Tuning of quantum entanglement in a superconductor with transition-metal and rare-earth impurities: Effect of potential scattering on quantum phase transitions

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Bipartite quantum entanglement of electron spins of a Cooper pair in a dirty *s*-wave superconductor is investigated at absolute zero temperature in terms of the exchange interaction and potential scattering as well as the relative distance between the two electrons. In the case of transition-metal impurities, we utilize the *T*-matrix approach to obtain the relevant Green's functions. We employ the two-electron spin-space density matrix, which is associated with the perturbed Green's functions. We show that a two-spin state can be described by the Werner state for dirty superconductors as well as for clean ones. We find that both the first and second quantum phase transitions (QPTs) are generated by the competition between potential scattering and other interactions. Meanwhile, it is not possible to determine the allowable value for each interaction individually. In the case of rare-earth impurities, concurrence, as a measure of bipartite entanglement, is discussed in terms of the relative distance and collision times for all finite and infinite Debye frequencies. The results confirm that quantum correlation and QPTs can be tuned by impurities.

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I. INTRODUCTION

Quantum entanglement of electron spins was investigated in a number of many-body systems [1–15]. In the case of *s*- and *d*-wave superconductors, bipartite entanglement of two electron spins and tripartite entanglement were overlooked using space-spin density matrix approach. Quantum phase transition (QPT) is defined by the abrupt change in the ground state caused by quantum fluctuations, and entanglement is an appropriate tool for finding QPTs in many-body systems [4,16]. Meanwhile, there is no one-to-one correspondence between QPT and the appearance of the critical point of concurrence, except under certain conditions [16–19], as in our case.

The physical properties of BCS *s*-wave superconductors with impurities including transition and rare-earth metals are described by the Shiba-Rusinov (SR) model and the Abrikosov-Gorkov (AG) theory [20–40]. According to the SR model, the scattering of classical impurity spins as well as a localized excited state in the energy gap, $\Delta(\vec{x})$, are taken into account [25–27]. There are some studies for dirty *s*-wave superconductors based on the SR model, which were addressed QPTs without quantum information considerations [41–43]. For instance, QPTs were investigated in terms of interimpurity distance, exchange interaction, J, and pair interaction [41,42]. Also, QPT and the hybridization of two bound states in the energy spectrum are modulated by the angle of magnetization [43].

Let us express how our study differs from previous ones. Previously, QPT was achieved without considering potential scattering in the SR model; but in the present study, this scattering plays a significant role. Here, if only exchange interaction or only potential scattering is taken into account, QPT does not occur. Compared to previous studies of QPT, we find different types of QPT from a quantum information point of view. Furthermore, von Neumann entropy was used to study single-site entanglement [39], while we rely on bipartite entanglement of two electron spins in a Cooper pair and the correlations associated to the distance between these electrons.

The plan of our paper is as follows. In Sec. II A, we discuss the case of transition-metal impurities, whereas in Sec. II B we address the case of rare-earth impurities. Section III is devoted to conclusions and an outlook.

II. MODEL AND METHOD

Basically, the entanglement arises from particles' statistics or the external/internal interactions [9]. We deal with a BCS *s*-wave superconductor as an interacting system. Mean-field approximation is a fundamental concept in BCS theory [8,9,15]. This approximation for obtaining the value of bipartite entanglement is accurate whenever the fluctuations associated to all interactions are small, as in our case. Furthermore, we assume that the energy gap is space independent and its average value can be used ($\Delta(\vec{x}) \simeq \Delta$). This assumption is established by Anderson's theorem for nonmagnetic impurity cases [19–23]. Also, for the transition-metal case, local variations of $\Delta(\vec{x})$ rapidly fade away from an impurity center at low-enough impurity concentrations [25–27]. In addition, this situation is satisfied for the rare-earth metal case because of

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the existence of a long-range coherence length [44,45]. The impurity part of the Hamiltonian is given by (throughout the paper, we use $\hbar = k_B = 1$)

$$H_{\text{imp}} = \sum_{\alpha\beta} \int d^3x J(\vec{x}) \psi_\alpha^\dagger(\vec{x}) S.\sigma_{\alpha\beta} \psi_\beta(\vec{x}) + \int d^3x \sum_\alpha \psi_\alpha^\dagger(\vec{x}) V(\vec{x}) \psi_\alpha(\vec{x}), \quad (1)$$

where S is the spin operator of the impurity, σ denotes the Pauli matrices, ψ and $\psi^\dagger$ are fermionic field creation and annihilation operators, respectively, and α and β denote spin indices. $J(\vec{x})$ is associated with the magnetic impurities and $V(\vec{x})$ refers to potential scattering, which is due to nonmagnetic impurities. The range of $J(\vec{x})$ is determined by the quantum mechanical structure of the electron cloud associated with the localized spin. Undoubtedly, $J(\vec{x})$ has different values for different kinds of impurities and hosts [46–48]. It should be mentioned that we assume that $J(\vec{x})$ has a constant value in momentum space. In the meantime, it is supposed that there is no correlation between different impurities.

