Elementary derivation of the stacking rules of invertible fermionic topological phases in one dimension

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(Received 3 May 2022; revised 21 June 2022; accepted 23 June 2022; published 12 July 2022)

Invertible fermionic topological (IFT) phases are gapped phases of matter with nondegenerate ground states on any closed spatial manifold. When open boundary conditions are imposed, nontrivial IFT phases support gapless boundary degrees of freedom. Distinct IFT phases in one-dimensional space with an internal symmetry group G_f have been characterized by a triplet of indices $([(\nu,\rho)],[\mu])$. Our main result is an elementary derivation of the fermionic stacking rules of one-dimensional IFT phases for any given internal symmetry group G_f from the perspective of the boundary, i.e., we give an explicit operational definition for the boundary representation $([(\nu_\wedge,\rho_\wedge)],[\mu_\wedge])$ obtained from stacking two IFT phases characterized by the triplets of boundary indices $([(\nu_1,\rho_1)],[\mu_1])$ and $([(\nu_2,\rho_2)],[\mu_2])$, respectively.

DOI: 10.1103/PhysRevB.106.035117

I. INTRODUCTION

Invertible topological phases of matter are described by Hamiltonians that are spatially local and support a nondegenerate and gapped ground state on *any closed* spatial manifold, after the thermodynamic limit has been taken. By convention, such a Hamiltonian is said to realize the trivial invertible topological phase if its gapped and nondegenerate ground state is a direct product of localized states, one localized state for each local degree of freedom. When space is not a closed manifold, any Hamiltonian realizing a *nontrivial* invertible topological phase must support gapless degrees of freedom that are localized at the boundaries. The problem of classifying the invertible topological phases in d-dimensional space has attracted a lot of interest in the last decade [1–4].

A continuous deformation of a local Hamiltonian is defined to include both the continuous change of short-range couplings between all existing local degrees of freedom or the addition (removal) of decoupled local degrees of freedom that realize a trivial invertible topological phase of their own. Any pair of Hamiltonians with nondegenerate and gapped ground states on any closed manifold are said to be equivalent if they can be continuously deformed into one another without closing the spectral gap. Invertible topological phases are then defined as the equivalence classes of such Hamiltonians under gap-preserving continuous deformations. Invertible topological phases display a group structure under a composition rule called the *stacking rule*. The stacking of any pair of invertible phases consists in creating a new invertible phase by defining the new local degrees of freedom to be the union of the local degrees of freedom from a representative of each invertible phase and by defining the new Hamiltonian acting on the new local degrees of freedom by taking the direct sum of the pair of representative Hamiltonians for each invertible phase. If the invertible topological phase resulting from stacking is the trivial one, then the pair of stacked invertible topological phases are inverse pairs.

The classification of invertible topological phases can be enriched by imposing an internal (independent of space) symmetry group G such that two invertible phases are equivalent only if they can be continuously deformed to one another without gap closing and without (neither explicitly nor spontaneously) breaking the G symmetry. Those invertible topological phases that are equivalent to the trivial phase under the continuous deformation that spontaneously or explicitly break the G symmetry are called the *symmetry protected topological* (SPT) phases [1,5]. When open boundary conditions are imposed, SPT phases support gapless degrees of freedom at the boundaries that are protected by the G symmetry, i.e., the boundary degrees of freedom cannot be gapped unless G symmetry is either explicitly or spontaneously broken.

Invertible topological phases are best understood in one-dimensional space. Their classification has been conjectured based on the study of translation-invariant and injective matrix product states (MPS) [6–10]. By taking advantage of the split property of nondegenerate gapped ground states in one-dimensional space [11], Bourne and Ogata in Ref. [12] have derived rigorously for any internal symmetry group G an exhaustive classification of invertible fermionic topological (IFT) phases that includes their stacking rules.

In this paper, we build on the seminal work by Fidkowski and Kitaev in Ref. [6] and provide an operational construction of the boundary representations of any internal G_f symmetry imposed on IFT phases in one-dimensional space. We find the counterparts that characterize the boundary representations to the topological indices used to classify one-dimensional IFT phases from a bulk perspective in Refs. [10,12]. Moreover, we explicitly derive their stacking rules by elementary methods. Our stacking rules differ from the ones derived in Ref. [9], but agree with the ones derived in Refs. [10,12].

This paper is organized as follows. In Sec. II we lay out our strategy and summarize the results. In Secs. III and IV, we define the boundary degrees of freedom and the representation of the internal symmetry group G_f acting on them, respectively, for any one-dimensional IFT phase. In Sec. V, we define a triplet of indices $([(\nu,\rho)],[\mu])$ that characterizes the boundary properties of a one-dimensional IFT phase. In Sec. VI, we derive the fermionic stacking rules of one-dimensional IFT phases by only using elementary means. In Sec. VIII, we relate some supersymmetric properties of the ground-state manifold when open boundary conditions are chosen to some values taken by the triplet $([(\nu,\rho)],[\mu])$. Our conclusions can be found in Sec. IX, while Appendices A, B, and C review group cohomology, central extension class, and the detailed definition of the triplet of boundary indices $([(\nu,\rho)],[\mu])$, respectively.

II. STRATEGY AND SUMMARY OF RESULTS

The classification of the IFT phases in one-dimensional space is intimately related to the classification of the projective representations of the fermionic symmetry group G_f, an internal symmetry acting globally on the fermionic Fock space. To illustrate this, we will first consider the representations of G_f on a closed one-dimensional chain and then investigate the consequences of imposing open boundary conditions.

We denote by Λ the set of points on a one-dimensional lattice. Given are the fermionic symmetry group G_f (Appendices A and B) and a global fermionic Fock space $\mathfrak{F}_\Lambda$ defined over Λ. We assume that there exists a faithful trivial representation $\widehat{U}_{\mathrm{bulk}}$ of the group G_f on the lattice Λ. In other words, there exists an injective map $\widehat{U}_{\mathrm{bulk}}: G_f \to \mathrm{Aut}(\mathfrak{F}_\Lambda)$ where $\mathrm{Aut}(\mathfrak{F}_\Lambda)$ is the set of automorphisms on the fermionic Fock space such that for any $g,h\in G_f$,

$$\widehat{U}_{\mathrm{bulk}}(g)\,\widehat{U}_{\mathrm{bulk}}(h)=\widehat{U}_{\mathrm{bulk}}(g\,h), \tag{2.1a}$$

where $g\,h$ denotes the composition of the elements $g,h\in G_f$. We define a group homomorphism

$$\mathfrak{c}: G_f \to \{0,1\} \tag{2.1b}$$

that specifies if an element $g\in G_f$ is to be represented by a unitary operator, in which case $[\mathfrak{c}(g)=0]$, or by an antiunitary operator, in which case $[\mathfrak{c}(g)=1]$. For any $g\in G_f$, its representation $\widehat{U}_{\mathrm{bulk}}(g)$ can be written as

$$\widehat{U}_{\mathrm{bulk}}(g)=\widehat{V}_{\mathrm{bulk}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \tag{2.1c}$$

where $\widehat{V}_{\mathrm{bulk}}(g)$ is a unitary operator acting on $\mathfrak{F}_\Lambda$ and K is the complex conjugation map.

For each point $j\in\Lambda$, we associate a set of Hermitian Majorana operators

$$\mathfrak{O}_j := \{\hat{\gamma}^{(j)}_1,\hat{\gamma}^{(j)}_2,\cdots,\hat{\gamma}^{(j)}_{n_j}\}, \tag{2.2a}$$

that realizes the Clifford algebra

$$C\ell_{n_j} := \mathrm{span}\left\{\prod_{\iota=1}^{n_j}\left(\hat{\gamma}^{(j)}_\iota\right)^{m_\iota}\ \middle|\ \{\hat{\gamma}^{(j)}_\iota,\hat{\gamma}^{(j)}_{\iota'}\}=2\delta_{\iota\iota'},\quad m_i=0,1,\quad \iota,\iota'=1,\cdots,n_j\right\}. \tag{2.2b}$$

The n_j Majorana operators (2.2a) span a local fermionic Fock space $\mathfrak{F}_j$ if n_j is an even integer. If n_j is odd, the Clifford algebra $C\ell_{n_j}$ contains a two-dimensional center, reason for which the n_j Majorana operators (2.2a) span a Hilbert space that cannot be interpreted as a fermionic Fock space [13]. The consistent definition of a global fermionic Fock space thus requires the total number of Majorana degrees of freedom to be even, i.e.,

$$\sum_j n_j = 0 \bmod 2. \tag{2.3}$$

We define a *local* [14] representation $\widehat{U}_j$ of the symmetry group G_f by demanding that on the degrees of freedom localized at site $j\in\Lambda$, $\widehat{U}_j$ acts in the same way as the global bulk representation $\widehat{U}_{\mathrm{bulk}}$ does, i.e., the consistency condition

$$\widehat{U}_j(g)\,\hat{\gamma}^{(j)}_\iota\,\widehat{U}^\dagger_j(g)=\widehat{U}_{\mathrm{bulk}}(g)\,\hat{\gamma}^{(j)}_\iota\,\widehat{U}^\dagger_{\mathrm{bulk}}(g), \tag{2.4}$$

for any $g\in G_f$ and $\iota=1,\dots,n_j$ must hold. Hereby, we assume that the bulk representation $\widehat{U}_{\mathrm{bulk}}$ is not anomalous in the sense that there are no obstructions that prevent decomposing $\widehat{U}_{\mathrm{bulk}}$ into the product of local representations $\widehat{U}_j$ (see Refs. [15,16] for examples when this is not possible). The definition (2.4) implies that the representation $\widehat{U}_j$ satisfies for any $g,h\in G_f$

$$\widehat{U}_j(g)\,\widehat{U}_j(h)=e^{\mathrm{i}\phi_j(g,h)}\,\widehat{U}_j(g\,h). \tag{2.5}$$

The phase factor $\phi_j(g,h)\in C^2(G_f,U(1))$ defines a 2-cochain (Appendix A). Its equivalence classes $[\phi_j]$ takes values in the second cohomology group $H^2(G_f,U(1)_{\mathfrak{c}})$ (Appendix A). The equivalence class $[\phi_j]$ characterizes the projective nature of the representation $\widehat{U}_j$. The value $[\phi_j]=0$ is assigned to the trivial projective representation for which the vanishing phase $\phi(g,h)=0$ for any $g,h\in G_f$ is a representative.

By definition, local Hamiltonians with the symmetry group G_f that realize IFT phases of matter must necessarily have nondegenerate and gapped ground states that transform as singlets under the symmetry group G_f with any closed boundary conditions. We restrict our attention to one-dimensional space and to IFT phases of matter with translation symmetry G_{trsl} in addition to the internal fermionic symmetry group G_f. In other words, the total symmetry group G_{tot} is by hypothesis the direct product

$$G_{\mathrm{tot}}\equiv G_{\mathrm{trsl}}\times G_f. \tag{2.6}$$

Imposing translation symmetry G_{trsl} requires the number n_j of Majorana degrees of freedom at each site to be independent of j with the same local representation $\widehat{U}_j(g)$ for any element $g\in G_f$. If so, the Lieb-Schulz-Mattis (LSM) theorems from Ref. [17] apply (see also Ref. [18]). A nondegenerate and gapped ground state that transforms as a singlet under the symmetry group G_{tot} is permissible if and only if:

(1) The number n_j of Majorana degrees of freedom at each site $j\in\Lambda$ is even, i.e., $n_j\equiv 2n$

(2) The local representation $\widehat{U}_j(g)$ realizes a trivial projective representation, i.e., $[\phi_j]\equiv[\phi]=0$.

The first condition requires that there exist a *local* fermionic Fock space $\mathfrak{F}_j$ spanned by the even number of local Majorana degrees of freedom (2.2a). Therefore, the global fermionic Fock space $\mathfrak{F}_\Lambda$ decomposes as a $\mathbb{Z}_2$-graded tensor

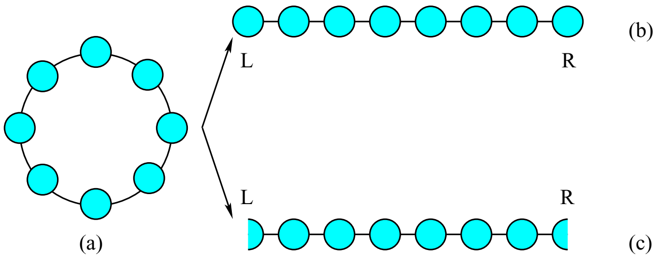

FIG. 1. The repeat unit cells of a one-dimensional lattices are pictured by colored discs. Each repeat unit cell hosts an even number $2n$ of Majorana degrees of freedom. Without loss of generality, the range of the couplings between Majorana degrees of freedom is one lattice spacing (the thick line between the repeat unit cells). Translation symmetry is imposed by choosing periodic boundary conditions, in which case the one-dimensional lattice is the discretization of a ring. Open boundary conditions break the translation symmetry. This can be achieved by cutting a thick line connecting two repeat unit cell or by cutting open a repeat unit cell. In the former case, the number of Majorana degrees of freedom on any one of the upmost left or right cells is the same even number $2n$ of Majorana degrees of freedom as that in a single repeat unit cell. In the latter case, the number of Majorana degrees of freedom on the upmost left cell is any integer $1 < n_{\rm L} < 2n$ while that on the upmost cell is $n_{\rm R} = 2n - n_{\rm L}$.

product $\otimes_{\mathfrak{g}}$ of the local Fock spaces $\mathfrak{F}_j$, i.e.,

$$\mathfrak{F}_\Lambda = \bigotimes_{j\in\Lambda}{}_{\mathfrak{g}}\,\mathfrak{F}_j. \tag{2.7}$$

The second condition requires that the local representation $\widehat{U}_j \in \mathrm{Aut}(\mathfrak{F}_j)$ is a representation in the trivial equivalence class $[\phi] = 0$. This implies that the global bulk representation $\widehat{U}_{\rm bulk}$ decomposes as the product of local representations $\widehat{U}_j$, i.e., for any $g \in G_f$

$$\widehat{U}_{\rm bulk}(g) = \left[\prod_{j\in\Lambda} \widehat{V}_j(g)\right] \mathsf{K}^{\mathfrak{c}(g)}. \tag{2.8}$$

Open boundary conditions break the hypothesis of translation symmetry in the LSM theorem from Ref. [17]. When a closed chain is opened up, the degrees of freedom localized in one or multiple repeat unit cells may be split into two disconnected components $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$ of the boundary $\Lambda_{\rm bd} := \partial\Lambda \equiv \Lambda_{\rm L} \cup \Lambda_{\rm R}$, as is illustrated in Fig. 1. If so, the two requirements of the LSM theorem need no longer hold at each disconnected component.

Any one-dimensional IFT phase of matter is thus characterized by the following data:

(1) There is a $\mathbb{Z}_2$-valued index $[\mu_{\rm B}] = \{0, 1\}$ (B = L, R) that measures the parity of the number of Majorana degrees of freedom that are localized on either one of the left (L) or right (R) boundaries (B) of the open chain $\Lambda_{\rm bd} = \Lambda_{\rm L} \cup \Lambda_{\rm R}$. The index $[\mu_{\rm B}]$ can be viewed as an element of the zeroth cohomology group $H^0(G_f, \mathbb{Z}_2) = \mathbb{Z}_2$.

(2) There is an equivalence class $[\phi_{\rm B}] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ of the second cohomology group (Appendix A) that characterizes the projective representation of the internal symmetry group G_f at either one of the left or right boundaries of an open chain $\Lambda_{\rm bd} = \Lambda_{\rm L} \cup \Lambda_{\rm R}$.

Given a disconnected component $\Lambda_{\rm B}$ of the boundary $\Lambda_{\rm bd}$, we assume the existence of a set of boundary Majorana degrees of freedom

$$\mathfrak{O}_{\rm B} := \left\{\hat{\gamma}^{(\rm B)}_1, \hat{\gamma}^{(\rm B)}_2, \dots, \hat{\gamma}^{(\rm B)}_{n_{\rm B}}\right\} \tag{2.9a}$$

that are associated with states exponentially localized in space at the boundary B. The pair of data $([\phi_{\rm B}], [\mu_{\rm B}]) \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}}) \times H^0(G_f, \mathbb{Z}_2)$ are assigned as follows. The index $[\mu_{\rm B}]$ is nothing but the parity of the number of Majorana degrees of freedom at $\Lambda_{\rm B}$, i.e., $[\mu_{\rm B}] = n_{\rm B} \bmod 2$. The equivalence class $[\phi_{\rm B}]$ of the projective phase $\phi_{\rm B}(g, h)$ is computed by constructing a boundary representation $\widehat{U}_{\rm B}$. This is done by demanding the consistency condition

$$\widehat{U}_{\rm B}(g)\, \hat{\gamma}^{(\rm B)}_\iota\, \widehat{U}^\dagger_{\rm B}(g) = \widehat{U}_{\rm bulk}(g)\, \hat{\gamma}^{(\rm B)}_\iota\, \widehat{U}^\dagger_{\rm bulk}(g), \tag{2.9b}$$

for any $g \in G_f$ and $\iota = 1, 2, \dots, n_{\rm B}$.

The index $[\phi_{\rm B}] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ depends both on $[\mu_{\rm B}] = 0, 1$ and the fermionic symmetry group G_f. This is so because G_f is the central extension of the internal symmetry group G by the fermion-parity symmetry group $\mathbb{Z}^{\rm F}_2$ with extension class $[\gamma] \in H^2(G, \mathbb{Z}^{\rm F}_2)$ (Appendix B), i.e., G is isomorphic to the group $G_f/\mathbb{Z}^{\rm F}_2$ [19]. As the center of the fermionic symmetry group G_f is the fermion-parity subgroup $\mathbb{Z}^{\rm F}_2$, its projective representations are sensitive to the values of $[\mu_{\rm B}]$. This sensitivity can be made explicit if, following Turzillo and You in Ref. [10], one trades the equivalence classes $[\phi_{\rm B}] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ for the equivalence classes $[(\nu_{\rm B}, \rho_{\rm B})] \in \ker \mathcal{D}^2_{\gamma,\mathfrak{c}}/\mathrm{im}\, \mathcal{D}^1_{\gamma,\mathfrak{c}}$ where $\mathcal{D}^2_{\gamma,\mathfrak{c}}$ and $\mathcal{D}^1_{\gamma,\mathfrak{c}}$ are modified coboundary operators (Appendix C).

The 2-cochain $\nu_{\rm B} \in C^2[G, \mathrm{U}(1)]$ encodes the projective representations of the group. The interpretation of the 1-cochain $\rho_{\rm B} \in C^1[G, \mathbb{Z}_2]$ depends on the value of $[\mu_{\rm B}] = 0, 1$. When $[\mu_{\rm B}] = 0$, the 1-cochain $\rho_{\rm B} \in C^1[G, \mathbb{Z}_2]$ encodes the relation between the representations of the elements of the group G and the representation of the fermion-parity from $\mathbb{Z}^{\rm F}_2$. When $[\mu_{\rm B}] = 1$, the 1-cochain $\rho_{\rm B} \in C^1[G, \mathbb{Z}_2]$ encodes the relation between the representations of the elements of the group G and the representation of the nontrivial central element of a Clifford algebra $C\ell_{2k+1}$ with an odd number of generators.

There are two possible scenarios for the fate of the set (2.9a) of boundary degrees of freedom on the boundary $\Lambda_{\rm B}$ that realize the triplet of boundary data $([(\nu_{\rm B}, \rho_{\rm B})], [\mu_{\rm B}])$ when the bulk is perturbed by local and continuous interactions that break neither explicitly nor spontaneously the G_f symmetry. In scenario I, the set (2.9a) is unchanged by the bulk perturbation. If so, the triplet of boundary data $([(\nu_{\rm B}, \rho_{\rm B})], [\mu_{\rm B}])$ does not change. In scenario II, the bulk perturbation changes the set (2.9a) by either the addition or removal of boundary degrees of freedom. If the degrees of freedom added to or removed from the boundary $\Lambda_{\rm B}$ realize the trivial triplet of data, then the resulting triplet of boundary data is unchanged according to the fermionic stacking rules, i.e., $([(\nu_{\rm B}, \rho_{\rm B})], [\mu_{\rm B}])$. If the degrees of freedom added to or removed from the boundary $\Lambda_{\rm B}$ realize a nontrivial triplet of data, then the triplet of boundary data is changed to $([(\nu'_{\rm B}, \rho'_{\rm B})], [\mu'_{\rm B}]) \neq ([(\nu_{\rm B}, \rho_{\rm B})], [\mu_{\rm B}])$ according to the fermionic stacking rules. If the bulk-boundary correspondence were to hold, then a gap-closing transition in the bulk that is induced by the bulk perturbations is required to change the boundary triplet of data. This hypothesis is plausible because Bourne and Ogata have shown rigorously in Ref. [12] the existence of triplets of bulk data that take values in the same cohomology groups as the triplets of boundary data $([(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})], [\mu_{\mathrm{B}}])$, obey the same stacking rules, and offer a bulk classification of IFT phases of matter. In this paper, we shall assume without proof this bulk-boundary correspondence.

