Geometric approach to Lieb-Schultz-Mattis theorem without translation symmetry under inversion or rotation symmetry

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We propose a geometric approach to the Lieb-Schultz-Mattis theorem for quantum many-body systems with discrete spin-rotation symmetries and lattice inversion or rotation symmetry, but without translation symmetry assumed. Under symmetry twisting on a $(d-1)$-dimensional plane, we find that any d-dimensional inversion-symmetric spin system possesses a doubly degenerate spectrum when it hosts a half-integer spin at the inversion-symmetric point. We also show that any rotation-symmetric generalized spin model with a projective representation at the rotation center has a similar degeneracy under symmetry twisting. We argue that these degeneracies imply that a unique symmetric gapped ground state that is smoothly connected to product states is forbidden in the original untwisted systems—generalized inversional or rotational Lieb-Schultz-Mattis theorems without lattice translation symmetry imposed. The traditional Lieb-Schultz-Mattis theorems with translations also fit in the proposed framework.

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I. INTRODUCTION

Strongly interacting many-body systems are central topics in condensed-matter and statistical physics. An important concept in the study of these systems is the Lieb-Schultz-Mattis (LSM) theorem [1] and its generalizations [2–5] that state an *ingappability*—an inability of having a unique gapped ground state that belongs to the phase of a trivial product state—of the systems respecting U(1) and translation symmetries with a fractional filling. Recent works study the ingappabilities from the interplay between translations and other symmetries, e.g., SU(2) [6,7] and SU(N) [2,8] symmetries, or even discrete subgroup symmetries [9–14].

The lattice translation symmetry is essential in these LSM-type theorems. Several generalizations have been recently proposed to systems with other lattice symmetries than translations [11–13,15–17]. Such extensions have been made in spin systems with rotation or inversion symmetry in low dimensions by employing lattice homotopy principles [15,17] and even proven in a rigorous manner in one dimension [12,13]. The ingappabilities of these systems are also related to the quantum anomaly of field theories [7,17,18]. Nevertheless, higher-dimensional generalizations are not well established and require further studies, e.g., inversions beyond one and two dimensions in a unifying way. Moreover, convincing lattice-based arguments are still lacking for systems with general lattice rotational and inversional symmetries.

In this paper, we propose a geometric picture to study LSM-type (in)gappabilities for generic lattice systems without translation symmetries. As concrete examples, we first focus on spin systems with discrete spin-rotation symmetries [rather than the full SO(3) for generality] and site-centered inversions in arbitrary dimensions, and general lattice rotations in two dimensions. We consider the closed geometry of finite lattice systems by identifying boundary spins, consistent with the lattice symmetries. We then twist the boundary condition using spin-rotation symmetries in a certain geometric pattern. Assuming that bulk properties are insensitive to such twisting at the boundary [14,19,20], we can extract ingappabilities of the original (untwisted) system from the interplay between the geometric pattern and lattice symmetries. This geometric picture is expected to apply to broader classes of systems, e.g., with nonsymmorphic symmetries [11,21].

II. GEOMETRIC PATTERNS OF SYMMETRY-TWISTED BOUNDARY CONDITIONS

An important indicator of the LSM-type ingappabilities is the ground-state degeneracies of a many-body system on a closed lattice that take values larger than unity in the presence of spontaneous symmetry breaking (SSB) or fractionalization. Quite often, periodic boundary conditions (PBCs) are assumed for Hamiltonians on square lattices with sizes L_i by identifying sites $\vec{r}_i \sim \vec{r}_i + L_i$. However, in the following discussion, except for certain symmetry requirements, we do not specify any concrete form of the Hamiltonian in advance, and we will choose boundary conditions that are compatible with all possible Hamiltonians respecting the required symmetries. For instance, when the imposed symmetry is the inversion $\vec{r} \to -\vec{r}$ about the origin $\vec{r} = \vec{0}$, an inversion-symmetric Hamiltonian cannot be closed by the PBC, as shown in Fig. 1(a); the PBC is inapplicable when only the inversion is imposed [22]. In contrast, the boundary closing shown in Fig. 1(b) is compatible with the inversion symmetry.

