# Diffusion of elastic waves in a continuum solid with a random array of pinned dislocations

Dmitry Churochkin [1] and Fernando Lund [2]
[1] *Institute of Physics, Saratov State University, Saratov 410012, Russia*
[2] *Departamento de Física and CIMAT, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile*

(Received 12 April 2022; revised 21 June 2022; accepted 27 June 2022; published 7 July 2022)

The propagation of incoherent elastic energy in a three-dimensional solid due to the scattering by many randomly placed and oriented, pinned dislocation segments is considered in a continuum mechanics framework. The scattering mechanism is that of an elastic string of length $L$ that re-radiates as a response to an incoming wave. The scatterers are thus not static but have their own dynamics. A Bethe-Salpeter (BS) equation is established and a Ward-Takahashi identity (WTI) is demonstrated. The BS equation is written as a spectral problem that, using the WTI, is solved in the diffusive limit. To leading order, a diffusion behavior indeed results and an explicit formula for the diffusion coefficient is obtained. It can be evaluated in an independent scattering approximation in the absence of intrinsic damping. It depends not only on the bare longitudinal and transverse wave velocities but also on the renormalized velocities as well as attenuation coefficients of the coherent waves. The influence of the length scale given by $L$, and of the resonant behavior for frequencies near the resonance frequency of the strings can be explicitly identified. A Kubo representation for the diffusion constant can be identified. Previous generic results, obtained with an energy transfer formalism, are recovered when the number of dislocations per unit volume is small. This includes the equipartition of diffusive energy density which, however, does not hold in general. The formalism bears a number of similarities with the behavior of electromagnetic waves in a medium with a random distribution of dielectric scatterers; the elastic interaction, however, is momentum dependent.

DOI: 10.1103/PhysRevB.106.024105

## I. INTRODUCTION

Dislocations have long been known to be a crucial component in the mechanical behavior of metals and alloys. In other areas of condensed-matter physics, however, they have often been considered rather a nuisance. Nevertheless, in recent years, increasing evidence has become available to the effect that dislocations, rather than an obstacle, can become a useful tool to increase the performance of functional materials. For example, dislocations have been shown to drive the amorphization of phase-change materials [1]; they can contribute to the control of polarization in bulk ferroelectrics [2] and they considerably alter the distribution of electronic and ionic defects in oxides [3,4]. Importantly, for optoelectronic devices, Massabuau *et al.* [5] reported evidence for carrier localization in the vicinity of dislocations in InGaN. However, progress along these lines has been hampered by a lack of understanding of the basic physics of dislocations, considered one-dimensional, extended, topological defects in a three-dimensional material.

Additionally, from a condensed matter physics point of view, surprisingly little appears to have been studied about the influence of dislocations on thermal transport, although experimental evidence of a measurable effect have been reported. Indeed, Kotchetkov *et al.* [6] showed, using a relaxation time approximation, that dislocations have a measurable effect on the thermal conductivity of GaN layers. Kamatagi *et al.* [7] and Ma *et al.* [8] studied the effect of point defects and dislocations on bulk wurtzite GaN, and found it to be significant. The same is true for freestanding GaN thin films [9]. A relaxation time approximation was also used by Singh *et al.* [10] to study the effect of stacking faults and dislocations on the phonon conductivity of plastically deformed LiF and Ge, with satisfactory results. Recently, the role of dislocations has become the focus of much attention and there is increasing quantitative evidence linking a decrease in thermal conductivity with an increase in dislocation density [11–16]. Additionally, a numerical experiment [17] has concluded that decorated dislocation engineering can lead to interesting fabrication strategies for thermoelectric devices.

Importantly, lack of a detailed understanding of phonon transport seriously hampers the fabrication of practical thermoelectric materials [18] and there is significant activity around this issue. It is worth mentioning here, for example, the calculation of thermal conductivity using first-principles atomistic simulations and the Boltzmann transport equation [19,20]. However, current simulation tools appear to still be insufficient to gauge the impact of defects, particularly extended, resonant defects such as dislocations, on phonon transport [21]. Molecular dynamics methods have also been used [22,23], but shortcomings have recently been pointed out by Bedoya-Martínez *et al.* [24]. Quite recently, and after decades of formulation of the traditionally used theoretical models for the phonon-dislocation interaction [25,26], dislocation dynamics such as used in the present paper has been incorporated into the understanding of thermal transport [27,28].

The interaction of acoustic waves—phonons—with dislocations has a long and distinguished history of scholarship [29–33]. However, only in recent years has it been possible to make sufficient quantitative progress to have, say, explicit formulas for the scattering cross section of an elastic wave by an oscillating dislocation segment in three dimensions for arbitrary wave polarization, dislocation, and Burgers vector orientation [34]. Use of the resulting formalism together with a multiple scattering approach has led to a nonintrusive way to characterize dislocation densities in metals and alloys through resonant ultrasound spectroscopy [35] and *in situ* time-of-flight measurements [36–38].

Maurel *et al.* [39], working within the framework of the continuum theory of elasticity, have developed a perturbation scheme for the propagation of elastic waves through a random array of pinned vibrating dislocations. On the grounds of that model, the problem of coherent propagation, and attenuation, has been investigated thoroughly in the independent scattering approximation (ISA) [40]. The coherent propagation regime carries only part of the information about the transport properties of a given physical system [41]. A complete treatment requires the investigation of incoherent behavior. Of special interest is the diffusive range, which is determined by the transfer of energy density and typically starts at transport distances a bit larger than a few attenuation lengths. The general approach to this problem is based on the asymptotic solution of the Bethe-Salpeter (BS) equation accompanied with the relevant Ward-Takahashi identity (WTI). In turn, the form of the WTI depends on the specifics of the system under consideration [41].

Diffusion techniques for incoherent waves were developed to treat the problem of electron localization [42–44] and were later used for the description of the localization of (scalar) acoustic waves moving through a random array of hard scatterers [45]. An eigenvalue method to solve the BS equation developed by Wölfle and Bhatt [43,44] was extended to the problem of light diffusion in a random medium of dielectric scatterers, which complies with the generalized WTI by Barabanenkov and Ozrin [46,47]. In a similar vein, the diffusion of light in a general anisotropic turbid media was studied by Stark and Lubensky [48].

The multiple scattering of acoustic and elastic waves has been dealt with in the literature: Kirkpatrick [45] studied the problem of the localization of scalar acoustic waves in a medium with hard scatterers, both in two and three dimensions, using a diagrammatic approach. A diffusion behavior appears in a Boltzmann approximation as a result of the summation of the ladder diagrams. Weaver [49] studied the diffusion of ultrasound in a polycrystalline material, introducing disorder through randomly fluctuating elastic constants, and obtained an equation of radiative transfer. Van Tiggelen and coworkers have studied the coherent backscattering of elastic waves in an infinite isotropic medium [50], their radiative transfer in a generalized diffusion approximation [51], and their multiple scattering within a plate [52]. The Schrödinger-like description used in the last work has been carried over by Trujillo *et al.* [53] to the description of elastic waves in dry granular media. The issue of localization of elastic waves, a phenomenon that may appear when the diffusion constant vanishes because of wave interference, has been addressed experimentally by Cobus *et al.* [54] and Goïcoechea *et al.* [55].

On a different perspective, the interaction of sound with the Volterra dislocations that are used in the present paper has been shown to lead to an improved understanding of the acoustic properties of glasses in the THz range [56,57]. The use of continuum mechanics, without an intrinsic length scale, offers a powerful tool since it applies to all glasses in the appropriate length scale. The same point of view can be helpful to advance our understanding of thermal transport in amorphous solids. Indeed, as emphasized, for example, by Beltukov *et al.* [58] through numerical simulations, there is a complex dynamics underlying energy transport by phonons in these materials.

The purpose of this paper is to address the above issues from a macroscopic point of view; specifically, to study the diffusion of elastic waves moving through a random array of vibrating dislocations. To this end, we describe the dynamics of a single dislocation following the Granato-Lücke vibrating string model [29]. It is assumed that we deal with an ensemble of noninteracting dislocations (or, more precisely, that they interact solely through the scattering of elastic waves). On this foundation, we extend the formalism developed by Barabanenkov and Ozrin [47] for electromagnetic waves to the case of elastic waves with different polarizations that interact with scatterers that obey the generalized Granato-Lücke string equation [34].

This paper is organized as follows: Section II sets up the formalism for the problem. It is an inhomogeneous wave equation in which the inhomogeneous term describes the interaction between wave and dislocation. This interaction term is dubbed the potential term by analogy with the case of de Broglie waves describing electrons. We shall use a perturbation approach, in which the potential term is considered a small perturbation. Previous results are briefly recalled. A BS equation is derived in Sec. III. Following the approach of Refs. [46,47] a WTI is obtained in Sec. IV. The eigenvalue problem for the BS equation is formulated and solved in Sec. V. A specific expression for the diffusion constant is obtained. This result is discussed in Sec. VI. It is shown that the diffusion constant can be cast in a Kubo-like expression [47] and that, in the low frequency and low density of scatterers limit, it reduces to the expression obtained in a radiation transfer formalism [59]. Section VII offers a final conclusion and outlook. A number of the more technical calculations are described in six appendices.

## II. PROBLEM SETUP AND PREVIOUS RESULTS

In the linear theory of elasticity, the dynamics of an isotropic medium with mass density $\rho$ and elastic constants $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $(\lambda, \mu)$ the Lamé constants is described by displacements $\mathbf{u}(\mathbf{x}, t)$ as a function of an equilibrium position $\mathbf{x}$ at time $t$. Velocity $\mathbf{v}$ is the time derivative, $\mathbf{v} = \partial\mathbf{u}/\partial t$. The speed of sound is $c_L \equiv \sqrt{\lambda + 2\mu/\rho}$, the speed of shear waves is $c_T \equiv \sqrt{\mu/\rho}$, and we shall denote their ratio by $\gamma \equiv c_L/c_T$. The vibration of edge dislocations of length $L$ that are pinned at the ends and characterized by the Burgers vector $\mathbf{b}$ with a local tangent oriented along $\hat{\tau}$ and situated in the equilibrium state at the point $\mathbf{X}_0$ perturbs the medium in such a way that the whole system is governed by the wave equation with a source [39,40],

$$\rho \frac{\partial^2}{\partial t^2} v_i(\mathbf{x}, t) - c_{ijkl} \frac{\partial^2}{\partial x_j \partial x_l} v_k(\mathbf{x}, t) = V_{ik} v_k(\mathbf{x}, t), \quad (1)$$

where the perturbation potential is defined as

$$V_{ik} = \mathcal{A}\, \mathsf{M}_{ij} \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{X}_0)\, \mathsf{M}_{lk} \frac{\partial}{\partial x_l}\bigg|_{\mathbf{x}=\mathbf{X}_0}, \quad (2)$$

with

$$\mathcal{A} \equiv \frac{8}{\pi^2} \frac{(\mu b)^2 L}{m} g(\omega). \quad (3)$$

$g(\omega) \equiv [\omega^2 + i\omega(B/m) - \omega_F^2]^{-1}$, $\hat{\mathbf{n}} \equiv \hat{\tau} \wedge \hat{\mathbf{t}}$, $\hat{\mathbf{t}} \equiv \mathbf{b}/|\mathbf{b}|$ is the unit Burgers vector that indicates the direction of glide, and $\mathsf{M}_{ij} \equiv t_i n_j + t_j n_i$, with

$$\omega_F \equiv \frac{\pi}{L} \sqrt{\frac{\Gamma}{m}} \quad (4)$$

the fundamental frequency of a vibrating string characterized by effective mass per unit length $m$, line tension $\Gamma$, and damping $B$, which represent the dislocation dynamics. Only glide motion, that is, along $\hat{t}$, is allowed, a fact that translates into $\tau_i V_{ik} \equiv 0$. Dislocation climb implies mass transport and is not allowed [60]. The medium is considered linear everywhere outside the dislocations core. Consequently, when more than one dislocation is present, their effect is obtained simply by addition of the individual terms. Note that the potential Eq. (2) involves two gradients, a feature that will lead, in momentum space, to a dependence on the square of the momentum. Care will have to be exercised then at short wavelengths.