The relation between the two-electron space-spin density matrix and the two-particle Green's function is described by [8]

$$\rho_2(\vec{x}_1, \vec{x}_2, \vec{x}_1', \vec{x}_2') = (-1/2) G(\vec{x}_1 t_1, \vec{x}_2 t_2, \vec{x}_1' t_1^+, \vec{x}_2' t_2^+), \quad (2)$$

where t^+ denotes an infinitesimal time after t. By rewriting the two-particle Green's function in terms of the fermionic single-particle Green's functions, we have

$$\begin{aligned} &\rho_{s_1,s_2;s_1',s_2'}(\vec{x}_1, \vec{x}_2, \vec{x}_1', \vec{x}_2') \\ &= (-1/2)(G_{s_1,s_1'}(\vec{x}_1 t_1, \vec{x}_1' t_1^+) G_{s_2,s_2'}(\vec{x}_2 t_2, \vec{x}_2' t_2^+) \\ &\quad - G_{s_1,s_2'}(\vec{x}_1 t_1, \vec{x}_2' t_2^+) G_{s_2,s_1'}(\vec{x}_2 t_2, \vec{x}_1' t_1^+)), \end{aligned} \quad (3)$$

where s_1, s_2, s_1', and s_2' are particles' spin indices. It merits mentioning that in the absence of impurity, the anomalous Green's function can be neglected compared to the single-particle Green's function [8]. Also, the modification due to the impurity on the anomalous Green's function is negligible. The normalized reduced space-spin density matrix in the presence of the impurity is written as

$$\widetilde{\rho} \equiv \frac{\rho}{Tr[\rho]} = (1 - \Lambda)\frac{I}{4} + \Lambda |\psi^{(-)}\rangle\langle\psi^{(-)}|, \quad (4)$$

where I is a 4×4 unit matrix and $|\psi^{(-)}\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ is the singlet state. By considering $\vec{x}_1 = \vec{x}_1'$, $\vec{x}_2 = \vec{x}_2'$ and $\vec{r} = \vec{x}_1 - \vec{x}_2$, parameter Λ is obtained as [9]

$$\Lambda = G(r)G(-r)/(2 - G(r)G(-r)), \quad (5)$$

where $G(r)$ is specified by $G_{s_1,s_2'}(\vec{x}_1 t_1, \vec{x}_2' t_1^+) \equiv \delta_{s_1 s_2'} G(r)$. Up to the first-order approximation, Λ is given by

$$\Lambda \approx \Lambda^{(0)} + \frac{4[g^{(0)2}(r)d(r) + q(r)]}{G^{(0)2}(0)[2 - g^{(0)2}(r)]^2}, \quad (6)$$

where $\Lambda^{(0)}$ is the nonimpurity part of Λ, and

$$\begin{aligned} g^{(0)}(r) &= G^{(0)}(r)/G^{(0)}(0), \\ d(r) &= \text{Im}[G^{(1)}(0)] \times \text{Im}[G^{(0)}(0)], \\ q(r) &= -2\text{Im}[G^{(1)}(r)]G^{(0)}(r), \end{aligned} \quad (7)$$

where $G^{(0)}(r)$ and $G^{(1)}(r)$ are the unperturbed Green's function and first-order perturbed Green's function due to the presence of impurities, respectively. The relation between concurrence and density matrix is given by $C = \max[0, (3\Lambda - 1)/2]$ [3,14]. According to Hill and Wootters [11,12], concurrence is established by operating the spin flip. Originally, concurrence was introduced for the mixed states of a two-qubit system as a scalar function to describe bipartite density matrices. Concurrence is a computationally manageable measure of entanglement [49].

Due to the nature of the system under investigation, we use a trick to convert the methodology used for calculating concurrence in noninteracting systems to interacting systems such as superconductors. This trick was used for both s- and d-wave superconductors in the absence of impurities and the only change was the use of new Green's functions [9]. Now, impurities in superconductors cause an additional interaction, which leads to new Green's functions. Thereby, the formulation of concurrence for noninteracting Green's functions can be survived. Now we proceed to obtain Green's functions and, subsequently, concurrence.