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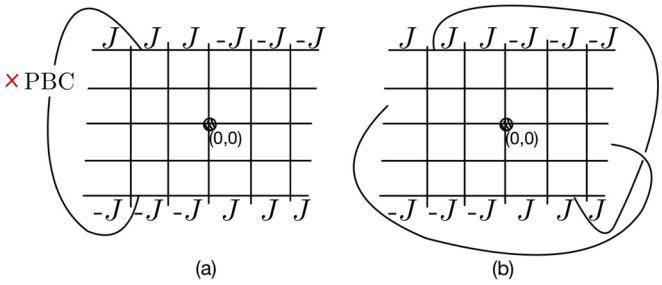

FIG. 1. (a) An inversion-symmetric Hamiltonian with coupling constants $\pm J$ has a boundary that cannot be closed by the PBC. (b) The boundary closing by identifications $\vec{r} \sim -\vec{r}$ is compatible with any inversion-symmetric Hamiltonian. The resultant manifold in the continuum limit is a real projective plane $\mathbb{RP}^2$.

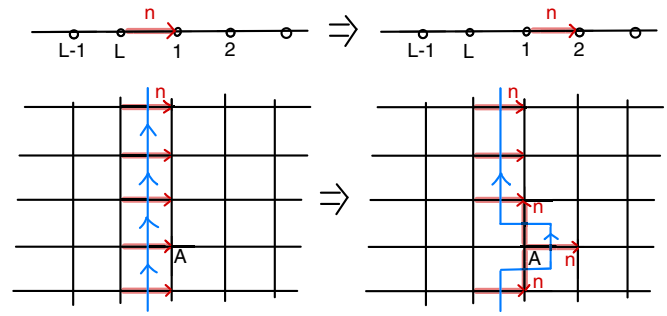

FIG. 3. The $n \in \mathbb{Z}_m$ twisting can be moved or deformed by a gauge transformation: $\mathcal{H}_{\rm tw} \to U^{-1}\mathcal{H}_{\rm tw}U$ with $U = \exp(i2\pi c^\dagger_{1(A)} c_{1(A)} n/m)$.

When the system possesses internal symmetries, such as U(1) or discrete $\mathbb{Z}_m$ symmetry, we can "mix" them into the boundary condition as follows. Let us take $\mathbb{Z}_m$ twisting in tight-binding models with charged operator $c_i^\dagger$ for illustration. In one dimension, angle-$(2\pi n/m)$ twisting with an integer n can be introduced by substituting the boundary hopping term $[c_L c_1^\dagger + \text{H.c.}]$ with $[\exp(i2\pi n/m) c_L c_1^\dagger + \text{H.c.}]$. For general coupling terms involving multiple sites crossing the bond between sites L and 1, we can apply the U(1) transformation $c_j^\dagger \to \exp(i2\pi n/m) c_j^\dagger$ *only* on the sites j on the "right" of the bond [23]. This twisting has two equivalent geometric presentations in Fig. 2(a). The first one is the obvious type, where a twisted boundary bond is marked in bold with an arrow pointing to the "right." The second type is the dual one: An arrowed dot is drawn on this bond to indicate that interaction terms across this dot get twisted by the phase specified with the integer n. These two dual pictures are almost the same in one dimension, but their differences will be clear in higher dimensions, making one of them more convenient than the other depending on the situation.

In two dimensions, we can similarly twist all bonds crossing the boundary line by the internal symmetries as shown in Fig. 2(b). The dual picture is exactly the boundary line which transversally intersects those bond centers, and the line is labeled by $n \in \mathbb{Z}_m$ with oriented arrows. In general d dimensions, we twist the bonds intersected by the $(d-1)$-dimensional boundary hypersurface. The dual picture has a certain hypersurface with an orientation and its symmetry label, which is a special case of the Poincaré duality.

Since the boundary of the boundary is always empty, the boundary line or face (transversal to all the twisted bonds)

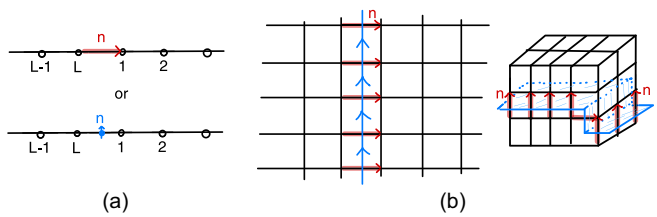

FIG. 2. Two equivalent representations of $n \in \mathbb{Z}_m$ twisting in one dimension (a) and higher dimensions (b). We will use either of them, whichever is convenient.

