dim t = 2$ and $s \in \partial_1(t)$, then s is the 1-simplex whose two vertices are the smallest and the greatest vertices of t. Therefore, $C'_1 = [v_0, v_1] + [v_0, v_3] + [v_1, v_2] + [v_2, v_3]$. If another branch structure is given on K so that the order between v_0 and v_1 is reversed while other orders remain the same (see Fig. 6), then C'_1 changes to $[v_0, v_1] + [v_0, v_2] + [v_1, v_3] + [v_2, v_3]$ and the difference with the original C'_1 is $[v_0, v_3] + [v_0, v_2] + [v_1, v_2] + [v_1, v_3]$ which is a boundary. So, C'_n depends on the branch structure and different choices of branch structures can only change C'_n by a boundary.

The Poincaré dual (see Appendix A 7) also depends on the branch structure. Thus, w_n depends on the branch structure and different choices of branch structures can only change w_n by a coboundary.

APPENDIX G: COMPUTE $w_2 w_3$ ON REAL MILNOR, WU, AND DOLD MANIFOLDS

The 5d real Milnor manifold [87] $Y^5 = H(2, 4)$ is the submanifold of $\mathbb{RP}^2 \times \mathbb{RP}^4$ given by

$$H(2,4) = \left\{ ([x_0, x_1, x_2], [y_0, \dots, y_4]) \in \mathbb{RP}^2 \times \mathbb{RP}^4 : \times \sum_{i=0}^{2} x_i y_i = 0 \right\}. \tag{G1}$$

In fact, $H(2,4)$ is the submanifold of $\mathbb{RP}^2 \times \mathbb{RP}^4$ Poincaré dual to $(a+b)$ where a and b are the generators of $H^*(\mathbb{RP}^2;\mathbb{Z}_2)$ and $H^*(\mathbb{RP}^4;\mathbb{Z}_2)$, respectively. Note that $a^3=0$ and $b^5=0$. The total Stiefel-Whitney class $\mathrm{w}(H(2,4))$ of $H(2,4)$ is given by the restriction to $H(2,4)$ of the expression

$$\frac{(1+a)^3(1+b)^5}{(1+a+b)}.$$

By direct computation, we find that $\mathrm{w}_2 = a^2+ab$ and $\mathrm{w}_3 = ab^2+a^2b$. So $\mathrm{w}_2\mathrm{w}_3 = a^2b^3$ and the Stiefel-Whitney number $\langle \mathrm{w}_2\mathrm{w}_3, [Y^5]\rangle = \langle (a+b)\mathrm{w}_2\mathrm{w}_3, [\mathbb{RP}^2\times\mathbb{RP}^4]\rangle = 1$.

The Wu manifold $W := \mathrm{SU}(3)/\mathrm{SO}(3)$ has cohomology ring $H^*(W;\mathbb{Z}_2) = \mathbb{Z}_2[z_2,z_3]/(z_2^2,z_3^2)$ with the total Stiefel-Whitney class $\mathrm{w}(W) = 1+z_2+z_3$, $\mathrm{Sq}(z_2) = z_2+z_3$, and $\mathrm{Sq}(z_3) = z_3+z_2z_3$ where $\mathrm{Sq} := \mathrm{Sq}^0+\mathrm{Sq}^1+\mathrm{Sq}^2+\cdots$ is the total Steenrod square. So the Stiefel-Whitney number $\langle \mathrm{w}_2\mathrm{w}_3, [W]\rangle = 1$.

The 5d Dold manifold [88] $P(1,2)$ is the quotient $S^1\times_\tau \mathbb{CP}^2$ where the involution τ sends $(x,[y])$ to $(-x,[\bar{y}])$. The ring structure of $H^*(P(1,2);\mathbb{Z}_2)$ is

$$H^*(P(1,2);\mathbb{Z}_2) = [\mathbb{Z}_2[c]/(c^2=0)]\otimes[\mathbb{Z}_2[d]/(d^3=0)],$$

and the total Stiefel-Whitney class of $P(1,2)$ is

$$\mathrm{w}(P(1,2)) = (1+c)(1+c+d)^3,$$

where $c\in H^1(P(1,2);\mathbb{Z}_2)$ and $d\in H^2(P(1,2);\mathbb{Z}_2)$. The Steenrod squares act by

$$\mathrm{Sq}^0 = \mathrm{id},\ \mathrm{Sq}^1(c) = 0,\ \mathrm{Sq}^1(d) = cd,\ \mathrm{Sq}^2(d) = d^2,$$

and all other Steenrod squares act trivially on c and d. By direct computation, we find that $\mathrm{w}_2 = d$ and $\mathrm{w}_3 = cd$. So the Stiefel-Whitney number $\langle \mathrm{w}_2\mathrm{w}_3, [P(1,2)]\rangle = 1$.

APPENDIX H: GENERALIZED WU RELATION

The classical Wu relation (31) expresses the action of a single Steenrod square Sq^n on a $\mathbb{Z}_2$-valued cocycle x_{d-n} in the top d dimension on a manifold M^d as the cup product $u_n x_{d-n}$ where u_n is the Wu class (32). In this Appendix, we generalize this Wu relation to other elements in the mod 2 Steenrod algebra $\mathcal{A}_2$.