An important quantity for the analysis is the Green's tensor, or impulse response function, for Eq. (1). Its average properties provide information about both coherent and incoherent wave behavior. In the frequency domain, it obeys the equation [39,40]

$$\rho\omega^2 G_{im}(\mathbf{x}, \mathbf{x}', \omega) + c_{ijkl} \frac{\partial^2}{\partial x_j \partial x_l} G_{km}(\mathbf{x}, \mathbf{x}', \omega) = -\sum_{\text{disloc. lines}} V_{ik} G_{km}(\mathbf{x}, \mathbf{x}', \omega) - \delta_{im}\delta(\mathbf{x} - \mathbf{x}'). \quad (5)$$

Equation (5) carries information about the asymptotic behavior of outgoing waves at large distances from the source. For convenience, we have not written explicitly the second argument in the Green's tensor: $G_{im}(\mathbf{x}, \omega)$ must be understood as $G_{im}(\mathbf{x}, \mathbf{x}', \omega)$ with $\mathbf{x}$ the detection point and $\mathbf{x}'$ the source point. The poles of the Fourier transformed averaged Green's tensor yield the modified spectrum of $T$ (transversal) and $L$ (longitudinal) modes present in the medium. A solution of Eq. (5) can be found perturbatively. Applying the ISA approach (i.e., that the random variables associated with each one of the dislocation segments are statistically independent of each other), we have found the averaged Green's tensor for outgoing waves $\langle \mathbf{G} \rangle^+(\mathbf{k}, \omega)$ as [40]

$$\langle \mathbf{G} \rangle^+(\mathbf{k}, \omega) = G_T(\mathbf{I} - P_{\hat{\mathbf{k}}}) + G_L P_{\hat{\mathbf{k}}}, \quad (6)$$

with

$$G_{T,L} = \frac{1}{\rho\omega^2 \left\{ \frac{k^2}{K_{T,L}^2} - 1 \right\}}$$

as well as the self-energy tensor $\mathbf{\Sigma}^+(\mathbf{k}, \omega)$ defined through the Dyson equation [39],

$$\langle \mathbf{G} \rangle^{-1} = (\mathbf{G}^0)^{-1} - \mathbf{\Sigma}, \quad (7)$$

with $\mathbf{G}^0$ the Green's tensor for free space and

$$\mathbf{\Sigma}^+(\mathbf{k}, \omega) = \Sigma_T(\mathbf{I} - P_{\hat{\mathbf{k}}}) + \Sigma_L P_{\hat{\mathbf{k}}}, \quad (8)$$

with

$$\Sigma_{T,L} = \rho \left( c_{T,L}^2 - \frac{\omega^2}{K_{T,L}^2} \right) k^2,$$
$$K_T = \frac{\omega}{c_T} \left[ 1 + \frac{n\mathcal{A}}{5\rho c_T^2 (1 + i\mathcal{A}I)} \right]^{-1/2}, \quad (9)$$
$$K_L = \frac{\omega}{c_L} \left[ 1 + \frac{4n\mathcal{A}}{15\rho c_L^2 (1 + i\mathcal{A}I)} \right]^{-1/2},$$

and

$$I = \frac{1}{30\pi} \left[ \frac{3\gamma^5 + 2}{\gamma^5} \right] \frac{\omega^3}{\rho c_T^5}, \quad (10)$$

where $P_{\hat{\mathbf{k}}} = \hat{\mathbf{k}}^t \hat{\mathbf{k}}$ and $\hat{\mathbf{k}}^t$ is the transposed unit vector along $\mathbf{k}$. The incoming waves, related to $\langle \mathbf{G} \rangle^-(\mathbf{k}, \omega)$ and $\mathbf{\Sigma}^-(\mathbf{k}, \omega)$, are described by the complex conjugate form of Eqs. (6) and (8).

The average $\langle \cdot \rangle$ is over dislocation position, orientation, and Burgers vector. It has been described in detail by Maurel *et al.* [39] On average, the medium is homogeneous and isotropic. The effective wave numbers $K_{T,L}$ define an effective phase velocity for wave propagation,

$$v_{T,L} \equiv \frac{\omega}{\text{Re}[K_{T,L}]}, \quad (11)$$

and attenuation length:

$$l_{T,L} \equiv \frac{1}{2\text{Im}[K_{T,L}]}. \quad (12)$$

These quantities will appear explicitly in the diffusion constant that will be discussed in Sec. V.

## III. BETHE-SALPETER EQUATION FOR AN ELASTIC MEDIUM WITH MANY VIBRATING DISLOCATION SEGMENTS

We have tested the methods of this paper in a simplified setting: that of the incoherent behavior of elastic waves in a two-dimensional continuum with a random distribution of screw dislocations [61] and edge dislocations [62]. The screw case is a scalar problem that keeps the whole basic physics of the diffusion behavior of elastic waves when propagating incoherently among a maze of dislocations. Being scalar, the algebra is much simpler. The edge case keeps the full vector nature of the three-dimensional problem but the algebra is still simpler in two dimensions, particularly since dislocations are points and not lines. The physics of the present problem is much richer because the dislocations have a finite length, a precise orientation, and Burgers vector, and the elastic waves have two polarizations that travel at different speeds. The algebra, however, is quite close to that of Ref. [62] and we shall refer to this reference for the details of the computation.

To track the wave transport after the phase coherence is lost, we have to focus on the evolution of the corresponding configurationally averaged intensity which is qualitatively represented in momentum space as the two-point correlation of the Green's tensor [41]

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega) &\equiv \Phi_{kl,mn}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)\\ &\equiv \langle G^{+}_{km}(\mathbf{k}^{+},\mathbf{k}'^{+},\omega^{+})G^{-}_{nl}(\mathbf{k}'^{-},\mathbf{k}^{-},\omega^{-})\rangle,\end{aligned} \tag{13}$$

with

$$\mathbf{k}^{\pm} = \mathbf{k} \pm \frac{\mathbf{q}}{2}, \quad \omega^{\pm} = \omega \pm \frac{\Omega}{2}. \tag{14}$$

The reciprocity of the Green's tensor, $G_{im}(\mathbf{x},\mathbf{x}',\omega) = G_{mi}(\mathbf{x}',\mathbf{x},\omega)$, implies

$$\Phi_{kl,mn}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega) = \Phi_{mn,kl}(\mathbf{k}',\mathbf{k};\mathbf{q},\Omega).$$

In this approach, diffusive behavior means that the two-point correlation tensor Eq. (13) has a specific pole structure in terms of the diffusive variables $\mathbf{q}$ and $\Omega$. Just as the Dyson equation yields the pole structure for the averaged Green's tensor, the BS equation yields the pole structure for the intensity [41]. Using the standard formalism [41,42,45,62], the BS equation for the elastic wave diffusion in the medium with dislocations is found to be (see Appendix A)

$$[\imath\omega\Omega\mathbf{E} + \mathbf{P}(\mathbf{k};\mathbf{q})] : \boldsymbol{\Phi}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega) + \int_{\mathbf{k}''} \mathbf{U}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega) : \boldsymbol{\Phi}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega) = \delta_{\mathbf{k},\mathbf{k}'}\boldsymbol{\Delta}\mathbf{G}(\mathbf{k};\mathbf{q},\Omega), \tag{15}$$

where

$$\mathbf{U}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega) \equiv U_{ij,kl}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega), \tag{16}$$

$$\equiv \Delta\Sigma_{ij,kl}(\mathbf{k};\mathbf{q},\Omega)\delta_{\mathbf{k},\mathbf{k}'} - \Delta G_{ij,mn}(\mathbf{k};\mathbf{q},\Omega)K_{mn,kl}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega), \tag{17}$$

$$\boldsymbol{\Delta}\mathbf{G}(\mathbf{k};\mathbf{q},\Omega) \equiv \Delta G_{ij,mn}(\mathbf{k};\mathbf{q},\Omega), \tag{18}$$

$$\equiv \frac{1}{2\imath\rho}(\delta_{im}\langle G\rangle^{-}_{nj}(\mathbf{k}^{-},\omega^{-}) - \langle G\rangle^{+}_{im}(\mathbf{k}^{+},\omega^{+})\delta_{nj}), \tag{19}$$

$$\boldsymbol{\Delta\Sigma}(\mathbf{k};\mathbf{q},\Omega) \equiv \Delta\Sigma_{ij,mn}(\mathbf{k};\mathbf{q},\Omega), \tag{20}$$

$$\equiv \frac{1}{2\imath\rho}(\delta_{im}\Sigma^{-}_{nj}(\mathbf{k}^{-},\omega^{-}) - \Sigma^{+}_{im}(\mathbf{k}^{+},\omega^{+})\delta_{nj}), \tag{21}$$

and

$$\mathbf{P}(\mathbf{k};\mathbf{q}) \equiv P_{ij,kl}(\mathbf{k};\mathbf{q}), \tag{22}$$

$$\equiv \frac{1}{2\imath\rho}(\delta_{ik}L_{lj}(\mathbf{k}^{-}) - L_{ik}(\mathbf{k}^{+})\delta_{lj}), \tag{23}$$

$$\mathbf{E} = E_{ij,kl} = \delta_{ik}\delta_{lj}, \tag{24}$$

$$L_{ij}(\mathbf{k}^{\pm}) = -c_{ikjl}k^{\pm}_{k}k^{\pm}_{l}. \tag{25}$$

Here, $K_{mn,kl}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)$ is the irreducible vertex, explicitly presented in Appendix A. We denote

$$\int_{\mathbf{p}''} = (2\pi)^{-3}\int d\mathbf{p}'', \tag{26}$$

and : is the inner tensor product defined in components for arbitrary fourth rank tensors as $\mathcal{E} : \mathcal{F} \equiv \mathcal{E}_{ij,kl}\mathcal{F}_{kl,mn}$.

## IV. WARD-TAKAHASHI IDENTITY

Energy conservation, formulated in the form of a WTI, underlies the theoretical description of incoherent transport of classical waves [63]. For specific forms of the perturbation potential, the WTI has been obtained on the basis of Lagrangian [63] as well as pre-WTI methods [46,47], an issue that was the object of some debate [64,65]. In this paper, we shall use the pre-WTI method [46,47] that deals directly with the equations of motion.

### A. Pre-WTI

We establish, as a preliminary step, a relation between the average Green's function and its two-point correlation that does not explicitly involve the interaction $V_{ik}$. To this end, we start with Eq. (5) written for Green's tensors at two different sets of variables, $G_{i_1m_1}(\mathbf{x}_1,\mathbf{x}'_1,\omega_1)$ and $G_{i_2m_2}(\mathbf{x}_2,\mathbf{x}'_2,\omega_2)$, and we take the two-sided Fourier transform of these relations with the definitions

$$\mathbf{F}(\mathbf{k},\mathbf{k}';\omega) = \iint d\mathbf{x}d\mathbf{x}' e^{-\imath\mathbf{k}\mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{x}';\omega)e^{\imath\mathbf{k}'\mathbf{x}'}, \tag{27}$$

$$\mathbf{G}(\mathbf{x},\mathbf{x}';\omega) = \int_{\mathbf{k}}\int_{\mathbf{k}'} e^{\imath\mathbf{k}\mathbf{x}}\mathbf{G}(\mathbf{k},\mathbf{k}';\omega)e^{-\imath\mathbf{k}'\mathbf{x}'}. \tag{28}$$

We now act on the first and second equations of the system from the right by $g^*(\omega_2)(G)^{-1}_{m1n1}(\mathbf{k}'_1,\mathbf{k}''_1;\omega_1)$ and $g(\omega_1)(G^*)^{-1}_{m2n2}(k'_2,\mathbf{k}''_2;\omega_2)$, respectively. The next step consists of subtraction of the second equation from the first, and evaluating at $i_1 = n_2 = i$, $n_1 = i_2 = n$; $\mathbf{k}_2 \to \mathbf{k}''_1$, $\mathbf{k}''_2 \to \mathbf{k}_1$. Otherwise, it is not possible to eliminate the remaining parts of the potentials in both equations that are subject to the substraction from each other, since the parts imply not only summation over defects but also contain components of the second rank tensor, i.e., to achieve identity of those parts between each other, the components must be also identical.