A. The transition-metal impurity case

The Green's function averaged over the positions and the spin directions of the impurities is described by Refs. [25–28,50] $\tilde{G}_{\vec{k}} = (a_1 + a_2\rho_2\sigma_2 + a_3\rho_3)$, where $a_{n(=1,2,3)} = \tilde{b}_n/(\tilde{\omega}^2 - \tilde{\Delta}^2 - \tilde{\varepsilon}_{\vec{k}}^2)$, $\tilde{b}_1 = \tilde{\omega}$, $\tilde{b}_2 = \tilde{\Delta}$, and $\tilde{b}_3 = \tilde{\varepsilon}_{\vec{k}}$. Here, $\tilde{\omega}$, $\tilde{\Delta}$,and $\tilde{\varepsilon}_{\vec{k}}$ are renormalized frequency, renormalized energy gap, and renormalized kinetic energy with respect to chemical potential, respectively. Additionally, $\tilde{b}_n$ is given by $\tilde{b}_n = b_n + (\Gamma_n \times \zeta_n(j, \nu, U(\omega/\Delta, j, \nu)))$ where $\Gamma_n = pt_n(j, \nu)/16\gamma j^2 \pi N(0)$. Meanwhile, $t_n(j, \nu)$ is always less than unity and is given by

$$\begin{aligned} t_1(j, \nu) &= (\nu^2 - j^2)^2 + j^2 + \nu^2, \\ t_2(j, \nu) &= \nu(1 + \nu^2 - j^2), \qquad t_3(j, \nu) = t_1(j, \nu) - 2j^2, \end{aligned} \quad (8)$$

and ζ_n is expressed by

$$\begin{aligned} \zeta_1(j, \nu, U(\omega/\Delta, j, \nu)) &= \frac{U\sqrt{(1 - U^2)}}{(\varepsilon_0^2 - U^2)}, \\ \zeta_{2,3}(j, \nu, U(\omega/\Delta, j, \nu)) &= \frac{\sqrt{(1 - U^2)}}{(\varepsilon_0^2 - U^2)}, \end{aligned} \quad (9)$$

where $\varepsilon_0 = (1 + \nu^2 - j^2)((1 + \nu^2 - j^2)^2 + 4j^2)^{-1/2}$ and U satisfies

$$\omega/\Delta = U - \frac{U\alpha\sqrt{(1 - U^2)}}{(\varepsilon_0^2 - U^2)}, \qquad \alpha = p/8\Delta\gamma\pi N(0), \quad (10)$$

where p is the normalized impurity concentration [30–33], $\ln\gamma$ stands for the Euler constant, and $N(0)$ is the density of single-particle states at the Fermi level. The dimensionless

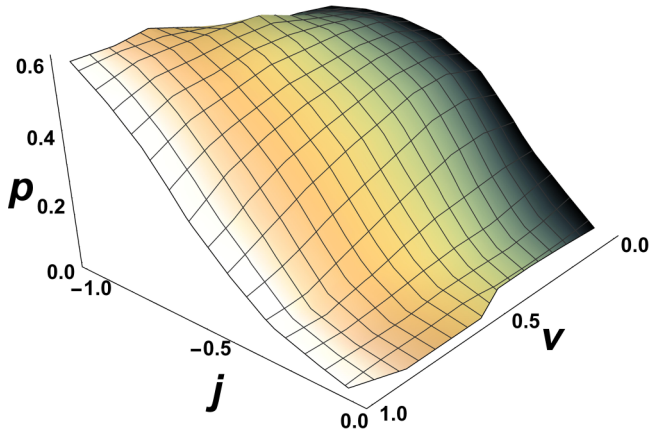

FIG. 1. The allowable range (under the surface) of the dimensionless parameters j, ν, and p regarding to the smallness of $\mathrm{Im}[G^{(1)}(r)]/\mathrm{Im}[G^{(0)}(0)]$.

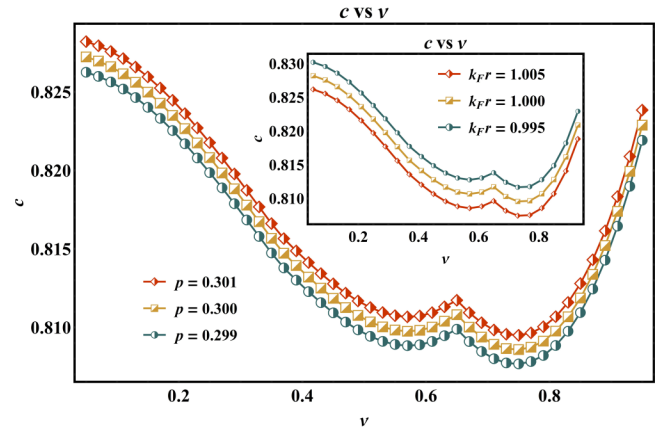

FIG. 2. For transition-metal case, concurrence versus ν for various values of p at $j = -0.8$ and $k_F r = 1$. Inset: Concurrence versus ν for various values of $k_F r$ at $j = -0.8$ and $p = 0.3$.

quantities j and ν are defined, respectively, by the exchange interaction as $J \equiv 2j/S\pi N(0)$ and by the potential scattering as $V \equiv \nu/\pi N(0)$. It merits mentioning that Γ_n, ε_0, and α (or, equivalently, j, ν, and p) enter the impurity effect on Green's functions and their values are small. It should be noted that the value of α can be controlled by p. Using these descriptions, we can assume that $U \approx \omega/\Delta$.