By Adem relation, $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$ and $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$. So the simplest element in $\mathcal{A}_2$ which is not a single Steenrod square is $\mathrm{Sq}^2\mathrm{Sq}^1$. We claim that $\mathrm{Sq}^2\mathrm{Sq}^1 x_{d-3} = (\mathrm{w}_1^3+\mathrm{w}_3)x_{d-3}$ on a manifold M^d where w_i is the Stiefel-Whitney class of M^d. In fact,

$$\begin{aligned}\mathrm{Sq}^2\mathrm{Sq}^1 x_{d-3} &= (\mathrm{w}_1^2+\mathrm{w}_2)(\mathrm{Sq}^1 x_{d-3})\\ &= \mathrm{Sq}^1(\mathrm{w}_1^2 x_{d-3}) + (\mathrm{Sq}^1\mathrm{w}_2)x_{d-3} + \mathrm{Sq}^1(\mathrm{w}_2 x_{d-3})\\ &= \mathrm{w}_1^3 x_{d-3} + (\mathrm{w}_1\mathrm{w}_2+\mathrm{w}_3)x_{d-3} + \mathrm{w}_1\mathrm{w}_2 x_{d-3}\\ &= (\mathrm{w}_1^3+\mathrm{w}_3)x_{d-3}.\end{aligned} \tag{H1}$$

In the first equality, we used the Wu relation (31) for Sq^2. In the second equality, we used the product formula for Steenrod square $\mathrm{Sq}^k(x\smile y) = \sum_{i+j=k}\mathrm{Sq}^i x\smile \mathrm{Sq}^j y$ and $\mathrm{Sq}^1(\mathrm{w}_1^2) = 0$. In the third equality, we used the Wu relation (31) for Sq^1 and $\mathrm{Sq}^1\mathrm{w}_2 = \mathrm{w}_1\mathrm{w}_2+\mathrm{w}_3$. This (H1) is a new generalized Wu relation, which is mentioned in Ref. .

APPENDIX I: PULLBACK CONSTRUCTION OF BRANCH-INDEPENDENT BOSONIC MODELS

In this Appendix, we are going to present a general systematic construction of branch-independent bosonic models. We will first construct a model with a finite-G symmetry, realizing a G-SPT order. The degrees of freedom in our model are described by $g_i\in G$ on each vertex i. The model on space-time M^d is defined by the path integral

$$Z(M^d) = \sum_{g_i} e^{-S(g_i)}. \tag{I1}$$

We can rewrite that model as

$$Z(M^d) = \sum_{g_i} e^{-S(a_{ij}^G)},\quad a_{ij}^G = g_i g_j^{-1}. \tag{I2}$$

We can add a background flat G gauge field $A_{ij}^G\in G$

$$A_{ij}^G A_{jk}^G = A_{ik}^G \tag{I3}$$

to describe the symmetry twist, and consider the following gauged model:

$$Z(M^d,A^G) = \sum_{g_i} e^{-S(a_{ij}^G)},\quad a_{ij}^G = g_i A_{ij}^G g_j^{-1}. \tag{I4}$$

Note that a_{ij}^G satisfy a flat condition

$$a_{ij}^G a_{jk}^G = a_{ik}^G. \tag{I5}$$

When $A_{ij}^G = 1$, the partition function in (I4) automatically has the G symmetry

$$g_i\to g_i h,\quad h\in G, \tag{I6}$$

even for space-time M^d with boundaries. This implies that the G symmetry is anomaly free.

We can choose a proper $S(a_{ij}^G)$ so that the bosonic model (I4) realized a bosonic SPT order. To do so, let us consider the classifying space of the group G, and choose a one-vertex triangulation of the classifying space. We denote the resulting simplicial complex as $\mathrm{B}G$. For the details about the one-vertex triangulation, see Ref. [89]. $\mathrm{B}G$ has the properties that $\pi_1(\mathrm{B}G) = G$ and $\pi_{n\neq 1}(\mathrm{B}G) = 0$. Since $\mathrm{B}G$ has only one vertex, the links in $\mathrm{B}G$ are labeled by $\bar{a}_{ij}^G\in G$, that satisfy the condition

$$\bar{a}_{ij}^G\bar{a}_{jk}^G = \bar{a}_{ik}^G \tag{I7}$$

for every triangle (ijk) in $\mathrm{B}G$.

a_{ij}^G on the links of space-time simplicial complex M^d defines a homomorphism of simplicial complex

$$M^d \xrightarrow{\phi(a_{ij}^G)} \mathrm{B}G. \tag{I8}$$

$\phi(a_{ij}^G)$ maps the vertices in M^d to the only vertex in $\mathrm{B}G$. $\phi(a_{ij}^G)$ maps the link in M^d with value a_{ij}^G to the link in $\mathrm{B}G$ labeled by $\bar{a}_{ik}^G = a_{ik}^G$. The two flat conditions (I5) and (I7) ensure that $\phi(a_{ij}^G)$ maps the triangles in M^d to triangles in $\mathrm{B}G$, etc. Thus, $\phi(a_{ij}^G)$ is a homomorphism of simplicial complex, and a_{ij}^G is the pullback of $\bar{a}_{ij}^G$ by $\phi(a_{ij}^G)$:

$$a_{ij}^G = \phi^*(a_{ij}^G)\,\bar{a}_{ij}^G. \tag{I9}$$

Let $\bar{\omega}_d$ be a $\mathbb{R}/\mathbb{Z}$-valued cocycle in BG:

$$\bar{\omega}_d \in H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z}). \tag{I10}$$