must be closed itself, i.e., closed loops or surfaces. It exactly means that the total net symmetry twisting around any plaquette must be zero. In the dual picture of symmetry twisting, we can argue that this mixture of twisting into the boundary condition does not introduce any boundary modes at the twisted bonds. It is a direct consequence of the following gauge invariance. We can use a gauge degree of freedom to rename the charged operator by a unitary operator. For example, in one dimension, the transformation $c_1^\dagger \to c_1^\dagger \exp(-i2\pi n/m)$ with a unitary operator $U = \exp(i2\pi c_1^\dagger c_1 n/m)$ undoes the earlier twisting of the bond between sites L and 1. However, it creates a new twisting between sites 1 and 2, and so effectively the twisting is moved by one site as shown in the upper panel of Fig. 3. Thus we can move the twisting freely by unitary transformations, which implies the absence of physical boundary modes there.

The above argument using gauge degrees of freedom applies to higher dimensions as shown in Fig. 3; the loop or surface transversal to the twisted bonds can be deformed arbitrarily by local unitary transformation, thereby unchanging any energy-spectrum property. It should be noted that such deformations are performed only locally in that they cannot make a noncontractible loop into a contractible one. These gauge properties of twisting induce the concept of bulk insensitivity, i.e., the insensitivity of the LSM-type gappability to the symmetry twisting. In other words, if the system has a unique symmetric gapped ground state that is in the same phase as trivial product states, it will still have a unique symmetric gapped ground state after the symmetry twisting. The bulk insensitivity is physically reasonable although not proven in general. In fact, it is only proven under a certain assumption on the first excited state in one dimension [19] and justified by a quantum-transfer-matrix formalism [14] in higher dimensions. In the following discussion, we will assume the insensitivity to twisted boundary conditions to derive LSM-type theorems.

A comment is here in order. When we impose the boundary condition of Fig. 1(b) on a ground state in an invertible or intrinsic topological ordered phase [24] (e.g., a quantum Hall phase or one of its higher-dimensional generalizations) that has (chiral) boundary modes, the chiral boundary modes remain gapless because two modes with the same chirality cannot be gapped. Therefore our theory with inversion

symmetry does not apply to invertible or intrinsic topological ordered phases.

III. INVERSIONAL LSM THEOREMS

Let us consider a spin chain $\mathcal{H}_0$ with length L respecting a site-centered inversion symmetry

$$I^{-1}\vec{S}_j I = \vec{S}_{-j} \tag{1}$$

and $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete spin-rotation symmetry

$$R_x^\pi = \exp\left(i\pi \sum_{j=1}^{L} S_j^x\right), \quad R_z^\pi = \exp\left(i\pi \sum_{j=1}^{L} S_j^z\right), \tag{2}$$

where the spin operator $\vec{S}_j \equiv [S_j^x, S_j^y, S_j^z]$ is not necessarily of the same spin representation for all sites j, except for those related by the inversion symmetry, and the spin of $\vec{S}_{j=0}$ at the inversion center is s. Considering the inversion symmetry, we can close the chain by identifying the boundary sites at the two ends that are related by inversion to each other. We then twist the chain by $\mathbb{Z}_2$ symmetry: R_z^π. Since the twisting position can be changed freely by gauge invariance, we put the twisting on the bond between sites -1 and 0 and denote the twisted Hamiltonian as $\mathcal{H}_{\rm tw}$. A typical example is the XYZ model:

$$\mathcal{H}_{\rm tw}^{XYZ} = \sum_{j\neq 0} h_j^{XYZ} - J_x S_{-1}^x S_0^x - J_y S_{-1}^y S_0^y + J_z S_{-1}^z S_0^z,$$

where $h_j^{XYZ} \equiv \sum_{\alpha=x,y,z} J_\alpha S_{j-1}^\alpha S_j^\alpha$. The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ is represented in the upper left panel of Fig. 4, where the arrows are irrelevant in the case of $\mathbb{Z}_2$ twisting. It still respects $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry but explicitly breaks the inversion symmetry I since the twisting is moved to the bond between sites 0 and 1 by I, as shown in the upper right panel of Fig. 4. Nevertheless, we can map it back by a gauge transformation $r_{z,0}^\pi \equiv \exp(i\pi S_{j=0}^z)$. It implies that $\mathcal{H}_{\rm tw}$ respects the modified inversion symmetry