Basically, the general range of ν as well as j are determined by $|\varepsilon_0| < 1$. It is worth noting that the system always remains in the singlet state, $j < 0$. Meanwhile, the allowable ranges of ν and j which are interdependent can be calculated numerically based on two hypotheses. First, the perturbed Green's function is small in comparison to the unperturbed Green's function. Second, the impurity terms, which exist in $\tilde{\Delta}$, $\tilde{\varepsilon}_{\vec{k}}$, and $\tilde{\omega}$, are small. Accordingly, the numerical results show that the allowable ranges are $p < 0.6$, $j < -0.2$ and $0 \leqslant \nu \leqslant 1$ (Fig. 1).

In addition, the appropriate ranges of j and ν for some special cases such as In-Fe [51], Pb-Mn [51], and Zn-Mn [52] are given in Table I. It should be noted that the relation between the three parameters j, ν, and p is very complicated; the value of each of these parameters is reevaluated whenever one of them is changed.

Basically, the Green's function in the presence of impurity scattering is given by

$$G_{\vec{k}\vec{k}'} = G^0_{\vec{k}}\delta_{\vec{k}\vec{k}'} + G^0_{\vec{k}} T G^0_{\vec{k}'} \tag{11}$$

where the scattering is described by a T matrix [50]. The general form of $G^0_{\vec{k}}$ is represented by $(\omega - \varepsilon_{\vec{k}}\rho_3 - \Delta\rho_2\sigma_2)^{-1}$,

TABLE I. The values of j, ν, and ε_0 for some special cases. Samples A and B are selected arbitrarily.

Sample	ε_0	j	ν
In-Fe	0.60	$-0.85 < j < -0.50$	$0 < \nu < 1$
Pb-Mn	0.55	$-0.90 < j < -0.54$	$0 < \nu < 1$
Sample A	0.45	$-0.99 < j < -0.61$	$0 < \nu < 1$
Sample B	0.35	$-1 < j < -0.69$	$0 < \nu < 0.86$
Zn-Mn	0.25	$-1 < j < -0.77$	$0 < \nu < 0.71$

which is a tensor product of both Pauli matrices, σ_i, and the electron-hole spin states, ρ_i. We neglect the α-dependent part of the T matrix. The first term of the series expansion for the T matrix around $\alpha = 0$ is obtained by $T^{(0)} = (S_1\Gamma_1 + S_2\Gamma_2\rho_2\sigma_2 + S_3\Gamma_3\rho_3)$, where

$$S_1 = \frac{-\omega\sqrt{\Delta^2-\omega^2}}{\Delta^2\varepsilon_0^2-\omega^2}, \qquad S_2 = \frac{\Delta\sqrt{\Delta^2-\omega^2}}{\Delta^2\varepsilon_0^2-\omega^2},$$
$$S_3 = \frac{\Delta^2-\omega^2}{\Delta^2\varepsilon_0^2-\omega^2}. \tag{12}$$

Then we obtain the perturbed Green's function as follows:

$$G^{(1)}(\vec{k},\omega) = \Xi(\vec{k},\omega)(\Gamma_1\omega((\omega+\varepsilon_{\vec{k}})^2+\Delta^2) - 2\Gamma_2\Delta^2(\omega+\varepsilon_{\vec{k}}) - \Gamma_3(\Delta^2-\omega^2)^{1/2} \times ((\omega+\varepsilon_{\vec{k}})^2-\Delta^2)),$$
$$\Xi(\vec{k},\omega) = \frac{(\Delta^2-\omega^2)^{1/2}}{\left(\omega^2-\Delta^2-\varepsilon_{\vec{k}}^2\right)^2\left(\omega^2-\Delta^2\varepsilon_0^2\right)}, \tag{13}$$

where $G^{(1)}$ is a linear combination of Γ_1, Γ_2, and Γ_3. Also, by rewriting k as $k \equiv k_F + \varepsilon_{\vec{k}}/v_F$, we have

$$G^{(1)}(r) = \frac{(2\pi)^{5/2}N(0)}{rk_F}\int_{-\omega_D}^{\omega_D} d\varepsilon_{\vec{k}}\int_{-\Delta}^{\Delta} d\omega G^{(1)}(\vec{k},\omega) \times \sin(k_F r + \varepsilon_{\vec{k}} r/v_F), \tag{14}$$

where r and k_F are the distance between the two electrons of the Cooper pair and the Fermi wave number, respectively. According to the numerical calculation, we find that the variation of Debye frequency in the Green's function is negligible, therefore we choose $\omega_D = 0.1$, which is measured with respect to the Fermi energy. Also, we use $\Delta/\varepsilon_F = 0.001$.