Noting the explicit expression of the bare Green's function,

$$(G^0)^{-1}_{ik}(\mathbf{k},\omega) = -(\rho\omega^2\delta_{ik} - c_{ijlk}k_jk_l), \tag{29}$$

we obtain

$$\lim_{\substack{\mathbf{k}_2\to\mathbf{k}_1''\\ \mathbf{k}_2''\to\mathbf{k}_1}} \big( -(G^0)^{-1}_{in}(\mathbf{k}_1,\omega_1)\delta_{\mathbf{k}_1,\mathbf{k}_1''}g^*(\omega_2) + g^*(\omega_2)(G)^{-1}_{in}(\mathbf{k}_1,\mathbf{k}_1'';\omega_1) + (G^{*0})^{-1}_{ni}(\mathbf{k}_2,\omega_2)\delta_{\mathbf{k}_2,\mathbf{k}_2''}g(\omega_1) - g(\omega_1)(G^*)^{-1}_{ni}(\mathbf{k}_2,\mathbf{k}_2'';\omega_2)\big) \equiv 0. \tag{30}$$

Now, multiplying this identity on the right by

$$\lim_{\substack{\mathbf{k}_2\to\mathbf{k}_1''\\ \mathbf{k}_2''\to\mathbf{k}_1}} G(\mathbf{k}_1'',\mathbf{k}_1''';\omega_1)_{nl}G^*(\mathbf{k}_2'',\mathbf{k}_2''';\omega_2)_{ij}, \tag{31}$$

averaging, and using the following notation:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{k}^+, & \mathbf{k}_2'' &= \mathbf{k}^-, & \mathbf{k}_1'' &= \mathbf{k}''^+, & \mathbf{k}_2 &= \mathbf{k}''^-,\\
\mathbf{k}_1''' &= \mathbf{k}'''^+, & \mathbf{k}_2''' &= \mathbf{k}'''^-, & \omega_1 &= \omega_+, & \omega_2 &= \omega_-,\\
G &= G^+, & G^* &= G^-, & G^0 &= G^{0+}, & G^{0*} &= G^{0-},
\end{aligned} \tag{32}$$

the following pre-WTI is obtained:

$$\int_{\mathbf{k}} \big((G^{0-})^{-1}_{ni}(\mathbf{k};\mathbf{q},\Omega)g(\omega_+) - (G^{0+})^{-1}_{ni}(\mathbf{k};\mathbf{q},\Omega)g^*(\omega_-)\big)\Phi_{ni,lj}(\mathbf{k},\mathbf{k}''';\mathbf{q},\Omega) + g^*(\omega_-)\langle G\rangle^-(\mathbf{k}''';\mathbf{q},\Omega)_{lj} - g(\omega_+)\langle G\rangle^+(\mathbf{k}''';\mathbf{q},\Omega)_{lj} \equiv 0. \tag{33}$$

If we use Eq. (13) and recall that

$$\begin{aligned}
(G^{0\pm})^{-1}_{in}(\mathbf{k}^\pm,\omega_\pm) &= (G^{0\pm})^{-1}_{in}(\mathbf{k};\mathbf{q},\Omega)\\
\langle G^\pm(\mathbf{k}'''^\pm,\mathbf{k}'''^\pm;\omega_\pm)_{jl}\rangle &= \langle G\rangle^\pm(\mathbf{k}''';\mathbf{q},\Omega)_{jl},
\end{aligned} \tag{34}$$

we see that the pre-WTI relates, in Fourier space, the averaged Green's function, with its two-point correlations without the explicit appearance of the interaction $V_{ij}$.

### B. WTI

The relation between averages obtained at the end of the last subsection is now turned into a relation between their irreducible parts, the irreducible vertex $\mathbf{K}$ and the mass operator $\mathbf{\Sigma}$. Multiplying Eq. (33) on the right by $\Phi^{-1}_{lj,mt}(\mathbf{k}''',\mathbf{k}'''';\mathbf{q},\Omega)$, using Eqs. (A2) and (7), the following WTI is obtained:

$$\begin{aligned}
&(\Sigma^-_{mt}(\mathbf{k}'''';\mathbf{q},\Omega)g(\omega_+) - \Sigma^+_{mt}(\mathbf{k}'''';\mathbf{q},\Omega)g^*(\omega_-))\\
&\equiv \int_{\mathbf{k}'''} (g^*(\omega_-)\langle G\rangle^-(\mathbf{k}''';\mathbf{q},\Omega)_{lj}\\
&\quad - g(\omega_+)\langle G\rangle^+(\mathbf{k}''';\mathbf{q},\Omega)_{lj})K_{lj,mt}(\mathbf{k}''',\mathbf{k}'''';\mathbf{q},\Omega).
\end{aligned} \tag{35}$$

In terms of the general, i.e., symbolical, representation of the WTI there are two differences compared to a well-known tensorial version of the WTI for electromagnetic waves [47]: First, $g$ is a complex valued resonance like function; second, the tensor rank of the WTI is two rather than four as in the case of electromagnetic waves [47]. In our case, this is all we need to solve the problem at hand.

The WTI can be written in the more compact form

$$\int_{\mathbf{k}''} \overline{\mathbf{U}}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega) = \frac{i}{2}\overline{\mathbf{A}}(\mathbf{k}';\mathbf{q},\Omega)(g(\omega_+) - g^*(\omega_-)), \tag{36}$$

with the following notation:

$$\overline{\mathbf{U}}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega) = U_{ii,mt}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega), \tag{37}$$

$$\begin{aligned}
\overline{\mathbf{A}}(\mathbf{k}';\mathbf{q},\Omega) &= A_{nn,mt}(\mathbf{k}';\mathbf{q},\Omega),\\
\mathbf{A}(\mathbf{k}';\mathbf{q},\Omega) &= \frac{2}{g(\omega_+) + g^*(\omega_-)}\bigg(\mathcal{R}\mathbf{\Sigma}(\mathbf{k}';\mathbf{q},\Omega)\\
&\quad + \int_{\mathbf{k}''} \mathcal{R}\mathbf{G}(\mathbf{k}'';\mathbf{q},\Omega) : \mathbf{K}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega)\bigg),
\end{aligned} \tag{38}$$

$$\mathcal{R}\mathbf{\Sigma}(\mathbf{k}';\mathbf{q},\Omega) = \frac{1}{2\rho}(\mathbf{I}\otimes\mathbf{\Sigma}^-(\mathbf{k}'^-,\omega^-) + \mathbf{\Sigma}^+(\mathbf{k}'^+,\omega^+)\otimes\mathbf{I}), \tag{39}$$

The tensor $\mathbf{U}$ is given by Eq. (16). The operation $\mathcal{R}$ is defined here for the self-energy tensor $\mathbf{\Sigma}$; it is similarly defined for the Green's tensor $\mathbf{G}$.

### C. Low Ω, low q behavior

The diffusion behavior appears in the limit $\Omega, \mathbf{q} \to 0$. In this case, the following relations for the self-energy and for the Green's function will prove useful:

$$\begin{aligned}
\mathbf{\Delta\Sigma}(\mathbf{k};\mathbf{0},0) = \mathbf{\Delta\Sigma}(\mathbf{k}) &= \Delta\Sigma_{im,tk}(\mathbf{k})\\
&= \frac{1}{2\imath\rho}(\delta_{it}\Sigma^*_{km}(\mathbf{k}) - \Sigma_{it}(\mathbf{k})\delta_{km}),
\end{aligned} \tag{40}$$

and its trace over two indices is given by

$$\Delta\Sigma_{ii,tk}(\mathbf{k}) = \frac{-1}{\rho}\big((\delta_{kt} - \hat{k}_k\hat{k}_t)\mathrm{Im}[\Sigma_T(\mathbf{k})] + \hat{k}_k\hat{k}_t\mathrm{Im}[\Sigma_L(\mathbf{k})]\big). \tag{41}$$

Similarly,

$$\mathbf{\Delta G}(\mathbf{k};\mathbf{0},0) = \mathbf{\Delta G}(\mathbf{k}) = \Delta G_{im,tk}(\mathbf{k}), \tag{42}$$

$$= \frac{1}{2\imath\rho}(\delta_{it}G^*_{km}(\mathbf{k}) - G_{it}(\mathbf{k})\delta_{km}), \tag{43}$$

so its trace is

$$\Delta G_{ii,tk}(\mathbf{k}) \approx \frac{-\pi(\delta_{kt} - \hat{k}_k\hat{k}_t)k^2}{\rho^2\omega^2}\delta\big(k^2 - \mathrm{Re}\big[K_T^2\big]\big), \tag{44}$$

$$+ \frac{-\pi\hat{k}_k\hat{k}_t k^2}{\rho^2\omega^2}\delta\big(k^2 - \mathrm{Re}\big[K_L^2\big]\big), \tag{45}$$

The last approximation holds in the limit $|\mathrm{Im}[K_{T,L}^2]| \ll |k^2 - \mathrm{Re}[K_{T,L}^2]|$. (The meaning of this inequality in terms of the dislocation parameters is explored in Sec. VI A 2). Also, an abbreviated notation has been introduced: $\langle G^+\rangle_{km}(\mathbf{k},\omega) = G_{km}(\mathbf{k})$, $\langle G^-\rangle_{km}(\mathbf{k},\omega) = G^*_{km}(\mathbf{k}))$ and similarly for $\Sigma^\pm_{km}(\mathbf{k},\omega)$.

### D. Lossless case, $B = 0$, and independent scattering approximation

When $B = 0$, i.e., when $g$ is real, $\mathbf{q}$, $\Omega$ tend to zero, and the standard ISA expressions for $\mathbf{\Sigma}$ and $\mathbf{K}$ tensors, Eq. (47) below, are taken (see Appendix C), the optical theorem is obtained. Explicitly, the WTI reads in this case

$$(\Sigma^*_{mt}(\mathbf{k}'''') - \Sigma_{mt}(\mathbf{k}''''))$$
$$\equiv \int_{\mathbf{k}'''}\big(G^{0*}(\mathbf{k}''')_{lj} - G^0(\mathbf{k}''')_{lj}\big)K_{lj,mt}(\mathbf{k}''',\mathbf{k}''''), \tag{46}$$

with the following expressions, valid to leading order in $n$, the density of scatterers:

$$\Sigma_{mt}(\mathbf{k}'''') = \Sigma_{mt}(\mathbf{k}'''';\mathbf{0},0)$$
$$\approx n\langle t\rangle_{mt}(\mathbf{k}''''), \tag{47}$$

$$K_{lj,mt}(\mathbf{k}''',\mathbf{k}'''';\mathbf{0},0) = K_{lj,mt}(\mathbf{k}''',\mathbf{k}''''),$$
$$\approx n\langle t_{lm}(\mathbf{k}''',\mathbf{k}'''')t^*_{tj}(\mathbf{k}'''',\mathbf{k}''')\rangle \tag{48}$$

$$\langle G\rangle(\mathbf{k}''';\mathbf{0},0)_{jl} = \langle G\rangle(\mathbf{k}''')_{jl}$$
$$\approx G^0(\mathbf{k}''')_{jl}. \tag{49}$$

## V. DIFFUSION BEHAVIOR

The similarity that has been established between the WTI for elastic and electromagnetic waves motivates us to employ the well-developed formalism [46–48,66] in the treatment of the diffusion problem. In that approach, we deal with the BS equation through the exploration of the eigenvalue problem for the operator with the kernel:

$$\mathbf{H} = [\imath\omega\Omega\mathbf{E} + \mathbf{P}(\mathbf{k};\mathbf{q})]\delta_{\mathbf{k}\mathbf{k}''} + \mathbf{U}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega). \tag{50}$$

In terms of $\mathbf{H}$, the BS Eq. (15) can be written as

$$\int_{\mathbf{k}''}\mathbf{H}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega) : \mathbf{\Phi}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega) = \mathbf{\Delta G}(\mathbf{k};\mathbf{q},\Omega)\delta_{\mathbf{k},\mathbf{k}'}. \tag{51}$$

Moreover, the definition of the kernel $\mathbf{H}$ ensures that it obeys the symmetry property

$$H_{ij,kl}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega)\Delta G_{kl,mn}(\mathbf{k}'';\mathbf{q},\Omega)$$
$$= H_{mn,kl}(\mathbf{k}'',\mathbf{k};\mathbf{q},\Omega)\Delta G_{kl,ij}(\mathbf{k};\mathbf{q},\Omega) \tag{52}$$

To see this, the explicit form of $\mathbf{U}$, and the reciprocity of the tensor $\mathbf{K}$, must be used.