There is no need to specify the triplets $([(\nu_{\mathrm{L}}, \rho_{\mathrm{L}})], [\mu_{\mathrm{L}}])$ and $([(\nu_{\mathrm{R}}, \rho_{\mathrm{R}})], [\mu_{\mathrm{R}}])$ associated with the disconnected components Λ_{L} and Λ_{R} independently. The triplet of data on the left boundary Λ_{L} fixes their counterparts on the right boundary Λ_{R}, owing to the condition that the ground state of a Hamiltonian realizing an IFT phase of matter must be nondegenerate and G_f symmetric when periodic boundary conditions are selected. Thus, we drop the subscripts when denoting the triplet of data $([(\nu, \rho)], [\mu])$ that characterize the IFT phases.

For any G_f that splits, i.e., G_f is isomorphic to the product $G \times \mathbb{Z}_2^F$, the index $[\mu]$ can take the values 0 or 1. If the group G_f is a nonsplit group, then $[\mu] = 0$ is the only possibility. When $[\mu] = 1$, the minimal degeneracy of the eigenspace for the ground states when open boundary conditions are selected is two for any split fermionic symmetry group G_f, including the smallest possible fermionic symmetry group $G_f = \mathbb{Z}_2^{\mathrm{F}}$. Hence, one-dimensional Hamiltonians realizing IFT phases of matter with $[\mu] = 1$ cannot be deformed adiabatically to a Hamiltonian realizing the trivial IFT phase of matter at the expense of breaking explicitly any of the protecting symmetries in G_f other than $\mathbb{Z}_2^{\mathrm{F}}$. *A forteriori*, these phases of matter are distinct from the fermionic SPT (FSPT) phases of matter in one-dimensional space. In one-dimensional space, FSPT phases of matter are only possible when $[\mu] = 0$.

Once the IFT phases in one-dimension are characterized by the triplet $([(\nu, \rho)], [\mu])$, it is imperative to derive the stacking rules, i.e., the group composition rules of the triplets $([(\nu, \rho)], [\mu])$ that are compatible with the $\mathbb{Z}_2$-graded tensor product between fermionic Fock spaces (in physics terminology, antisymmetrization). Stacking rules can be derived by considering the topological indices $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ of an IFT phase of matter that is constructed by combining the boundary degrees of freedoms of any representatives of two other IFT phases with topological indices $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$, respectively. The stacking rules are essential properties of IFT phases of matter. They enforce a group composition law between IFT phases of matter sharing the same fermionic symmetry group G_f. This group composition law can be interpreted as the physical operation by which two blocks of matter, each realizing IFT phases of matter sharing the same fermionic symmetry group G_f, are brought into contact so as to form a single larger block of matter sharing the same fermionic symmetry group G_f. This group composition law is also needed to implement a consistency condition corresponding to changing from open to closed boundary conditions. Topological data associated with the left and the right disconnected components of the one-dimensional boundary must be the inverse of each other with respect to the stacking rules, i.e., one should obtain the trivial data $([(0, 0)], 0)$ if the change from open to periodic boundary conditions is interpreted as the stacking of opposite boundaries.

The main result of this paper is the derivation of the stacking rules of any IFT phase of matter in one-dimensional space from the perspective of the boundaries. Working on the boundaries allows to use elementary tools of quantum mechanics and mathematics.

To achieve this goal, we first define the set of boundary degrees of freedom $\mathfrak{D}_{\mathrm{B}}$ and the corresponding representation $\widehat{U}_{\mathrm{B}}$ that satisfy the consistency condition (2.9b) for the cases of $[\mu] = 0$ and $[\mu] = 1$ separately in Secs. III and IV, respectively. In doing so, we give the explicit representation for the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$ of the fermion parity symmetry. In Sec. V, we define the 2-cochains $\phi(g, h) \in C^2(G_f, U(1))$ and $\nu(g, h) \in C^2(G, U(1))$, and the 1-cochain $\rho(g) \in H^1(G, \mathbb{Z}_2)$ for the cases of $[\mu] = 0$ and $[\mu] = 1$ separately.

Second, we derive the fermionic stacking rules using elementary means in Sec. VI. To this end, we consider two boundary representations $\widehat{U}_1$ and $\widehat{U}_2$ together with their associated triplets $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$, respectively. We then explicitly construct the *stacked* representation $\widehat{U}_\wedge$ that acts on the combined degrees of freedom of the two representations. This is done by demanding that the stacked representation $\widehat{U}_\wedge$ satisfies the two conditions that are the counterparts to the consistency condition (2.9b).

When constructing the stacked representation $\widehat{U}_\wedge$, we shall consider the following four cases separately: (i) even-even $([\mu_1] = [\mu_2] = 0)$ stacking, (ii) even-odd $([\mu_1] = 0, [\mu_2] = 1)$ stacking, (iii) odd-even $([\mu_1] = 1, [\mu_2] = 0)$ stacking, (iv) and odd-odd $([\mu_1] = [\mu_2] = 1)$ stacking. We find the four fermionic stacking rules

$$\begin{aligned}&([(\nu_1,\ \rho_1)],\ 0) \wedge ([(\nu_2,\ \rho_2)],\ 0)\\ &\quad = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2),\ \rho_1 + \rho_2)],\ 0),\end{aligned} \tag{2.10a}$$

$$\begin{aligned}&([(\nu_1,\ \rho_1)],\ 0) \wedge ([(\nu_2,\ \rho_2)],\ 1)\\ &\quad = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2 + \rho_1 \smile \mathfrak{c}),\ \rho_1 + \rho_2)],\ 1),\end{aligned} \tag{2.10b}$$

$$\begin{aligned}&([(\nu_1,\ \rho_1)],\ 1) \wedge ([(\nu_2,\ \rho_2)],\ 0)\\ &\quad = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2 + \rho_2 \smile \mathfrak{c}),\ \rho_1 + \rho_2)],\ 1),\end{aligned} \tag{2.10c}$$

$$\begin{aligned}&([(\nu_1,\ \rho_1)],\ 1) \wedge ([(\nu_2,\ \rho_2)],\ 1)\\ &\quad = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2),\ \rho_1 + \rho_2 + \mathfrak{c})],\ 0),\end{aligned} \tag{2.10d}$$

respectively. The derivation of the stacking rules (2.10) is the main result of this paper. Here, we denote the stacking operation with the symbol $\wedge$. We also made use of the cup product $\smile$ defined in Appendix A to construct a 2-cochains out of two 1-cochains. If the group G_f only contains unitary symmetries, i.e., $\mathfrak{c}(g) = 0$ for any $g \in G_f$, then the stacking rules (2.10) reduce to

$$\begin{aligned}&([(\nu_1,\ \rho_1)],\ [\mu_1]) \wedge ([(\nu_2,\ \rho_2)],\ [\mu_2])\\ &\quad = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2),\ \rho_1 + \rho_2)],\ [\mu_1] + [\mu_2]).\end{aligned} \tag{2.11}$$

Finally, we discuss the protected ground-state degeneracy of representatives of an IFT phase with the index $([(\nu, \rho)], [\mu])$ when open boundary conditions are imposed in Sec. VIII. This exercise allows us to give the generic conditions on the index $([(\nu, \rho)], [\mu])$ that imply an emergent boundary supersymmetry, as was recently explored in Refs. [20,21].

As an illustration of this general result, we turn our attention to the symmetry class BDI from the tenfold way (the protecting symmetries are fermion parity and spinless time reversal) in one-dimensional space [22]. In the Supplemental Material [23], we derive in detail (i) the explicit values of $([(\nu_{\mathrm{B}}, \rho_{\mathrm{B}})], [\mu_{\mathrm{B}}])$ for the left (B = L) and the right (B = R) boundaries, (ii) their stacking rules, (iii) and the protected ground-state degeneracies for a class of Hamiltonians that we call Majorana c chains with $c \in \mathbb{Z}$. The asymmetry between the left and right boundaries is made explicit. We then use the Jordan-Wigner transformation to bosonize the Majorana c chains into a family of spin-1/2 cluster c chains [24,25]. These spin-1/2 cluster c chains are shown to realize bosonic symmetry-protected topological phases of matter characterized by a doublet $([\nu_{\mathrm{B}}], [\rho_{\mathrm{B}}])$ of indices that quantify which projective representations of the protecting symmetries (a global rotation in Pauli space by the angle π around some given direction in Pauli space and spinless time-reversal symmetry) is realized on the boundaries. Hereto, we derive (i) the explicit values of $([\nu_{\mathrm{B}}], [\rho_{\mathrm{B}}])$ for the left (B = L) and the right (B = R) boundaries, (ii) their stacking rules, (iii) and the protected ground-state degeneracies. The differences with the Majorana c chains are explained. In particular, it is shown that the left and right boundaries share the same projective representations of the protecting symmetries for the spin-1/2 cluster c chains.

III. ASSUMPTIONS AND DEFINITIONS

We start from a given internal symmetry group G_f acting on Majorana degrees of freedom. The subscript f is attached to emphasize the "fermionic" nature of the group G_f as we shall now explain. For quantum systems built out of Majorana degrees of freedom, any Hamiltonian that dictates the quantum dynamics is built from products of even numbers of Majorana operators. In other words, the fermion parity operator $(-1)^{\widehat{F}}$ necessarily commutes with the Hamiltonian, where $\widehat{F}$ denotes the total fermion number operator. We denote by $\mathbb{Z}_2^{\mathrm{F}} := \{e, p\}$ the cyclic group generated by the abstract element $p \in \mathbb{Z}_2^{\mathrm{F}}$ that we shall interpret as the fermion parity. The superscript F is attached to the cyclic group $\mathbb{Z}_2^{\mathrm{F}}$ to distinguish its role as the fermion parity symmetry. We assume that the fermion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$ can be neither explicitly nor spontaneously broken and is a subgroup of the center of G_f. We denote by $G \cong G_f/\mathbb{Z}_2^{\mathrm{F}}$ the group consisting of all symmetries other than the fermion parity symmetry $\mathbb{Z}_2^{\mathrm{F}}$.

The internal symmetry group G_f is specified by two pieces of data. The first piece is the central extension class $[\gamma] \in H^2(G, \mathbb{Z}_2^{\mathrm{F}})$ that characterize how the group G and the fermion parity symmetry group $\mathbb{Z}_2^{\mathrm{F}}$ are glued together to produce the group G_f. This is to say that, G_f is not restricted to be the direct product $G_f = G \times \mathbb{Z}_2^{\mathrm{F}}$. The group G_f is such that (i) $\mathbb{Z}_2^{\mathrm{F}}$ is a subgroup of the center of G_f (ii) and G is isomorphic to $G_f/\mathbb{Z}_2^{\mathrm{F}}$. We assign the equivalence class $[\gamma] = 0$ to the case of G_f being isomorphic to the direct product $G \times \mathbb{Z}_2^{\mathrm{F}}$ and say that G_f splits (Appendices A and B). The second piece is the group homomorphism $\mathfrak{c} : G_f \to \{0, 1\}$ that specifies if an element $g \in G_f$ is to be represented by a unitary $[\mathfrak{c}(g) = 0]$ operator or by an antiunitary $[\mathfrak{c}(g) = 1]$ operator [by definition, $\mathfrak{c}(p) = 0$].

We denote by Λ the set of points on a one-dimensional lattice that we shall call the bulk. We assume that there exists a nonvanishing boundary

$$\Lambda_{\mathrm{bd}} \equiv \partial\Lambda \tag{3.1a}$$

of the bulk Λ. The boundary Λ_{bd} is the union of two disconnected components Λ_{L} or Λ_{R} of the one-dimensional universe Λ,

$$\Lambda_{\mathrm{bd}} = \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{R}}, \quad \Lambda_{\mathrm{L}} \cap \Lambda_{\mathrm{R}} = \emptyset. \tag{3.1b}$$

The hypothesis that states bound to Λ_{L} or Λ_{R} do not overlap in space only holds for all fermionic invertible topological phases after the thermodynamic limit has been taken.

We assume that there exists a faithful representation of the group G_f acting on the bulk Λ, i.e., an injective map $\widehat{U}_{\mathrm{bulk}} : G_f \to \mathrm{Aut}(\mathfrak{F}_\Lambda)$ where $\mathrm{Aut}(\mathfrak{F}_\Lambda)$ is the set of automorphisms on the fermionic Fock space $\mathfrak{F}_\Lambda$ of the one-dimensional universe. We impose that the map $\widehat{U}_{\mathrm{bulk}}$ forms an ordinary representation of the group G_f on the bulk Λ, i.e., it satisfies Eq. (2.1). This might not be so anymore when restricting the action of G_f to any one of the disconnected components Λ_{L} or Λ_{R} on the boundary Λ_{bd}, in which case the existence of degenerate ground states must follow when open boundary conditions are selected.

Without loss of generality, we consider any one of Λ_{L} and Λ_{R}, which we denote Λ_{B}. We are going to construct a projective representation of the symmetry group G_f on this component Λ_{B} of the boundary Λ_{bd}, while the opposite component of the boundary must then always be represented by the "inverse" projective representation.

On the boundary Λ_{B}, we assume the existence of a set of n Hermitian Majorana operators

$$\mathfrak{O}_n := \{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_n\} \tag{3.2a}$$

that realizes the Clifford algebra

$$\mathrm{C}\ell_n := \mathrm{span}\left\{\prod_{i=1}^{n} (\hat{\gamma}_i)^{m_i} \,\middle|\, \{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij},\right. \\ \left. m_i = 0, 1, \quad i, j = 1, \cdots, n\right\}. \tag{3.2b}$$

We call these operators Majorana operators. We assign the index $[\mu] \in \{0, 1\}$ to the parity of n, i.e.,

$$[\mu] = n \bmod 2. \tag{3.2c}$$

We consider the cases of even and odd n separately.

When $[\mu] = 0$, the even number n of Majorana operators from the set (3.2a) span the fermionic Fock space

$$\mathfrak{F}_{\Lambda_{\mathrm{B}},0} := \mathrm{span}\left\{\prod_{\alpha=1}^{n/2} \left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)^{m_\alpha} |0\rangle \,\middle|\, \right. \\ \left. \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)|0\rangle = 0, \quad m_\alpha = 0, 1\right\} \tag{3.3a}$$

of dimension [26]

$$\dim \mathfrak{F}_{\Lambda_{\mathrm{B}},0} = 2^{n/2}. \tag{3.3b}$$

When $[\mu] = 1$, the odd number n of Majorana operators from the set (3.2a) span a vector space that is not a fermionic Fock space. In order to recover a fermionic Fock space, we add to the set (3.2a) made of an odd number n of Majorana operators the Majorana operator $\hat{\gamma}_\infty$ [6],

$$\mathfrak{O}_{n,\infty} := \{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{2\lfloor n/2\rfloor}, \hat{\gamma}_n, \hat{\gamma}_\infty\}, \tag{3.4}$$

thereby defining the Clifford algebra $C\ell_{n+1}$. Here, the lower floor function $\lfloor\cdot\rfloor$ returns the largest integer $\lfloor x\rfloor$ smaller than the positive real number x. We may then define the fermionic Fock space

$$\mathfrak{F}_{\Lambda_{\mathrm{B}},1} := \mathrm{span}\left\{\prod_{\alpha=1}^{(n+1)/2}\left(\frac{\hat{\gamma}_{2\alpha-1} - \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)^{m_\alpha}|0\rangle \,\middle|\, \left(\frac{\hat{\gamma}_{2\alpha-1} + \mathrm{i}\hat{\gamma}_{2\alpha}}{2}\right)|0\rangle = 0,\ m_\alpha = 0, 1\right\} \tag{3.5a}$$

of dimension

$$\dim \mathfrak{F}_{\Lambda_{\mathrm{B}},1} = 2^{(n+1)/2}, \tag{3.5b}$$

where it is understood that $\hat{\gamma}_{n+1} \equiv \hat{\gamma}_\infty$. In this fermionic Fock space, all creation and annihilation fermion operators are local, except for one pair. The pair of creation and annihilation operator built out of the pair $\hat{\gamma}_n$ and $\hat{\gamma}_\infty$ of Majorana operators is nonlocal as $\hat{\gamma}_\infty$ originates from the opposite component of the boundary of one-dimensional space owing to the open boundary conditions, a distance infinitely far away after the thermodynamic limit has been taken. The same is true of the two-dimensional fermionic Fock space

$$\mathfrak{F}_{\mathrm{LR}} := \mathrm{span}\left\{\left(\frac{\hat{\gamma}_n - \mathrm{i}\hat{\gamma}_\infty}{2}\right)^{m_\alpha}|0\rangle \,\middle|\, \left(\frac{\hat{\gamma}_n + \mathrm{i}\hat{\gamma}_\infty}{2}\right)|0\rangle = 0\right\} \tag{3.6}$$

spanned by the pair $\hat{\gamma}_n$ and $\hat{\gamma}_\infty$.

Finally, it is assumed that the component Λ_{B} of the boundary Λ_{bd} defined in Eq. (3.1b) is symmetric under the action of G_f in the sense that

$$\widehat{U}_{\mathrm{bulk}}(g)\, C\ell_n\, \widehat{U}^\dagger_{\mathrm{bulk}}(g) \subset C\ell_n, \quad \forall g \in G_f. \tag{3.7}$$

IV. BOUNDARY PROJECTIVE REPRESENTATION OF G_f

We assume that, for any $g \in G_f$, there exists a norm-preserving operator $\widehat{U}_{\mathrm{B}}(g)$ acting on the Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]}$ as domain of definition such that

$$\widehat{U}_{\mathrm{B}}(g)\, \hat{\gamma}_i\, \widehat{U}^\dagger_{\mathrm{B}}(g) = \widehat{U}_{\mathrm{bulk}}(g)\, \hat{\gamma}_i\, \widehat{U}^\dagger_{\mathrm{bulk}}(g), \tag{4.1}$$

for $i = 1, 2, \cdots, n$. The boundary representation $\widehat{U}_{\mathrm{B}}(g)$ of any element $g \neq e,\ p$ is not unique since Eq. (4.1) is left invariant by the multiplication from the right of $\widehat{U}_{\mathrm{B}}(g)$ with any norm-preserving element from the center of the Clifford algebra $C\ell_n$. When n is even this center is trivial and one dimensional. When n is odd ($[\mu] = 1$) this center is nontrivial and two dimensional. In contrast, irrespective of $[\mu]$ the representations $\widehat{U}_{\mathrm{B}}(e)$ and $\widehat{U}_{\mathrm{B}}(p)$ of the identity and fermion parity acting on the fermionic Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]}$ are uniquely determined up to a multiplicative phase factor.

Finally, we observe two consequences of Eq. (4.1). First, the boundary representation $\widehat{U}_{\mathrm{B}}$ inherits the injectivity of the bulk representation $\widehat{U}_{\mathrm{bulk}}$ of the fermionic symmetry group G_f. Second, for any element $g \in G_f$, the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ has a definite fermion parity. However, unlike the representation $\widehat{U}_{\mathrm{bulk}}$, the representation $\widehat{U}_{\mathrm{B}}$ can be projective as we shall explain.

A. The case of $[\mu] = 0$

When the number n of Majorana operators on the boundary Λ_{B} is even, $[\mu] = 0$. We denote the identity on the local fermionic Fock space (3.3a) by $\mathbb{1}_{\mathrm{B},0}$. The boundary representation of element $p \in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}_{\mathrm{B}}(p) := \prod_{\alpha=1}^{n/2} \widehat{P}_\alpha, \quad \widehat{P}_\alpha := \mathrm{i}\hat{\gamma}_{2\alpha-1}\,\hat{\gamma}_{2\alpha}. \tag{4.2a}$$

The parity operators $\widehat{P}_1, \ldots, \widehat{P}_{n/2}$ are Hermitian, square to the identity, and are pairwise commuting. Hence, $\widehat{U}_{\mathrm{B}}(p)$ is Hermitian and squares to the identity. Since operators $\widehat{P}_1, \ldots, \widehat{P}_{n/2}$ are pairwise commuting, we can simultaneously diagonalize them and choose any one of them to be even under complex conjugation K,

$$\mathsf{K}\,\widehat{P}_\alpha\,\mathsf{K} = \widehat{P}_\alpha, \tag{4.2b}$$

for $\alpha = 1, \ldots, n/2$. The most general form of a representation of element $g \in G_f$ is

$$\widehat{U}_{\mathrm{B}}(g) := \widehat{V}_{\mathrm{B}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \tag{4.3}$$

where $\widehat{V}_{\mathrm{B}}(g)$ is a unitary operator that belongs tow $C\ell_n$ defined in Eq. (3.2).