$$\tilde{I} \equiv I \cdot r_{z,0}^\pi, \quad [\tilde{I}, \mathcal{H}_{\rm tw}] = 0, \quad \tilde{I} R_x^\pi = (-1)^{2s} R_x^\pi \tilde{I} \tag{3}$$

and $\mathcal{H}_{\rm tw}$ still respects $R_{x,z}^\pi$. Here, the factor $(-1)^{2s}$ comes from the commutator

$$r_{z,0}^\pi r_{x,0}^\pi = (-1)^{2s} r_{x,0}^\pi r_{z,0}^\pi. \tag{4}$$

It implies that $\mathcal{H}_{\rm tw}$ must have a doubly degenerate energy spectrum if $s = 1/2, 3/2, \ldots$, i.e., a half-integer spin at the origin. Thus the original Hamiltonian $\mathcal{H}_0$ cannot have a unique gapped ground state when $s = 1/2, 3/2, \ldots$: Either a gapless or degenerate ground state should result in the thermodynamic limit, because, otherwise, the bulk insensitivity would mean that $\mathcal{H}_{\rm tw}$ could also have a unique gapped ground state that contradicts the above double degeneracy. Note that the twofold degeneracy discussed above is not due to the Kramers theorem (applicable only when the total spin is a half-integer) but comes from the projective representation of $\vec{S}_0$.

This approach can be generalized to arbitrary higher dimensions with $\mathbb{Z}_2 \times \mathbb{Z}_2$ and inversion I about the origin $\vec{r} \to -\vec{r}$, but the boundary closing needs more consideration. For instance, in two dimensions, the PBC is not consistent as shown in Fig. 1(a), and the only sensible way of closing is identifying boundary sites $\vec{S}_{\vec{r}} \sim \vec{S}_{-\vec{r}}$ in Fig. 1(b), which is compatible with any inversion-symmetric Hamiltonian $\mathcal{H}_0$. Then we twist $\mathcal{H}_0$ by $\mathbb{Z}_2 : R_\pi^z$ as in the middle left panel of Fig. 4 with a series of twisted bonds or, in the dual picture, a closed loop. The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ softly breaks I and has the modified inversion $\tilde{I}$ in the same form as Eq. (3). Therefore, by employing a similar argument as above, we see that $\mathcal{H}_0$ cannot have a unique gapped ground state when the spin s at $\vec{r} = 0$ is a half-integer.

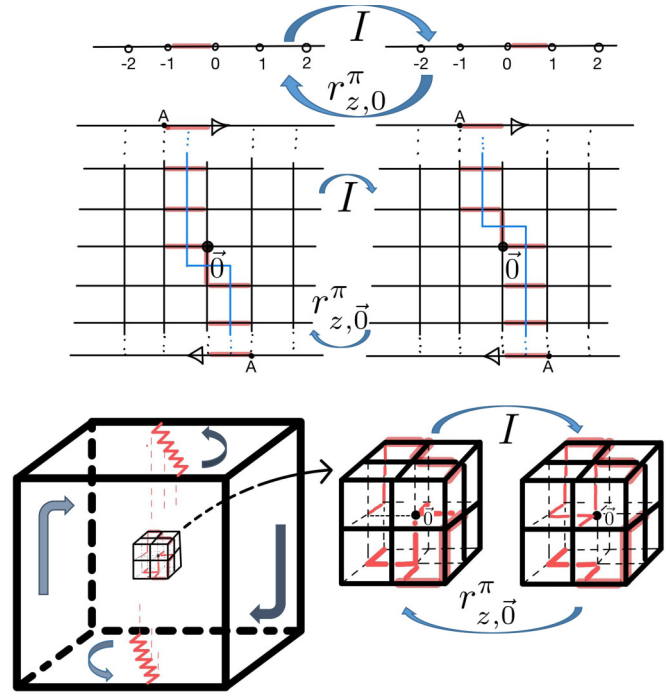

FIG. 4. The twisted Hamiltonians $\mathcal{H}_{\rm tw}$ on the left side are transformed by inversions to $I^{-1}\mathcal{H}_{\rm tw}I$ on the right side. $I^{-1}\mathcal{H}_{\rm tw}I$ can be transformed back by the gauge transformation $(r_{z,0}^\pi)^{-1}(I^{-1}\mathcal{H}_{\rm tw}I)r_{z,0}^\pi = \mathcal{H}_{\rm tw}$, which implies that $\mathcal{H}_{\rm tw}$ is symmetric under $\tilde{I}$ in Eq. (3). The sites on the boundary are identified by $\vec{r} \sim -\vec{r}$, so the continuum limit of the lattice in d dimensions is a real projective hyperplane $\mathbb{RP}^d$.