In accordance with the general formalism [46–48,66], the solution of Eq. (51) should be found through the consideration of the spectral problem for the corresponding homogeneous equation with $\mathbf{f}^{\mathbf{r}n}_{kl}(\mathbf{k}'';\mathbf{q},\Omega)$ (respectively, $\mathbf{f}^{\mathbf{l}n}_{kl}(\mathbf{k}'';\mathbf{q},\Omega)$) as right (respectively, left) eigentensors and $\lambda_n(\mathbf{q},\Omega)$ as eigenvalue:

$$\int_{\mathbf{k}''}H_{ij,kl}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega)\mathbf{f}^{\mathbf{r}n}_{kl}(\mathbf{k}'';\mathbf{q},\Omega) = \lambda_n(\mathbf{q},\Omega)\mathbf{f}^{\mathbf{r}n}_{ij}(\mathbf{k};\mathbf{q},\Omega). \tag{53}$$

Following Refs. [46–48,66], we assume the eigentensors in Eq. (53) obey completeness and orthogonality conditions:

$$\int_{\mathbf{k}}\mathbf{f}^{\mathbf{r}m}_{ij}(\mathbf{k};\mathbf{q},\Omega)\mathbf{f}^{\mathbf{l}n}_{ij}(\mathbf{k};\mathbf{q},\Omega) = \delta_{mn},$$
$$\sum_n \mathbf{f}^{\mathbf{r}n}_{ij}(\mathbf{k};\mathbf{q},\Omega)\mathbf{f}^{\mathbf{l}n}_{kl}(\mathbf{k}';\mathbf{q},\Omega) = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ik}\delta_{lj}. \tag{54}$$

The left and right eigentensors are related, as a consequence of the symmetry properties (52) of the operator $\mathbf{H}$, as follows:

$$\mathbf{f}^{\mathbf{r}n}_{mn}(\mathbf{k};\mathbf{q},\Omega) = \Delta G_{mn,kl}(\mathbf{k};\mathbf{q},\Omega)\mathbf{f}^{\mathbf{l}n}_{kl}(\mathbf{k};\mathbf{q},\Omega). \tag{55}$$

A set of properties for the eigentensors reflected in Eqs. (54) and (55) enable us to form the basis for the representation of the solution $\mathbf{\Phi}$ as a series over the states $n$ [46–48,66]:

$$\Phi_{ij,kl} = \sum_n \frac{\mathbf{f}^{\mathbf{r}n}_{ij}(\mathbf{k};\mathbf{q},\Omega)\mathbf{f}^{\mathbf{r}n}_{kl}(\mathbf{k}';\mathbf{q},\Omega)}{\lambda_n(\mathbf{q},\Omega)}. \tag{56}$$

The concept of diffusion assumes that in the limit $\mathbf{q}\to 0$, $\Omega\to 0$ the function $\mathbf{\Phi}$ has a pole structure, dictating the lowest eigenvalue asymptotics $\lambda_0(\mathbf{q}\to 0,\Omega\to 0)\to 0$, and being separated from a regular part [46,47,66]. Therefore, the whole problem is reduced to the determination of coefficients of perturbative expansion for $\lambda_0(\mathbf{q},\Omega)$ with regard to $\mathbf{q}$ and $\Omega$ up to the second and first order, respectively, taken around the point $\mathbf{q}=0$, $\Omega=0$. To do this, Eq. (53) has to be treated perturbatively, with the condition that Eqs. (36) and (52) hold at every order of the perturbation in $\mathbf{q}$, and $\Omega$ [46,47,66].

### A. Perturbation approach to the eigenvalue problem

The solution to Eq. (53) is developed in a successive approximation scheme for small $\Omega$ and small $\mathbf{q}$:

$$\begin{aligned}\mathbf{H}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega) &= \mathbf{H}(\mathbf{k},\mathbf{k}'';\mathbf{0},0)+\mathbf{H}^{1\Omega}(\mathbf{k},\mathbf{k}'';\mathbf{0},\Omega)\\&\quad+\mathbf{H}^{1\mathbf{q}}(\mathbf{k},\mathbf{k}'';\mathbf{q},0)+\mathbf{H}^{2\mathbf{q}}(\mathbf{k},\mathbf{k}'';\mathbf{q},0)+\cdots,\\ \mathbf{f}^{\mathbf{r}0}(\mathbf{k}'';\mathbf{q},\Omega) &= \mathbf{f}(\mathbf{k}'';\mathbf{0},0)+\mathbf{f}^{1\Omega}(\mathbf{k}'';\mathbf{0},\Omega)\\&\quad+\mathbf{f}^{1\mathbf{q}}(\mathbf{k}'';\mathbf{q},0)+\mathbf{f}^{2\mathbf{q}}(\mathbf{k}'';\mathbf{q},0)+\cdots,\\ \lambda_0(\mathbf{q},\Omega) &= \lambda^{1\Omega}(\mathbf{0},\Omega)+\lambda^{1\mathbf{q}}(\mathbf{q},0)+\lambda^{2\mathbf{q}}(\mathbf{q},0)+\cdots.\end{aligned} \tag{57}$$

and, by deploying the perturbative scheme in detail (see Appendix D), the following set of coupled integral equations is obtained:

$$\int_{\mathbf{k}''} H_{ij,kl}(\mathbf{k},\mathbf{k}'')\mathbf{f}_{kl}(\mathbf{k}'')=0, \tag{58}$$

$$\int_{\mathbf{k}''}\left(H_{ij,kl}(\mathbf{k},\mathbf{k}'')\mathbf{f}^{1\Omega}_{kl}(\mathbf{k}'')+H^{1\Omega}_{ij,kl}(\mathbf{k},\mathbf{k}'')\mathbf{f}_{kl}(\mathbf{k}'')\right)=\lambda^{1\Omega}\mathbf{f}_{ij}(\mathbf{k}), \tag{59}$$

$$\int_{\mathbf{k}''}\left(H_{ij,kl}(\mathbf{k},\mathbf{k}'')\mathbf{f}^{1\mathbf{q}}_{kl}(\mathbf{k}'')+H^{1\mathbf{q}}_{ij,kl}(\mathbf{k},\mathbf{k}'')\mathbf{f}_{kl}(\mathbf{k}'')\right)=0, \tag{60}$$

$$\int_{\mathbf{k}} \mathcal{B}P_{ii,kl}(\mathbf{k};\mathbf{q})\mathbf{f}^{1\mathbf{q}}_{kl}(\mathbf{k})=\lambda^{2\mathbf{q}}, \tag{61}$$

where the arguments $\mathbf{q}$ and $\Omega$ have been omitted. As shown in Appendix D, the first-order-in-wave-number contribution to the eigenvalue vanishes:

$$\lambda^{1\mathbf{q}}=0. \tag{62}$$

This result ensures the existence of a diffusion regime for the problem at hand.

Using Eqs. (36) and (58), the eigentensor $\mathbf{f}^{\mathbf{r}0}$ at $\mathbf{q}=0$, $\Omega=0$ is found to be

$$f_{ij}(\mathbf{k}'')=\mathcal{B}\Delta G_{ij,kk}(\mathbf{k}'') \tag{63}$$

with

$$\mathcal{B}^{-2}=\int_{\mathbf{v}}\Delta G_{jj,kk}(\mathbf{v}). \tag{64}$$

Integrating Eq. (59) over $\mathbf{k}$ and using the WTI, Eq. (36), at the corresponding order, the eigenvalue $\lambda^{1\Omega}$ is obtained:

$$\lambda^{1\Omega}=i\omega\Omega(1+a), \tag{65}$$

with

$$a=\frac{1}{\int_{\mathbf{k}} f_{ss}(\mathbf{k})}\times\int_{\mathbf{k}''}\frac{(A_{ii,kl}(\mathbf{k}'';\mathbf{0},\Omega)(g(\omega_+)-g^*(\omega_-)))^{1\Omega}}{2\omega\Omega}f_{kl}(\mathbf{k}''). \tag{66}$$

A similar parameter appears in the diffusion of light and, since it is positive, it renormalizes the phase velocity to a value that is smaller than the transport velocity [46,47,67,68]. To see that our $a$ is indeed positive, replace Eqs. (37) and (63) into Eq. (66) to obtain

$$a=\frac{-\int_{\mathbf{k}}\mathrm{Im}[\Sigma_{mn}(\mathbf{k})G_{mn}(\mathbf{k})]}{\rho^2(\omega_{r1}^2-\omega^2)\int_{\mathbf{v}}\Delta G_{ii,jj}(\mathbf{v})}, \tag{67}$$

$$\approx\frac{2R_{2T}^{3/2}(c_T^2R_{2T}-\omega^2)+R_{2L}^{3/2}(c_L^2R_{2L}-\omega^2)}{(\omega_F^2-\omega^2)(2R_{2T}^{3/2}+R_{2L}^{3/2})}, \tag{68}$$

where $R_{2L,T}\equiv\mathrm{Re}[K_{L,T}^2]$ and $I_{2L,T}\equiv\mathrm{Im}[K_{L,T}^2]$. The last approximation is obtained in the limit of small $\mathrm{Im}[K_{T,L}^2]$, as explained in Appendix F. Clearly, $a>0$ for wave frequencies $\omega$ smaller that the first fundamental mode of the vibrating stringlike dislocation $\omega<\omega_F$.

### B. Diffusion constant

From Eqs. (55)–(61), the following leading order expression for the singular part of the intensity, $\mathbf{\Phi}^{\text{sing}}$ is obtained:

$$\Phi^{\text{sing}}_{ij,kl}=\frac{f^{\mathbf{r0}}_{ij}(\mathbf{k};\mathbf{q},\Omega)f^{\mathbf{r0}}_{kl}(\mathbf{k}';\mathbf{q},\Omega)}{\lambda^{1\Omega}+\lambda^{2\mathbf{q}}} =\frac{f^{\mathbf{r0}}_{ij}(\mathbf{k};\mathbf{q},\Omega)f^{\mathbf{r0}}_{kl}(\mathbf{k}';\mathbf{q},\Omega)}{\frac{\lambda^{1\Omega}}{-i\Omega}\left(-i\Omega+\frac{-i\Omega\lambda^{2\mathbf{q}}}{\lambda^{1\Omega}q^2}q^2\right)}. \tag{69}$$

Then, using Eqs. (65) and (69), the diffusion constant can be simply read off. It is

$$D\equiv-\frac{i\Omega\lambda^{2\mathbf{q}}}{q^2\lambda^{1\Omega}}, \tag{70}$$

$$\equiv D^{\mathcal{R}}+D_{\Delta G^{1\mathbf{q}}}, \tag{71}$$

with

$$D^{\mathcal{R}}\equiv\frac{\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}}P_{ss,kl}(\mathbf{k};\mathbf{q})\times\int_{\mathbf{k}_2}\Phi_{kl,ij}(\mathbf{k},\mathbf{k}_2)P_{ij,tt}(\mathbf{q};\mathbf{k}_2), \tag{72}$$

$$D_{\Delta G^{1\mathbf{q}}}\equiv-\frac{\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}}P_{ss,kl}(\mathbf{k};\mathbf{q})\Delta G^{1\mathbf{q}}_{kl,tt}(\mathbf{k}). \tag{73}$$

To obtain Eq. (71), in which the diffusion constant is written as the sum of two terms, we have substituted the values for $\lambda^{2\mathbf{q}}$, $\lambda^{1\Omega}$ given by Eqs. (61) and (65). The first one ensued from the form of $f^{1\mathbf{q}}(\mathbf{k})$ (see Appendix E). Thus, the expression for the diffusion constant in Eq. (71) is the sum of two contributions, as defined in Eqs. (72) and (73). The computation, sketched in Appendix F, is laborious but a fairly straightforward generalization of a similar computation carried out in two dimensions for elastic waves diffusing among many edge dislocations [62]. The result is the limit of small $\mathrm{Im}[K_{T,L}^2]$,

$$D^{\text{lead}}\approx\frac{1}{(1+a)}\frac{\left(c_L^4\frac{R_{2L}^{7/2}}{I_{2L}}+2c_T^4\frac{R_{2T}^{7/2}}{I_{2T}}\right)}{3\omega^3(2R_{2T}^{3/2}+R_{2L}^{3/2})}, \tag{74}$$

with $a$ given by Eq. (68). In the limit of small frequencies, this becomes

$$D^{\text{lead}}_{\omega\to0}\approx\left(\frac{v_T^3c_L^4}{(2v_L^3+v_T^3)v_L^4}\frac{v_Ll_L}{3}+\frac{2v_L^3c_T^4}{(2v_L^3+v_T^3)v_T^4}\frac{v_Tl_T}{3}\right), \tag{75}$$

where $v_{T,L}$ and $l_{T,L}$ are the effective velocities and attenuation lengths introduced in Sec. II, Eqs. (11) and (12).