B. The case of $[\mu] = 1$

When the number n of Majorana operators on the boundary Λ_{B} is odd, $[\mu] = 1$. We denote the identity on the nonlocal fermionic Fock space (3.5a) by $\mathbb{1}_{\mathrm{B},1}$. The boundary representation of element $p \in G_f$ that generates the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ is chosen to be

$$\widehat{U}_{\mathrm{B}}(p) := \widehat{P}\,\widehat{P}_{\mathrm{nonloc}}, \tag{4.4a}$$

$$\widehat{P} := \prod_{\alpha=1}^{(n-1)/2} \widehat{P}_\alpha, \quad \widehat{P}_\alpha := \mathrm{i}\hat{\gamma}_{2\alpha-1}\,\hat{\gamma}_{2\alpha}, \tag{4.4b}$$

$$\widehat{P}_{\mathrm{nonloc}} := \mathrm{i}\hat{\gamma}_n\,\hat{\gamma}_\infty, \tag{4.4c}$$

for $\widehat{U}_{\mathrm{B}}(p)$ is proportional to the product $\hat{\gamma}_1\ \ldots\ \hat{\gamma}_n\,\hat{\gamma}_\infty$ of all the generators in $C\ell_{n+1}$. As such, $\widehat{U}_{\mathrm{B}}(p)$ anticommutes with all the Majorana operators that span the nonlocal fermionic Fock space (3.5a). The parity operators $\widehat{P}_1, \ldots, \widehat{P}_{(n-1)/2}, \widehat{P}_{\mathrm{nonloc}}$ are Hermitian, square to the identity, and are pairwise commuting. We choose to diagonalize them simultaneously and choose each of them to be even under complex conjugation K,

$$\mathsf{K}\,\widehat{P}_\alpha\,\mathsf{K} = \widehat{P}_\alpha, \quad \mathsf{K}\,\widehat{P}_{\mathrm{nonloc}}\,\mathsf{K} = \widehat{P}_{\mathrm{nonloc}}, \tag{4.4d}$$

for $\alpha, \alpha' = 1, \cdots, (n-1)/2$.

In addition to defining a representation of the fermion parity p, we need to account for the fact that the center of the Clifford algebra $\mathrm{C}\ell_n$ is two-dimensional when n is odd. We choose to represent the nontrivial element of this center by

$$\widehat{Y}_{\mathrm{B}} := \widehat{P}\,\hat{\gamma}_n, \quad \widehat{Y}_{\mathrm{B}}^{\dagger} = \widehat{Y}_{\mathrm{B}}, \quad \widehat{Y}_{\mathrm{B}}^{2} = \hat{\mathbb{1}}_{\mathrm{B},1}. \tag{4.5}$$

By construction, $\widehat{Y}_{\mathrm{B}}$ is proportional to the product $\hat{\gamma}_1 \cdots \hat{\gamma}_n \neq \hat{\mathbb{1}}_{\mathrm{B},1}$. It commutes with the Majorana operators $\hat{\gamma}_1, \dots, \hat{\gamma}_n$, while it anticommutes with the Majorana operator $\hat{\gamma}_\infty$. The operator $\widehat{Y}_{\mathrm{B}}$ is of odd fermion parity for it anticommutes with the fermion parity operator (4.4). Because $\widehat{Y}_{\mathrm{B}}$ commutes with all the elements of $\mathrm{C}\ell_n$, it follows that the left-hand side of Eq. (4.1) is invariant under the G_f-resolved transformation

$$\widehat{U}_{\mathrm{B}}(g) \mapsto \widehat{U}_{\mathrm{B}}(g)\,\widehat{Y}_{\mathrm{B}} \tag{4.6}$$

under which the fermion parity of $\widehat{U}_{\mathrm{B}}(g)$ is reversed.

Since the Clifford algebra $\mathrm{C}\ell_n$ is closed under the action of the boundary representation $\widehat{U}_{\mathrm{bulk}}(g)$, the same must be true for the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ [recall Eqs. (3.7) and (4.1)]. In other words, $\widehat{U}_{\mathrm{B}}(g)$ preserves locality in that its action on those operators whose nontrivial actions are limited to Λ_{B} is merely to mix them. This locality is guaranteed only if the condition

$$[\widehat{U}_{\mathrm{B}}(g)\,\hat{\gamma}_i\,\widehat{U}^{\dagger}_{\mathrm{B}}(g), \widehat{Y}_{\mathrm{B}}] = 0 \tag{4.7}$$

is satisfied for any $g \in G_f$ and $i = 1, \dots, n$. In turn, condition (4.7) implies that $\widehat{U}_{\mathrm{B}}(g)$ either commutes or anticommutes with the center $\widehat{Y}_{\mathrm{B}}$ of $\mathrm{C}\ell_n$, i.e.,

$$\widehat{Y}_{\mathrm{B}}\,\widehat{U}_{\mathrm{B}}(g) = \pm \widehat{U}_{\mathrm{B}}(g)\,\widehat{Y}_{\mathrm{B}}. \tag{4.8}$$

Furthermore, this is true only if the decomposition

$$\widehat{U}_{\mathrm{B}}(g) := \widehat{V}_{\mathrm{B}}(g)\,\widehat{Q}_{\mathrm{B}}(g)\,\mathsf{K}^{\mathfrak{c}(g)} \tag{4.9}$$

holds. Here, $\widehat{V}_{\mathrm{B}}(g) \in \mathrm{C}\ell_n \subset \mathrm{C}\ell_{n+1}$ is a unitary operator with well-defined fermion parity and the operator $\widehat{Q}_{\mathrm{B}}(g)$ is either proportional to the identity operator in $\mathrm{C}\ell_{n+1}$ or to the operator $\hat{\gamma}_\infty$.

The invariance of Eq. (4.1) under the G_f-resolved transformation (4.6) allows to fix the fermion parity of $\widehat{U}_{\mathrm{B}}(g)$ to be even for all $g \in G_f$. In this "gauge",

$$\widehat{U}_{\mathrm{B}}(g) = \widehat{V}_{\mathrm{B}}(g)\,\widehat{Q}_{\mathrm{B}}(g)\,\mathsf{K}^{\mathfrak{c}(g)}, \quad \widehat{Q}_{\mathrm{B}}(g) = [\hat{\gamma}_\infty]^{q(g)}, \tag{4.10}$$

where $q(g) = 0, 1$ denotes the fermion parity of the unitary operator $\widehat{V}_{\mathrm{B}}(g)$. Equation (4.10) together with Eqs. (4.4) and (4.5) define the realization of the symmetry group G_f on the boundary Λ_{B} when $[\mu] = 1$.

V. DEFINITION OF INDICES

We consider a boundary representation $\widehat{U}_{\mathrm{B}} : G_f \to \mathrm{Aut}(\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]})$, where $\mathrm{Aut}(\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]})$ denotes the set of automorphisms on the fermionic Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}},[\mu]}$. We demand that this map satisfies, for any $g, h, f \in G_f$,

$$\widehat{U}_{\mathrm{B}}(g)\,\widehat{U}_{\mathrm{B}}(h) = e^{\mathrm{i}\phi(g,h)}\,\widehat{U}_{\mathrm{B}}(gh), \tag{5.1a}$$

where gh denotes the composition of the elements $g, h \in G_f$. The map $\phi(\cdot,\cdot) \in C^2(G_f, \mathrm{U}(1))$ is a U(1)-valued 2-cochain [27] (Appendix A). Furthermore, to ensure the compatibility with the associativity of the composition law of G_f, we demand that, for any $g, h, f \in G_f$,

$$\phi(g,h) + \phi(gh,f) = (-1)^{\mathfrak{c}(g)}\,\phi(h,f) + \phi(g,hf). \tag{5.1b}$$

The 2-cochains that satisfy this condition are called 2-cocycles (Appendix A). The map (5.1) defines a projective representation of the symmetry group G_f.

Under the gauge transformation

$$\widehat{U}_{\mathrm{B}}(g) \mapsto e^{\mathrm{i}\xi(g)}\,\widehat{U}_{\mathrm{B}}(g), \tag{5.2a}$$

the phase $\phi(g,h)$ entering any projective representation of the symmetry group G_f changes by

$$\phi'(g,h) - \phi(g,h) = \xi(gh) - \xi(g) - (-1)^{\mathfrak{c}(g)}\,\xi(h) \tag{5.2b}$$

for any $g, h \in G_f$. Two 2-cochains ϕ and ϕ' are equivalent if they are related by a gauge transformation. The 2-cochains ϕ that vanish under a gauge transformation, i.e., the identity

$$\phi(g,h) = \xi(gh) - \xi(g) - (-1)^{\mathfrak{c}(g)}\,\xi(h) \tag{5.2c}$$

for any $g, h \in G_f$ holds, are called 2-coboundaries. The set of equivalence classes $[\phi]$ of 2-cocycles under the gauge transformations is the second cohomology group $H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ (Appendix A).

Elements of G_f were referred to, so far, by single letters g, with e reserved for the identity and p reserved from the fermion parity. When we want to emphasize that elements of G_f are elements of the set $G \times \mathbb{Z}_2^{\mathrm{F}}$ as is done in Appendix B, we will denote an element of G_f as (g, h) with $g \in G$, $h \in \mathbb{Z}_2^{\mathrm{F}}$. With this convention, (id, e) is the identity, (id, p) is the fermion parity, and the projection (g, e) of (g, h) on G defines the inclusion map $G \subset G_f$. Here, the 2-cochain $\nu \in C^2(G, \mathrm{U}(1))$ captures the projective representation (5.1) for those elements of G_f of the form (g, e). When $[\mu] = 0$, the 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ measures if an operator representing an element of G commutes or anticommutes with the operator representing the fermion parity p. When $[\mu] = 1$, the 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ measures if an operator representing an element of G commutes or anticommutes with the central element $\widehat{Y}_{\mathrm{B}}$ of the Clifford algebra $\mathrm{C}\ell_n$ when $[\mu] = 1$. Indeed, it is possible to organize $C^2(G, \mathrm{U}(1)) \times C^1(G, \mathbb{Z}_2)$ into a coset of equivalence classes $\{[(\nu, \rho)]\}$ such that there is a one-to-one correspondence between any element $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ and $[(\nu, \rho)]$, as was shown in Ref. [10] and is reviewed in Appendix C. When defining the indices (ν, ρ), we made an implicit choice for which elements of G_f are mapped to the pairs $(g, e) \in G \times \mathbb{Z}_2^F$. Different choices are related to each other by group isomorphisms on G_f. We explain in Appendix D how the pair (ν, ρ) changes under isomorphisms relating different representatives of the central extension class $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$.

A. The case of $[\mu] = 0$

When the number n of Majorana operators on the boundary Λ_{B} is even, $[\mu] = 0$. The 2-cochain $\nu \in C^2(G, \mathrm{U}(1))$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_f to G,

$$\nu(g,h) := \phi((g,e),(h,e)), \tag{5.3}$$

for any $g, h \in G$.

The 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ is defined by the relation

$$e^{\mathrm{i}\pi\rho((g,h))} \equiv (-1)^{\rho((g,h))} := \begin{cases} \widehat{U}_{\mathrm{B}}((g,h))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p))\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,h))\,\widehat{U}^{\dagger}_{\mathrm{B}}((\mathrm{id},p)), & \text{if } \mathfrak{c}((g,h))=0,\\ \widehat{U}_{\mathrm{B}}((g,h))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p))\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,h))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p)), & \text{if } \mathfrak{c}((g,h))=1, \end{cases} \tag{5.4}$$

for any $(g,h) \in G_f$. The 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ takes the values 0 or 1. The 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ is a group homomorphism from G_f to $\mathbb{Z}_2 = \{0, 1\}$, since it has a vanishing coboundary and, hence, is a 1-cocycle [28] (Appendix A). It measures the fermion parity of the operator $\widehat{U}_{\mathrm{B}}((g,h))$. As expected we have $\rho((\mathrm{id}, p)) = 0$.

The 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ is defined by restricting the domain of definition of $\rho \in C^1(G_f, \mathbb{Z}_2)$ from G_f to G, i.e.,

$$e^{\mathrm{i}\pi\rho(g)} \equiv (-1)^{\rho(g)} := \begin{cases} \widehat{U}_{\mathrm{B}}((g,e))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p))\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,e))\,\widehat{U}^{\dagger}_{\mathrm{B}}((\mathrm{id},p)), & \text{if } \mathfrak{c}((g,e))=0,\\ \widehat{U}_{\mathrm{B}}((g,e))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p))\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,e))\,\widehat{U}_{\mathrm{B}}((\mathrm{id},p)), & \text{if } \mathfrak{c}((g,e))=1, \end{cases} \tag{5.5a}$$

for any $g \in G$. In terms of the 2-cocycle ϕ, $\rho \in C^1(G, \mathbb{Z}_2)$ is, for any $g \in G$, given by

$$\rho(g) = \frac{1}{\pi}[\phi((g,e),(\mathrm{id},p)) - \phi((\mathrm{id},p),(g,e)) + \mathfrak{c}(g,e)\,\phi((\mathrm{id},p),(\mathrm{id},p))]. \tag{5.5b}$$

The definitions (5.4) and (5.5) are made so that the 1-cochain ρ is invariant under the gauge transformation (5.2a). We note that when a gauge choice is made by choosing the representation $\widehat{U}((\mathrm{id}, p))$ to be Hermitian, the two cases in the definitions (5.4) and (5.5) are equivalent.

B. The case of $[\mu] = 1$

When the number n of Majorana operators on the boundary Λ_{B} is odd, $[\mu] = 1$. The 2-cochain $\nu \in C^2(G, \mathrm{U}(1))$ is defined by restricting the domain of definition of the 2-cochain ϕ from G_f to G,

$$\nu(g,h) := \phi((g,e),(h,e)), \tag{5.6}$$

for any $g, h \in G$.

When $[\mu] = 1$, the Clifford algebra $\mathrm{C}\ell_n$ spanned by the Majorana operators (3.2) has a two-dimensional center, in which case the fermion parity of the boundary representation $\widehat{U}_{\mathrm{B}}((g,h))$ for any element $(g,h) \in G_f$ can be reversed by multiplying it with the generator $\widehat{Y}_{\mathrm{B}}$ of the two-dimensional center of the Clifford algebra $\mathrm{C}\ell_n$. Moreover, any $\widehat{U}_{\mathrm{B}}((g,h))$ must either commute or anticommute with $\widehat{Y}_{\mathrm{B}}$ according to Eq. (4.8).

For this reason, we define the 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ through

$$e^{\mathrm{i}\pi\rho((g,h))} \equiv (-1)^{\rho((g,h))} := \begin{cases} \widehat{U}_{\mathrm{B}}((g,h))\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,h))\,\widehat{Y}^{\dagger}_{\mathrm{B}}, & \text{if } \mathfrak{c}((g,h))=0,\\ \widehat{U}_{\mathrm{B}}((g,h))\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,h))\,\widehat{Y}_{\mathrm{B}}, & \text{if } \mathfrak{c}((g,h))=1, \end{cases} \tag{5.7}$$

for any $(g,h) \in G_f$. The 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ takes the value 0 and 1. The 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ is a group homomorphism from G_f to $\mathbb{Z}_2 = \{0, 1\}$ since it has a vanishing coboundary and, hence, is a 1-cocycle (Appendix A). Since $\widehat{Y}_{\mathrm{B}}$ is of odd fermion parity by definition (4.5), it anticommutes with the representation $\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))$. This implies that $\rho(\mathrm{id}, p) = 1$. More generally, the 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ measures if the representation $\widehat{U}_{\mathrm{B}}(g,h)$ of $(g,h) \in G_f$ commutes or anticommutes with $\widehat{Y}_{\mathrm{B}}$.

The 1-cochain $\rho \in C^1(G, \mathbb{Z}_2)$ is defined by restricting the domain of definition of $\rho \in C^1(G_f, \mathbb{Z}_2)$ from G_f to G, i.e.,

$$e^{\mathrm{i}\pi\rho(g)} \equiv (-1)^{\rho(g)} := \begin{cases} \widehat{U}_{\mathrm{B}}((g,e))\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,e))\,\widehat{Y}^{\dagger}_{\mathrm{B}}, & \text{if } \mathfrak{c}((g,e))=0,\\ \widehat{U}_{\mathrm{B}}((g,e))\,\widehat{Y}_{\mathrm{B}}\,\widehat{U}^{\dagger}_{\mathrm{B}}((g,e))\,\widehat{Y}_{\mathrm{B}}, & \text{if } \mathfrak{c}((g,e))=1, \end{cases} \tag{5.8}$$

for any $g \in G$.

The definitions (5.7) and (5.8) are made so that the 1-cochain ρ is invariant under the gauge transformation (5.2a). We note that when a gauge choice is made by choosing the representation $\widehat{Y}_{\mathrm{B}}$ to be Hermitian, the two cases in the definitions (5.7) and (5.8) are equivalent.

The fact that the 1-cochain $\rho \in C^1(G_f, \mathbb{Z}_2)$ defined in Eq. (5.7) is a group homomorphism puts constraints on the structure of the internal symmetry group G_f. Compatibility between the existence of the group homomorphism $\rho \in C^1(G_f, \mathbb{Z}_2)$, which is defined in Eq. (5.7) and the group composition rule in G_f [see Eq. (B3)] requires that the central extension class $[\gamma] \in H^2(G_f, \mathbb{Z}_2^{\mathrm{F}})$ is trivial, i.e., $[\gamma] = 0$. This is so because, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the homomorphism $\rho \in C^1(G_f, \mathbb{Z}_2)$ is a group isomorphism [29]. It can then be used to construct a group isomorphism from G_f to the direct product $G \times \mathbb{Z}_2^{\mathrm{F}}$. In other words, the only internal symmetry groups G_f compatible with boundaries supporting an odd number of Majorana degrees of freedom ($[\mu]=1$) are those that split, i.e, $G_f = G\times\mathbb{Z}_2^{\mathrm{F}}$.

For simplicity, we revert back to the single letters (e.g., g or h) to denote elements of the group G_f from now on. We will use this notation as long as there are no ambiguities. Whenever used, the appropriate definition for 1-cochain ρ [definitions (5.4) or (5.5) and definitions (5.7) or (5.8)] should be understood from the context.

We close Sec. V B by spelling out two identities that will be convenient when deriving the stacking rules in Sec. VI. We note that definition (5.7) involves conjugation of the central element $\widehat{Y}_{\mathrm{B}}$ by the boundary representation $\widehat{U}_{\mathrm{B}}(g)$ of some element $g\in G_f$. By definitions (4.4) and (4.5), $\widehat{Y}_{\mathrm{B}}$ can be written as

$$\widehat{Y}_{\mathrm{B}} = -\mathrm{i}\,\widehat{U}_{\mathrm{B}}(p)\,\hat{\gamma}_{\infty}. \tag{5.9a}$$

Using this identity in definition (5.7) allows one to express the complex conjugation of $\hat{\gamma}_\infty$ in terms of group homomorphisms $\mathfrak{c}$, q, and ρ. Since Eq. (4.4d) implies that the Majorana operators $\hat{\gamma}_\infty$ and $\hat{\gamma}_n$ transform oppositely under complex conjugation, one finds the pair of identities

$$\mathsf{K}^{\mathfrak{c}(g)}\,\hat{\gamma}_\infty\,\mathsf{K}^{\mathfrak{c}(g)} = (-1)^{\mathfrak{c}(g)+q(g)+\rho(g)}\hat{\gamma}_\infty, \tag{5.9b}$$

$$\mathsf{K}^{\mathfrak{c}(g)}\,\hat{\gamma}_n\,\mathsf{K}^{\mathfrak{c}(g)} = (-1)^{q(g)+\rho(g)}\hat{\gamma}_n, \tag{5.9c}$$

for any $g\in G_f$

VI. FERMIONIC STACKING RULES

Given the two triplets $((\nu_1,\rho_1),[\mu_1])$ and $((\nu_2,\rho_2),[\mu_2])$ associated to the pair $\widehat{U}_1$ and $\widehat{U}_2$ of boundary representations, respectively, we shall construct the triplet $((\nu_\wedge,\rho_\wedge),[\mu_\wedge])$ that is associated with the representation $\widehat{U}_\wedge$, whereby $\widehat{U}_\wedge$ must be compatible with the symmetry group G_f and is obtained from taking the tensor product of the two set of boundary degrees of freedom. We call this operation stacking.