Concurrence $[c \equiv c(p, j, \nu, k_F r)]$ is calculated as a function of internal/external parameters. Changing p (Fig. 2) or $k_F r$ (the inset in Fig. 2) only changes the value of concurrence. All critical points (including the local maximum point and the turning point) occur at the same value of ν.

Figure 3 illustrates how increasing $|j|$ leads to moving the discontinuous point, which appears in the local maximum

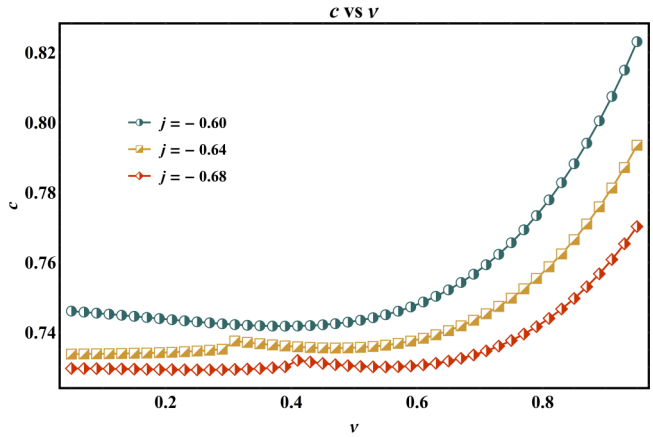

FIG. 3. For transition-metal case, concurrence versus ν for various values of j at $p = 0.2$ and $k_F r = 1$.

point. Based on our investigation, critical points do not appear for those values of j in the interval $(-0.60, -0.05)$. By increasing $|j|$, critical points move toward high values of ν. Furthermore, the turning (local maximum) point disappears for $j > -0.64 (j > -0.60)$. We show how the different hosts and their energy gap can affect not only the value of concurrence but also the occurrence of the critical points (Fig. 4). By increasing the value of Δ (or pair interaction), the turning (local maximum) point moves toward the lower (higher) values of ν.

Partial derivatives of concurrence versus ν provide valuable insight into critical points (Fig. 5). The first partial derivative reaches a discontinuity (continuity) at the turning (local maximum) point, but concurrence remains continuous (discontinuous) and we assert that this point corresponds the second (first)-order QPT. For second-order QPT, ν does not have a widely varying range based on the variation of $|j|$ or Δ.

Concurrence versus the relative distance for different values of p, ν, and j is shown in Fig. 6. The value of concurrence in the presence of transition-metal impurity is larger than that in the nonimpurity case. It can be seen that the curves for different values of ν are almost overlapped. At higher values of p, the zero value of concurrence occurs at larger values of $k_F r$. Finally, for higher values of $|j|$, the variation of concurrence does not significantly change.

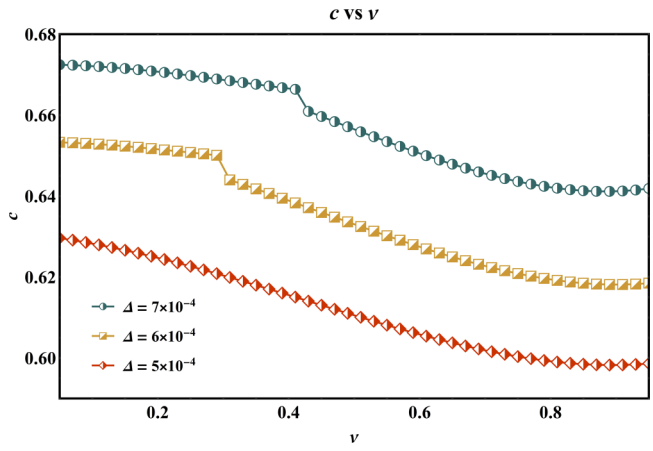

FIG. 4. For transition-metal case, concurrence versus ν for various values of Δ (measured with respect to Fermi energy) at $j = -0.8$, $p = 0.2$, and $k_F r = 1$.