## VI. DISCUSSION

The main result of this paper is expression Eq. (74) for the diffusion coefficient for elastic waves traveling in a continuum elastic medium populated with many, randomly placed and oriented, dislocation segments, and the simpler expression Eq. (75), its value in the limit of low frequencies. It is valid (see below) for frequencies that are not too close to the fundamental string frequency $\omega_F$. It is the sum of two terms, each one characterized by an attenuation length that appears because the imaginary part of the effective wave vector $\mathrm{Im}[K_T^2]$ does not vanish. It has an overall factor $(1+a)$ with $a$ given by Eq. (67). Similar factors have been identified in the diffusion of sound in a layer with a rough interface [66] and of light waves in media with microstructure [68], in association with resonant scattering, as here. Indeed, if $\omega = \omega_F$, the diffusion coefficient vanishes. Having a frequency exactly equal to $\omega_F$, however, takes us outside the domain of validity of the approximations employed in this paper. In any case, it is allowed for the frequency to approach the resonant frequency, and the associated diffusion constant does get smaller. This raises the question of looking more closely at this regime (see below). The aforementioned models [66,68] also allow the possibility of an additional factor $(1+\Delta)$, associated with the extended nature of the scatterers present. Our formalism allows for the presence of this factor as well—it appears in Eq. (F17). In our specific example, however, the analog of $\Delta$ vanishes because we have taken scatterers that are effectively pointlike.

### A. Restrictions placed by approximations employed

#### *1. Long wavelength by comparison with dislocation segment length*

At the outset, in Sec. II we have formulated the wave-dislocation interaction problem in an approximation in which the whole interaction takes place at a single point, the dislocation center, although the specific interaction Eq. (2) does contain the information that the dislocation segment is a vibrating string of length $L$, with a specific eigenfrequency, at which a resonant interaction may occur.

Also, while the coherent behavior of the elastic waves has been studied including internal losses, encapsulated by the constant $B$ in Eq. (3), the diffusion coefficient, Eqs. (74) and (75), considers $B=0$ because our derivation of the diffusion regime necessitates that conservation of energy holds. We expect these internal losses to become significant for frequencies near the first fundamental mode for the string, but not at the low frequencies considered here, where wavelength is large compared to dislocation length.

#### *2.* $|\mathrm{Im}[\mathbf{K}^2_{T,L}]| \ll |\mathbf{k}^2 - \mathrm{Re}[\mathbf{K}^2_{T,L}]|$

This approximation has been repeatedly used in the algebra, with $K_{T,L}$ the transverse ($T$) and longitudinal ($L$) effective wave vectors Eq, (9) characterizing the coherent propagation of waves. Using Eq. (9) for $B=0$, the case with no internal losses for which we have carried out the computations in the ISA, the inequality of this subsection translates into

$$\left|\omega^2-\omega_F^2\right| \gg \frac{1}{\pi^2}\frac{\omega^3}{\omega_F}, \tag{76}$$

so the working frequency $\omega$ can be close, but not equal to, the resonant string frequency $\omega_F$.

#### *3. Independent scattering approximation*

The ISA means that the random variables characterizing the dislocation segments, position, and orientation are statistically independent. It simplifies the computation of statistical averages, keeping only leading order terms in $n$, the number of dislocation segments per unit volume, in Eqs. (47). To have a rough estimate of what this means in terms of dimensionless variables, consider the value of the $t$ matrix at low frequencies [40] and the following inequality results: $nL^3 \ll 1$. That is, the two length scales of the model, $L$ and $n^{-1/3}$, are related by the requirement that the separation among dislocation segments be larger that their length.

### B. Kubo representation for the diffusion constant

We have obtained an explicit form for the diffusion constant of elastic wave energy when traveling through an elastic medium full of vibrating dislocation segments by use of a perturbation approach to the solution of the BS equation, regarded as an eigenvalue problem. In this subsection, we will show that the diffusion constant, given by Eq. (70), admits a Kubo representation similar to that for diffusion of electromagnetic waves [47].

To achieve a Kubo representation for the diffusion constant, we have to focus on the transformation of the $\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k})$ from Eq. (E2). According to Ref. [47], this implies, first, the construction of the equation similar to Eq. (33) but for

$$\begin{aligned}\mathbf{\Phi}^{--}&(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)\\ &\equiv \Phi^{--}_{kl,mn}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)\\ &\equiv \langle G^{-}_{km}(\mathbf{k}^{+},\mathbf{k}'^{+},\omega^{-})G^{-}_{nl}(\mathbf{k}'^{-},\mathbf{k}^{-},\omega^{-})\rangle.\end{aligned} \tag{77}$$

It should be noted that the subtraction trick, briefly mentioned in Sec. IV A, is rather general and can be implemented without loss of generality to get the equation for $\mathbf{\Phi}^{--}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)$. Passing through similar steps, one can obtain

$$\int_{\mathbf{k}}\big((G^{0-})^{-1}_{ni}(\mathbf{k}^{-},\omega_{-})-(G^{0-})^{-1}_{ni}(\mathbf{k}^{+},\omega_{-})\big)\Phi^{--}_{ni,lj}(\mathbf{k},\mathbf{k}''';\mathbf{q},\Omega)+\langle G^{-}(\mathbf{k}'''^{-},\mathbf{k}'''^{-};\omega_{-})_{lj}\rangle-\langle G^{-}(\mathbf{k}'''^{+},\mathbf{k}'''^{+};\omega_{-})_{lj}\rangle\equiv 0 \tag{78}$$

or, at $\Omega\to 0$,

$$\int_{\mathbf{k}}\frac{\partial L_{ni}(\mathbf{k})}{\partial\mathbf{k}}\cdot\mathbf{q}\,\mathrm{Re}[\Phi^{--}_{ni,lj}(\mathbf{k},\mathbf{k}''';\mathbf{q},0)]\equiv \mathrm{Re}[-\langle G^{-}(\mathbf{k}'''^{-},\mathbf{k}'''^{-};\omega)_{lj}\rangle+\langle G^{-}(\mathbf{k}'''^{+},\mathbf{k}'''^{+};\omega)_{lj}\rangle], \tag{79}$$

and, to first order in $\mathbf{q}$, the identity reduces to

$$\frac{\imath}{\rho}\int_{\mathbf{k}}\frac{\partial L_{ni}(\mathbf{k})}{2\partial\mathbf{k}}\cdot\mathbf{q}\mathrm{Re}[\Phi^{--}_{ni,lj}(\mathbf{k},\mathbf{k}''';\mathbf{0},0)]\equiv\Delta G^{1\mathbf{q}}_{lj,mm}(\mathbf{k}'''). \tag{80}$$

Hence, we get for $\mathbf{f}^{1\mathbf{q}}_{kl}(\mathbf{k}'')$ :

$$\mathbf{f}^{1\mathbf{q}}_{kl}(\mathbf{k}'')=-\frac{\imath B}{\rho}\int_{\mathbf{k}_2}\left(\langle G^{+}_{kk_1}(\mathbf{k}'',\mathbf{k}_2;\omega)G^{-}_{l_1l}(\mathbf{k}_2,\mathbf{k}'';\omega)\rangle-\mathrm{Re}\big[\langle G^{-}_{kk_1}(\mathbf{k}'',\mathbf{k}_2;\omega)G^{-}_{l_1l}(\mathbf{k}_2,\mathbf{k}'';\omega)\rangle\big]\right)\frac{\partial L_{k_1l_1}(\mathbf{k}_2)}{2\partial\mathbf{k}_2}\cdot\mathbf{q}. \tag{81}$$

In Eq. (81), we deal with the difference of products of complex numbers that may be symbolically presented in the form

$$X^{*}_{kk_1}X_{l_1l}-\mathrm{Re}\big[X_{kk_1}X_{l_1l}\big]=2\mathrm{Im}[X]_{kk_1}\mathrm{Im}[X]_{l_1l}+\imath\big(\mathrm{Re}[X]_{kk_1}\mathrm{Im}[X]_{l_1l}-\mathrm{Im}[X]_{kk_1}\mathrm{Re}[X]_{l_1l}\big). \tag{82}$$

Using Eqs. (81), (82), and (61), we get

$$\lambda^{2\mathbf{q}}=\frac{2B^2}{\rho^2}\int_{\mathbf{k}}\int_{\mathbf{k}_2}\mathbf{q}\cdot\frac{\partial L_{kl}(\mathbf{k})}{2\partial\mathbf{k}}\big\langle\mathrm{Im}\big[G^{-}_{kk_1}(\mathbf{k},\mathbf{k}_2;\omega)\big]\mathrm{Im}\big[G^{-}_{l_1l}(\mathbf{k}_2,\mathbf{k};\omega)\big]\big\rangle\frac{\partial L_{k_1l_1}(\mathbf{k}_2)}{2\partial\mathbf{k}_2}\cdot\mathbf{q}, \tag{83}$$

so, from Eqs. (65), (70), and (83) the diffusion constant reads

$$D=\frac{-2B^2}{\rho^2q^2\omega(1+a)}\int_{\mathbf{k}}\int_{\mathbf{k}_2}\mathbf{q}\cdot\frac{\partial L_{kl}(\mathbf{k})}{2\partial\mathbf{k}}\big\langle\mathrm{Im}\big[G^{-}_{kk_1}(\mathbf{k},\mathbf{k}_2;\omega)\big]\mathrm{Im}\big[G^{-}_{l_1l}(\mathbf{k}_2,\mathbf{k};\omega)\big]\big\rangle\frac{\partial L_{k_1l_1}(\mathbf{k}_2)}{2\partial\mathbf{k}_2}\cdot\mathbf{q}, \tag{84}$$

which is the desired Kubo representation.

### C. Transport equation approach and equipartition of energy

Ryzhik *et al.* [59] studied the transport of elastic energy density in a random medium. They showed that diffusive behavior occurs on long time and distance scales, and they have determined a diffusion coefficient. They, however, dealt with continuous random media and not, as in our case, with discrete scatterers that are randomly distributed in a medium. It is still of interest to compare our result Eq. (75) with the value they give for the diffusion constant, which is their Eq. (5.46) (in their notation):

$$D^{\mathrm{el}}=\frac{1}{\left(2/v_S^3+1/v_P^3\right)}\left(\frac{l_Pv_P}{3v_P^3}+\frac{2l_Sv_S}{3v_S^3}\right). \tag{85}$$

Here $P$ means primary, or longitudinal ($L$) in our language, and $S$ means secondary, or transverse ($T$) in our case. The quantities $l_P$ and $l_S$ are longitudinal and transverse mean-free paths that are determined by unspecified scattering cross sections. We find there is a strong resemblance to Eq. (75). One important difference, however, is that Eq. (75), based as it is on a solution to the BS equation, involves not one phase velocity for each polarization but two: the velocity in the absence of scatterers and the velocity of coherent waves in the presence of scatterers. The latter quantity appears because of the relation between mass operator and irreducible kernel provided by the WTI. These considerations are absent in a transport equation approach. Both approaches coincide, however, in the limit of a very small density of dislocations, in which case $v_{L,T}\approx c_{L,T}$.

Ryzhik *et al.* [59] also noted that, in their diffusive limit, the energy of elastic waves is "equipartitioned" in the sense that, if $\mathcal{E}_L$ (respectively, $\mathcal{E}_T$) is the longitudinal (respectively, transverse) energy density so the total energy $\mathcal{E}=\mathcal{E}_T+\mathcal{E}_L$, then

$$\frac{\mathcal{E}_T}{\mathcal{E}_L}=2\gamma^3. \tag{86}$$

Earlier, Weaver [69] obtained this result, taking as the definition of the diffuse field a state in which energy is equipartitioned among all normal modes available to the elastic solid, and using the Debye density of states to compute the ratio between longitudinal and transverse modes.

In our formulation, the diffuse field energy tensor is defined by

$$\mathcal{E}(\mathbf{q},\Omega)_{ij,kl}=\lim_{\mathbf{q}\to\mathbf{0},\Omega\to 0}\int_{\mathbf{k}}\int_{\mathbf{k}'}\Phi_{ij,kl}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega). \tag{87}$$

It is a straightforward calculation, using the solution Eq. (56) to lowest order, Eqs. (57) and (63), to show that

$$\mathcal{E}(\mathbf{0},\Omega)_{ij,kl}=\frac{i}{\Omega}\frac{\delta_{ij}\delta_{kl}}{36\pi\rho^2\omega^3}\frac{\left[2\mathrm{Re}\big[K_T^2\big]^{3/2}+\mathrm{Re}\big[K_L^2\big]^{3/2}\right]}{(1+a)}, \tag{88}$$

$$\longrightarrow\frac{i}{\Omega}\frac{\delta_{ij}\delta_{kl}}{36\pi\rho^2}\left(\frac{2}{c_T^3}+\frac{1}{c_L^3}\right), \tag{89}$$

where the last limit is obtained when the density of dislocations is very small. Note that, in general, the diffuse energy density does not split into a sum of longitudinal and transverse terms, because of the $(1+a)$ denominator which, as we have discussed, is a consequence of the timescale introduced into the problem by the fundamental mode of the vibrating strings that are doing the scattering of the elastic waves.