Now we can construct a bosonic model using $\bar{\omega}_d$:

$$Z(M^d, A^G) = \sum_{g_i} e^{i2\pi \int_{M^d} \phi^*(a^G_{ij})\bar{\omega}_d}, \quad a^G_{ij} = g_i A^G_{ij} g_j^{-1}. \tag{I11}$$

We refer to this construction as pullback construction. Clearly, the resulting bosonic model is branch independent since during the whole construction, the branch structure is not even specified. We also note that the action amplitude $e^{i2\pi \int_{M^d} \phi^*(a^G_{ij})\bar{\omega}_d}$ does not depend on the Stiefel-Whitney cocycle w_n.

The branch-independent bosonic model (I11) realizes the SPT order labeled by $\bar{\omega}_d \in H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z})$, and described by the SPT invariant [32,33]

$$Z^{\mathrm{top}}(M^d, A^G) = e^{i2\pi \int_{M^d} \phi^*(A^G_{ij})\bar{\omega}_d}. \tag{I12}$$

This is consistent with the group cohomology classification of SPT order [10].

The above construction of branch-independent bosonic models can be generalized to compact continuous group G with less rigor, where the branch-independent bosonic model is given by the following path integral:

$$Z(M^d, A^G) = \sum_{\phi} e^{i2\pi \int_{M^d} \phi^*\bar{\omega}_d} e^{-\int_{M^d} \mathcal{L}(\phi)}. \tag{I13}$$

Here we give the classifying space of G a triangulation and denote the resulting simplicial complex as BG. The trianglation temporarily breaks the G symmetry. ϕ is a homomorphism from space-time simplicial complex M^d to classifying space simplicial complex BG:

$$M^d \xrightarrow{\phi} \mathrm{B}G. \tag{I14}$$

The path integral is a sum over the homomorphisms ϕ. This is similar to the definition of nonlinear σ model on a continuum space-time, where the path integral is a sum over the continuous maps from space-time manifold to target space. The term $e^{-\int_{M^d} \mathcal{L}(\phi)}$ is chosen such that the model (I13) describes a disordered state. In the limit of fine triangulation the model has a G symmetry, and the disordered state is G symmetric.

Also, $\bar{\omega}_d$ in the path integral (I13) is a $\mathbb{R}/\mathbb{Z}$-valued cocycle in BG:

$$\bar{\omega}_d \in H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z}). \tag{I15}$$

If $H^n(\mathrm{B}G; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_n \oplus \cdots$, then $H^n(\mathrm{B}G; \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_n \oplus \cdots$ according to the universal coefficient theorem

$$H^d(X; \mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} H^d(X; \mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{M}, H^{d+1}(X; \mathbb{Z})). \tag{I16}$$

We see that the torsion part of $H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z})$ and $H^{d+1}(\mathrm{B}G; \mathbb{Z})$ coincide:

$$\mathrm{Tor} H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z}) = \mathrm{Tor} H^{d+1}(\mathrm{B}G; \mathbb{Z}). \tag{I17}$$

The cocycle that generates $\mathbb{R}/\mathbb{Z}$ part of $H^n(\mathrm{B}G; \mathbb{R}/\mathbb{Z})$ does not have a quantized coefficient and can be continuously changed to zero. So the $\mathbb{R}/\mathbb{Z}$ part of $H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z})$ does not characterize a topological phase. Only $\bar{\omega}_d \in \mathrm{Tor} H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z}) = \mathrm{Tor}(\mathbb{R}/\mathbb{Z}, H^{d+1}(\mathrm{B}G; \mathbb{Z}))$ gives rise to distinct topological phase via the model (I13), which is a G-SPT phase.

When G is continuous, some G-SPT order can belong to $\mathbb{Z}$ class which is not a torsion. To construct branch-independent bosonic model to realize this kind of SPT order, we need to generalize the above model (I13) to the form

$$Z(M^d, A^G) = \sum_{\phi} e^{-\int_{M^d} \mathcal{L}(\phi)} e^{i2\pi [\int_{M^d} \phi^*\bar{\omega}_d + \int_{N^{d+1}} \phi_N^* \bar{\nu}_{d+1}]},$$
$$M^d = \partial N^{d+1}. \tag{I18}$$