The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ in three dimensions is sketched in the bottom panel of Fig. 4. The twisting satisfies the closed-form condition that the total twisting around any plaquette is zero. The same result as in the lower-dimensional cases follows since $\mathcal{H}_{\rm tw}$ respects $\tilde{I}$. The geometric situation in arbitrary dimensions is that when we perform the inversion transformation, the spin at $\vec{r} = 0$ is wrapped by the hypersurface (in the dual picture) spanned by the centers of twisted bonds together with the hypersurface inverted by I. The wrapping is exactly canceled by the gauge transformation $r_{z,0}^\pi$, which gives the commutator as in Eq. (3). It follows that $\mathcal{H}_0$ respecting $\mathbb{Z}_2 \times \mathbb{Z}_2$ and I is LSM-type ingappable in arbitrary dimensions when s is not an integer.

IV. ROTATIONAL LSM THEOREMS

In two dimensions, the inversion $\vec{r} \to -\vec{r}$ is equivalent to lattice rotation by 180°. It is natural to generalize I to be N-fold rotations C_N generated by a lattice rotation by $2\pi/N$

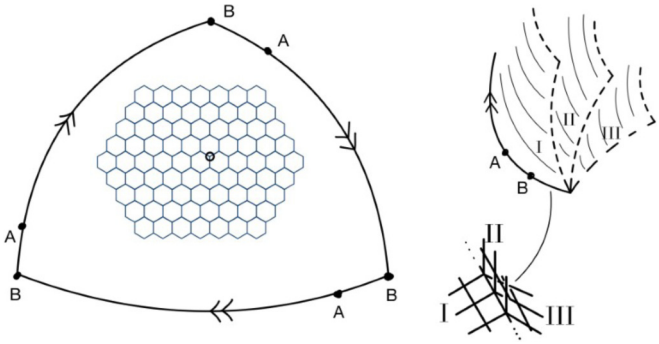

FIG. 5. The only boundary closing compatible with any C_3-symmetric Hamiltonian is to identify three boundary edges. The continuum limit is a pseudo projective plane of order 3 with a three-page intersection (upper right) which is realized by a simple identification of three edges at the lattice scale (bottom right).

around the origin $\vec{r} = \vec{0}$, e.g., C_3 on a honeycomb lattice:

$$C_N : \vec{r} \to \begin{bmatrix} \cos(2\pi/N) & -\sin(2\pi/N) \\ \sin(2\pi/N) & \cos(2\pi/N) \end{bmatrix} \vec{r}. \tag{5}$$

However, even with translation symmetries, spin-1/2 systems on the honeycomb lattice admit a unique gapped ground state with C_3 and SO(3) spin-rotation symmetries [7,25]. This motivates us to consider SU(3) "spin" degrees of freedom or generally SU(N) "spins" obeying an su(N) algebra [26,27]. However, the only condition to be used here is that there is a discrete "spin"-rotation symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$ (known as shift symmetries [28,29]) by

$$V_N = \prod_{\vec{r}} v_{N,\vec{r}} \quad \text{and} \quad W_N = \prod_{\vec{r}} w_{N,\vec{r}}, \tag{6}$$

which generalize R_z^π and R_x^π of the $N = 2$ case in Eq. (2) [30]. There is a number b called the Young-tableau box number [31], analogous to the spin s above, to characterize the SU(N) "spin" at the origin by the following commutator:

$$v_{N,\vec{r}=0} w_{N,\vec{r}=0} = \exp\left(i\frac{2\pi b}{N}\right) w_{N,\vec{r}=0} v_{N,\vec{r}=0}. \tag{7}$$

Thus $b = 2s$ when $N = 2$, and we do not need detailed forms of $v_{N,\vec{r}}$ and $w_{N,\vec{r}}$.