Since the number of boundary Majorana degrees of freedom on which $\widehat{U}_\wedge$ acts is obtained by adding the boundary Majorana degrees of freedom

$$\mathfrak{D}_1 := \left\{\hat{\gamma}^{(1)}_1,\ \hat{\gamma}^{(1)}_2,\ \cdots,\ \hat{\gamma}^{(1)}_{n_1}\right\} \tag{6.1}$$

on which $\widehat{U}_1$ acts to the boundary Majorana degrees of freedom

$$\mathfrak{D}_2 := \left\{\hat{\gamma}^{(2)}_1,\ \hat{\gamma}^{(2)}_2,\ \cdots,\ \hat{\gamma}^{(2)}_{n_2}\right\} \tag{6.2}$$

on which $\widehat{U}_2$ acts, we define the index $[\mu_\wedge]$ of the stacked representation to be

$$[\mu_\wedge] := [\mu_1]+[\mu_2] \bmod 2. \tag{6.3}$$

For any $g\in G_f$, we define the stacked representation $\widehat{U}_\wedge(g)$ to be a norm preserving operator that satisfies the identities

$$\widehat{U}_\wedge(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_\wedge(g) := \widehat{U}_1(g)\,\hat{\gamma}^{(1)}_i\,\widehat{U}^\dagger_1(g), \tag{6.4a}$$

$$\widehat{U}_\wedge(g)\,\hat{\gamma}^{(2)}_j\,\widehat{U}^\dagger_\wedge(g) := \widehat{U}_2(g)\,\hat{\gamma}^{(2)}_j\,\widehat{U}^\dagger_2(g), \tag{6.4b}$$

for $i=1,\dots,n_1$ and $j=1,\dots,n_2$. This definition is the natural generalization of Eq. (4.1). Because $\widehat{U}_1(g)$ and $\widehat{U}_2(g)$ act on single Majorana operators in the same way as the bulk representation of the element $g\in G_f$ does, the same is true for the stacked representation $\widehat{U}_\wedge(g)$. The stacked representation $\widehat{U}_\wedge(g)$ is not unique since Eqs. (6.4a) and (6.4b) are left invariant by the multiplications from the right of $\widehat{U}_\wedge(g)$ with any norm-preserving element from the center of the Clifford algebra $C\ell_{n_1+n_2}$.

When constructing an explicit representation of $\widehat{U}_\wedge(g)$ for any $g\in G_f$, we shall consider the three cases: (i) even-even stacking, $[\mu_1]=[\mu_2]=0$, (ii) even-odd stacking, $[\mu_1]=0$, $[\mu_2]=1$, (iii) and odd-odd stacking, $[\mu_1]=[\mu_2]=1$. The case of odd-even stacking is to be treated analogously to the case of even-odd stacking.

As is done in Sec. III, we begin with the construction of a representation of the fermion parity $p\in G_f$. When $[\mu_\wedge]=0$, the stacked representation of $\widehat{U}_\wedge(p)$ follows from combining Eq. (6.4) with the counterpart to Eq. (4.2). When $[\mu_\wedge]=1$, the stacked representation of $\widehat{U}_\wedge(p)$ follows from combining Eq. (6.4) with the counterparts to Eqs. (4.4). More precisely, the stacked representation $\widehat{U}_\wedge(p)$ of the fermion parity p is defined to be

$$\widehat{U}_\wedge(p) := \begin{cases} \widehat{U}_1(p)\,\widehat{U}_2(p), & \text{if } [\mu_1]=[\mu_2]=0, \\ \widehat{U}_1(p)\,\widehat{U}_2(p), & \text{if } [\mu_1]=0,\ [\mu_2]=1, \\ \widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, & \text{if } [\mu_1]=[\mu_2]=1. \end{cases} \tag{6.5}$$

By construction, we have chosen a Hermitian representation $\widehat{U}_\wedge(p)$ of the fermion parity p.

Next, we fix the action of the stacked complex conjugation $\mathsf{K}_\wedge$ on the single Majorana operators spanning the fermionic Fock space of the stacked boundary by demanding that some set of mutually commuting fermion parity operators are left invariant under complex conjugation [recall Eqs. (4.2) and (4.4)]. For the cases of even-even ($[\mu_1]=[\mu_2]=0$) and even-odd stacking ($[\mu_1]=0$, $[\mu_2]=1$), we define $\mathsf{K}_\wedge$ by

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_\wedge := \mathsf{K}_1\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_1, \tag{6.6a}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_\wedge := \mathsf{K}_2\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_2, \tag{6.6b}$$

for $i=1,\dots,n_1$ and $j=1,\dots,n_2$. For the case of odd-odd stacking, we define $\mathsf{K}_\wedge$ by

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_\wedge := \mathsf{K}_1\,\hat{\gamma}^{(1)}_i\,\mathsf{K}_1, \tag{6.7a}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_\wedge := \mathsf{K}_2\,\hat{\gamma}^{(2)}_j\,\mathsf{K}_2, \tag{6.7b}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(1)}_{n_1}\,\mathsf{K}_\wedge := +\hat{\gamma}^{(1)}_{n_1}, \tag{6.7c}$$

$$\mathsf{K}_\wedge\,\hat{\gamma}^{(2)}_{n_2}\,\mathsf{K}_\wedge := -\hat{\gamma}^{(2)}_{n_2}, \tag{6.7d}$$

for $i=1,\dots,n_1-1$ and $j=1,\dots,n_2-1$. One verifies that, by construction, the fermion parity operator $\widehat{U}_\wedge(p)$ is invariant under conjugation by $\mathsf{K}_\wedge$.

For the stacked representation with $[\mu_\wedge]=1$ that is achieved with an even-odd stacking, we define the central element $\widehat{Y}_\wedge$ by

$$\widehat{Y}_\wedge := \widehat{U}_1(p)\,\widehat{Y}_2, \tag{6.8}$$

where $\widehat{Y}_2$ is the central element inherited from the representation $\widehat{U}_2$, which by assumption has $[\mu_2]=1$ for the case of even-odd stacking.

In what follows, we give explicit representations of $\widehat{U}_\wedge(g)$ in terms of the pair $\widehat{U}_1(g)$ and $\widehat{U}_2(g)$ and of $([(\nu_\wedge,\ \rho_\wedge)], [\mu_\wedge])$ in terms of the pairs $([(\nu_1,\ \rho_1)], [\mu_1])$ and $([(\nu_2,\ \rho_2)], [\mu_2])$.

A. Even-even stacking

For even-even stacking, we have $[\mu_1] = [\mu_2] = 0$. We define

$$[\mu_\wedge] := [\mu_1] + [\mu_2] = 0. \tag{6.9}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (4.3), i.e., for any $g \in G_f$,

$$\widehat{U}_1(g) = \widehat{V}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)}, \quad \widehat{U}_2(g) = \widehat{V}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{6.10}$$

with the pair of unitary operators $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$. The naive guess $\widehat{V}_1(g)\,\widehat{V}_2(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_\wedge(g)$, for one verifies that it fails to satisfy Eq. (6.4). Instead, for any $g \in G_f$, we define

$$\widehat{U}_\wedge(g) := \widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_1(p)]^{\rho_2(g)}\,[\widehat{U}_2(p)]^{\rho_1(g)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}. \tag{6.11}$$

One verifies that this definition satisfies Eq. (6.4) and, *a forteriori*, Eq. (4.1). The parity operators $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$ in definition (6.11) ensure that no additional minus signs are introduced when Majorana operators $\hat{\gamma}_i^{(1)}$ and $\hat{\gamma}_i^{(2)}$ are conjugated by $\widehat{U}_\wedge(g)$. This is because, by definition (5.5), the values $\rho_1(g)$ and $\rho_2(g)$ encode the fermion parity of the unitary operators $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$, respectively, and the parity operators $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$ correct for any additional minus signs arising from fermionic algebra between operators from representation $\widehat{U}_1$ and $\widehat{U}_2$ in compliance with Eq. (6.4).

As a sanity check, one verifies that when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the definition (6.11) of the stacked representation together with definition (4.2) deliver the Hermitian representations

$$\widehat{U}_\wedge(e) = \hat{\mathbb{1}}_{\wedge,0}, \quad \widehat{U}_\wedge(p) = \widehat{U}_1(p)\,\widehat{U}_2(p), \tag{6.12}$$

that are consistent with the definition (6.5).

When the representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of two elements g and h of G_f are composed, we obtain from definition (6.11)

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) =\ & \widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_1(p)]^{\rho_2(g)}\,[\widehat{U}_2(p)]^{\rho_1(g)} \\ & \times \overline{\widehat{V}_1(h)}^{\wedge,g}\,\overline{\widehat{V}_2(h)}^{\wedge,g}\,[\widehat{U}_1(p)]^{\rho_2(h)}\,[\widehat{U}_2(p)]^{\rho_1(h)} \\ & \times \mathsf{K}_\wedge^{\mathfrak{c}(gh)},\end{aligned} \tag{6.13a}$$

where we have introduced the notation

$$\overline{\widehat{O}}^{1,g} := \mathsf{K}_1^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_1^{\mathfrak{c}(g)}, \tag{6.13b}$$

$$\overline{\widehat{O}}^{2,g} := \mathsf{K}_2^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{6.13c}$$

$$\overline{\widehat{O}}^{\wedge,g} := \mathsf{K}_\wedge^{\mathfrak{c}(g)}\,\widehat{O}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}, \tag{6.13d}$$

for any operator $\widehat{O}$, used the reality condition obeyed by $\widehat{U}_1(p)$ and $\widehat{U}_2(p)$, and the fact that $\mathfrak{c}$ is a group homomorphism. We rearrange the terms in Eq. (6.13a) to obtain

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = (-1)^{\rho_1(g)\rho_2(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g} \\ \times [\widehat{U}_1(p)]^{\rho_2(gh)}[\widehat{U}_2(p)]^{\rho_1(gh)}\mathsf{K}_\wedge^{\mathfrak{c}(gh)},\end{aligned} \tag{6.14}$$

where $(-1)^{\rho_1(g)\rho_2(h)}$ is the total multiplicative phase factor arising due to fermionic algebra. In reaching the last line, we have used definition (6.6) to trade the complex conjugation by $\mathsf{K}_\wedge$ with complex conjugations by K_1 and K_2, and the fact that ρ_1 and ρ_2 are group homomorphisms. To proceed, we observe that definition (5.1) implies

$$\widehat{V}_i(g)\,\overline{\widehat{V}_i(h)}^{i,g} = e^{\mathrm{i}\phi_i(g,h)}\,\widehat{V}_i(gh), \quad i = 1, 2. \tag{6.15}$$

Inserting these identities to Eq. (6.14) delivers

$$\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(gh), \tag{6.16a}$$

where we have defined

$$\phi_\wedge(g, h) := \phi_1(g, h) + \phi_2(g, h) + \pi\,\rho_1(g)\,\rho_2(h). \tag{6.16b}$$

The construction of the indices $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ in terms of the indices $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$ is achieved as follows.

According to definition (5.3), the 2-cochain $\nu_\wedge$ is simply obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\nu_\wedge(g, h) = \nu_1(g, h) + \nu_2(g, h) + \pi(\rho_1 \smile \rho_2)(g, h). \tag{6.17a}$$

In the last step we have used the cup product $\smile$ to construct a 2-cochain $\rho_1 \smile \rho_2$ out of the pair of one cochains ρ_1 and ρ_2. For the 1-cochain $\rho_\wedge$, definition (5.5) delivers

$$\rho_\wedge(g) = \rho_1(g) + \rho_2(g), \tag{6.17b}$$

which is nothing but the total fermion parity of the stacked representation $\widehat{U}_\wedge(g)$ of element $g \in G$.

B. Even-odd stacking

For even-odd stacking, we have $[\mu_1] = 0$, $[\mu_2] = 1$. Hence, we define

$$[\mu_\wedge] := [\mu_1] + [\mu_2] = 1. \tag{6.18}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (4.3) and (4.10), respectively, i.e., for any $g \in G_f$,

$$\widehat{U}_1(g) = \widehat{V}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)}, \tag{6.19a}$$

$$\widehat{U}_2(g) = \widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{6.19b}$$

$$\widehat{Q}_2(g) = \left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)}. \tag{6.19c}$$

The naive guess $\widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_\wedge(g)$, for one verifies that it fails to satisfy Eq. (6.4) and to be of even fermion parity. Instead, we define the stacked representation to be

$$\begin{aligned}\widehat{U}_\wedge(g) &:= \widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\left[\widehat{U}_1(p)\,\hat{\gamma}_\infty^{(2)}\right]^{\rho_1(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)} \\ &\equiv \widehat{V}_\wedge(g)\,\widehat{Q}_\wedge(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)},\end{aligned} \tag{6.20a}$$

$$\widehat{V}_\wedge(g) := \widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_1(p)\,]^{\rho_1(g)}, \tag{6.20b}$$

$$\widehat{Q}_\wedge(g) := \widehat{Q}_2(g)\left[\hat{\gamma}_\infty^{(2)}\right]^{\rho_1(g)} = \left[\hat{\gamma}_\infty^{(2)}\right]^{q_2(g)+\rho_1(g)}. \tag{6.20c}$$

One verifies that this definition satisfies Eq. (6.4) and, *a forteriori*, Eq. (4.1). For any $g \in G_f$, the definition (6.20) guarantees that $\widehat{U}_\wedge(g)$ is of even fermion parity. This property is inherited from the fact that $\widehat{U}_2(g)$ is of even fermion parity according to Eq. (4.10) and the factor $\widehat{U}_1(p)\,\hat{\gamma}_\infty^{(2)}$ compensates for the fermion parity of the operator $\widehat{V}_1(g)$. The product

$\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}$ also compensates for additional minus signs arising from fermionic algebra between the operators from representations $\widehat{U}_1$ and $\widehat{U}_2$ in compliance with Eq. (6.4).

As a sanity check, one verifies that, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the definition (6.20) of the stacked representation together with definitions (4.2) and (4.4) deliver the Hermitian representations

$$\widehat{U}_\wedge(e) = \hat{\mathbb{1}}_{\wedge,1}, \quad \widehat{U}_\wedge(p) = \widehat{U}_1(p)\,\widehat{U}_2(p), \tag{6.21}$$

that are consistent with the definition (6.5).

When representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of two elements $g, h \in G_f$ are composed, we obtain from definition (6.20)

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = {}&\widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\left[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\\ &\times \overline{\widehat{V}_1(h)}^{1,g}\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\\ &\times \left[\widehat{U}_1(p)\,\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}\mathsf{K}_\wedge^{\mathfrak{c}(gh)},\end{aligned} \tag{6.22}$$

where we have traded the complex conjugation $\mathsf{K}_\wedge$ by complex conjugations K_1 and K_2 through Eq. (6.6). We rearrange the terms in Eq. (6.22) to obtain

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = {}&(-1)^{\rho_1(g)\,q_2(h)}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\\ &\times \widehat{V}_2(g)\,\widehat{Q}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g}\\ &\times \left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(g)}\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)}[\widehat{U}_1(p)]^{\rho_1(gh)}\,\mathsf{K}_\wedge^{\mathfrak{c}(gh)}.\end{aligned} \tag{6.23}$$

Hereby, the multiplicative phase factor $(-1)^{\rho_1(g)\,q_2(h)}$ is induced by the fermionic algebra between $\hat{\gamma}^{(2)}_{\infty}$ and $\widehat{V}_2(h)$ [recall that by definition (4.10) $\widehat{V}_2(h)$ has fermion parity $q_2(h)$]. Using Eqs. (5.1) and (5.9b), we obtain the identities

$$\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g} = e^{\mathrm{i}\phi_1(g,h)}\widehat{V}_1(gh), \tag{6.24a}$$

$$\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\overline{\widehat{Q}_2(h)}^{2,g} = e^{\mathrm{i}\phi_2(g,h)}\widehat{V}_2(gh)\widehat{Q}_2(gh), \tag{6.24b}$$

$$\left[\overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\right]^{\rho_1(h)} = (-1)^{\rho_1(h)[\mathfrak{c}(g)+q_2(g)+\rho_2(g)]}\left[\hat{\gamma}^{(2)}_{\infty}\right]^{\rho_1(h)}. \tag{6.24c}$$

Inserting these identities to Eq. (6.23), one is left with

$$\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(gh), \tag{6.25a}$$

where

$$\begin{aligned}\phi_\wedge(g,h) := {}&\phi_1(g,h) + \phi_2(g,h) + \pi\,\rho_1(g)\,q_2(h)\\ &+ \pi\,\rho_1(h)[\mathfrak{c}(g) + q_2(g) + \rho_2(g)].\end{aligned} \tag{6.25b}$$

The projective phase (6.25b) can be simplified by noting that terms that contain the 1-cochain q_2 can be gauged away under the transformation (5.2). More concretely, for any two $\mathbb{Z}_2$ valued 1-cochains $\alpha, \beta \in C^1(G_f, \mathbb{Z}_2)$, the equivalence relations

$$\begin{aligned}&\alpha(g)\beta(h) \sim \alpha(h)\beta(g),\\ &\alpha(g)\,\beta(h) + \alpha(h)\,\beta(g) \sim 0 \bmod 2,\end{aligned} \tag{6.26}$$

hold. Therefore, the 2-cochain $\phi_\wedge(g,h)$ defined in (6.25b) is gauge equivalent to

$$\begin{aligned}\phi'_\wedge(g,h) := {}&\phi_1(g,h) + \phi_2(g,h) + \pi\,\rho_1(g)\,\rho_2(h)\\ &+ \pi\,\rho_1(g)\,\mathfrak{c}(h),\end{aligned} \tag{6.27}$$

where in reaching the last line we have used the equivalence (6.26) to trade $\rho_1(h)\,\rho_2(g)$ and $\rho_1(h)\,\mathfrak{c}(g)$ with $\rho_1(g)\,\rho_2(h)$ and $\rho_1(g)\,\mathfrak{c}(h)$, respectively.

The construction of the indices $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ in terms of the indices $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$ is achieved as follows.

According to definition (5.6), the 2-cochain $\nu_\wedge$ is simply obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\begin{aligned}\nu_\wedge(g,h) := {}&\nu_1(g,h) + \nu_2(g,h) + \pi(\rho_1 \smile \rho_2)(g,h)\\ &+ \pi(\rho_1 \smile \mathfrak{c})(g,h),\end{aligned} \tag{6.28a}$$

where we introduced the cup product $\smile$ to construct a 2-cochain out of 1-cochains.

Since the stacked representation has index $[\mu_\wedge] = 1$, the 1-cochain $\rho_\wedge$ can be either determined by the definition (5.8) or by the identity (5.9b). The definition (6.6) implies

$$\overline{\hat{\gamma}^{(2)}_{\infty}}^{\wedge,g}\,\hat{\gamma}^{(2)}_{\infty} = \overline{\hat{\gamma}^{(2)}_{\infty}}^{2,g}\,\hat{\gamma}^{(2)}_{\infty}. \tag{6.28b}$$

Using identity (5.9b) for the left and right hand sides separately, and comparing the two we find

$$\rho_\wedge(g) = q_2(g) + q_\wedge(g) + \rho_2(g) = \rho_1(g) + \rho_2(g) \bmod 2, \tag{6.28c}$$

where the value of the 1-cochain $q_\wedge(g) = \rho_1(g) + q_2(g)$ is read off from the fermion parity of the unitary operator $\widehat{V}_\wedge(g)$ defined in Eq. (6.20).

C. Odd-odd stacking

For odd-odd stacking, we have $[\mu_1] = [\mu_2] = 1$. Hence, we define

$$[\mu_\wedge] := [\mu_1] + [\mu_2] = 0. \tag{6.29}$$

The representations $\widehat{U}_1$ and $\widehat{U}_2$ of the group G_f are of the form (4.10), i.e., for any $g \in G_f$,

$$\widehat{U}_1(g) = \widehat{V}_1(g)\,\widehat{Q}_1(g)\,\mathsf{K}_1^{\mathfrak{c}(g)}, \tag{6.30a}$$

$$\widehat{Q}_1(g) = \left[\hat{\gamma}^{(1)}_{\infty}\right]^{q_1(g)}, \tag{6.30b}$$

$$\widehat{U}_2(g) = \widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_2^{\mathfrak{c}(g)}, \tag{6.30c}$$

$$\widehat{Q}_2(g) = \left[\hat{\gamma}^{(2)}_{\infty}\right]^{q_2(g)}. \tag{6.30d}$$

The naive guess $\widehat{V}_1(g)\,\widehat{Q}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}$ is not a satisfactory definition of $\widehat{U}_\wedge(g)$, for one verifies that it fails to satisfy Eq. (6.4). Instead, we define the stacked representation to be

$$\begin{aligned}\widehat{U}_\wedge(g) := {}&(-\mathrm{i})^{\delta_{g,p}}\widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_\wedge(p)]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)},\end{aligned} \tag{6.31a}$$

$$\widehat{U}_\wedge(p) := \widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, \tag{6.31b}$$

where $\widehat{P}_1$ and $\widehat{P}_2$ are the fermion parity operators constructed out of the Majorana operators $\hat{\gamma}^{(1)}_1, \ldots, \hat{\gamma}^{(1)}_{n_1-1}$ and $\hat{\gamma}^{(2)}_1, \ldots, \hat{\gamma}^{(2)}_{n_2-1}$, respectively [recall definitions (4.4) and (6.5)]. The exponent $\delta_{g,p}$ of the multiplicative phase factor $(-\mathrm{i})^{\delta_{g,p}}$ is the Kronecker delta defined over the group G_f.