The term $e^{i2\pi \int_{N^{d+1}} \phi_N^* \bar{\nu}_{d+1}}$, living in one higher dimension, is a Wess-Zumino-Witten–type term, and ϕ_N is a homomorphism of simplicial complex

$$N^{d+1} \xrightarrow{\phi_N} \mathrm{B}G \tag{I19}$$

such that at the boundary $M^d = \partial N^{d+1}$, $\phi_N = \phi$. $\bar{\nu}_{d+1}$ is a $\mathbb{R}$-valued cocycle that satisfies a quantization condition

$$\int_{N^{d+1}} \bar{\nu}_{d+1} \in \mathbb{Z}, \tag{I20}$$

for all closed N^{d+1} in BG. In other words, the $\mathbb{R}$-valued cocycle $\bar{\nu}_{d+1}$ represents a cohomology class in the free part of $H^{d+1}(\mathrm{B}G; \mathbb{Z})$:

$$\bar{\nu}_{d+1} \in \mathrm{Free}(H^{d+1}(\mathrm{B}G; \mathbb{Z})). \tag{I21}$$

In the disordered phase, (I18) realizes a G-SPT order characterized by $(\bar{\omega}_d, \bar{\nu}_{d+1})$ in $\mathrm{Tor}(H^d(\mathrm{B}G; \mathbb{R}/\mathbb{Z})) = \mathrm{Tor}(H^{d+1}(\mathrm{B}G; \mathbb{Z}))$ and $\mathrm{Free}(H^{d+1}(\mathrm{B}G; \mathbb{Z}))$. In other words, the G-SPT order is characterized by the elements in $H^{d+1}(\mathrm{B}G; \mathbb{Z})$, which agree with the group cohomology theory of SPT order for symmetries described by compact groups.

When $G = \mathrm{SO}(\infty)$, the term $e^{i2\pi \int_{N^{d+1}} \phi_N^* \bar{\nu}_{d+1}}$ gives rise to the SO(∞) Chern-Simons term on $M^d = \partial N^{d+1}$, whose generator is the Pontryagin class (for $d+1 = 0$ mod 4). The pullback of different maps ϕ,

$$M^d \xrightarrow{\phi} \mathrm{BSO}(\infty), \tag{I22}$$

gives rise to different SO(∞) bundle over M^d. If we restrict $\sum_\phi$ in (I18) to a subset of maps ϕ, such that the SO(∞) bundle over M^d is the same as the stabilized tangent bundle of M^d, the model (I18) may realize a bosonic invertible topological order. The $\mathbb{Z}$ class of invertible topological orders are described by gravitational Chern-Simons term, which is also SO(∞) Chern-Simons term. We see that the model (I18) can only realize gravitational Chern-Simons terms generated by Pontryagin classes, which have no framing anomaly. Thus, in 3d, the model (I18) can only realize invertible topological orders generated by E_8^3 topological order. The E_8 topological order is characterized by a gravitational Chern-Simons term that corresponds to $\frac{1}{3}$ of the first Pontryagin class, which has a framing anomaly.

APPENDIX J: BACKGROUND VS DYNAMICAL GAUGE TRANSFORMATIONS

In Sec. III B 1, we describe the invariance or noninvariance of path integral in terms of the change of coboundaries and branch structures. Here we fill in some additional terminology more accessible for quantum field theorists: in terms of *background gauge transformations* vs *dynamical gauge transformations*.

For 4d $\mathbb{Z}_2$ gauge theory (24) described by $\mathbb{Z}_2$-valued 1-cochain and 2-cochain dynamical fields $a^{\mathbb{Z}_2}$ and $b^{\mathbb{Z}_2}$,

$$Z = \sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}} e^{i\pi \int_{B^4} a^{\mathbb{Z}_2} db^{\mathbb{Z}_2}},$$

where $\sum_{a^{\mathbb{Z}_2}, b^{\mathbb{Z}_2}}$ is a summation over $\mathbb{Z}_2$-valued 1- and 2-cochains. The above theory has *dynamical gauge transformations* for dynamical fields:

$$a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + du_0, \quad b^{\mathbb{Z}_2} \to b^{\mathbb{Z}_2} + du_1, \tag{J1}$$

where $u_0 \in C^0(B^4; \mathbb{Z}_2)$ and $u_1 \in C^1(B^4; \mathbb{Z}_2)$ are $\mathbb{Z}_2$-valued 0- and 1-cochain fields.

Now let us discuss another interpretation of (25). The Stiefel-Whitney classes $w_2 \in H^2(M^5; \mathbb{Z}_2)$ and $w_3 \in H^3(M^5; \mathbb{Z}_2)$ are special cohomology classes satisfying the extra axioms A1–A4 listed earlier, with the base manifold M^5 for the real vector bundle. Since M^5 is orientable, we have $w_1 \overset{2}{=} 0$ and $w_3 \overset{2}{=} \mathrm{Sq}^1 w_2$.

When M^5 has a boundary, the partition function (25) depends on the choice of the coboundaries in w_2 and w_3. i.e., under the following *background gauge transformation* for nondynamical fields:

$$\begin{aligned} &w_2 \to w_2 + dv_1, \\ &w_3 \to w_3 + \mathrm{Sq}^1 dv_1 + dv_2 \to w_3 + dv_2. \end{aligned} \tag{J2}$$

Although the Stiefel-Whitney classes have the relation $\mathrm{Sq}^1 w_2 = w_3$ so that the transformation dv_1 can be related to $\mathrm{Sq}^1 dv_1$, but they can differ by a coboundary dv_2 which thus absorbs $\mathrm{Sq}^1 dv_1$.