Let us first consider an SU(3) "spin" system on a honeycomb lattice respecting C_3 and $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetries with a "spin" of b at the rotation center $\vec{r} = 0$. In closing the boundary, the PBC is incompatible for a similar reason to the one given before, and the only consistent way is to identify or paste the boundary sites by C_3 as in the left panel of Fig. 5. In the continuum limit, the resultant space is called a pseudo projective plane of order 3, and it has a threefold intersection at the boundary closing, which is realized at the lattice scale by gluing the three edges together, i.e., boundary-site identifications $\vec{r} \sim C_3(\vec{r})$ as in Fig. 5. Then we twist the Hamiltonian by $\mathbb{Z}_3, V_3$ symmetry and deform the twisting configuration as shown in the upper left panel of Fig. 6. The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ preserves $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry but breaks C_3. Nevertheless, as shown in the upper right panel of

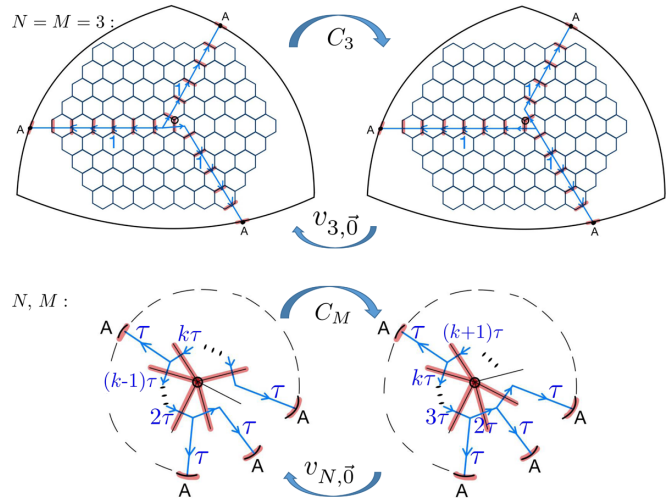

FIG. 6. The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ is presented on the left side, which is transformed to $C_M^{-1}\mathcal{H}_{\rm tw}C_M$ on the right side and can be transformed back by $(v_{N,0})^\tau$: $(v_{N,0})^{-\tau}C_M^{-1}\mathcal{H}_{\rm tw}C_M(v_{N,0})^\tau = \mathcal{H}_{\rm tw}$ with $\tau = N/\text{g.c.d.}(M,N)$. The twisting numbers should be understood as mod$N$, so a p twisting is equal to an $(N-p)$ twisting with the opposite arrow direction.

Fig. 6 it preserves a modified rotation

$$\widetilde{C}_3 \equiv C_3 v_{N=3,\vec{r}=0}, \tag{8}$$

which implies a nontrivial commutator between symmetries $\widetilde{C}_3$ and W_3:

$$\widetilde{C}_3 W_3 = \exp(i2\pi b/3) W_3 \widetilde{C}_3. \tag{9}$$

Thus, when 3 does not divide b, $\mathcal{H}_{\rm tw}$ must possess a triply degenerate spectrum, and the bulk insensitivity implies that the original Hamiltonian $\mathcal{H}_0$ cannot have a unique gapped ground state when the central "spin" b is not a multiple of 3.

For general N, we consider an SU(N) spin system with $\mathbb{Z}_N \times \mathbb{Z}_N$ and general C_M symmetries, where M and N are unnecessarily equal and there is an SU(N) spin of box b at $\vec{r} = 0$. As shown in the bottom left panel of Fig. 6, we close the lattice to form a pseudo projective plane of order M and introduce M lines of V_N twisting to the τth power radiating from $\vec{r} = 0$ with $\tau \equiv N/\text{g.c.d.}(M,N)$ to have a well-defined twisting, i.e., twisting around any plaquette is 0 mod$N$. Here, "g.c.d." denotes the greatest common divisor. The modified $\widetilde{C}_M = C_M(V_N)^\tau$ is respected by $\mathcal{H}_{\rm tw}$ and

$$\widetilde{C}_M W_N = \exp\left(i\frac{2\pi b}{\text{g.c.d.}(M,N)}\right) W_N \widetilde{C}_M. \tag{10}$$

Together with the bulk insensitivity, the above phase factor gives the rotational LSM theorem: When a two-dimensional SU(N) system preserves $\mathbb{Z}_N \times \mathbb{Z}_N$ and C_M symmetries, a unique gapped ground state is forbidden when the box number b of the "spin" at the rotational center is not a multiple of g.c.d.(M,N). This explains why C_3 and spin-rotation symmetries cannot ensure any ingappability on the spin-1/2 honeycomb lattice [7,25] where $b = 2s \in \mathbb{Z}$ and $N = 2$, $M = 3$ coprime.