Additional insight into these results can be obtained noting that, using the result Eq. (6) for the coherent Green's function, it is straightforward to verify that, in the limit $|\mathrm{Im}[K^2_{T,L}]|\ll|k^2-\mathrm{Re}[K^2_{T,L}]|$ already discussed in previous sections,

$$\begin{aligned}\mathrm{Tr}[\mathrm{Im}[\langle\mathbf{G}\rangle^{+}(\mathbf{k},\omega)]]&=-\Delta G_{ii,mm}(\mathbf{k},\omega)\\&\approx\frac{\pi k^2}{\rho\omega^2}\big(2\delta\big(k^2-\mathrm{Re}\big[K_T^2\big]\big)\\&\quad+\delta\big(k^2-\mathrm{Re}\big[K_L^2\big]\big)\big).\end{aligned} \tag{90}$$

Now, if we consider the diffusive energy as being carried by the coherent waves whose states are labeled by three polarizations and three real numbers, the components of a wave vector $\mathbf{k}$, we see that

$$g_{T,L}\big(\mathrm{Re}\big[K_{T,L}^2\big]\big) = \sum_{\mathbf{k}} \delta\big(k^2 - \mathrm{Re}\big[K_{T,L}^2\big]\big) \quad (91)$$

counts the number of states that have the same $\mathrm{Re}[K_{T,L}^2]$, and

$$g_{T,L}(\omega) = \frac{1}{V} g_{T,L}\big(\mathrm{Re}\big[K_{T,L}^2\big]\big) \frac{\partial \mathrm{Re}\big[K_{T,L}^2\big]}{\partial \omega} = \frac{1}{6\pi^2} \frac{\partial\big(\mathrm{Re}\big[K_{T,L}^2\big]^{3/2}\big)}{\partial \omega} \quad (92)$$

is the density of states per unit frequency $\omega$ and unit volume $V$. The second equality follows from Eq. (91). The ratio of transverse states to longitudinal states is then

$$\frac{2g_T(\omega)}{g_L(\omega)} = \frac{2\frac{\partial\left(\mathrm{Re}\left[K_T^2\right]^{3/2}\right)}{\partial\omega}}{\frac{\partial\left(\mathrm{Re}\left[K_L^2\right]^{3/2}\right)}{\partial\omega}} \longrightarrow 2\gamma^3, \quad (93)$$

where the limiting behavior is obtained for a small density of dislocations. We see that, in general, diffuse energy density, given by Eq. (88), at a given frequency is not proportional to the density of states at that same frequency, given by Eq. (92). However, said proportionality (equipartition) is recovered in the limit of very few dislocations.

### D. Low-frequency behavior of the diffusion coefficient

As noted in Sec. VI A 3, our ISA approximation allows us to keep terms that are linear in the density of scatterers $n$ only. However, we can look at what happens to the diffusion coefficient Eq. (75) as a function of frequency, for low frequencies, $\omega \ll \omega_F$. In this case, Eq. (9) leads to

$$\mathrm{Re}[K_{L,T}] \approx \frac{\omega}{c_{L,T}}\left[1 + \frac{8nLc_T^4\mathcal{C}_{L,T}\rho b^2}{10\pi^2 c_{L,T}^2 m\omega_F^2}\left(1 + \frac{\omega^2}{\omega_F^2} + \dots\right)\right], \quad (94)$$

$$\mathrm{Im}[K_{L,T}] \approx \frac{16nL^2\omega^4\mathcal{C}_{L,T}c_T^3}{75c_{L,T}^3\pi^5\omega_F^4}\left(\frac{\rho b^2}{m}\right)^2 \times \left(\frac{3\gamma^5+2}{\gamma^5}\right)\left(1 + 2\frac{\omega^2}{\omega_F^2} + \dots\right), \quad (95)$$

where $\mathcal{C}_L = 4/3$, $\mathcal{C}_T = 1$. Substitution of Eqs. (11) and (12) into (75) yields

$$D_{\omega\to 0}^{\mathrm{lead}} \approx \frac{\gamma^3(3\gamma+8)}{4(1+2\gamma^3)}\frac{c_T l_T}{3} \approx \frac{25\pi^5\gamma^3(3\gamma+8)c_T}{128(1+2\gamma^3)}\left(\frac{m}{\rho b^2}\right)^2 \times \left(\frac{\gamma^5}{3\gamma^5+2}\right)\frac{\omega_F^4}{nL^2\omega^4}\left(1 - 2\frac{\omega^2}{\omega_F^2} + \cdots\right). \quad (96)$$

We see that the attenuation length, and consequently the diffusion coefficient, are inversely proportional to frequency to the fourth power, indicating that there is a diffusion that is due to loss of coherence originating in Rayleigh scattering, as it should for long wavelengths and in the absence of internal losses. Also, it is inversely proportional to dislocation density $n$ and length $L$, as one would expect.

## VII. CONCLUSIONS AND OUTLOOK

We have studied the diffusive behavior of elastic waves in a continuum that is populated by many edge-dislocation segments of length $L$, pinned at their ends. Their position is random, as well as the orientation of their tangent and Burgers vectors. The dislocations are modeled as elastic strings with internal losses and are dynamical objects in their own right. The study relies on the existence of a regime where coherent wave behavior occurs, previously studied [40]. The elastic waves are assumed to be monochromatic, with a frequency that is small compared to the first resonant frequency of the stringlike pinned dislocations and computations are actually carried out in an ISA, that is, when the random variables, position, and orientation, characterizing the dislocations, are statistically independent. In this case, the coherent wave has an effective velocity and an attenuation that are, to leading order, proportional to the number $n$ of dislocation segments per unit volume, the small dimensionless parameter being $nL^3$.

The diffusion behavior is studied using a BS equation, supplemented by a WTI. Both equations hold in the presence of internal losses by the strings. However, to use the ISA, a necessary requirement for the actual computation of a diffusion coefficient, it is necessary to assume that these losses vanish. If this were not the case, the diffusive behavior would be influenced not only by the incoherent diffusion induced by the disordered dislocation segments but also by a decay induced by the internal losses. It should be of interest to explore this regime, especially in view of the possible experimental measurements of the diffusion reported here.

Alternatively, one may ask about the origin of the internal losses. If they are due to inelastic scattering of the dislocation with phonons, a complete calculation of the phonon-dislocation interaction has been recently carried out [27] for phonons of arbitrary frequency. That is, without the requirement that their wavelength be long compared to dislocation length $L$. It should be of interest then to explore a BS equation and attendant WTI in this case, since the inelastic effects would be explicitly considered from the very beginning.

A study of the diffusion problem without the restriction of dislocation lengths small compared to wavelength would have the added benefit to clarify the role played by the vibrating string resonances. As indicated in the previous section, the diffusion constant that has been computed in the present paper can, formally, vanish when the wave frequency coincides with the resonant frequency. A similarly strong effect that resonances can have upon the diffusion of light has been considered by Lubatsch *et al.* [70]. This regime is outside the frame of approximations employed to carry out our computations, however, it would be of interest in the future to explore in some detail the actual behavior of the diffusion coefficient for frequencies comparable to the resonant string frequency.

It should be possible to experimentally verify our results, for example, the low-frequency approximation to the diffusion coefficient, Eq. (96), using setups as those described by Weaver *et al.* [71] using aluminum, or by Sotelo *et al.* [72]

using stainless steel. It should suffice to consider samples that differ only in dislocation content, such as can be achieved by cold rolling or annealing. This approach has been successfully followed to experimentally verify the influence of dislocations on the coherent propagation of elastic waves [35,37,38].

The continuum mechanics approach employed in the present paper has the advantage of being applicable to any homogeneous solid material at all length scales down to several interatomic spacings. This is true even if the atomic structure does not have long-range order, and it has been established [57] that the coherent wave behavior already alluded to provides an adequate understanding of the behavior of amorphous materials in the THz range. Recently, Beltukov *et al.* [58] performed a numerical study of wave packet behavior in amorphous silicon, and detected a transition from propagating to diffusive regimes, depending on the frequency of the waves. This phenomenology is relevant to the understanding of heat transport in amorphous solids, one of the significant unknowns in contemporary condensed-matter physics, and it looks tempting to apply the methods presented in this paper to try and elucidate the nature of heat propagation in glasses.

## ACKNOWLEDGMENT

This work was supported in part by Fondecyt Grant No. 1191179.

## APPENDIX A: BETHE-SALPETER EQUATION

The key idea of the BS equation is the existence of an analogy of the Dyson equation for the intensity $\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle$. To get it explicitly, we use the representations

$$\begin{aligned}\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle &= \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle + (\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle - \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle)\\ &= \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle + \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle : \langle \mathbf{G}^+ \rangle^{-1} \otimes \langle \mathbf{G}^- \rangle^{-1}\\ &: (\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle - \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle) : \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle^{-1} : \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle\\ &= \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle + \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle : \left(\langle \mathbf{G}^+ \rangle^{-1} \otimes \langle \mathbf{G}^- \rangle^{-1} - \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle^{-1}\right) : \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle. \end{aligned} \tag{A1}$$

From the last equality in Eq. (A1), it is easy to introduce the pole structure for the intensity by defining the irreducible vertex $\mathbf{K}$ as

$$\mathbf{K} = \langle \mathbf{G}^+ \rangle^{-1} \otimes \langle \mathbf{G}^- \rangle^{-1} - \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle^{-1}. \tag{A2}$$

The BS equation in the form Eq. (A2) clearly corroborates the pole specificity of the intensity $\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle$ in the sense that the $\mathbf{K}$ plays the same role as the self-energy $\mathbf{\Sigma}$ for both averaged $\langle \mathbf{G} \rangle$ and free medium $\mathbf{G}_0$ Green's tensors in the Dyson equation:

$$\mathbf{\Sigma} = \mathbf{G}_0^{-1} - \langle \mathbf{G} \rangle^{-1}. \tag{A3}$$

Replacing Eq. (A2) into the last equality of Eq. (A1), the BS equation takes the widely accepted form

$$\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle = \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle + \langle \mathbf{G}^+ \rangle \otimes \langle \mathbf{G}^- \rangle : \mathbf{K} : \langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle. \tag{A4}$$

If we now define the Fourier transforms as [41]

$$\begin{aligned} G^+_{i_1 m_1}(\mathbf{x}_1, \mathbf{x}_1'; \omega^+) &= \int_{\mathbf{k}_1} \int_{\mathbf{k}_1'} e^{\imath \mathbf{k}_1 \mathbf{x}_1} G^+_{i_1 m_1}(\mathbf{k}_1, \mathbf{k}_1'; \omega^+) e^{-\imath \mathbf{k}_1' \mathbf{x}_1'},\\ G^-_{i_2 m_2}(\mathbf{x}_2, \mathbf{x}_2'; \omega^-) &= \int_{\mathbf{k}_2} \int_{\mathbf{k}_2'} e^{-\imath \mathbf{k}_2 \mathbf{x}_2} G^-_{i_2 m_2}(\mathbf{k}_2', \mathbf{k}_2; \omega^-) e^{\imath \mathbf{k}_2' \mathbf{x}_2'}, \end{aligned} \tag{A5}$$

then

$$\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle = \int_{\mathbf{k}} \int_{\mathbf{k}'} \int_{\mathbf{q}} \mathbf{\Phi}(\mathbf{k}, \mathbf{k}'; \mathbf{q}, \Omega) e^{\imath (\mathbf{k}\mathbf{r} - \mathbf{k}'\mathbf{r}' + \mathbf{q}(\mathbf{R} - \mathbf{R}'))}. \tag{A6}$$