An anomalous $\mathbb{Z}_2$ gauge theory (that has 't Hooft anomaly of space-time SO diffeomorphism) on the boundary $B^4 = \partial M^5$ of the topological state is described by (25) which has not only the *dynamical gauge transformations* [involving u_0 and u_1 in (J1)] but also additional background gauge transformations [involving v_1 in (J1)]:

$$\begin{aligned} &a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + du_0 + v_1, \quad b^{\mathbb{Z}_2} \to b^{\mathbb{Z}_2} + du_1 + v_2, \\ &w_2 \to w_2 + dv_1, \quad w_3 \to w_3 + dv_2. \end{aligned} \tag{J3}$$

It turns out that the background gauge transformations at the lattice scale of the simplicial complex are important to ensure the anomaly inflow or anomaly cancellation between the bulk and boundary for the 't Hooft anomaly of global symmetries. In contrast, the dynamical gauge transformations at the lattice scale of the simplicial complex are not so crucial or fundamental: the dynamical gauge invariance at the lattice scale, even if we break them locally, the dynamical gauge invariance can reemerge at a larger length scale. So, only the emergent dynamical gauge invariance is crucial.

APPENDIX K: $\mathbb{Z}_2$ TOPOLOGICAL ORDER WITH EMERGENT FERMION AND HIGHER-DIMENSIONAL BOSONIZATION

In this Appendix, we review and summarize the higher-dimensional bosonization following [71]. In $(d+1)$-dimensional space-time, a bosonic model that realizes a $\mathbb{Z}_2$ topological order is described by the following path integral:

$$Z(M^{d+1}) = \sum_{da^{\mathbb{Z}_2} \overset{2}{=} 0} 1, \tag{K1}$$

where $\sum_{da^{\mathbb{Z}_2} \overset{2}{=} 0}$ sums over all $\mathbb{Z}_2$-valued 1-cocycles $a^{\mathbb{Z}_2}$. The low-energy effective theory of the $\mathbb{Z}_2$ topological order is a $\mathbb{Z}_2$ gauge theory where the pointlike $\mathbb{Z}_2$ charge is a boson. Such a $\mathbb{Z}_2$ topological order has another realization in terms of $\mathbb{Z}_2$-valued $(d-1)$-cocyles $l^{\mathbb{Z}_2}$:

$$Z(M^{d+1}) = \sum_{dl^{\mathbb{Z}_2}_{d-1} \overset{2}{=} 0} 1. \tag{K2}$$

There is a twisted $\mathbb{Z}_2$ topological order [59], whose low-energy effective theory is a twisted $\mathbb{Z}_2$ gauge theory where the pointlike $\mathbb{Z}_2$ charge is a fermion. Such a twisted $\mathbb{Z}_2$ topological order is realized by the following bosonic model:

$$Z(M^{d+1}) = \sum_{dl^{\mathbb{Z}_2}_{d-1} \overset{2}{=} 0} e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}. \tag{K3}$$

The above path integral does not contain a $\mathbb{Z}_2$ charge. To include a $\mathbb{Z}_2$ charge, we note that the world line of a particle can be described by its Poincaré dual f_d, which is a $\mathbb{Z}_2$-valued d-coboundary. The path integral including such a world line is given by

$$Z(M^{d+1}) = \sum_{dl^{\mathbb{Z}_2}_{d-1} \overset{2}{=} f_d} e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}. \tag{K4}$$

The term $e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}$ gives the world line (described by f_{d-1}) a Fermi statistics.

The $(d+1)$d twisted $\mathbb{Z}_2$ topological order has a d-dimensional boundary, formed by condensing the $\mathbb{Z}_2$ flux. Such a boundary contains only pointlike topological excitations, which are fermions, coming from the bulk fermionic $\mathbb{Z}_2$ charge. Such a boundary is the canonical boundary of the path integral (K4), which is described by the following path integral:

$$Z(M^{d+1}) = \sum_{dl^{\mathbb{Z}_2}_{d-1} \overset{2}{=} 0} e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}, \quad B^d = \partial M^{d+1}. \tag{K5}$$

The cocycle $l^{\mathbb{Z}_2}_{d-1}$ on the boundary B^d can be viewed as the Poincaré dual of the world line of boundary fermions.

Now, let us try to view the above path integral (K5) as a path integral on the boundary B^d for the field $l^{\mathbb{Z}_2}_{d-1}$, and view the term $e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}$ as a Wess-Zumino-Witten–type term defined in one higher dimension. But, such a viewpoint is not quite correct since the $e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}$ not only depends on $l^{\mathbb{Z}_2}_{d-1}$ on the boundary $B^d = \partial M^{d+1}$, but also depends on M^{d+1}, i.e., how we extend B^d. However, when M^{d+1} is oriented and spin, $w_1 \overset{2,d}{=} 0$ and $w_2 \overset{2,d}{=} 0$, $\mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1} \overset{2,d}{=} 0$. In this case, $e^{i\pi \int_{M^{d+1}} \mathbb{S}\mathrm{q}^2 l^{\mathbb{Z}_2}_{d-1}}$ only depends on $l^{\mathbb{Z}_2}_{d-1}$ on the boundary $B^d = \partial M^{d+1}$, and can indeed be viewed as a Wess-Zumino-Witten–type term.