B. Rare-earth metal and nonmagnetic impurity cases

In the following, we present the results for the concurrence based on the perturbed Green's functions, which will be explained in the Appendix. At a fixed collision time, τ, concurrence is reduced by increasing the value of $k_F r$ [Fig. 7(a)]. For the sake of comparison, we also present the curve of concurrence for nonimpurity case [8,9]. The curves of concurrence for nonimpurity case and nonmagnetic impurity with finite ω_D case are overlapped. Although, for the case of infinite ω_D, at a fixed $k_F r$, concurrence takes smaller values than those in the finite case. With increasing $1/\tau$, the concurrence decreases as a result of increasing the impurity effect [Fig. 7(b)]. At a fixed value of $1/\tau$, the concurrence for infinite ω_D is smaller than the finite one.

Investigating concurrence in the presence of the rare-earth impurity in terms of $k_F r$, τ_1, and τ_2 for both finite and infinite ω_D cases has been done. The appropriate ranges of τ_1 and τ_2 are tuned by considering different values of ω_D, Δ, and ω. After lengthy calculations, we conclude that the values of τ_1 and τ_2 are extremely large (without any physical meaning).

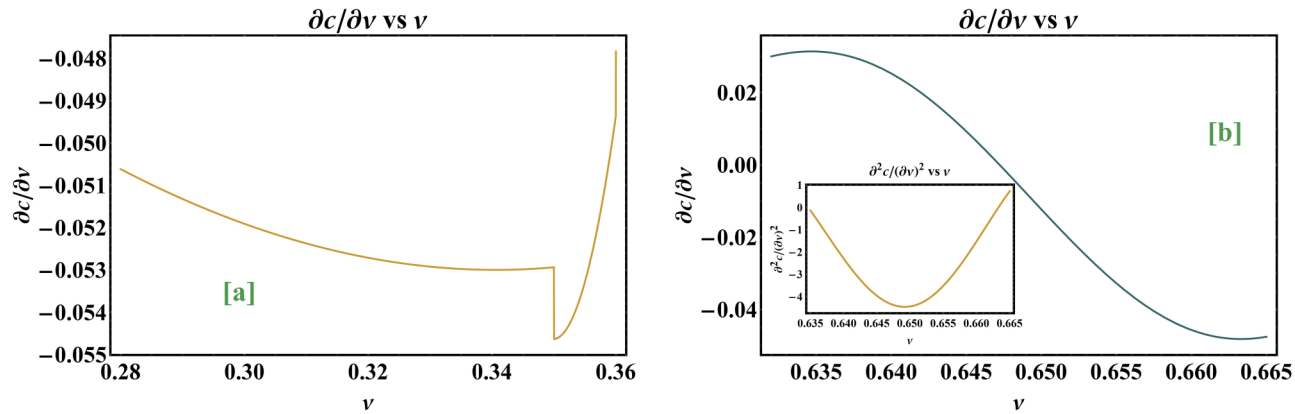

FIG. 5. Partial derivatives of concurrence versus ν at $j = -0.8$, $p = 0.3$, and $k_F r = 1$ (a) for the values of ν located in the interval (0.28,0.37) (b) for the values of ν located in the interval (0.635,0.665).

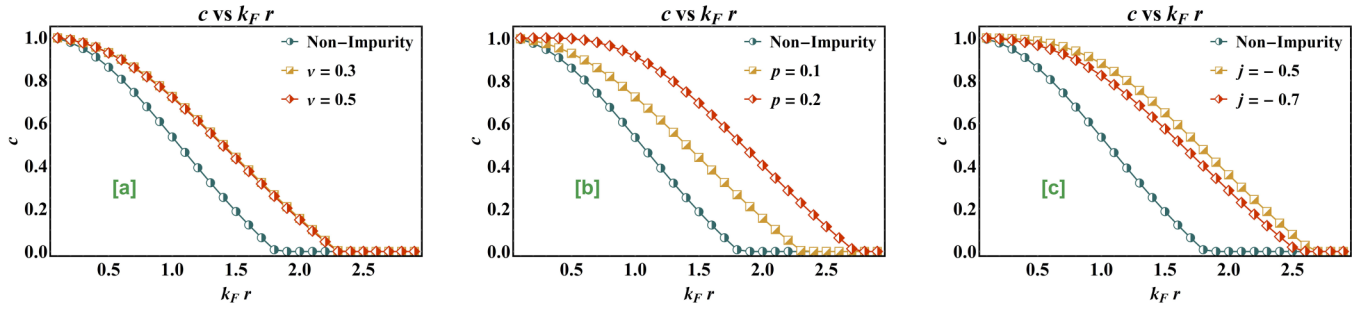

FIG. 6. For transition-metal case, (a) concurrence versus $k_F r$ for various values of ν at $j = -0.8$ and $p = 0.2$, (b) concurrence versus $k_F r$ for various values of p at $j = -0.3$ and $\nu = 0.3$, and (c) concurrence versus $k_F r$ for various values of j at $p = 0.3$ and $\nu = 0.3$.