As a sanity check, one verifies that, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, the definition (6.31) of the stacked representation together with the definition (4.4) deliver the Hermitian representations

$$\widehat{U}_\wedge(e) = \hat{\mathbb{1}}_{\wedge,1}, \quad \widehat{U}_\wedge(p) = \widehat{P}_1\,\widehat{P}_2\,\mathrm{i}\hat{\gamma}^{(1)}_{n_1}\,\hat{\gamma}^{(2)}_{n_2}, \tag{6.32}$$

that are consistent with the definition (6.5). The choice of the multiplicative phase factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (6.31) is not unique since representation $\widehat{U}(g)$ of any element $g \in G_f$ is defined up to a multiplicative $U(1)$ phase. We observe that the multiplicative factor $(-\mathrm{i})^{\delta_{g,p}}$ in Eq. (6.31) ensures that the stacked representation $\widehat{U}_\wedge(p)$ is Hermitian in compliance with the "gauge" choice made in definition (4.4).

Several comments are due. First, one verifies that the definition (6.31) satisfies Eq. (6.4) and, *a forteriori*, Eq. (4.1). Second, the Majorana operators $\hat{\gamma}^{(1)}_\infty$ and $\hat{\gamma}^{(2)}_\infty$ do not enter the definition (6.31) of the stacked representation $\widehat{U}_\wedge$. This is expected as the stacked representation $\widehat{U}_\wedge$ has $[\mu_\wedge] = 0$. Accordingly, $\widehat{U}_\wedge$ is constructed solely out of the even number $n_1 + n_2$ of Majorana operators spanning the fermionic Fock space of the stacked boundary [recall definition (4.3)]. Third, the definition (6.31) is not symmetric under exchange of the labels 1 and 2, as is to be expected by inspection of Eq. (6.7).

Before computing the stacked 2-cochain $\phi_\wedge$, we shall derive two useful identities that relate complex conjugation by $\mathsf{K}_\wedge$ to complex conjugation by K_1 and K_2 for any pair $g, h \in G_f$. This is needed as definition (6.7) of $\mathsf{K}_\wedge$ is not completely fixed from the definitions of K_1 and K_2 as in the case of even-even and even-odd stacking, recall Eq. (6.6). The consistency conditions (4.1) and (6.4) imply the identities

$$\widehat{U}_\wedge(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_\wedge(g) = \widehat{U}_1(g)\,\widehat{V}_1(h)\,\widehat{U}^\dagger_1(g), \tag{6.33a}$$

$$\widehat{U}_\wedge(g)\,\widehat{V}_2(h)\,\widehat{U}^\dagger_\wedge(g) = \widehat{U}_2(g)\,\widehat{V}_2(h)\,\widehat{U}^\dagger_2(g), \tag{6.33b}$$

for any $g, h \in G_f$. If g is to be represented antiunitarily, then complex conjugation is denoted by $\mathsf{K}_\wedge$, K_1, and K_2 for $\widehat{U}_\wedge(g)$, $\widehat{U}_1(g)$, and $\widehat{U}_2(g)$, respectively. Comparing the two sides delivers the pair of identities

$$\begin{aligned}\overline{\widehat{V}_1(h)}^{\wedge,g} &= (-1)^{q_1(h)[\rho_1(g)+q_1(g)]}\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(g)+\rho_1(g)}\,\overline{\widehat{V}_1(h)}^{1,g}\\ &\quad\times\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(g)+\rho_1(g)},\end{aligned} \tag{6.34a}$$

$$\begin{aligned}\overline{\widehat{V}_2(h)}^{\wedge,g} &= (-1)^{q_2(h)[\mathfrak{c}(g)+\rho_2(g)+q_2(g)]}\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\ &\quad\times\overline{\widehat{V}_2(h)}^{2,g}\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}.\end{aligned} \tag{6.34b}$$

These pair of identities are not symmetric under exchange of the labels 1 and 2 as the definitions (6.7) and (6.7) are not symmetric as well. The phase factors multiplying the right-hand side are due to the fermionic algebra between the operators.

We are now ready to compute the 2-cochain $\phi_\wedge(g, h)$ associated with the stacked representation $\widehat{U}_\wedge$. Composing the representations $\widehat{U}_\wedge(g)$ and $\widehat{U}_\wedge(h)$ of any pair $g, h \in G_f$ delivers

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = &\ (-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}}(-1)^{\mathfrak{c}(g)[\mathfrak{c}(h)+q_2(h)+\rho_2(h)]}\widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_\wedge(p)]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(g)+\rho_1(g)}\\ &\times\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\overline{\widehat{V}_1(h)}^{\wedge,g}\overline{\widehat{V}_2(h)}^{\wedge,g}[\widehat{U}_\wedge(p)]^{\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)}[\hat{\gamma}^{(1)}_{n_1}]^{q_1(h)+\rho_1(h)}\,\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\,\mathsf{K}^{\mathfrak{c}(gh)}_\wedge,\end{aligned} \tag{6.35}$$

where the multiplier $(-1)^{\mathfrak{c}(g)}$ in the phase factor $(-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}}$ arises when the complex conjugation $\mathsf{K}^{\mathfrak{c}(g)}_\wedge$ is passed through $(-\mathrm{i})^{\delta_{h,p}}$. The multiplicative phase factor $(-1)^{\mathfrak{c}(g)[\mathfrak{c}(h)+q_2(h)+\rho_2(h)]}$ is due to complex conjugation of Majorana operators $\hat{\gamma}^{(1)}_{n_1}$ and $\hat{\gamma}^{(2)}_{n_2}$, through Eq. (6.7).

We rearrange the terms in Eq. (6.35) to obtain

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = &\ (-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}}\,(-1)^{\chi_1(g,h)}\,\widehat{V}_1(g)\,[\hat{\gamma}^{(1)}_{n_1}]^{q_1(g)+\rho_1(g)}\,\overline{\widehat{V}_1(h)}^{\wedge,g}\,\widehat{V}_2(g)\,\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\,\overline{\widehat{V}_2(h)}^{\wedge,g}\\ &\times[\widehat{U}_\wedge(p)]^{\mathfrak{c}(gh)+\rho_1(gh)+\rho_2(gh)}\,\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(h)+\rho_1(h)}\,\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\,\mathsf{K}^{\mathfrak{c}(gh)}_\wedge,\end{aligned} \tag{6.36a}$$

where

$$\begin{aligned}\chi_1(g,h) := &\ \mathfrak{c}(g)[1+\mathfrak{c}(h)+q_1(g)+q_2(g)+\rho_2(h)]+\rho_1(g)[1+q_1(g)+q_1(h)+q_2(h)]\\ &+\rho_2(g)[1+q_1(g)+q_2(g)+q_2(h)]+q_1(g)q_2(g),\end{aligned} \tag{6.36b}$$

is the total multiple phase factor that is due to the fermionic algebra when rearranging the terms together with the multiplicative phase factor in Eq. (6.35).

We use the identity (6.34) in Eq. (6.35) to obtain

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = &\ (-1)^{\chi_1(g,h)}\,(-1)^{\chi_{\mathrm{conj}}(g,h)}\,(-\mathrm{i})^{\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}}\,\widehat{V}_1(g)\,\overline{\widehat{V}_1(h)}^{1,g}\,\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(g)+\rho_1(g)}\widehat{V}_2(g)\,\overline{\widehat{V}_2(h)}^{2,g}\,\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\\ &\times[\widehat{U}_\wedge(p)]^{\mathfrak{c}(gh)+\rho_1(gh)+\rho_2(gh)}\,\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(h)+\rho_1(h)}\,\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\,\mathsf{K}^{\mathfrak{c}(gh)}_\wedge,\end{aligned} \tag{6.37a}$$

where we have consolidated the two multiplicative phase factors on the right-hand sides of Eqs. (6.34a) and (6.34b) into the multiplicative phase factor

$$\chi_{\mathrm{conj}}(g,h) := q_1(h)[\rho_1(g)+q_1(g)]+q_2(h)[\mathfrak{c}(g)+\rho_2(g)+q_2(g)]. \tag{6.37b}$$

To proceed, we observe that the definitions (5.1) and (4.10) can be cast into

$$\widehat{V}_i(g)\overline{\widehat{V}_i(h)}^{i,g} = e^{\mathrm{i}\phi_i(g,h)+\mathrm{i}\pi q_i(h)[\mathfrak{c}(g)+\rho_i(g)]}\,\widehat{V}_i(gh), \tag{6.38}$$

for $i = 1, 2$. Inserting Eq. (6.38) into Eq. (6.37) delivers

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = {}& e^{\mathrm{i}\phi_{\mathrm{comp}}(g,h)+\mathrm{i}\pi\chi_1(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\frac{3\pi}{2}(\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p})}\widehat{V}_1(gh)\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(g)+\rho_1(g)}\widehat{V}_2(gh)\\ &\times\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}[\widehat{U}_\wedge(p)]^{\mathfrak{c}(gh)+\rho_1(gh)+\rho_2(gh)}\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(h)+\rho_1(h)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(h)+q_2(h)+\rho_2(h)}\mathsf{K}_\wedge^{\mathfrak{c}(gh)},\end{aligned} \tag{6.39a}$$

where we have defined the phase factor accumulated from the group composition rule (6.38)

$$\phi_{\mathrm{comp}}(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\, q_1(h)[\mathfrak{c}(g)+\rho_1(g)] + \pi\, q_2(h)[\mathfrak{c}(g)+\rho_2(g)]. \tag{6.39b}$$

It remains to reorder operators on the right-hand side of Eq. (6.39a) with the goal to isolate the operator $\widehat{U}_\wedge(gh)$, whose definition is given by Eq. (6.31). Doing so, one finds

$$\begin{aligned}\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h) = {}& e^{\mathrm{i}\phi_{\mathrm{comp}}(g,h)+\mathrm{i}\pi\chi_1(g,h)+\mathrm{i}\pi\chi_{\mathrm{conj}}(g,h)+\mathrm{i}\pi\chi_{\mathrm{ord}}(g,h)+\mathrm{i}\chi_{\mathrm{gag}}(g,h)}(-\mathrm{i})^{\delta_{gh,p}}\,\widehat{V}_1(gh)\,\widehat{V}_2(gh)\,[\widehat{U}_\wedge(p)]^{\mathfrak{c}(gh)+\rho_1(gh)+\rho_2(gh)}\\ &\times\left[\hat{\gamma}^{(1)}_{n_1}\right]^{q_1(gh)+\rho_1(gh)}\left[\hat{\gamma}^{(2)}_{n_2}\right]^{\mathfrak{c}(gh)+q_2(gh)+\rho_2(gh)}\mathsf{K}_\wedge^{\mathfrak{c}(gh)},\\ \equiv{}& e^{\mathrm{i}\phi_\wedge(g,h)}\,\widehat{U}_\wedge(gh),\end{aligned} \tag{6.40a}$$

where we have defined the phase factors

$$\chi_{\mathrm{ord}}(g,h) := [\mathfrak{c}(gh)+\rho_1(g)+\rho_2(gh)+q_1(h)][\mathfrak{c}(g)+q_2(g)+\rho_2(g)] + [\mathfrak{c}(gh)+\rho_1(gh)+\rho_2(gh)+q_2(gh)][q_1(g)+\rho_1(g)], \tag{6.40b}$$

$$\chi_{\mathrm{gag}}(g,h) := \frac{3\pi}{2}(\delta_{g,p}+(-1)^{\mathfrak{c}(g)}\delta_{h,p}-\delta_{gh,p}). \tag{6.40c}$$

$$\phi_\wedge(g,h) := \phi_{\mathrm{comp}}(g,h) + \pi\,\chi_1(g,h) + \pi\,\chi_{\mathrm{conj}}(g,h) + \pi\,\chi_{\mathrm{ord}}(g,h) + \chi_{\mathrm{gag}}(g,h), \tag{6.40d}$$

and used the definition (6.31) for $\widehat{U}_\wedge(gh)$. The phase factor $\chi_{\mathrm{gag}}(g,h)$ that appears in Eq. (6.40d) is an artifact of the particular gauge choice we have made when defining an Hermitian representation for the fermion parity operator in Eq. (4.2). Indeed, we observe that $\chi_{\mathrm{gag}}(g,h)$ is nothing but a pure gauge under the gauge transformation (5.2a), i.e., the right-hand side of Eq. (5.2b) is identical to $\chi_{\mathrm{gag}}(g,h)$ if we choose $\xi(g) = -3\pi\,\delta_{g,h}/2$. Under such a gauge transformation, the representation $\widehat{U}_\wedge(p)$ of fermion parity p is no longer Hermitian. However, by definition, the equivalence classes $[\phi_\wedge]$ of the stacked 2-cochain $\phi_\wedge(g,h)$ are invariant under the gauge transformations (5.2a). Therefore, the stacked 2-cochain $\phi_\wedge(g,h)$ is gauge equivalent to

$$\phi_\wedge(g,h) \sim \phi_1(g,h) + \phi_2(g,h) + \pi\,\chi(g,h). \tag{6.41}$$

We have reserved the phase $\chi(g,h)$ for all phases other than the 2-cochains $\phi_1(g,h)$, $\phi_2(g,h)$, and $\chi_{\mathrm{gag}}(g,h)$ in Eq. (6.40d), i.e.,

$$\begin{aligned}\chi(g,h) :={}& \frac{1}{\pi}(\phi_{\mathrm{comp}}(g,h)-\phi_1(g,h)-\phi_2(g,h)) + \chi_1(g,h) + \chi_{\mathrm{conj}}(g,h) + \chi_{\mathrm{ord}}(g,h)\\ ={}& \rho_1(g)[q_1(h)+\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)] + \rho_2(g)[q_1(h)+q_2(h)+\mathfrak{c}(h)+\rho_2(h)]\\ &+ q_1(h)\,q_1(g) + q_2(h)\,q_2(g) + q_2(g)[\mathfrak{c}(g)+\rho_2(h)+q_1(h)] + q_1(g)[\mathfrak{c}(h)+\rho_1(h)+\rho_2(h)+q_2(h)].\end{aligned} \tag{6.42}$$

In defining $\chi(g,h)$ we have simplified all the contributions from Eqs. (6.36b), (6.37b), (6.39b), and (6.40b) by using the facts that 1-cochains $\mathfrak{c}$, ρ_i, and q_i for $i = 1, 2$ are all group homomorphisms and $\chi(g,h)$ is only defined modulo 2. Further simplifications in Eq. (6.42) can be made by using the gauge equivalence (6.26) of products of $\mathbb{Z}_2$-valued 1-cochains. Consequently, one is left with

$$\chi(g,h) \sim \rho_1(g)\,\rho_2(h) + \sum_{i=1}^{2}\varphi_i(g,h), \tag{6.43a}$$

$$\varphi_i(g,h) := \rho_i(g)[\mathfrak{c}(h)+\rho_i(h)] + q_i(g)\,[\mathfrak{c}(h)+q_i(h)], \tag{6.43b}$$

where $\varphi_i(g,h)$ is a 2-cochain for $i = 1, 2$.

Finally, we will show that φ_i defined by Eq. (6.43b) vanishes for $i = 1, 2$. To this end, we define

$$\overline{\hat{\gamma}^{(i)}_\infty}^{i} = (-1)^{\zeta_i}\,\hat{\gamma}^{(i)}_\infty, \tag{6.44a}$$

for $\zeta_i, = 0, 1$ and $i = 1, 2$, which, together with Eq. (5.9), deliver the following two identities

$$q_i(g) = \mathfrak{c}(g)(1+\zeta_i) + \rho_i(g) \bmod 2, \tag{6.44b}$$

for any $g \in G_f$ and $i = 1, 2$. Inserting identity (6.44b) to Eq. (6.43b) delivers the desired result

$$\varphi_i(g,h) = \mathfrak{c}(g)\,\mathfrak{c}(h)\,\zeta_i\big(1+\zeta_i\big) = 0 \bmod 2. \tag{6.45}$$

We therefore obtained the stacked 2-cochain $\phi_\wedge(g,h)$

$$\phi_\wedge(g,h) := \phi_1(g,h) + \phi_2(g,h) + \pi\,\rho_1(g)\,\rho_2(h). \tag{6.46}$$

The construction of the indices $([(\nu_\wedge, \rho_\wedge)], [\mu_\wedge])$ in terms of the indices $([(\nu_1, \rho_1)], [\mu_1])$ and $([(\nu_2, \rho_2)], [\mu_2])$ is achieved as follows.

The 2-cochain $\nu_\wedge$ is obtained by restricting $\phi_\wedge$ to the elements of G, i.e.,

$$\nu_\wedge(g,h) := \nu_1(g,h) + \nu_2(g,h) + \pi(\rho_1 \smile \rho_2)(g,h), \tag{6.47a}$$

where we introduced the cup product $\smile$ to construct a 2-cochain out of 1-cochains.

Since $[\mu_\wedge] = 0$, we identify the 1-cochain $\rho_\wedge(g)$ as the total fermion parity of the representation of element $g \in G_f$ [recall definition (5.5)]. From the definition (6.31), we thus find

$$\rho_\wedge(g) = \rho_1(g) + \rho_2(g) + \mathfrak{c}(g), \tag{6.47b}$$

where the first two terms originate from $\widehat{V}_1(g)$ and $\widehat{V}_2(g)$, the next two terms originate from $\hat{\gamma}^{(1)}_{n_1}$, and the last three terms originate from $\hat{\gamma}^{(2)}_{n_2}$.

D. Summary of the fermionic stacking rules

In Secs. VI A, VI B, and VI C, we have explicitly constructed the stacked representation $\widehat{U}_\wedge$ given two representations $\widehat{U}_1$ and $\widehat{U}_2$ in Eqs. (6.11), (6.20), and (6.31). This was achieved by defining for any $g \in G_f$

$$\widehat{U}_\wedge(g) := \begin{cases} \widehat{V}_1(g)\,\widehat{V}_2(g)\,[\widehat{U}_1(p)]^{\rho_2(g)}\,[\widehat{U}_2(p)]^{\rho_1(g)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}, & \text{if } [\mu_1]=[\mu_2]=0,\\ \widehat{V}_1(g)\,\widehat{V}_2(g)\,\widehat{Q}_2(g)\,\big[\widehat{U}_1(p)\,\hat{\gamma}^{(2)}_\infty\big]^{\rho_1(g)}\,\mathsf{K}_\wedge^{\mathfrak{c}(g)}, & \text{if } [\mu_1]=0, [\mu_2]=1,\\ (-\mathrm{i})^{\delta_{g,p}}\widehat{V}_1(g)\widehat{V}_2(g)[\widehat{U}_\wedge(p)]^{\mathfrak{c}(g)+\rho_1(g)+\rho_2(g)}\big[\hat{\gamma}^{(1)}_{n_1}\big]^{q_1(g)+\rho_1(g)}\big[\hat{\gamma}^{(2)}_{n_2}\big]^{\mathfrak{c}(g)+q_2(g)+\rho_2(g)}\mathsf{K}_\wedge^{\mathfrak{c}(g)}, & \text{if } [\mu_1]=[\mu_2]=1, \end{cases} \tag{6.48}$$

and deriving Eqs. (6.17), (6.28), and (6.47) by comparing $\widehat{U}_\wedge(g)\,\widehat{U}_\wedge(h)$ to $\widehat{U}_\wedge(gh)$ for any pair $g, h \in G_f$. We collect these equations into the fermionic stacking rules of one-dimensional IFT phases

$$([(\nu_1,\ \rho_1)],\ 0) \wedge ([(\nu_2,\ \rho_2)],\ 0) = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2),\ \rho_1 + \rho_2)],\ 0), \tag{6.49a}$$

$$([(\nu_1,\ \rho_1)],\ 0) \wedge ([(\nu_2,\ \rho_2)],\ 1) = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2 + \rho_1 \smile \mathfrak{c}),\ \rho_1 + \rho_2)],\ 1), \tag{6.49b}$$

$$([(\nu_1,\ \rho_1)],\ 1) \wedge ([(\nu_2,\ \rho_2)],\ 0) = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2 + \rho_2 \smile \mathfrak{c}),\ \rho_1 + \rho_2)],\ 1), \tag{6.49c}$$

$$([(\nu_1,\ \rho_1)],\ 1) \wedge ([(\nu_2,\ \rho_2)],\ 1) = ([(\nu_1 + \nu_2 + \pi\,\rho_1 \smile \rho_2,\ \rho_1 + \rho_2 + \mathfrak{c})],\ 0). \tag{6.49d}$$

They correspond to the even-even, even-odd, odd-even, and odd-odd stacking, respectively. The stacking rules (6.49) agree with the ones derived in Refs. [10,12]. We note that the even-odd stacking rule derived in Ref. [10] contains the term $\rho_1 \smile \rho_1$ instead of the term $\rho_1 \smile \mathfrak{c}$. These two terms are gauge equivalent to each other under the transformation (5.2b) with $\xi = \pi\,\rho_1 \smile \mathfrak{c} - \frac{\pi}{2}\,\rho_1 \smile \rho_1$. The presentation in Eq. (6.49) makes the role of antiunitary symmetries in the stacking rules explicit. If the group G_f consist of only unitary symmetries, i.e., $\mathfrak{c}(g) = 0$ for any $g \in G_f$, the stacking rules (6.49) reduce to

$$([(\nu_1,\ \rho_1)],\ [\mu_1]) \wedge ([(\nu_2,\ \rho_2)],\ [\mu_2]) = ([(\nu_1 + \nu_2 + \pi(\rho_1 \smile \rho_2),\ \rho_1 + \rho_2)],\ [\mu_1] + [\mu_2]). \tag{6.50}$$

The stacking rules (6.49) dictate the group structure of IFT phases that are symmetric under the group G_f. This group structure encodes the physical operation by which two open chains realizing IFT phases that are symmetric under group G_f are brought adiabatically into contact so as to realize an IFT phases that is symmetric under group G_f. The stacking rules (6.49a) and (6.49d) each encodes how the left and right boundaries of an open chain realizing an IFT phase that is symmetric under the group G_f are glued back together in such a way that the resulting chain obeying periodic boundary conditions supports a nondegenerate gapped ground state.