We can modify the path integral (K5) to relax the requirement for M^{d+1} to be spin:

$$Z(M^{d+1}) = \sum_{dl^{\mathbb{Z}_2}_{d-1} \overset{2}{=} 0} e^{i\pi \left(\int_{B^d} l^{\mathbb{Z}_2}_{d-1} A^{\mathbb{Z}_2^f} + \int_{M^{d+1}} \mathbb{Sq}^2 l^{\mathbb{Z}_2}_{d-1} + \mathrm{w}_2 l^{\mathbb{Z}_2}_{d-1} \right)},$$

$$B^d = \partial M^{d+1}, \quad \mathrm{w}_2 \overset{2}{=} dA^{\mathbb{Z}_2^f} \text{ on } B^d. \tag{K6}$$

In the above, M^{d+1} is orientable but may not be spin, and B^d is spin such that $\mathrm{w}_2 \overset{2}{=} dA^{\mathbb{Z}_2^f}$ on B^d. In this case, $e^{i\pi \int_{M^{d+1}} \mathbb{Sq}^2 l^{\mathbb{Z}_2}_{d-1} + \mathrm{w}_2 l^{\mathbb{Z}_2}_{d-1}}$ is a Wess-Zumino-Witten–type term, that only depends on $l^{\mathbb{Z}_2}_{d-1}$ and w_2 on B^d. Also, (K6) is invariant under the "gauge" transformation

$$\mathrm{w}_2 \to \mathrm{w}_2 + dv_1^{\mathbb{Z}_2}, \quad A^{\mathbb{Z}_2^f} \to A^{\mathbb{Z}_2^f} + v_1^{\mathbb{Z}_2}, \tag{K7}$$

so that it is branch independent.

$A^{\mathbb{Z}_2^f}$ is a $\mathbb{Z}_2$-valued 1-cochain, which corresponds to the spin structure on B^d. The relation $\mathrm{w}_2 \overset{2}{=} dA^{\mathbb{Z}_2^f}$ tells us that the world line $l^{\mathbb{Z}_2}_{d-1}$ couples to the SO(n) tangent bundle of B^d in such a way that the world line corresponds to a half-integer spin.

[1] B. Zeng and X.-G. Wen, Gapped quantum liquids and topological order, stochastic local transformations and emergence of unitarity, Phys. Rev. B **91**, 125121 (2015).
[2] B. Swingle and J. McGreevy, Renormalization group constructions of topological quantum liquids and beyond, Phys. Rev. B **93**, 045127 (2016).
[3] X.-G. Wen, Topological orders in rigid states, Int. J. Mod. Phys. B **04**, 239 (1990).
[4] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. **121**, 351 (1989).
[5] X.-G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus Riemann surfaces, Phys. Rev. B **41**, 9377 (1990).
[6] X. Chen, Z.-C. Gu, and X.-G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, Phys. Rev. B **82**, 155138 (2010).
[7] Z.-C. Gu and X.-G. Wen, Tensor-entanglement-filtering renormalization approach and symmetry protected topological order, Phys. Rev. B **80**, 155131 (2009).
[8] A. Y. Kitaev, talk at KITP UCSB 2011 (unpublished).
[9] X. Chen, Z.-X. Liu, and X.-G. Wen, Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations, Phys. Rev. B **84**, 235141 (2011).
[10] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, Phys. Rev. B **87**, 155114 (2013).
[11] Y.-M. Lu and A. Vishwanath, Theory and classification of interacting integer topological phases in two dimensions: A chern-simons approach, Phys. Rev. B **86**, 125119 (2012).
[12] A. Vishwanath and T. Senthil, Physics of Three-Dimensional Bosonic Topological Insulators: Surface-Deconfined Criticality and Quantized Magnetoelectric Effect, Phys. Rev. X **3**, 011016 (2013).
[13] Z.-C. Gu and X.-G. Wen, Symmetry-protected topological orders for interacting fermions–fermionic topological non-linear sigma-models and a group super-cohomology theory, Phys. Rev. B **90**, 115141 (2014).
[14] T. Senthil, Annu. Rev. Condens. Matter Phys. **6**, 299 (2015).
[15] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, Fermionic symmetry protected topological phases and cobordisms, J. High Energy Phys. 12 (2015) 052.
[16] D. Freed and M. Hopkins, Reflection positivity and invertible topological phases, Geom. Topol. **25**, 1165 (2021)
[17] A. Kapustin and R. Thorngren, Fermionic SPT phases in higher dimensions and bosonization, J. High Energy Phys. 10 (2017) 080.
[18] Q.-R. Wang and Z.-C. Gu, Towards a Complete Classification of Symmetry-Protected Topological Phases for Interacting Fermions in Three Dimensions and a General Group Supercohomology Theory, Phys. Rev. X **8**, 011055 (2018).
[19] M. Barkeshli, Y.-A. Chen, P.-S. Hsin, and N. Manjunath, Classification of (2+1)D invertible fermionic topological phases with symmetry, Phys. Rev. B **105**, 235143 (2022).
[20] X.-G. Wen, Quantum orders and symmetric spin liquids, Phys. Rev. B **65**, 165113 (2002).
[21] X.-G. Wen, Quantum Orders in an Exact Soluble Model, Phys. Rev. Lett. **90**, 016803 (2003).
[22] A. Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. **321**, 2 (2006).
[23] F. Wang and A. Vishwanath, Spin-liquid states on the triangular and kagomé lattices: A projective-symmetry-group analysis of Schwinger boson states, Phys. Rev. B **74**, 174423 (2006).
[24] S.-P. Kou, M. Levin, and X.-G. Wen, Mutual Chern-Simons theory for z_2 topological order, Phys. Rev. B **78**, 155134 (2008).
[25] S.-P. Kou and X.-G. Wen, Translation-symmetry-protected topological orders in quantum spin systems, Phys. Rev. B **80**, 224406 (2009).
[26] Y.-M. Lu and A. Vishwanath, Classification and Properties of Symmetry Enriched Topological Phases: A Chern-Simons approach with applications to Z2 spin liquids, Phys. Rev. B **93**, 155121 (2016).
[27] A. M. Essin and M. Hermele, Classifying fractionalization: Symmetry classification of gapped Z2 spin liquids in two dimensions, Phys. Rev. B **87**, 104406 (2013).
[28] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, Phys. Rev. B **100**, 115147 (2019).
[29] T. Lan, L. Kong, and X.-G. Wen, Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries, Phys. Rev. B **95**, 235140 (2017).
[30] D. Bulmash and M. Barkeshli, Fermionic symmetry fractionalization in (2+1)D, Phys. Rev. B **105**, 125114 (2022).
[31] D. Aasen, P. Bonderson, and C. Knapp, Characterization and classification of fermionic symmetry enriched topological phases, arXiv:2109.10911.
[32] X.-G. Wen, Symmetry-protected topological invariants of symmetry-protected topological phases of interacting bosons and fermions, Phys. Rev. B **89**, 035147 (2014).