III. CONCLUSIONS

Quantum correlation in many-body systems is very desirable for quantum information protocols. Within the BCS theory (by assuming the smallness of fluctuations of all interactions), we have considered an s-wave superconductor in the presence of a rare-earth element and transition-metal described by AG theory and SR model, respectively. Then we have examined quantum correlation and bipartite entanglement as well as one of their most important applications known as QPT. We have proved that there is a one-to-one correspondence between QPT and entanglement. Also, we have shown that the system describes the Werner state as a noninteracting Fermi gas. For the SR model, we have shown that not only the occurrence of the first and second QPTs depends on the potential scattering, but also it does not require to have higher values of $|j|$, which still plays a role on localized excited states. The first-order QPT occurs at higher values of ν when $|j|$ or Δ increases. In the case of second-order QPT, the enhancement of $|j|(\Delta)$ leads to the displacement of QPT toward higher (lower) values of ν. It merits mentioning that there is no QPT for some values of $|j|$. At a fixed $k_F r$, the value of concurrence in the presence of the transition-metal impurity is larger than that in the nonimpurity case. As we have shown, entanglement can exist for large $k_F r$, if p has a high value. Meanwhile, under AG theory, we have demonstrated that dirty superconductors don't show any QPT, even by considering all possible ω_D cases. Finally, quantum information allows us to adjust the physical properties of such systems.

ATTRIBUTIONS

All authors have equally contributed to the calculations, discussing the results, and to writing and reviewing the paper.

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APPENDIX

Through a straightforward but lengthy calculation, the Green's functions are determined by considering different values of ω_D, which are selected based on the comparison between Δ and ω. Here, nonmagnetic and magnetic impurities are abbreviated by indices NM and M, respectively. It should be mentioned that the impurity effect enters via $\eta_{\omega_1}(\tilde{\omega} = \omega\eta_{\omega_1})$ and $\eta_{\omega_2}(\tilde{\Delta} = \Delta\eta_{\omega_2})$ that consist of τ_1 and τ_2, respectively (for nonmagnetic case $\tau_1 = \tau_2$). In addition, τ_1 and τ_2 are represented by $1/\tau_i \propto V^2 \pm S(S+1)J^2/4$.

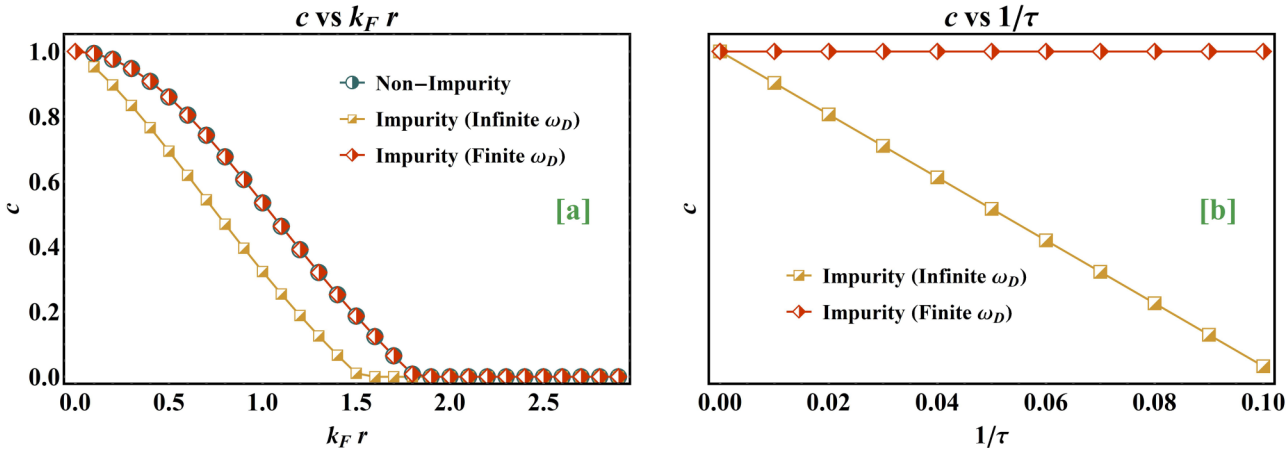

FIG. 7. For nonmagnetic impurity case, (a) concurrence versus $k_F r$ for infinite and finite Debye frequency and (b) concurrence versus inverse collision time (in arbitrary units) for infinite and finite Debye frequency at a fixed $k_F r (= 1 \times 10^{-3})$.