VII. BULK REPRESENTATIONS DERIVED FROM STACKING RULES

Invertible fermionic topological phases of matter in one-dimensional space have an internal symmetry group G_f that is represented in the bulk by the faithful representation $\widehat{U}_{\mathrm{bulk}}$ given in Eq. (2.1). Because these symmetries are internal, they induce for any site j of any one-dimensional lattice Λ a faithful representation $\widehat{U}_j$. However, representatives of IFT phases can also accommodate projective representations of the internal symmetry group G_f on the left and right boundaries of Λ provided the stacking of these two boundary representations is gauge-equivalent to a faithful representation of G_f, as is captured by Fig. 1.

Instead of deducing the existence of local projective representations for the internal symmetry group G_f from a faithful bulk representation $\widehat{U}_{\mathrm{bulk}}$, we are going to construct a bulk representation $\widehat{U}_{\mathrm{bulk}}$ of the symmetry group G_f out of a given set of projective representations $\widehat{U}_j$ acting on the Clifford algebra

$$C\ell_{n_j} := \mathrm{span}\left\{\hat{\gamma}^{(j)}_1,\ \hat{\gamma}^{(j)}_2,\ \cdots,\ \hat{\gamma}^{(j)}_{n_j}\right\} \tag{7.1}$$

spanned by n_j Majorana degrees of freedom for any site $\boldsymbol{j}$ from a d-dimensional lattice Λ provided

$$\sum_{j\in\Lambda} n_j = 0 \bmod 2. \tag{7.2}$$

To this end, we use the fact that the definition (6.48) and the stacking rules (6.49) are associative [30]. If so, for any $g \in G_f$ and any labeling $\boldsymbol{j}_1, \boldsymbol{j}_2, \cdots, \boldsymbol{j}_{|\Lambda|}$ with $|\Lambda|$ the cardinality of Λ, we can define $\widehat{U}_{\text{bulk}}(g)$ by stacking $\widehat{U}_{\boldsymbol{j}_1}(g)$ with $\widehat{U}_{\boldsymbol{j}_2}(g)$, which we then stack with $\widehat{U}_{\boldsymbol{j}_3}(g)$, and so on. By construction, it follows that

$$\widehat{U}_{\boldsymbol{j}}(g)\,\hat{\gamma}^{(\boldsymbol{j})}_{\iota}\,\widehat{U}^{\dagger}_{\boldsymbol{j}}(g) = \widehat{U}_{\text{bulk}}(g)\,\hat{\gamma}^{(\boldsymbol{j})}_{\iota}\,\widehat{U}^{\dagger}_{\text{bulk}}(g), \tag{7.3}$$

for any $\iota = 1, \cdots, n_{\boldsymbol{j}}$, $\boldsymbol{j} \in \Lambda$, and $g \in G_f$. Equation (7.3) is the counterpart to the consistency condition (4.1) that we used to construct boundary representations. It also follows that the representation

$$\widehat{U}_{\text{bulk}}(g) = \left[\prod_{\boldsymbol{j}\in\Lambda} \widehat{V}_{\boldsymbol{j}}(g)\right] \mathsf{K}^{\mathfrak{c}(g)}, \tag{7.4}$$

for any $g \in G_f$ holds if and only if the local representation $\widehat{U}_{\boldsymbol{j}}(g)$ has the indices $\rho_{\boldsymbol{j}}(g) = 0$ and $[\mu_{\boldsymbol{j}}] = 0$ for any $\boldsymbol{j} \in \Lambda$. This implies that the local representation $\widehat{U}_{\boldsymbol{j}}(g)$ of any element $g \in G_f$ is of even fermion parity and the number of Majorana degrees of freedom $n_{\boldsymbol{j}}$ is an even integer for any site $\boldsymbol{j} \in \Lambda$. It is then appropriate to call a local representation $\widehat{U}_{\boldsymbol{j}}$ that has nontrivial indices $\rho_{\boldsymbol{j}}$ and $[\mu_{\boldsymbol{j}}]$ an *intrinsically fermionic* representation. In other words, the decomposition (7.4) is possible if and only if the local representation $\widehat{U}_{\boldsymbol{j}}$ for any site $\boldsymbol{j} \in \Lambda$ is not intrinsically fermionic. In particular, if all local degrees of freedom are bosonic, then the decomposition (7.4) is always valid. However, instead of the decomposition (7.4), $\widehat{U}_{\text{bulk}}(g)$ is obtained in all generality by iterating Eq. (6.48) for any $g \in G_f$.

VIII. GROUND-STATE DEGENERACIES

In Secs. III, IV, and V we have shown that the distinct IFT phases are characterized by the projective character of the boundary representation $\widehat{U}_{\text{B}}$. In turn this projective character is captured by the triplet of indices $([(\nu,\rho)],[\mu])$. Let us now consider the implications of this triplet being nontrivial, i.e., $([(\nu,\rho)],[\mu]) \neq ([(0,0)],[0])$, for the spectral degeneracy of the boundary states.

The foremost consequence of the nontrivial indices $([(\nu,\rho)],[\mu])$ is the robustness of the boundary degeneracy that is protected by a combination of the symmetry group G_f being represented projectively and the existence of a nonlocal boundary Fock space, denoted $\mathfrak{F}_{\text{LR}}$ in Eq. (3.6), whenever opposite boundaries host odd numbers of Majorana degrees of freedom.

Recently, a robust quantum mechanical supersymmetry [31] was shown in Refs. [20,21,32] to be generically present in nontrivial IFT phases. We are going to recast these results by showing how the quantum mechanical supersymmetry present at the boundaries can be deduced from the indices $([(\nu,\rho)],[\mu])$.

In what follows, we consider the two cases $[\mu]=0$ and $[\mu]=1$ separately. For each case, we first discuss the degeneracies associated with nontrivial pair $[(\nu,\rho)]$ on general grounds.

A. The case of $[\mu] = 0$

When $[\mu]=0$, there always are even numbers of Majorana degrees of freedom localized on each disconnected component Λ_{L} and Λ_{R} of the boundary Λ_{bd} [recall definition (3.1b)]. In this case, the boundary Fock space $\mathfrak{F}_{\Lambda_{\text{bd}}}$ spanned by the Majorana degrees of freedom supported on Λ_{bd} decomposes as

$$\mathfrak{F}_{\Lambda_{\text{bd}}} = \mathfrak{F}_{\Lambda_{\text{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\text{R}}}, \tag{8.1}$$

where $\otimes_{\mathfrak{g}}$ denotes a $\mathbb{Z}_2$ graded tensor product, while $\mathfrak{F}_{\Lambda_{\text{L}}}$ and $\mathfrak{F}_{\Lambda_{\text{R}}}$ are the Fock spaces spanned by the Majorana degrees of freedom localized at the disconnected components Λ_{L} and Λ_{R}, respectively. The Fock spaces $\mathfrak{F}_{\Lambda_{\text{L}}}$ and $\mathfrak{F}_{\Lambda_{\text{R}}}$ are defined by Eq. (3.3a). We denote with $\widehat{H}_{\text{L}}$ and $\widehat{H}_{\text{R}}$ the Hamiltonians that act on Fock spaces $\mathfrak{F}_{\Lambda_{\text{L}}}$ and $\mathfrak{F}_{\Lambda_{\text{R}}}$ and govern the dynamics of the local Majorana degrees of freedom localized at boundaries Λ_{L} and Λ_{R}, respectively. By assumption, the Hamiltonians $\widehat{H}_{\text{L}}$ and $\widehat{H}_{\text{R}}$ are invariant under the representations (possibly projective) $\widehat{U}_{\text{L}}$ and $\widehat{U}_{\text{R}}$ of the given symmetry group G_f, respectively.

Since $[\mu]=0$, the only nontrivial IFT phases are those with nontrivial equivalence classes $[(\nu,\rho)] \neq [(0,0)]$, i.e., the FSPT phases. By definition, the indices $([\nu_{\text{L}},\rho_{\text{L}}],0)$ and $([\nu_{\text{R}},\rho_{\text{R}}],0)$ associated with the representations (possibly projective) $\widehat{U}_{\text{L}}$ and $\widehat{U}_{\text{R}}$, respectively, satisfy

$$([(\nu_{\text{L}},\rho_{\text{L}})],0) \wedge ([(\nu_{\text{R}},\rho_{\text{R}})],0) = ([(0,0)],0) \tag{8.2}$$

under the stacking rule (6.49a).

If we focus on a single boundary (denoted by B), the equivalence class $[(\nu_{\text{B}},\rho_{\text{B}})]$ characterizes the nontrivial projective nature of the boundary representation $\widehat{U}_{\text{B}}$. Whenever $[(\nu_{\text{B}},\rho_{\text{B}})] \neq [(0,0)]$, it is guaranteed that there is no state that is invariant under the action of $\widehat{U}_{\text{B}}(g)$ for all $g \in G_f$. In other words, there is no state in the Fock space $\mathfrak{F}_{\Lambda_{\text{B}}}$ that transforms as a singlet under the representation $\widehat{U}_{\text{B}}$. Any eigenenergy of a G_f-symmetric boundary Hamiltonian $\widehat{H}_{\text{B}}$ must be degenerate. The degeneracy is protected by the particular representation $\widehat{U}_{\text{B}}$ of the symmetry group G_f and cannot be lifted without breaking the G_f symmetry. The minimal degeneracy that is protected by the G_f symmetry depends on the explicit structure of the group G_f and the equivalence class $[(\nu_{\text{B}},\rho_{\text{B}})]$ of the boundary representation $\widehat{U}_{\text{B}}$.

Since for $[\mu]=0$, the boundary representations $\widehat{U}_{\text{L}}$ and $\widehat{U}_{\text{R}}$ act on two independent Fock spaces $\mathfrak{F}_{\Lambda_{\text{L}}}$ and $\mathfrak{F}_{\Lambda_{\text{R}}}$, the total protected ground-state degeneracy $\text{GSD}^{[\mu]=0}_{\text{bd}}$ when open boundary conditions are imposed is nothing but the product of the protected ground-state degeneracies $\text{GSD}^{[\mu]=0}_{\text{L}}$ and $\text{GSD}^{[\mu]=0}_{\text{R}}$ of the Hamiltonians $\widehat{H}_{\text{L}}$ and $\widehat{H}_{\text{R}}$, respectively, i.e.,

$$\text{GSD}^{[\mu]=0}_{\text{bd}} = \text{GSD}^{[\mu]=0}_{\text{L}} \times \text{GSD}^{[\mu]=0}_{\text{R}}. \tag{8.3}$$

When $[\mu]=0$, the 1-cochain $\rho_{\text{B}}(g)=0,1$ encodes the commutation relation between the representations $\widehat{U}_{\text{B}}(g)$ of group element $g \in G_f$ and $\widehat{U}_{\text{B}}(p)$ of fermion parity $p \in G_f$. A nonzero second entry in the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ implies that there exists at least one group element $g\in G_f$ with $\rho_{\mathrm{B}}(g)=1$, i.e., the operator $\widehat{U}_{\mathrm{B}}(g)$ is of odd fermion parity. If this is so, the boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ must possess an emergent quantum mechanical supersymmetry. The supercharges associated with the boundary supersymmetry are constructed following Ref. [32]. Assume without loss of generality that all energy eigenvalues ε_α of a boundary Hamiltonian $\widehat{H}_{\mathrm{B}}$ are shifted to the positive energies, i.e., $\varepsilon_\alpha>0$. Also assume that there exists a group element $g\in G_f$ with $\rho_{\mathrm{B}}(g)=1$. For any orthonormal eigenstate $|\psi_\alpha\rangle$ of $\widehat{H}_{\mathrm{B}}$ with energy ε_α, the state

$$|\psi'_\alpha\rangle := \widehat{U}_{\mathrm{B}}(g)\,|\psi_\alpha\rangle, \tag{8.4a}$$

is also an orthonormal eigenstate of $\widehat{H}_{\mathrm{B}}$ with the same energy but opposite fermion parity. Since the fermion parities of $|\psi'_\alpha\rangle$ and $|\psi_\alpha\rangle$ are different, they are orthogonal. Two supercharges can then be defined as

$$\widehat{Q}_1 := \sum_\alpha \sqrt{\varepsilon_\alpha}\big[\big(\widehat{U}_{\mathrm{B}}(g)\,|\psi_\alpha\rangle\big)\langle\psi_\alpha| + |\psi_\alpha\rangle\big(\langle\psi_\alpha|\,\widehat{U}^\dagger_{\mathrm{B}}(g)\big)\big], \tag{8.4b}$$

$$\widehat{Q}_2 := \sum_\alpha \mathrm{i}\sqrt{\varepsilon_\alpha}\big[\big(\widehat{U}_{\mathrm{B}}(g)\,|\psi_\alpha\rangle\big)\langle\psi_\alpha| - |\psi_\alpha\rangle\big(\langle\psi_\alpha|\,\widehat{U}^\dagger_{\mathrm{B}}(g)\big)\big]. \tag{8.4c}$$

Operators $\widehat{Q}_1$ and $\widehat{Q}_2$ are Hermitian, carry odd fermion parity, and satisfy the defining properties

$$\{\widehat{Q}_i,\widehat{Q}_j\} = 2\widehat{H}_{\mathrm{B}}\,\delta_{i,j},\quad [\widehat{Q}_i,\widehat{H}_{\mathrm{B}}]=0,\quad i,j=1,2, \tag{8.4d}$$

of fermionic supercharges. The precise number of supercharges on the boundary Λ_{B} depends on the pair $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ that characterizes the number of symmetry operators $\widehat{U}_{\mathrm{B}}(g)$ that carry odd fermion parity and their mutual algebra.

B. The case of $[\mu]=1$

When $[\mu]=1$, there are odd number of Majorana degrees of freedom localized on each disconnected component Λ_{L} and Λ_{R} of the boundary Λ_{bd} [recall definition (3.1b)]. In this case, the boundary Fock space $\mathfrak{F}_{\Lambda_{\mathrm{bd}}}$ spanned by Majorana degrees of freedom supported on Λ_{bd} decomposes as

$$\mathfrak{F}_{\Lambda_{\mathrm{bd}}} = \mathfrak{F}_{\Lambda_{\mathrm{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{R}}}, \tag{8.5}$$

where $\otimes_{\mathfrak{g}}$ denotes a $\mathbb{Z}_2$ graded tensor product. The Fock spaces $\mathfrak{F}_{\Lambda_{\mathrm{B}}}$ with $\mathrm{B}=\mathrm{L},\mathrm{R}$ is spanned by all the Majorana operators localized at the disconnected components Λ_{B} except one. The two-dimensional Fock space $\mathfrak{F}_{\Lambda_{\mathrm{LR}}}$ is spanned by the two remaining Majorana operators with one localized on the left boundary Λ_{L} and the other localized on the right boundary Λ_{R} of the open chain. Correspondingly, the pair of fermionic creation and annihilation operators that span $\mathfrak{F}_{\Lambda_{\mathrm{LR}}}$ are nonlocal in the sense that they are formed by Majorana operators supported on opposite boundaries. One can define Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ that are constructed out of Majorana operators localized at the boundaries Λ_{L} and Λ_{R}. If so, the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ act on Fock spaces

$$\mathfrak{F}_{\Lambda_{\mathrm{L}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}}, \tag{8.6a}$$

and

$$\mathfrak{F}_{\Lambda_{\mathrm{R}}} \otimes_{\mathfrak{g}} \mathfrak{F}_{\Lambda_{\mathrm{LR}}}, \tag{8.6b}$$

respectively. By assumption, the Hamiltonians $\widehat{H}_{\mathrm{L}}$ and $\widehat{H}_{\mathrm{R}}$ are invariant under the representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$ of a given symmetry group G_f, respectively.

On each boundary Λ_{B}, there exists a local Hermitian and unitary operator $\widehat{Y}_{\mathrm{B}}$ that commutes with any other local operator supported on Λ_{B}. The operator $\widehat{Y}_{\mathrm{B}}$ is defined by Eq. (4.5) and is the representation of the nontrivial central element of a Clifford algebra $C\ell_n$ with n an odd number of generators. It therefore carries an odd fermion parity and anticommutes with the representation $\widehat{U}_{\mathrm{B}}(p)$ of fermion parity. It follows that $\widehat{Y}_{\mathrm{B}}$ must commute with $\widehat{H}_{\mathrm{B}}$. We label the simultaneous eigenstates of $\widehat{H}_{\mathrm{B}}$ and $\widehat{Y}_{\mathrm{B}}$ by $|\psi_{\mathrm{B},\alpha,\pm}\rangle$, i.e.,

$$\widehat{Y}_{\mathrm{B}}\,|\psi_{\mathrm{B},\alpha,\pm}\rangle = \pm|\psi_{\mathrm{B},\alpha,\pm}\rangle,\quad \widehat{H}_{\mathrm{B}}\,|\psi_{\mathrm{B},\alpha,\pm}\rangle = \varepsilon_\alpha\,|\psi_{\mathrm{B},\alpha,\pm}\rangle, \tag{8.7}$$

where ε_α is the corresponding energy eigenvalue, which we assume without loss of generality to be strictly positive. Hence, all eigenstates of $\widehat{H}_{\mathrm{B}}$ are at least twofold degenerate. Since $\widehat{Y}_{\mathrm{B}}$ carries odd fermion parity, the eigenstates $|\psi_{\mathrm{B},\alpha,\pm}\rangle$ do not have definite fermion parities. The simultaneous eigenstates of $\widehat{H}_{\mathrm{B}}$ and $\widehat{U}_{\mathrm{B}}(p)$ must be the bonding and anti-bonding linear combinations of $|\psi_{\mathrm{B},\alpha,+}\rangle$ and $|\psi_{\mathrm{B},\alpha,-}\rangle$ that are exchanged under the action of $\widehat{Y}_{\mathrm{B}}$. The twofold degeneracy of $\widehat{H}_{\mathrm{B}}$ when $[\mu]=1$ is due to the presence of the two-dimensional Fock space $\mathfrak{F}_{\mathrm{LR}}$. This twofold degeneracy is of supersymmetric nature and the associated supercharges are

$$\widehat{Q}_1 := \sum_\alpha \sqrt{\varepsilon_\alpha}\,(\,|\psi_{\alpha,+}\rangle\langle\psi_{\alpha,+}| - |\psi_{\alpha,-}\rangle\langle\psi_{\alpha,-}|\,), \tag{8.8a}$$

$$\widehat{Q}_2 := \sum_\alpha \mathrm{i}\sqrt{\varepsilon_\alpha}\,(\,|\psi_{\alpha,+}\rangle\langle\psi_{\alpha,-}| - |\psi_{\alpha,-}\rangle\langle\psi_{\alpha,+}|\,). \tag{8.8b}$$

Operators $\widehat{Q}_1$ and $\widehat{Q}_2$ are Hermitian. They carry odd fermion parity since the operator $\widehat{U}_{\mathrm{B}}(p)$ exchanges the states $|\psi_{\alpha,\pm}\rangle$ with $|\psi_{\alpha,\mp}\rangle$. They satisfy the defining properties

$$\{\widehat{Q}_i,\widehat{Q}_j\} = 2\widehat{H}_{\mathrm{B}}\,\delta_{i,j},\quad [\widehat{Q}_i,\widehat{H}_{\mathrm{B}}]=0,\quad i,j=1,2, \tag{8.8c}$$

of fermionic supercharges.

There may be other supercharges in addition to the ones defined in Eq. (8.8) due to the representation $\widehat{U}_{\mathrm{B}}$ of the group G_f. The precise number of these additional supercharges on the boundary Λ_{B} depends on the pair $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ that characterizes the number of symmetry operators $\widehat{U}_{\mathrm{B}}(g)$ that carry odd fermion parity and their mutual algebra. They can be constructed in the same fashion as in Eq. (8.4).