[33] L.-Y. Hung and X.-G. Wen, Universal symmetry-protected topological invariants for symmetry-protected topological states, Phys. Rev. B **89**, 075121 (2014).
[34] A. Kapustin, Bosonic Topological insulators and paramagnets: A view from cobordisms, arXiv:1404.6659.
[35] L. Kong and X.-G. Wen, Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions, arXiv:1405.5858.
[36] J. C. Wang, Z.-C. Gu, and X.-G. Wen, Field Theory Representation of Pure Gauge and Mixed Gauge-Gravity Symmetry-Protected Topological Invariants, Group Cohomology and Beyond, Phys. Rev. Lett. **114**, 031601 (2015).
[37] E. Witten, Fermion path integrals and topological phases, Rev. Mod. Phys. **88**, 035001 (2016).
[38] M. Guo, P. Putrov, and J. Wang, Time reversal, SU(N) Yang-Mills and cobordisms: Interacting topological superconductors/insulators and quantum spin liquids in 3 + 1 D, Ann. Phys. **394**, 244 (2018).
[39] C. Wang, A. C. Potter, and T. Senthil, Classification of interacting electronic topological insulators in three dimensions, Science **343**, 6171 (2014).
[40] S. Kravec, J. McGreevy, and B. Swingle, All-fermion electrodynamics and fermion number anomaly inflow, Phys. Rev. D **92**, 085024 (2015).
[41] R. Thorngren, Framed Wilson operators, fermionic strings, and gravitational anomaly in 4d, J. High Energ. Phys. 02 (2015) 152.
[42] R. Thorngren, Topological terms and phases of sigma models, arXiv:1710.02545.
[43] J. Wang, X.-G. Wen, and E. Witten, Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions, Phys. Rev. X **8**, 031048 (2018).
[44] J. Wang, X.-G. Wen, and E. Witten, A new SU(2) anomaly, J. Math. Phys. **60**, 052301 (2019).
[45] J. Wang and Y.-Z. You, Gauge enhanced quantum criticality beyond the standard model, Phys. Rev. D **106**, 025013 (2022).
[46] J. Wang and Y.-Z. You, Gauge enhanced quantum criticality between grand unifications: Categorical higher symmetry retraction, arXiv:2111.10369.
[47] Y.-Z. You and J. Wang, Deconfined quantum criticality among grand unified theories, arXiv:2202.13498.
[48] Y.-A. Chen and P.-S. Hsin, Exactly solvable lattice Hamiltonians and gravitational anomalies, arXiv:2110.14644.
[49] L. Fidkowski, J. Haah, and M. B. Hastings, Gravitational anomaly of 3+1 dimensional Z_2 toric code with fermionic charges and fermionic loop self-statistics, arXiv:2110.14654
[50] X.-G. Wen, Construction of bosonic symmetry-protected-trivial states and their topological invariants via $G \times SO(\infty)$ nonlinear σ-models, Phys. Rev. B **91**, 205101 (2015).
[51] M. DeMarco and X.-G. Wen, Compact $U^k(1)$ Chern-Simons Theory as a Local Bosonic Lattice Model with Exact Discrete 1-Symmetries, Phys. Rev. Lett. **126**, 021603 (2021).
[52] X.-G. Wen and A. Zee, Classification of abelian quantum hall states and matrix formulation of topological fluids, Phys. Rev. B **46**, 2290 (1992).
[53] X.-G. Wen, Topological orders and edge excitations in fractional quantum hall states, Adv. Phys. **44**, 405 (1995).
[54] D. S. Freed, Short-range entanglement and invertible field theories, arXiv:1406.7278.
[55] Z. Wan and J. Wang, Beyond standard models and grand unifications: Anomalies, topological terms, and dynamical constraints via cobordisms, J. High Energy Phys. 07 (2020) 062.
[56] X.-G. Wen, Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders, Phys. Rev. D **88**, 045013 (2013).
[57] L. Alvarez-Gaume and E. Witten, Gravitational anomalies, Nucl. Phys. B **234**, 269 (1984).
[58] E. Witten, Global gravitational anomalies, Commun. Math. Phys. **100**, 197 (1985).
[59] M. Levin and X.-G. Wen, Fermions, strings, and gauge fields in lattice spin models, Phys. Rev. B **67**, 245316 (2003).
[60] G. 't Hooft, Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, Recent Developments in Gauge Theories, Proceedings, Nato Advanced Study Institute, Cargese, France, August 26–September 8, 1979, NATO Sci. Ser. B **59**, 135 (1980).
[61] C. G. Callan Jr. and J. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. B **250**, 427 (1985).
[62] F. Costantino, A calculus for branched spines of 3-manifolds, Math. Z. **251**, 427 (2005).
[63] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry-protected topological orders in interacting bosonic systems, Science **338**, 1604 (2012).
[64] R. Z. Goldstein and E. C. Turner, A formula for Stiefel-Whitney homology classes, Proc. Am. Math. Soc. **58**, 339 (1976).
[65] X.-G. Wen, Exactly soluble local bosonic cocycle models, statistical transmutation, and simplest time-reversal symmetric topological orders in 3+1 dimensions, Phys. Rev. B **95**, 205142 (2017).
[66] X.-G. Wen and Z. Wang, Volume and topological invariants of quantum many-body systems, Phys. Rev. Research **2**, 033030 (2020).
[67] D. Gaiotto and A. Kapustin, Spin TQFTs and fermionic phases of matter, Int. J. Mod. Phys. A **31**, 1645044 (2016).
[68] T. Banks and N. Seiberg, Symmetries and strings in field theory and gravity, Phys. Rev. D **83**, 084019 (2011).
[69] P. Putrov, J. Wang, and S.-T. Yau, Braiding statistics and link invariants of bosonic/fermionic topological quantum matter in 2+1 and 3+1 dimensions, Ann. Phys. **384**, 254 (2017).
[70] L. Bhardwaj, D. Gaiotto, and A. Kapustin, State sum constructions of spin-TFTs and string net constructions of fermionic phases of matter, J. High Energy Phys. 04 (2017) 096.
[71] T. Lan, C. Zhu, and X.-G. Wen, Fermion decoration construction of symmetry protected trivial orders for fermion systems with any symmetries g_f and in any dimensions, Phys. Rev. B **100**, 235141 (2019).
[72] Y.-A. Chen, Exact bosonization in arbitrary dimensions, Phys. Rev. Research **2**, 033527 (2020).
[73] L. Pontrjagin, C. R. (Doklady) Acad. Sci. URSS (N. S.) **34**, 35 (1942).
[74] J. Whitehead, On simply connected, 4-dimensional polyhedra, Commentarii Mathematici Helvetici **22**, 48 (1949).
[75] H. Georgi and S. L. Glashow, Unity of All Elementary Particle Forces, Phys. Rev. Lett. **32**, 438 (1974).
[76] H. Fritzsch and P. Minkowski, Unified interactions of leptons and hadrons, Ann. Phys. **93**, 193 (1975).
[77] J. Wang and X.-G. Wen, A non-perturbative definition of the standard models, Phys. Rev. Research **2**, 023356 (2020).