With space and momentum variables being specified as

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{R} + \frac{\mathbf{r}}{2}, & \mathbf{x}_2 &= \mathbf{R} - \frac{\mathbf{r}}{2},\\ \mathbf{x}_1' &= \mathbf{R}' + \frac{\mathbf{r}'}{2}, & \mathbf{x}_2' &= \mathbf{R}' - \frac{\mathbf{r}'}{2},\\ \mathbf{k}_1 &= \mathbf{k}^+ = \mathbf{k} + \frac{\mathbf{q}}{2}, & \mathbf{k}_1' &= \mathbf{k}'^+ = \mathbf{k}' + \frac{\mathbf{q}}{2},\\ \mathbf{k}_2' &= \mathbf{k}'^- = \mathbf{k}' - \frac{\mathbf{q}}{2}, & \mathbf{k}_2 &= \mathbf{k}^- = \mathbf{k} - \frac{\mathbf{q}}{2}. \end{aligned} \tag{A7}$$

and applying the inverse Fourier transform [48]

$$\int d(\mathbf{R}-\mathbf{R}')d\mathbf{r}d\mathbf{r}'e^{-\imath(\mathbf{kr}-\mathbf{k}'\mathbf{r}'+\mathbf{q}(\mathbf{R}-\mathbf{R}'))} \tag{A8}$$

to Eq. (A4), the BS equation in momentum space is obtained:

$$\mathbf{\Phi}(\mathbf{k},\mathbf{k}';\mathbf{q},\Omega)=\langle\mathbf{G}^+\rangle\otimes\langle\mathbf{G}^-\rangle(\mathbf{k};\mathbf{q},\Omega)\delta_{\mathbf{k},\mathbf{k}'}+\langle\mathbf{G}^+\rangle\otimes\langle\mathbf{G}^-\rangle(\mathbf{k};\mathbf{q},\Omega):\mathbf{K}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega):\mathbf{\Phi}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega), \tag{A9}$$

where $\delta_{\mathbf{k},\mathbf{k}'}=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$ and the internal momentum variables, i.e., $\mathbf{k}''$, are integrated over. To modify further Eq. (A9) to its kinetic form, we use the following identity for the outer product of the averaged Green's tensors:

$$(\langle\mathbf{G}^+\rangle^{-1}\otimes\mathbf{I}-\mathbf{I}\otimes\langle\mathbf{G}^-\rangle^{-1}):\langle\mathbf{G}^+\rangle\otimes\langle\mathbf{G}^-\rangle=\mathbf{I}\otimes\langle\mathbf{G}^-\rangle-\langle\mathbf{G}^+\rangle\otimes\mathbf{I}, \tag{A10}$$

where $\mathbf{I}$ is a unit tensor. Acting from the left on Eq. (A9) with the tensor $(\langle\mathbf{G}^+\rangle^{-1}\otimes\mathbf{I}-\mathbf{I}\otimes\langle\mathbf{G}^-\rangle^{-1})$ and using the property Eq. (A10), the following relation is obtained:

$$(\langle\mathbf{G}^+\rangle^{-1}\otimes\mathbf{I}-\mathbf{I}\otimes\langle\mathbf{G}^-\rangle^{-1}):\mathbf{\Phi}=(\mathbf{I}\otimes\langle\mathbf{G}^-\rangle-\langle\mathbf{G}^+\rangle\otimes\mathbf{I}):(\mathbf{I}\otimes\mathbf{I}\delta_{\mathbf{k},\mathbf{k}'}+\mathbf{K}(\mathbf{k},\mathbf{k}'';\mathbf{q},\Omega):\mathbf{\Phi}(\mathbf{k}'',\mathbf{k}';\mathbf{q},\Omega)). \tag{A11}$$

Finally, substituting Eqs. (A3), (16), and (18) into (A11) as well as the explicit form of the Green's tensor for the free medium [39], we obtain the BS equation in the form of Eq. (15).

## APPENDIX B: INTEGRATION OVER SOLID ANGLES IN 3D

The developed approach requires evaluation of the following integrals over a $n$-dimensional solid angle $\Omega^n$, comprised of the product of radial unit $n$-dimensional vectors $\hat{r}$ ($\hat{r}^2=1$):

$$I^{nk}=\int d\Omega_{\hat{r}}^{(n)}\hat{r}^{i_1}\cdots\hat{r}^{i_k}. \tag{B1}$$

In a previous paper [62], we were interested in the diffusion of waves in a two-dimensional continuum. Now we have a problem in three dimensions and we are led to the evaluation of integrals

$$\begin{aligned} I^{32}&=\int d\Omega_{\hat{\mathbf{r}}}^{(3)}\hat{r}^{i_1}\hat{r}^{i_2},\\ I^{34}&=\int d\Omega_{\hat{\mathbf{r}}}^{(3)}\hat{r}^{i_1}\hat{r}^{i_2}\hat{r}^{i_3}\hat{r}^{i_4}, \end{aligned} \tag{B2}$$

with $\Omega^{(3)}=4\pi$, $d\Omega_{\hat{r}}^{(3)}=\sin\theta d\theta d\phi$ and $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$ are azimuthal and polar angles of a 3D spherical frame. To do this, we use the results of Ref. [73], according to which the tensor integral of the product of $k$ radial unit $n$-dimensional vectors

$$\langle\hat{r}^{i_1}\cdots\hat{r}^{i_k}\rangle_{\hat{r}}=\frac{1}{\Omega^{(n)}}\int d\Omega_{\hat{\mathbf{r}}}^{(n)}\hat{r}^{i_1}\cdots\hat{r}^{i_k}=\frac{I^{nk}}{\Omega^{(n)}} \tag{B3}$$

vanishes when $k$ is odd, and is equal to a totally symmetric isotropic tensor when it is even,

$$\langle\hat{r}^{i_1}\cdots\hat{r}^{i_{2k}}\rangle_{\hat{r}}=\tilde{\mathcal{L}}_{(2k)}^{i_1\cdots i_{2k}}, \tag{B4}$$

that is, defined recursively,

$$\begin{aligned} \tilde{\mathcal{L}}_{(2k)}^{i_1\cdots i_{2k}}=&\frac{1}{n+2k-2}\big(\delta_{i_1i_2}\tilde{\mathcal{L}}_{(2k-2)}^{i_3\cdots i_{2k}}\\ &+\delta_{i_1i_3}\tilde{\mathcal{L}}_{(2k-2)}^{i_2i_4\cdots i_{2k}}+\cdots+\delta_{i_1i_{2k}}\tilde{\mathcal{L}}_{(2k-2)}^{i_2\cdots i_{2k-1}}\big), \end{aligned} \tag{B5}$$

with initial condition $\tilde{\mathcal{L}}_0=1$.

These formulas provide us with the values we need for the integrals in Eq. (B2):

$$\begin{aligned} \langle\hat{r}^i\hat{r}^j\rangle&=\frac{\delta_{ij}}{n}=\frac{I^{32}}{4\pi},\\ \langle\hat{r}^i\hat{r}^j\hat{r}^k\hat{r}^l\rangle&=\frac{1}{n+2}\big(\delta_{ij}\tilde{\mathcal{L}}_{(2)}^{kl}+\delta_{ik}\tilde{\mathcal{L}}_{(2)}^{jl}+\delta_{il}\tilde{\mathcal{L}}_{(2)}^{jk}\big)\\ &=\frac{1}{n(n+2)}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\\ &=\frac{I^{34}}{4\pi}, \end{aligned} \tag{B6}$$

where $n=3$ in Eq. (B6) for three dimensions, the case of interest here. It should be noted that the meaning of an averaging symbol $\langle\,\rangle$ is a bit different from the orientation averaging in the main text. The latter suggests averaging that includes integration over three Euler angles, whereas the former is just averaging over a solid angle defined by two angles of a spherical frame. At some limiting cases, the integration over Euler angles might be reduced to the integration over spherical angles only.

## APPENDIX C: OPTICAL THEOREM

We need to show that Eq. (46),

$$\begin{aligned} &\big(\Sigma_{ij}^*(\mathbf{k})-\Sigma_{ij}(\mathbf{k})\big)\\ &\quad=\int_{\mathbf{k}_1}\big(G^{0*}(\mathbf{k}_1)_{mn}-G^0(\mathbf{k}_1)_{mn}\big)K_{mn,ij}(\mathbf{k}_1,\mathbf{k}), \end{aligned} \tag{C1}$$

holds, in the ISA, when $B=0$. In this case, the mass and irreducible vertex operators are related to the $t$ matrix by Eq. (47) and the $t$ matrix itself is given by Eq. (29) from Ref. [40]. We have then, for the left-hand side,

$$\begin{aligned} \big(\Sigma_{ij}^*(\mathbf{k})-\Sigma_{ij}(\mathbf{k})\big)&=2\mathrm{in}\mathrm{Im}\left[\frac{\mathcal{A}}{1+\mathcal{A}I}\right]\langle\mathsf{M}_{ik}\mathsf{M}_{lj}\rangle k_kk_l\\ &=\frac{-2\mathrm{in}\mathcal{A}^2\mathrm{Im}[I]}{[1+\mathcal{A}I][1+\mathcal{A}I]^*}\langle\mathsf{M}_{ik}\mathsf{M}_{lj}\rangle k_kk_l, \end{aligned} \tag{C2}$$

and, for the right-hand side,

$$\int_{\mathbf{k}_1}(G^{0*}(\mathbf{k}_1)_{mn}-G^0(\mathbf{k}_1)_{mn})K_{mn,ij}(\mathbf{k}_1,\mathbf{k}) \tag{C3}$$

$$=\frac{-\text{in}}{4\pi^2}\int_{k_1}\int_{\hat{\mathbf{k}}_1}k_1^4\left(\frac{\delta(k_1^2-k_T^2)}{\rho c_T^2}\delta_{mn}\hat{k}_{1s}\hat{k}_{1t}+\left(\frac{\delta(k_1^2-k_L^2)}{\rho c_L^2}-\frac{\delta(k_1^2-k_T^2)}{\rho c_T^2}\right)\hat{k}_{1m}\hat{k}_{1n}\hat{k}_{1s}\hat{k}_{1t}\right)\frac{\mathcal{A}}{1+\mathcal{A}I}\left(\frac{\mathcal{A}}{1+\mathcal{A}I}\right)^*\langle \mathsf{M}_{ms}\mathsf{M}_{ki}\mathsf{M}_{jl}\mathsf{M}_{tn}\rangle k_k k_l$$

$$\equiv\frac{-2in\mathcal{A}^2\text{Im}[I]}{[1+\mathcal{A}I][1+\mathcal{A}I]^*}\langle \mathsf{M}_{ki}\mathsf{M}_{jl}\rangle k_k k_l, \tag{C4}$$

which coincides with the left-hand side given by Eq. (C2). We have used results of Appendix B, properties of tensor M, as well as the explicit expressions for tensors, which are included into Eq. (C1). This calculation, being three-dimensional, differs from the analogous computation carried out in Ref. [62] in two dimensions.

## APPENDIX D: PERTURBATION SCHEME FOR THE SPECTRAL PROBLEM

To build up the system of equations for the determination of the diffusive pole structure, we have to substitute the series from Eqs. (57) into Eq. (53) and gather together all terms of the same order, either in $\Omega$ or in $\mathbf{q}$. Moreover, we assume that at every order of the perturbation scheme both WTI from Eq. (36) and symmetry constraints from Eq. (52) are valid. This yields (omitting the $\Omega$ and $\mathbf{q}$ arguments, as well as indices for brevity)

$$\int_{\mathbf{k}''}(\mathbf{H}(\mathbf{k},\mathbf{k}'')+\mathbf{H}^{1\Omega}(\mathbf{k},\mathbf{k}'')+\mathbf{H}^{1\mathbf{q}}(\mathbf{k},\mathbf{k}'')+\mathbf{H}^{2\mathbf{q}}(\mathbf{k},\mathbf{k}'')+\dots)(\mathbf{f}(\mathbf{k}'')+\mathbf{f}^{1\Omega}(\mathbf{k}'')+\mathbf{f}^{1\mathbf{q}}(\mathbf{k}'')+\mathbf{f}^{2\mathbf{q}}(\mathbf{k}'')+\dots)$$
$$=(\lambda^{1\Omega}+\lambda^{1\mathbf{q}}+\lambda^{2\mathbf{q}}+\dots)(\mathbf{f}(\mathbf{k})+\mathbf{f}^{1\Omega}(\mathbf{k})+\mathbf{f}^{1\mathbf{q}}(\mathbf{k})+\mathbf{f}^{2\mathbf{q}}(\mathbf{k})+\dots). \tag{D1}$$

At first order in $\Omega$ and zero order in $\mathbf{q}$, Eq. (D1) easily leads to Eqs. (58) and (59) in the text. In a similar manner, collecting the first order in $\mathbf{q}$ terms from Eq. (D1), we obtain the following equation for $\lambda^{1\mathbf{q}}$:

$$\int_{\mathbf{k}''}(\mathbf{H}(\mathbf{k},\mathbf{k}'')\mathbf{f}^{1\mathbf{q}}(\mathbf{k}'')+\mathbf{H}^{1\mathbf{q}}(\mathbf{k},\mathbf{k}'')\mathbf{f}(\mathbf{k}''))=\lambda^{1\mathbf{q}}\mathbf{f}(\mathbf{k}). \tag{D2}$$

Integrating Eq. (D2) over $\mathbf{k}$ and subsequently summing over the external indices cancels the contribution from the first term on its left-hand side because of the WTI, so using (63) we have

$$\int_{\mathbf{k}}\int_{\mathbf{k}''}H^{1\mathbf{q}}_{ii,kl}(\mathbf{k},\mathbf{k}'')\Delta G_{kl,mm}(\mathbf{k}'')=\lambda^{1\mathbf{q}}\int_{\mathbf{k}}\Delta G_{ii,mm}(\mathbf{k}). \tag{D3}$$

The left-hand side of Eq. (D3) is equal to zero because of the WTI written to first order in $\mathbf{q}$, as well as the odd in $\mathbf{k}$ character of the tensor $P_{ii,kl}$ defined in Eq. (22). Therefore, we obtain

$$\lambda^{1\mathbf{q}}=0. \tag{D4}$$

To complete the set of equations for the reconstruction of $\lambda_0(\mathbf{q},\Omega)$, we need $\lambda^{2\mathbf{q}}$. To second order in $\mathbf{q}$, Eq. (D1) gives

$$\int_{\mathbf{k}''}(\mathbf{H}(\mathbf{k},\mathbf{k}'')\mathbf{f}^{2\mathbf{q}}(\mathbf{k}'')+\mathbf{H}^{1\mathbf{q}}(\mathbf{k},\mathbf{k}'')\mathbf{f}^{1\mathbf{q}}(\mathbf{k}'')+\mathbf{H}^{2\mathbf{q}}(\mathbf{k},\mathbf{k}'')\mathbf{f}(\mathbf{k}''))=\lambda^{2\mathbf{q}}\mathbf{f}(\mathbf{k}). \tag{D5}$$

Then, Eq. (61) of the text is obtained integrating Eq. (D5) over $\mathbf{k}$, summing over the external indices and using the explicit form of the WTI at corresponding orders.