By definition, the indices $([(\nu_{\mathrm{L}},\rho_{\mathrm{L}})],1)$ and $([(\nu_{\mathrm{R}},\rho_{\mathrm{R}})],1)$ associated to the representations $\widehat{U}_{\mathrm{L}}$ and $\widehat{U}_{\mathrm{R}}$, respectively, satisfy

$$\big([(\nu_{\mathrm{L}},\rho_{\mathrm{L}})],1\big)\wedge\big([(\nu_{\mathrm{R}},\rho_{\mathrm{R}})],1\big)=\big([(0,0)],0\big) \tag{8.9}$$

under the stacking rule (6.49d). If we focus on a single boundary (denoted by B), the equivalence class $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]$ characterizes the nontrivial projective nature of the boundary representation $\widehat{U}_{\mathrm{B}}$. Whenever $[(\nu_{\mathrm{B}},\rho_{\mathrm{B}})]\neq[(0,0)]$, it is guaranteed that there is no state that is invariant under the action of $\widehat{U}_{\mathrm{B}}(g)$ for all $g\in G_f$. In other words, there is no state in the Fock space $\mathfrak{F}_{\Lambda_{\mathrm{B}}}$ that transforms as a singlet under the representation $\widehat{U}_{\rm B}$. Each eigenstate of a symmetric boundary Hamiltonian $\widehat{H}_{\rm B}$ must carry degeneracies in addition to the twofold degeneracy due to $[\mu]=1$. The degeneracy is protected by the particular representation $\widehat{U}_{\rm B}$ of the symmetry group G_f and cannot be lifted without breaking the G_f symmetry. The exact degeneracy protected by the representation depends on the explicit form of the group G_f, and the boundary representation $\widehat{U}_{\rm B}$ with the equivalence class $[(\nu_{\rm B},\rho_{\rm B})]$.

Since for $[\mu]=1$, the boundary representations $\widehat{U}_{\rm L}$ and $\widehat{U}_{\rm R}$ do not act on two decoupled Fock spaces. The total protected ground-state degeneracy $\mathrm{GSD}^{[\mu]=1}_{\rm bd}$ when open boundary conditions are imposed cannot be computed by taking the products of degeneracies associated with the Hamiltonians $\widehat{H}_{\rm L}$ and $\widehat{H}_{\rm R}$ separately. However, $\mathrm{GSD}^{[\mu]=1}_{\rm bd}$ can be computed by multiplying the "naive" protected ground state degeneracies of the Hamiltonians at the two boundaries and modding out the twofold degeneracy due to $\mathfrak{F}_{\rm LR}$ shared by the two Hamiltonians, i.e.,

$$\mathrm{GSD}^{[\mu]=1}_{\rm bd}=\tfrac{1}{2}\times\mathrm{GSD}^{[\mu]=1}_{\rm L}\times\mathrm{GSD}^{[\mu]=1}_{\rm R}, \tag{8.10}$$

where $\mathrm{GSD}^{[\mu]=1}_{\rm L}$ and $\mathrm{GSD}^{[\mu]=1}_{\rm R}$ are the protected ground state degeneracies of $\widehat{H}_{\rm L}$ and $\widehat{H}_{\rm R}$, respectively.

IX. CONCLUSIONS

In this paper, we have studied one-dimensional invertible fermionic topological (IFT) phases. By extending ideas presented in Ref. [6], we have explicitly constructed the boundary representations of any internal fermionic symmetry group G_f. To this end, we have defined a triplet $([(\nu,\rho)],[\mu])$ that characterizes all inequivalent boundary representations of G_f. This index classifies all distinct invertible fermionic topological phases with the internal fermionic symmetry group G_f. We have also given an elementary derivation of the fermionic stacking rules. These stacking rules dictate the group structure of one-dimensional invertible fermionic topological phases given a symmetry group G_f. They agree with the stacking rules derived in Refs. [10,12], but disagree with the ones derived in Ref. [9].

Given an IFT phase in one-dimensional space characterized by the triplet $([\nu,\rho]),[\mu])$, we have deduced the protected ground-state degeneracies on general grounds. In doing so, we have identified that an emergent supersymmetry at the boundaries is implied whenever the IFT phase is intrinsically fermionic, i.e., either $\rho(g)=1$ for some $g\in G_f$ or $[\mu]=1$ for the boundary projective representation.

Finally, we have given a concrete application of these results by working out the IFT phases in symmetry class BDI from the tenfold way in the Supplemental Material. By applying the Jordan-Wigner transformation we can map the Majorana c chains that we chose from the symmetry class BDI to spin-1/2 cluster c chains. We can then explain how IFT phases are turned into bosonic symmetry protected topological phases of matter by the nonlocal Jordan-Wigner transformation.

ACKNOWLEDGMENTS

We thank Alex Turzillo for many valuable comments. Ö.M.A. was supported by the Swiss National Science Foundation (SNSF) under Grant No. 200021 184637.

APPENDIX A: GROUP COHOMOLOGY

Given two groups G and M, an *n-cochain* is the map

$$\phi: G^n\to M,\quad (g_1,g_2,\cdots,g_n)\mapsto\phi(g_1,g_2,\cdots,g_n), \tag{A1}$$

that maps an n-tuple $(g_1,g_2,\cdots,g_n)$ to an element $\phi(g_1,g_2,\dots,g_n)\in M$. The set of all n-cochains from G^n to M is denoted by $C^n(G,M)$. We define an M-valued 0-cochain to be an element of the group M itself, i.e., $C^0(G,M)=M$. Henceforth, we will denote the group composition rule in G by $\cdot$ and the group composition rule in M additively by $+$ ($-$ denoting the inverse element).

Given the group homomorphism $\mathfrak{c}: G\to\{0,1\}$, for any $g\in G$, we define the group action

$$\begin{aligned}\mathfrak{C}_g: M&\to M,\\ m&\mapsto(-1)^{\mathfrak{c}(g)}\,m.\end{aligned} \tag{A2}$$

The homomorphism $\mathfrak{c}$ indicates whether and element $g\in G$ is represented unitarily $[\mathfrak{c}(g)=0]$ or antiunitarily $[\mathfrak{c}(g)=1]$. We define the map $\delta^n_{\mathfrak{c}}$

$$\delta^n_{\mathfrak{c}}: C^n(G,M)\ \to\ C^{n+1}(G,M),\quad \phi\mapsto\left(\delta^n_{\mathfrak{c}}\phi\right), \tag{A3a}$$

from n-cochains to $(n+1)$-cochains such that

$$\begin{aligned}\left(\delta^n_{\mathfrak{c}}\phi\right)(g_1,\cdots,g_{n+1}) &:= \mathfrak{C}_{g_1}(\phi(g_2,\cdots,g_n,g_{n+1}))\\ &+\sum_{i=1}^{n}(-1)^i\phi(g_1,\cdots,g_i\cdot g_{i+1},\cdots,g_{n+1})\\ &-(-1)^n\,\phi(g_1,\cdots,g_n).\end{aligned} \tag{A3b}$$

The map $\delta^n_{\mathfrak{c}}$ is called a *coboundary operator*.

Example $n=2$. The coboundary operator $\delta^2_{\mathfrak{c}}$ is defined by

$$\begin{aligned}\left(\delta^2_{\mathfrak{c}}\phi\right)(g_1,g_2,g_3) &= \mathfrak{C}_{g_1}(\phi(g_2,g_3))+(-1)^1\phi(g_1\cdot g_2,g_3)\\ &\quad+(-1)^2\phi(g_1,g_2\cdot g_3)-(-1)^2\,\phi(g_1,g_2)\\ &=(-1)^{\mathfrak{c}(g_1)}\,\phi(g_2,g_3)-\phi(g_1\cdot g_2,g_3)\\ &\quad+\phi(g_1,g_2\cdot g_3)-\phi(g_1,g_2).\end{aligned} \tag{A4}$$

We observe that

$$\begin{aligned}\left(\delta^2_{\mathfrak{c}}\phi\right)(g_1,g_2,g_3)=0 \iff \phi(g_1,g_2)+\phi(g_1\cdot g_2,g_3)\\ =\phi(g_1,g_2\cdot g_3)+(-1)^{\mathfrak{c}(g_1)}\,\phi(g_2,g_3)\end{aligned} \tag{A5}$$

is nothing but the 2-cocycle condition (5.1b) obeyed by ϕ.

Example $n=1$. The coboundary operator $\delta^1_{\mathfrak{c}}$ is defined by

$$\begin{aligned}\left(\delta^1_{\mathfrak{c}}\phi\right)(g_1,g_2) &= \mathfrak{C}_{g_1}(\phi(g_2))+(-1)^1\phi(g_1\cdot g_2)-(-1)^1\phi(g_1)\\ &=(-1)^{\mathfrak{c}(g_1)}\,\phi(g_2)-\phi(g_1\cdot g_2)+\phi(g_1).\end{aligned} \tag{A6}$$

One verifies the important identity

$$\Phi(g_1,g_2):=\left(\delta^1_{\mathfrak{c}}\phi\right)(g_1,g_2)\ \Rightarrow\ \left(\delta^2_{\mathfrak{c}}\Phi\right)(g_1,g_2,g_3)=0. \tag{A7}$$

Using the coboundary operator, we define two sets

$$Z^n(G, M_{\mathfrak{c}}) := \ker\left(\delta^n_{\mathfrak{c}}\right) = \left\{\phi \in C^n(G, M) \mid \delta^n_{\mathfrak{c}}\phi = 0\right\}, \tag{A8a}$$

and

$$B^n(G, M_{\mathfrak{c}}) := \mathrm{im}\left(\delta^{n-1}_{\mathfrak{c}}\right) = \left\{\phi \in C^n(G, M) \mid \phi = \delta^{n-1}_{\mathfrak{c}}\phi', \phi' \in C^{n-1}(G, M)\right\}. \tag{A8b}$$

The cochains in $Z^n(G, M_{\mathfrak{c}})$ are called *n-cocycles*. The cochains in $B^n(G, M_{\mathfrak{c}})$ are called *n-coboundaries*. The action of the boundary operator on the elements of the group M is sensitive to the homomorphism $\mathfrak{c}$. For this reason, we label M by $\mathfrak{c}$ in $Z^n(G, M_{\mathfrak{c}})$ and $B^n(G, M_{\mathfrak{c}})$. The importance of the coboundaries is that the identity (A7) generalizes to

$$\phi = \delta^{n-1}_{\mathfrak{c}}\phi' \ \Rightarrow\ \delta^n_{\mathfrak{c}}\phi = 0. \tag{A9}$$

The nth cohomology group is defined as the quotient of the n-cocycles by the n-coboundaries, i.e.,

$$H^n(G, M_{\mathfrak{c}}) := Z^n(G, M_{\mathfrak{c}})/B^n(G, M_{\mathfrak{c}}). \tag{A10}$$

The nth cohomology group $H^n(G, M_{\mathfrak{c}})$ is an additive Abelian group. We denote its elements by $[\phi] \in H^n(G, M_{\mathfrak{c}})$, i.e., the equivalence class of the n-cocycle ϕ.

Finally, we define the following operation on the cochains. Given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$, we produce the cochain $(\phi \cup \theta) \in C^{n+m}(G, N \times M)$ through

$$(\phi \cup \theta)(g_1, \cdots, g_n, g_{n+1}, \cdots, g_m) := \left(\phi(g_1, \cdots, g_n), \mathfrak{C}_{g_1\cdot g_2\cdots g_n}\left(\theta(g_{n+1}, \cdots, g_{n+m})\right)\right). \tag{A11a}$$

If we compose operation (A11a) with the pairing map $f : N \times M \to M'$ where M' is an Abelian group, we obtain the cup product

$$(\phi \smile \theta)(g_1, \cdots, g_n, g_{n+1}, \cdots, g_m) := f\left(\left(\phi(g_1, \cdots, g_n), \mathfrak{C}_{g_1\cdot g_2\cdots g_n}\left(\theta(g_{n+1}, \cdots, g_{n+m})\right)\right)\right). \tag{A11b}$$

Hence, $(\phi \smile \theta) \in C^{n+m}(G, M')$. For our purposes, both N and M are subsets of the integer numbers, $M' = \mathbb{Z}_2$, while the pairing map f is

$$f\left(\left(\phi(g_1, \cdots, g_n), \mathfrak{C}_{g_1\cdot g_2\cdots g_n}\left(\theta(g_{n+1}, \cdots, g_{n+m})\right)\right)\right) := \phi(g_1, \cdots, g_n)\, \mathfrak{C}_{g_1\cdot g_2\cdots g_n}\left(\theta(g_{n+1}, \cdots, g_{n+m})\right) \bmod 2 \tag{A12}$$

where multiplication of cochains ϕ and θ is treated as multiplication of integers numbers modulo 2. For instance, for the cup product of a 1-cochain $\alpha \in C^1(G, \mathbb{Z}_2)$ and a 2-cochain $\beta \in C^2(G, \mathbb{Z}_2)$, we write

$$(\alpha \smile \beta)(g_1, g_2, g_3) = \alpha(g_1)\, \mathfrak{C}_{g_1}(\beta(g_2, g_3)) = \alpha(g_1)\, \beta(g_2, g_3), \tag{A13}$$

where the cup product takes values in $\mathbb{Z}_2 = \{0, 1\}$ and multiplication of α and β is the multiplication of integers. In reaching the last equality, we have used the fact that the 2-cochain $\beta(g_2, g_3)$ takes values in $\mathbb{Z}_2$ for which $\mathfrak{C}_{g_1}(\beta(g_2, g_3)) = \beta(g_2, g_3)$ for any g_1. The cup product defined in Eq. (A11b) satisfies

$$\delta^{n+m}_{\mathfrak{c}}(\phi \smile \theta) = \left(\delta^n_{\mathfrak{c}}\phi \smile \theta\right) + (-1)^n \left(\phi \smile \delta^m_{\mathfrak{c}}\theta\right), \tag{A14}$$

given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$. Hence, the cup product of two cocycles is again a cocycle as the right-hand side of Eq. (A14) vanishes.

APPENDIX B: CONSTRUCTION OF THE FERMIONIC SYMMETRY GROUP G_f

For quantum systems built out of Majorana degrees of freedom the parity (evenness or oddness) of the total fermion number is always a constant of the motion. If $\widehat{F}$ denotes the operator whose eigenvalues counts the total number of local fermions in the Fock space, then the parity operator $(-1)^{\widehat{F}}$ necessarily commutes with the Hamiltonian that dictates the quantum dynamics, even though $\widehat{F}$ might not, as is the case in any mean-field treatment of superconductivity.

We denote the group of two elements e and p

$$\mathbb{Z}^{\mathrm{F}}_2 := \{e, p \,|\, e\, p = p\, e = p, \quad e = e\, e = p\, p\}, \tag{B1}$$

whereby e is the identity element and we shall interpret the quantum representation of p as the fermion parity operator. It is because of this interpretation of the group element p that we attach the upper index F to the cyclic group $\mathbb{Z}_2$. In addition to the symmetry group $\mathbb{Z}^{\mathrm{F}}_2$, we assume the existence of a second symmetry group G with the composition law $\cdot$ and the identity element id. We would like to construct a new symmetry group G_f out of the two groups G and $\mathbb{Z}^{\mathrm{F}}_2$. Here, the symmetry group G_f inherits the "fermionic" label f from its center $\mathbb{Z}^{\mathrm{F}}_2$. One possibility is to consider the Cartesian product

$$G \times \mathbb{Z}^{\mathrm{F}}_2 := \left\{(g, h) \mid g \in G, \quad h \in \mathbb{Z}^{\mathrm{F}}_2\right\} \tag{B2a}$$

with the composition rule

$$(g_1, h_1) \circ (g_2, h_2) := (g_1 \cdot g_2, h_1\, h_2). \tag{B2b}$$

The resulting group G_f is the direct product of G and $\mathbb{Z}^{\mathrm{F}}_2$. However, the composition rule (B2b) is not the only one compatible with the existence of a neutral element, inverse, and associativity. To see this, we assume first the existence of the map

$$\gamma : G \times G \to \mathbb{Z}^{\mathrm{F}}_2, (g_1, g_2) \mapsto \gamma(g_1, g_2), \tag{B3a}$$

whereby we impose the conditions

$$\gamma(\mathrm{id}, g) = \gamma(g, \mathrm{id}) = e, \gamma(g^{-1}, g) = \gamma(g, g^{-1}), \tag{B3b}$$

for all $g \in G$ and

$$\gamma(g_1, g_2)\, \gamma(g_1 \cdot g_2, g_3) = \gamma(g_1, g_2 \cdot g_3)\, \gamma(g_2, g_3), \tag{B3c}$$

for all $g_1, g_2, g_3 \in G$. Second, we define G_f to be the set of all pairs (g, h) with $g \in G$ and $h \in \mathbb{Z}^{\mathrm{F}}_2$ obeying the composition rule

$$\underset{\gamma}{\circ} : \left(G \times \mathbb{Z}^{\mathrm{F}}_2\right) \times \left(G \times \mathbb{Z}^{\mathrm{F}}_2\right) \to G \times \mathbb{Z}^{\mathrm{F}}_2, \quad ((g_1, h_1), (g_2, h_2)) \mapsto (g_1, h_1) \underset{\gamma}{\circ} (g_2, h_2), \tag{B3d}$$

where

$$(g_1, h_1) \underset{\gamma}{\circ} (g_2, h_2) := (g_1 \cdot g_2, h_1\, h_2\, \gamma(g_1, g_2)). \tag{B3e}$$

One verifies the following properties. First, the order within the composition $h_1\, h_2\, \gamma(g_1, g_2)$ is arbitrary since $\mathbb{Z}_2^{\mathrm{F}}$ is Abelian. Second, conditions (B3b) and (B3c) ensure that G_f is a group with the neutral element

$$(\mathrm{id}, e), \tag{B4a}$$

the inverse to (g, h) is

$$(g^{-1}, [\gamma(g, g^{-1})]^{-1}\, h^{-1}), \tag{B4b}$$

and the center (those elements of the group that commute with all group elements) given by

$$\left(\mathrm{id}, \mathbb{Z}_2^{\mathrm{F}}\right), \tag{B4c}$$

i.e., the group G_f is a central extension of G by $\mathbb{Z}_2^{\mathrm{F}}$. Third, the map γ can be equivalent to a map γ' of the form (B3a) in that they generate two isomorphic groups. This is true if there exists the one-to-one map

$$\begin{aligned}\widetilde{\kappa} : G \times \mathbb{Z}_2^{\mathrm{F}} &\to G \times \mathbb{Z}_2^{\mathrm{F}},\\ (g, h) &\mapsto (g, \kappa(g)\, h)\end{aligned} \tag{B5a}$$

induced by the map

$$\kappa : G \to \mathbb{Z}_2^{\mathrm{F}}, \quad g \mapsto \kappa(g), \tag{B5b}$$

such that the condition

$$\widetilde{\kappa}((g_1, h_1) \underset{\gamma}{\circ} (g_2, h_2)) = \widetilde{\kappa}((g_1, h_1)) \underset{\gamma'}{\circ} \widetilde{\kappa}((g_2, h_2)) \tag{B6}$$

holds for all $(g_1, h_1), (g_2, h_2) \in G \times \mathbb{Z}_2^{\mathrm{F}}$. In other words, γ and γ' generate two isomorphic groups if the identity

$$\kappa(g_1 \cdot g_2) \cdot \gamma(g_1, g_2) = \kappa(g_1) \cdot \kappa(g_2) \cdot \gamma'(g_1, g_2) \tag{B7}$$

holds for all $g_1, g_2 \in G$. This group isomorphism defines an equivalence relation. We say that the group G_f obtained by extending the group G with the group $\mathbb{Z}_2^{\mathrm{F}}$ through the map γ splits when a map (B5b) exists such that

$$\kappa(g_1 \cdot g_2) \cdot \gamma(g_1, g_2) = \kappa(g_1) \cdot \kappa(g_2) \tag{B8}$$

holds for all $g_1, g_2 \in G$, i.e., G_f splits when it is isomorphic to the direct product (B2).

The task of classifying all the non-equivalent central extensions of G by $\mathbb{Z}_2^{\mathrm{F}}$ through γ is achieved by enumerating all the elements of the second cohomology group $H^2(G, \mathbb{Z}_2^{\mathrm{F}})$. We define an index $[\gamma] \in H^2(G, \mathbb{Z}_2^{\mathrm{F}})$ to represent such an equivalence class, whereby the index $[\gamma] = 0$ is assigned to the case when G_f splits.