[78] N. E. Steenrod, Products of cocycles and extensions of mappings, Ann. Math. **48**, 290 (1947).

[79] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms I: Classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory, Ann. Math. Sci. Appl. **4**, 107 (2019).

[80] F. Wilczek and A. Zee, Families from spinors, Phys. Rev. D **25**, 553 (1982).

[81] Y. Fujimoto, SO(18) unification, Phys. Rev. D **26**, 3183 (1982).

[82] E. Witten, An SU(2) anomaly, Phys. Lett. B **117**, 324 (1982).

[83] R. Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvetici **28**, 17 (1954).

[84] J.-P. Serre, Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, Commentarii Mathematici Helvetici **27**, 198 (1953).

[85] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76 (Princeton University Press, Princeton, NJ, 1974), pp. vii+331.

[86] H. Whitney, On the theory of sphere-bundles, Proc. Natl. Acad. Sci. USA **26**, 148 (1940).

[87] J. Milnor, On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds, Topology **3**, 223 (1965).

[88] A. Dold, Erzeugende der thomschen algebra N, Math. Z. **65**, 25 (1956).

[89] C. Zhu, T. Lan, and X.-G. Wen, Topological nonlinear σ-model, higher gauge theory, and a systematic construction of 3+1D topological orders for boson systems, Phys. Rev. B **100**, 045105 (2019).