For example, the SU(4) Kugel-Khomskii (KK) model with $b = 1$ on the square lattice [32,33] is ingappable from Eq. (10) with $M = 4$ or 2, which is consistent with the C_M-SSB plaquette phase obtained by numerics [34,35]. In contrast, the same SU(4) KK model is found to be ingappable (gappable) on the triangular lattice [36] from Eq. (10) with $M = 2$ ($M = 3$), which favors SSB of C_2 symmetry while C_3 symmetry is kept intact; see Ref. [37].

V. CONCLUDING REMARKS

Our geometric paradigm suggests the following general framework in arbitrary dimensions: (1) We close the lattice tentatively in all compatible ways with lattice symmetries; (2) we do the symmetry twisting on the bonds transversal to a *noncontractible* hypersurface of codimension one; and (3) the ingappability is extracted from the algebra of the modified lattice symmetry. Thus it is not restricted to the symmetries discussed in this paper, and we expect that it can be applied to more general settings, e.g., the systems with nonsymmorphic symmetries [11,21] and other rotationlike symmetries such as space dihedral symmetries [7,15,17]. It is applicable to traditional LSM theorems with translations as shown in the Appendix.

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APPENDIX: GEOMETRIC DERIVATION OF LSM THEOREMS WITH TRANSLATION SYMMETRY

In the main text, we have focused on LSM-type theorems for lattice Hamiltonians without any translation symmetry imposed. We have developed a geometric approach using symmetry-twisted boundary conditions to studying ingappability of spin systems or SU(N) generalizations from rotation, inversion, and discrete spin-rotation symmetries. In fact, such a geometric approach can also be applied to traditional LSM theorems with translation symmetries and b boxes per unit cell, reviewed in Fig. 7, where we close the lattice by a tilted boundary condition which is compatible with translations. The SU(N) symmetry per unit cell at $\vec{r}$ is generated by $v_{N,\vec{r}}$ and $w_{N,\vec{r}}$ satisfying

$$v_{N,\vec{r}} w_{N,\vec{r}} = \exp\left(i\frac{2\pi b}{N}\right) w_{N,\vec{r}} v_{N,\vec{r}}, \tag{A1}$$

where b is $\vec{r}$ independent due to the imposed translation symmetry.

The modified translation symmetry $\widetilde{T}_1 = T_1 v_{N,\vec{C}}$, where $\vec{C}$ is the spin position imposed by the gauge transformation $v_{N,\vec{C}}$, is preserved by the twisted Hamiltonian and has a commutator with W_N symmetry:

$$\widetilde{T}_1 W_N = \exp\left(i2\pi\frac{b}{N}\right) W_N \widetilde{T}_1. \tag{A2}$$

When b is indivisible by N, the LSM-type ingappability is concluded from the bulk insensitivity.

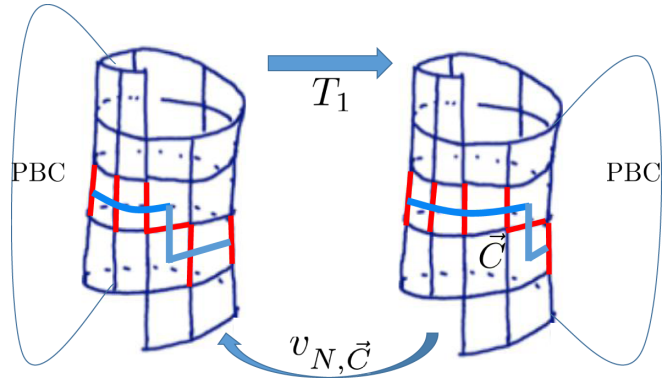

FIG. 7. The twisted Hamiltonian $\mathcal{H}_{\rm tw}$ is presented on the left side, where we suppress the arrows and numbers of twisting for simplicity. $\mathcal{H}_{\rm tw}$ is transformed by the translation T_1 and can be mapped back by a following gauge transformation $v_{N,\vec{C}}$ with $\vec{C}$ indicated above. This argument is generalizable to arbitrary dimensions.

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