APPENDIX C: CLASSIFICATION OF PROJECTIVE REPRESENTATIONS OF G_f

It was described in Appendix B, how a global symmetry group G_f for a fermionic quantum system naturally contains the fermion-number parity symmetry group $\mathbb{Z}_2^{\mathrm{F}}$ in its center, i.e., it is a central extension of a group G by $\mathbb{Z}_2^{\mathrm{F}}$. Such group extension are classified by prescribing an element $[\gamma] \in H^2(G, \mathbb{Z}_2^{\mathrm{F}})$, such that we may think of G_f as the set of tuples $(g, h) \in G \times \mathbb{Z}_2^{\mathrm{F}}$ with composition rule as in Eq. (B3e). From this perspective there is an implicit choice of trivialization $\tau : G_f \to \mathbb{Z}_2^{\mathrm{F}}$ and projection $b : G_f \to G$ such that

$$\tau[(g, h)] = h, \quad b[(g, h)] = g. \tag{C1}$$

Importantly, τ is related to the extension class γ that defines the group extension via the relation

$$b^{\star}\gamma = \delta_{\mathfrak{c}}^{1}\tau \tag{C2}$$

where $b^{\star}\gamma \in C^2(G_f, \mathbb{Z}_2^{\mathrm{F}})$ is the pullback of γ via b.

As explained in Sec. V, we shall trade the 2-cocycle $\phi(g, h) \in \mathbb{Z}^2(G, \mathrm{U}(1)_{\mathfrak{c}})$ with the tuple $(\nu, \rho) \in C^2[G, \mathrm{U}(1)] \times C^1(G, \mathbb{Z}_2)$ that satisfy certain cocycle and coboundary conditions. To this end, it is convenient to define the modified 2-coboundary operator

$$\mathcal{D}^2_{\gamma}(\nu,\ \rho) := \left(\delta^2_{\mathfrak{c}}\nu - \pi\, \rho \smile \gamma, \delta^1_{\mathfrak{c}}\rho\right), \tag{C3}$$

acting on a tuple of cochains $(\nu, \rho) \in C^2[G, \mathrm{U}(1)] \times C^1(G, \mathbb{Z}_2)$ together with the modified 1-coboundary operator

$$\mathcal{D}^1_{\gamma}(\alpha, \beta) := \left(\delta^1_{\mathfrak{c}}\alpha + \pi\, \beta \smile \gamma, \delta^0_{\mathfrak{c}}\beta\right) \tag{C4}$$

acting on a tuple of cochains $(\alpha,\ \beta) \in C^1(G, \mathrm{U}(1)) \times C^0(G, \mathbb{Z}_2)$. Being a 0-cochain β does not take any arguments and takes values in $\mathbb{Z}_2$, i.e., $\beta \in \mathbb{Z}_2$. Note that for the 0-cochain β, the coboundary operator (A3b) acts as

$$\left(\delta^0_{\mathfrak{c}}\beta\right)(g) = \mathfrak{C}_g(\beta) - \beta, \tag{C5}$$

which in fact vanishes for any $g \in G$ since β takes values in $\mathbb{Z}_2$ and $\mathfrak{C}_g(\beta) = \beta$. Using Eq. (A14) and the fact that γ is a cocycle, i.e., $\delta^2_{\mathfrak{c}}\gamma = 0$, one verifies that

$$\mathcal{D}^2_{\gamma}\, \mathcal{D}^1_{\gamma}(\alpha, \beta) = (0, 0) \tag{C6}$$

for any tuple $(\alpha,\ \beta) \in C^1(G, \mathrm{U}(1)) \times C^0(G, \mathbb{Z}_2)$.

It was proved in Ref. [10] that one may assign to any 2-cocycle $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ an equivalence class $[(\nu, \rho)]$ of those tuples $(\nu, \rho) \in C^2[G, \mathrm{U}(1)] \times C^1(G, \mathbb{Z}_2)$ that satisfy the cocycle condition under the modified 2-coboundary operator (2.3) given by

$$\mathcal{D}^2_{\gamma}(\nu, \rho) = \left(\delta^2_{\mathfrak{c}}\nu - \pi\, \rho \smile \gamma,\ \delta^1_{\mathfrak{c}}\rho\right) = (0, 0). \tag{C7}$$

Indeed, two tuples (ν, ρ) and (ν', ρ') that satisfy Eq. (C7) are said to be equivalent if there exists a tuple $(\alpha,\ \beta) \in C^1(G, \mathrm{U}(1)) \times C^0(G, \mathbb{Z}_2)$ such that

$$(\nu,\ \rho) = (\nu',\ \rho') + \mathcal{D}^1_{\gamma}(\alpha,\ \beta) = \left(\nu' + \delta^1_{\mathfrak{c}}\alpha + \pi\, \beta \smile \gamma,\ \delta^0_{\mathfrak{c}}\beta\right). \tag{C8}$$

In other words, using this equivalence relation we define an equivalence class $[(\nu, \rho)]$ of the tuple (ν, ρ) as an element of the set

$$[(\nu, \rho)] \in \frac{\ker\left(\mathcal{D}^2_{\gamma}\right)}{\mathrm{im}\left(\mathcal{D}^1_{\gamma}\right)}. \tag{C9}$$

The proof of the one-to-one correspondence between $[\phi]$ and $[(\nu, \rho)]$ then follows in three steps, which we sketch out below. We refer the reader to Ref. [10] for more details.

(1) First, given a cocycle $\phi \in Z^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$, one can define $\rho \in Z^1(G, \mathbb{Z}_2)$ via Eqs. (5.5) or (5.8). The fact that ρ is a cocycle follows from that fact that ϕ is a cocycle.

(2) Next, one can always find a representative ϕ in every cohomology class $[\phi] \in H^2(G_f, \mathrm{U}(1)_{\mathfrak{c}})$ that satisfies the relation $\phi = \nu + \pi\rho \smile \tau$.

(3) Finally, the fact that $\delta^2_{\mathfrak{c}}\phi = 0$ implies that $\delta^2_{\mathfrak{c}}\nu = \pi\rho \smile \gamma$.

We note that when the $[\gamma] = 0$, i.e., the group G_f splits as $G_f = G \times \mathbb{Z}_2^{\mathrm{F}}$, the modified coboundary operators (C3) and (C4) reduce to the coboundary operator (A3b) with $n = 2$ and $n = 1$, respectively. If so the cochains ν and ρ are both cocycles, i.e., $(\nu,\ \rho) \in Z^2(G, \mathrm{U}(1)_{\mathfrak{c}}) \times Z^1(G, \mathbb{Z}_2)$. The equivalence classes $[(\nu, \rho)]$ of the tuple (ν, ρ) is then equal to the equivalence cohomology classes of each of its components, i.e.,

$$[(\nu, \rho)] = ([\nu], [\rho]) \in H^2(G, \mathrm{U}(1)_{\mathfrak{c}}) \times H^1(G, \mathbb{Z}_2). \quad \text{(C10)}$$

We use the notation $([\nu], [\rho])$ for the two indices whenever the group G_f splits ($[\gamma] = 0$). The notation $[(\nu, \rho)]$ applies whenever the group G_f does not split ($[\gamma] \neq 0$).

APPENDIX D: CHANGE IN INDICES (ν, ρ) UNDER GROUP ISOMORPHISMS

As explained in the Appendix B, the fermionic symmetry group G_f can be constructed as the set of pairs $(g, h) \in G \times \mathbb{Z}_2^F$ with the composition rule (B3) specified by the 2-cochain $\gamma \in C^2(G, \mathbb{Z}_2^F)$. The distinct central extensions G_f of G are then classified by the equivalence classes $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$. In other words, the central extension G_f is determined up to the group isomorphisms (B5a) under which the equivalence class $[\gamma]$ is invariant.

In Sec. V, we defined the pair of indices $(\nu, \rho) \in C^2(G, U(1)) \times C^1(G, \mathbb{Z}_2)$ for a given index $[\mu] = 0, 1$. The definitions (5.3) and (5.6) of ν and the definitions (5.5) and (5.8) are not invariant under group isomorphisms. In particular, when restricting the domain of definition of the 2-cochain ϕ from G_f to G, we made an implicit choice of γ. This choice is inherited from a given bulk representation $\widehat{U}_{\mathrm{bulk}}$ through the consistency condition (4.1) since under nontrivial group isomorphisms the transformation rules implemented by at least one group element $g \in G_f$ would change. In this Appendix, we discuss how the pair $(\nu, \rho) \in C^2(G, U(1)) \times C^1(G, \mathbb{Z}_2)$ is shifted under the group isomorphism (B5a) for the cases $[\mu] = 0, 1$.

Let G_f be a fermionic symmetry group obtained by centrally extending the symmetry group G by $\mathbb{Z}_2^F$ through the 2-cochain γ. We denote the elements of G_f by the pairs $(g, h) \in G \times \mathbb{Z}_2^F$. Let G'_f be a fermionic symmetry group isomorphic to G_f through the group isomorphism

$$\begin{aligned}\widetilde{\kappa} : G_f &\to G'_f,\\ (g, h) &\mapsto (g', h') = (g, p^{\kappa(g)}\, h),\end{aligned} \quad \text{(D1a)}$$

where $\kappa(g) = 0, 1$ for any $g \in G$ and we introduced the shorthand notation $p^0 = e$ and $p^1 = p$ for the elements in $\mathbb{Z}_2^F$. In other words, G'_f is the central extension of G by $\mathbb{Z}_2^F$ through the 2-cochain γ' such that

$$\gamma'(g_1, g_2) = \gamma(g_1, g_2)\, p^{\kappa(g_1)+\kappa(g_2)+\kappa(g_1 g_2)}, \quad \text{(D1b)}$$

for any $g_1, g_2 \in G$.

One verifies that the pairs $(g', h') \in G'_f$ are identified with the pairs $(g = g', p^{\kappa(g)}h') \in G_f$ under the group isomorphism $\widetilde{\kappa}$. The identity

$$(g, h) \underset{\gamma}{\circ} (\mathrm{id}, p^{\kappa(g)}) = (g, h\, p^{\kappa(g)}), \quad \text{(D2a)}$$

which holds for any $g \in G$ and $h \in \mathbb{Z}_2^F$, then suggests that the boundary representation $\widehat{U}'_{\mathrm{B}}$ of element $(g', h') \in G'_f$ is related to the boundary representation $\widehat{U}_{\mathrm{B}}$ of element $(g = g', p^{\kappa(g)}\, h') \in G_f$ via the relation

$$\widehat{U}'_{\mathrm{B}}((g', h')) \propto \widehat{U}_{\mathrm{B}}((g = g', h'))\, [\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g)}, \quad \text{(D2b)}$$

i.e., the operator $\widehat{U}'_{\mathrm{B}}((g', h'))$ must act up to a multiplicative phase factor as the operator $\widehat{U}_{\mathrm{B}}((g, h'))$ composed with the fermion parity operator $\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))$ if $\kappa(g) = 1$. Hereby, the exponent $\kappa(g)$ ensures that the operators $\widehat{U}'_{\mathrm{B}}((g', h'))$ and $\widehat{U}_{\mathrm{B}}((g, h'))$ act identically, if $\kappa(g) = 0$. Without loss of generality, we take the proportionality in (D2b) to be equality. We shall treat the cases of $[\mu] = 0$ and $[\mu] = 1$ separately.

1. The case of $[\mu] = 0$

On the one hand, invoking the definition (5.3) for the 2-cochain ν' associated with the group G'_f delivers

$$\begin{aligned}&\widehat{U}'_{\mathrm{B}}((g'_1, e))\, \widehat{U}'_{\mathrm{B}}((g'_2, e))\\ &\quad = e^{\mathrm{i}\nu'(g'_1, g'_2)}\, \widehat{U}'_{\mathrm{B}}((g'_1\, g'_2, \gamma'(g'_1, g'_2)))\\ &\quad = e^{\mathrm{i}\nu'(g_1, g_2)}\, \widehat{U}_{\mathrm{B}}((g_1\, g_2, \gamma(g_1, g_2) p^{\kappa(g_1)+\kappa(g_2)+\kappa(g_1 g_2)}))\\ &\qquad \times [\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_1 g_2)},\end{aligned} \quad \text{(D3a)}$$

where in reaching the last line we have used Eqs. (D1b) and (D2b). Applying the identity (D2a), we find

$$\begin{aligned}\widehat{U}'_{\mathrm{B}}((g'_1, e))\, \widehat{U}'_{\mathrm{B}}((g'_2, e)) &= e^{\mathrm{i}\nu'(g_1, g_2)}\, \widehat{U}_{\mathrm{B}}((g_1\, g_2, \gamma(g_1, g_2)))\\ &\quad \times [\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_1)+\kappa(g_2)},\end{aligned} \quad \text{(D3b)}$$

where the equality holds up to a multiplicative phase factor that can be gauged away, reason for which it is omitted for convenience. On the other hand, inserting Eq. (D2b) on the left-hand side delivers

$$\begin{aligned}\widehat{U}'_{\mathrm{B}}((g'_1, e))\, \widehat{U}'_{\mathrm{B}}((g'_2, e)) &= \widehat{U}_{\mathrm{B}}((g_1, e))\, [\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_1)}\, \widehat{U}_{\mathrm{B}}((g_2, e))\, [\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_2)}\\ &= e^{\mathrm{i}\nu(g_1, g_2)+\mathrm{i}\pi\,\kappa(g_1)\rho(g_2)}\, \widehat{U}_{\mathrm{B}}((g_1\, g_2, \gamma(g_1, g_2)))[\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_1)+\kappa(g_2)},\end{aligned} \quad \text{(D3c)}$$

where the phase factor $e^{\mathrm{i}\nu(g_1, g_2)}$ arises from the definition (5.3) of 2-cochain ν and the phase factor $e^{\mathrm{i}\kappa(g_1)\rho(g_2)}$ arises when the operators $\widehat{U}_{\mathrm{B}}((g_2, e))$ and $[\widehat{U}_{\mathrm{B}}((\mathrm{id}, p))]^{\kappa(g_1)}$ are interchanged. Comparing Eqs. (D3b) and (D3c), we make the identification

$$\nu'(g_1, g_2) = \nu(g_1, g_2) + \pi\, (\kappa \smile \rho)(g_1, g_2). \quad \text{(D4)}$$

The index ρ by definition (5.4) measures the fermion parity of the representation of the element $(g, h) \in G_f$. One notes that the relation (D2b) implies that the representations $\widehat{U}_{\rm B}$ and $\widehat{U}'_{\rm B}$ have the same fermion fermion parity since $\widehat{U}_{\rm B}((\mathrm{id}, p))$ is fermion parity even. Hence, the indices ρ and ρ' associated with G_f and G'_f, respectively, coincide.

We conclude that under the isomorphism (D1a) the pair of indices $((\nu', \rho'), 0)$ and $((\nu, \rho), 0)$ are related as

$$((\nu', \rho'),\ 0)\ = ((\nu + \pi\,(\kappa \smile \rho),\ \rho), 0). \tag{D5}$$

2. The case of $[\mu] = 1$

When $[\mu] = 1$, the definition (5.6) of the index ν is the same as it is when $[\mu] = 0$. Hence, the argument in the previous section follows through, i.e., Eq. (D4) holds. However, the definition (5.7) of index ρ differs from its definition in Eq. (5.4) when $[\mu] = 0$. From Eqs. (5.7) and (D2b), one observes that under the isomorphism (D1a) the index ρ gets shifted by κ, i.e.,

$$\rho'(g) = \rho(g) + \kappa(g). \tag{D6}$$

This is because computation of the index ρ' involves an additional conjugation of $\widehat{Y}_{\rm B}$ by fermion parity operator $\widehat{U}_{\rm B}((\mathrm{id}, p))$, which brings an additional factor of $(-1)^{\kappa(g)}$. We conclude that under the isomorphism (D1a) the pair of indices $((\nu', \rho'), 1)$ and $((\nu, \rho), 1)$ are related as

$$((\nu', \rho'),\ 1) = ((\nu + \pi\,(\kappa \smile \rho),\ \rho + \kappa), 1). \tag{D7}$$

Under the group isomorphism (D1a) the values of the indices (ν, ρ) and their respective equivalence classes may change [according to Eqs. (D5) and (D7)]. However, the number of equivalence classes $([(\nu, \rho)], [\mu])$ and their stacking rules remain the same, i.e., Eqs. (6.48) commute with the relations (D5) and (D7).

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[13] The Hilbert space on which we choose to define a representation of the Clifford algebra $C\ell_{2n}$ spanned by $2n$ Majorana operators is the fermionic Fock space of dimension 2^n defined as follows. First, n pairs of Majorana operators are chosen. Second, any one of these n pairs of Majorana operators defines a conjugate pair of creation and annihilation fermion operators. Third, the vacuum state that is annihilated by all fermion operators is the highest weight state from which $2^n - 1$ orthonormal states with the fermion number $n_{\rm f} = 1, 2, \ldots, n-1, n$ descend by acting on the vacuum state with the product of $n_{\rm f}$ distinct fermionic creation operators. By construction, $C\ell_{2n}$ has a nontrivial complex irreducible representation of dimension 2^n for which each element of the basis that defines the fermionic Fock space has a well-defined fermion parity. Because the center of the Clifford algebra $C\ell_{2n}$ is spanned by the identity alone, a redefinition of any one of the $2n$ Majorana operators by multiplication with an element of the center of $C\ell_{2n}$ changes any basis element of the fermionic Fock space by at most a multiplicative $\mathbb{C}$ number. A redefinition of any one of the $2n$ Majorana operators by multiplication with an element of the center of $C\ell_{2n}$ thus leaves the fermion parity of each element of the fermionic Fock basis unchanged. This is not so any more for a representation of the Clifford algebra $C\ell_{2n+1}$ spanned by $2n+1$ Majorana operators. Even though it is still possible to define a Hilbert space of dimension 2^{n+1} on which the Clifford algebra $C\ell_{2n+1}$ has a nontrivial irreducible representation [33], the center of $C\ell_{2n+1}$ is a two-dimensional subalgebra for the product of all $2n+1$ Majorana operators commutes with any one of these $2n+1$ Majorana operators. It follows that there is no element in $C\ell_{2n+1}$, which anticommutes with all the Majorana generators of $C\ell_{2n+1}$, i.e., it is not possible to distinguish an element in $C\ell_{2n+1}$, which assigns odd fermion parity to all $2n+1$ Majorana generators. The best one can do is to construct a 2^n-dimensional fermionic Fock space using the generators of a $C\ell_{2n}$ subalgebra of $C\ell_{2n+1}$ and a two-dimensional Hilbert space in which states do not have any assigned fermion parity or fermion number. Consequently, the action on $C\ell_{2n+1}$ of the symmetries inherited from the bulk does not determine uniquely, i.e., up to a phase factor, the action on the representation of $C\ell_{2n+1}$ of this symmetry.

[14] When n_j is an odd integer, it is not always possible to construct a local representation $\widehat{U}_j(g)$ for any $g \in G_f$ only out of the

Majorana degrees of freedom in the set $\mathfrak{O}_j$ defined in (2.2a). However, we still call $\widehat{U}_j(g)$ a local representation in the sense that it can always be constructed by supplementing the Clifford algebra (2.2b) by an additional Majorana degree of freedom $\hat{\gamma}_{j,\infty}$ that is localized at some other site j' with the number of Majorana operators $n_{j'}$ being an odd integer.

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[27] Note that we denote the elements of the set of 2-cochains $(C^2(G_f, U(1))$ by the phase $\phi(g_1, g_2)$ as opposed to its exponential as is the usual convention. This is because we impose an additive composition rule on the group $U(1)$ as opposed to a multiplicative one. See Appendix A for the definition of the set $C^2(G_f, U(1))$ and details on the convention we adopt.

[28] A $\mathbb{Z}_2$ valued 1-cocycle $\rho \in Z^1(G_f, \mathbb{Z}_2)$ satisfies by definition $(\delta^1_{\mathfrak{c}}\rho)(g,h) = \rho(g) + \mathfrak{c}(g)\rho(h) - \rho(gh) = 0$ for any $g, h \in G_f$. Since in the group $\mathbb{Z}_2$, $\rho(h) = \pm\rho(h)$ for any $h \in G_f$, the 1-cocycle ρ is a group homomorphism.

[29] This is not true when $[\mu] = 0$. The group homomorphism $\rho \in C^1(G_f, \mathbb{Z}_2)$ defined in Eq. (5.4) takes the values $\rho((\mathrm{id}, e)) = \rho((\mathrm{id}, p)) = 0$. Therefore, when restricted to the center $\mathbb{Z}_2^{\mathrm{F}} \subset G_f$, it is not an isomorphism.

[30] Stacking three representations $\widehat{U}_1$, $\widehat{U}_2$, and $\widehat{U}_3$ using the definition (6.48) is associative, i.e., it is independent of which two of the three representations are first stacked. This associativity follows from the consistency condition (6.4). This is because, for a Clifford algebra $C\ell_{2n}$ with an even number of generators, specifying the transformation rules on its generators together with the action of the complex conjugation uniquely (up to a phase factor) determines the representation $\widehat{U}(g)$ of any element $g \in G_f$. For a Clifford algebra $C\ell_{2n+1}$ with an odd number of generators, this is no longer true since $C\ell_{2n+1}$ has a two-dimensional center spanned by $\hat{\mathbb{1}}$ and $\widehat{Y}$. We removed the ambiguity consisting in multiplying $\widehat{U}(g)$ by the central element $\widehat{Y}$ by demanding that $\widehat{U}(g)$ is of even fermion parity.

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