1. Infinite Debye frequency case

For the nonmagnetic case, the exact Green's function is given by

$$G_{NM_1}(r) = \lim_{t\to 0^-} G_{NM_1}(r,t) = \left(\lim_{t\to 0^-}\int d\omega G^{(0)}(r,\omega)e^{i\omega t}\right)e^{\frac{-r}{2\tau v_F}} \equiv G^{(0)}(r)e^{\frac{-r}{2\tau v_F}}, \tag{A1}$$

where v_F is the Fermi velocity and $G^{(0)}(r)$ was given in Refs. [8,9]. For s-wave superconductors, the angular dependence does not exist in the Green's functions. It is worth mentioning that Eq. (15) is equal to the product of the Green's function of a clean superconductor and $\exp(-r/2\tau v_F)$. For magnetic cases, we obtain the perturbed Green's functions as follows:

$$G^{(1)}_{M_1}(\vec{k},\omega) = \frac{\omega+\varepsilon_{\vec{k}}+i/2\tau_1}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\left(1-\frac{\frac{i\omega}{\tau_1}-\frac{i|b|\Delta}{\tau_2}}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\right) \qquad \omega^2>\Delta^2, \tag{A2}$$

$$G^{(1)}_{M_2}(\vec{k},\omega) = \frac{\omega+\varepsilon_{\vec{k}}+|a|/2\tau_1}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\left(1-\frac{\frac{\omega|a|}{\tau_1}-\frac{\Delta}{\tau_2}}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\right) \qquad \omega^2<\Delta^2, \tag{A3}$$

where $a^2 \equiv \omega^2/\Delta^2$ and $b^2 \equiv 1/a^2$.

2. Finite Debye frequency case

In some works [21], the value of ω_D is taken to be infinite to simplify the calculations. However, in real materials, ω_D has finite values. For nonmagnetic cases, we have

$$G^{(1)}_{NM_2}(\vec{k},\omega) = \frac{\omega}{\tau}\left(\frac{\omega_D\left(\omega^2-\Delta^2+\varepsilon^2_{\vec{k}}\right)}{\pi\left(\omega^2-\Delta^2\right)\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)^2}\right) \qquad \omega^2>\Delta^2,\ \omega_D<\sqrt{\omega^2-\Delta^2}, \tag{A4}$$

$$G^{(1)}_{NM_3}(\vec{k},\omega) = \frac{1}{\tau}\left(\frac{\omega(-i\pi\omega_D+2\sqrt{\omega^2-\Delta^2})\left(\omega^2-\Delta^2+\varepsilon^2_{\vec{k}}\right)}{2\pi\omega_D\sqrt{\omega^2-\Delta^2}\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)^2}\right) \qquad \omega^2>\Delta^2,\ \omega_D\gg\sqrt{\omega^2-\Delta^2}. \tag{A5}$$

For magnetic cases, we have

$$G^{(1)}_{M_3}(\vec{k},\omega) \approx \frac{\omega+\varepsilon_{\vec{k}}-\omega_D/\tau_1\pi\omega}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\left(1-\frac{\frac{2\omega_D}{\tau_1\pi}+\frac{2\omega_D b^2}{\tau_2\pi}}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\right) \qquad \omega^2>\Delta^2,\ \omega_D<\sqrt{\omega^2-\Delta^2}, \tag{A6}$$

$$G^{(1)}_{M_4}(\vec{k},\omega) \approx \left(\frac{1}{\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}}\right)\left((\omega+\varepsilon_{\vec{k}})\left(1-\frac{\frac{1}{\tau_1}\left(i\omega-\frac{2\omega^2}{\pi\omega_D}\right)+\frac{1}{\tau_2}\left(\frac{2\Delta^2}{\pi\omega_D}-i|b|\Delta\right)}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\right)-\frac{\omega}{\pi\tau_1\omega_D}+\frac{i}{2\tau_1}\right)$$
$$\omega^2>\Delta^2,\ \omega_D\gg\sqrt{\omega^2-\Delta^2}, \tag{A7}$$

$$G^{(1)}_{M_5}(\vec{k},\omega) \approx \left(\frac{1}{\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}}\right)\left((\omega+\varepsilon_{\vec{k}})\left(1-\frac{\frac{1}{\tau_1}\left(|a|\omega-\frac{2\omega^2}{\pi\omega_D}\right)+\frac{1}{\tau_2}\left(\frac{2\Delta^2}{\pi\omega_D}-\Delta\right)}{\left(\omega^2-\Delta^2-\varepsilon^2_{\vec{k}}\right)}\right)-\frac{\omega}{\pi\tau_1\omega_D}+\frac{|a|}{2\tau_1}\right)$$
$$\omega^2<\Delta^2,\ \omega_D\gg\sqrt{\Delta^2-\omega^2}. \tag{A8}$$

We do not take into account the conditions $\omega^2<\Delta^2$ and $\omega_D<\sqrt{\Delta^2-\omega^2}$, since they contradict the fact that Δ is always less than ω_D.

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