## APPENDIX E: SOLUTION FOR $\mathbf{f}^{1\mathbf{q}}(\mathbf{k})$

$f^{1\mathbf{q}}(\mathbf{k})$ is obtained by replacing Eq. (63) into Eq. (59), using the symmetry property from Eq. (52), applying the WTI to $H^{1\mathbf{q}}(\mathbf{k},\mathbf{k}'')$, and substituting $\delta_{\mathbf{k}'',\mathbf{k}}\Delta G_{kl,mm}(\mathbf{k})$ by its value given by Eq. (15) to get

$$\int_{\mathbf{k}''}H^{1\mathbf{q}}_{ii,kl}(\mathbf{k},\mathbf{k}'')\mathcal{B}\Delta G_{kl,mm}(\mathbf{k}'')=\int_{\mathbf{k}''}\mathcal{B}\big(P_{ii,kl}(\mathbf{q};\mathbf{k}'')\delta_{\mathbf{k}'',\mathbf{k}}\Delta G_{kl,mm}(\mathbf{k})-H_{ii,kl}(\mathbf{k},\mathbf{k}'')\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k}'')\big)$$
$$=\mathcal{B}\int_{\mathbf{k}''}H_{ii,kl}(\mathbf{k},\mathbf{k}'')\left[\int_{\mathbf{k}_2}\Phi_{kl,k_1l_1}(\mathbf{k}'',\mathbf{k}_2)P_{k_1l_1,ii}(\mathbf{q};\mathbf{k}_2)-\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k}'')\right].$$

Hence,

$$f^{1\mathbf{q}}_{kl}(\mathbf{k}'')=-\mathcal{B}\left(\int_{\mathbf{k}_2}\Phi_{kl,k_1l_1}(\mathbf{k}'',\mathbf{k}_2)P_{k_1l_1,ii}(\mathbf{q};\mathbf{k}_2)-\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k}'')\right), \tag{E1}$$

with

$$\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k}) = \mathbf{q}\cdot\frac{\partial\Delta G_{kl,mm}(\mathbf{k};\mathbf{q}',0)}{\partial\mathbf{q}'}|_{\mathbf{q}'=0} \tag{E2}$$

$$= -\frac{1}{2\imath\rho}\mathbf{q}\cdot\frac{\partial(\mathrm{Re}[G_{kl}(\mathbf{k})])}{\partial\mathbf{k}} \tag{E3}$$

$$= -\frac{q_t\mathrm{Re}[G_L-G_T]}{2\imath\rho}\frac{\partial P_{\hat{\mathbf{k}}}}{\partial k_t} - \frac{q_t}{2\imath\rho}\left(\frac{\partial(\mathrm{Re}[G_T])}{\partial k_t}(\mathbf{I}-P_{\hat{\mathbf{k}}}) + \frac{\partial(\mathrm{Re}[G_L])}{\partial k_t}P_{\hat{\mathbf{k}}}\right) \tag{E4}$$

and

$$\frac{\partial P_{\hat{\mathbf{k}}}}{\partial k_t} = \frac{\partial\left(\frac{k_k k_l}{k^2}\right)}{\partial k_t} = \left(\frac{k_l\delta_{kt}+k_k\delta_{lt}}{k^2}\right) - \frac{2k_k k_l k_t}{k^4} \tag{E5}$$

$$\mathrm{Re}[G_{T,L}] = \frac{F_{T,L}(\omega,k)}{\rho\omega^2\mathrm{Im}\left[K^2_{T,L}\right]}\left(\mathrm{Re}\left[K^2_{T,L}\right]\left(k^2-\mathrm{Re}\left[K^2_{T,L}\right]\right) - \mathrm{Im}\left[K^2_{T,L}\right]^2\right) \tag{E6}$$

$$\frac{\partial(\mathrm{Re}[G_{T,L}])}{\partial k_t} = \frac{2k_t F_{T,L}(\omega,k)}{\rho\omega^2\mathrm{Im}\left[K^2_{T,L}\right]}\left(2k^2F_{T,L}(\omega,k)\mathrm{Im}\left[K^2_{T,L}\right] - \mathrm{Re}\left[K^2_{T,L}\right]\right) \tag{E7}$$

$$F_{T,L}(\omega,k) = \left(\frac{\mathrm{Im}\left[K^2_{T,L}\right]}{\left(k^2-\mathrm{Re}\left[K^2_{T,L}\right]\right)^2+\mathrm{Im}\left[K^2_{T,L}\right]^2}\right). \tag{E8}$$

## APPENDIX F: CALCULATION OF $D$

The calculation reported herein follows very closely an analogous computation in two dimensions [62], for which the reader is referred for a more detailed presentation. As we noted in Eq. (71), the diffusion constant is the sum of two terms: $D = D^{\mathcal{R}} + D_{\Delta G^{1\mathbf{q}}}$, and we sketch how to compute each term.

### 1. $D_{\Delta G^{1\mathbf{q}}}$

Using Eqs. (10), (E2), and (E5), Eq. (73) turns into

$$\begin{aligned} D_{\Delta G^{1\mathbf{q}}} &= \frac{-\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}} P_{ii,kl}(\mathbf{k};\mathbf{q})\Delta G^{1\mathbf{q}}_{kl,mm}(\mathbf{k}) \\ &= \frac{\mathcal{B}^2}{4\rho^2q^2\omega(1+a)}\int_{\mathbf{k}} q_s\frac{\partial L_{kl}(\mathbf{k})}{\partial k_s}\frac{\partial(\mathrm{Re}[G^-_{kl}(\mathbf{k})])}{\partial k_t}q_t \\ &= \frac{-\mathcal{B}^2q_sq_t\left(c_L^2-c_T^2\right)}{2\rho q^2\omega(1+a)}\int_{\mathbf{k}}\left(\mathrm{Re}[G_L-G_T]\left(\delta_{st}-\frac{k_sk_t}{k^2}\right)\right) + \frac{-B^2q_sq_t}{2\rho q^2\omega(1+a)}\int_{\mathbf{k}}\left(2c_T^2\frac{\partial(\mathrm{Re}[G_T])}{\partial k_t}+c_L^2\frac{\partial(\mathrm{Re}[G_L])}{\partial k_t}\right)k_s. \end{aligned} \tag{F1}$$

The following two types of integrals have to be considered in Eq. (F1):

$$\mathbb{I}^{st}_{T,L} = \int_{\mathbf{k}}\mathrm{Re}[G_{T,L}]\left(\delta_{st}-\frac{k_sk_t}{k^2}\right) = \frac{\delta_{st}}{3\pi^2}\int_{-\infty}^{\infty}k^2\Theta(k)\mathrm{Re}[G_{T,L}]dk \tag{F2}$$

$$\mathbb{J}^{st}_{T,L} = \int_{\mathbf{k}}\left(\frac{\partial(\mathrm{Re}[G_{T,L}])}{\partial k_t}\right)k_s.$$

Using Eqs. (E6), (E7) we have

$$\mathbb{I}^{st}_{T,L} = \int_{-\infty}^{\infty}\frac{\delta_{st}k^2\Theta(k)F_{T,L}(\omega,k)\left(\mathrm{Re}\left[K^2_{T,L}\right]\left(k^2-\mathrm{Re}\left[K^2_{T,L}\right]\right)-\mathrm{Im}\left[K^2_{T,L}\right]^2\right)}{3\pi^2\rho\omega^2\mathrm{Im}\left[K^2_{T,L}\right]}dk,$$

$$\begin{aligned} \mathbb{J}^{st}_{T,L} &= \int_{\mathbf{k}}\frac{2k_sk_t\left(2k^2F^2_{T,L}(\omega,k)\mathrm{Im}\left[K^2_{T,L}\right]-\mathrm{Re}\left[K^2_{T,L}\right]F_{T,L}(\omega,k)\right)}{\rho\omega^2\mathrm{Im}\left[K^2_{T,L}\right]} \\ &= \int_{-\infty}^{\infty}\delta_{st}\Theta(k)k^4dk\left(\frac{2k^2F^2_{T,L}(\omega,k)\mathrm{Im}\left[K^2_{T,L}\right]-\mathrm{Re}\left[K^2_{T,L}\right]F_{T,L}(\omega,k)}{3\pi^2\rho\omega^2\mathrm{Im}\left[K^2_{T,L}\right]}\right). \end{aligned} \tag{F3}$$

The integral $\mathbb{J}^{st}_{T,L}$ in Eq. (F3) includes an ill-defined term, proportional to $F^2_{T,L}(\omega,k)$, that can be regularized [74] when $|\mathrm{Im}[K^2_{T,L}]| \ll |k^2 - \mathrm{Re}[K^2_{T,L}]|$ to obtain

$$F_{T,L}(\omega,k) = \pi\delta\left(k^2 - \mathrm{Re}\left[K^2_{T,L}\right]\right),$$
$$F_{T,L}(\omega,k)^2 = \frac{\pi\delta\left(k^2 - \mathrm{Re}\left[K^2_{T,L}\right]\right)}{2\mathrm{Im}\left[K^2_{T,L}\right]}. \tag{F4}$$

Consequently, Eqs. (F1)–(F4) yield

$$\mathbb{I}^{st}_{T,L} = \frac{-\delta_{st}\mathrm{Re}\left[K^2_{T,L}\right]^{\frac{1}{2}}\mathrm{Im}\left[K^2_{T,L}\right]}{6\pi\rho\omega^2}, \tag{F5}$$
$$\mathbb{J}^{st}_{T,L} = 0, \tag{F6}$$

so

$$D_{\Delta G^{\mathrm{lq}}} = \frac{1}{3}\frac{-B^2\left(c_L^2 - c_T^2\right)}{4\pi\rho^2\omega^3(1+a)}\left(\mathrm{Re}\left[K_T^2\right]^{\frac{1}{2}}\mathrm{Im}\left[K_T^2\right] - \mathrm{Re}\left[K_L^2\right]^{\frac{1}{2}}\mathrm{Im}\left[K_L^2\right]\right). \tag{F7}$$

### 2. $D^{\mathcal{R}}$

In this calculation, which is similar to the analogous one carried out in two dimensions [62], we apply a method introduced in the treatment of light diffusion [75], introducing an auxiliary tensor function $\mathbf{\Psi}_{,s}(\mathbf{k})$ defined by

$$\mathbf{\Psi}_{,s}(\mathbf{k})q_s \equiv \Psi_{kl,s}(\mathbf{k})q_s, \tag{F8}$$
$$\equiv \int_{\mathbf{k}'} \Phi_{kl,mn}(\mathbf{k},\mathbf{k}')P_{mn,tt}(\mathbf{q};\mathbf{k}'), \tag{F9}$$
$$= -\int_{\mathbf{k}'} \Phi_{kl,mn}(\mathbf{k},\mathbf{k}')\frac{1}{2i\rho}\frac{\partial L_{mn}(\mathbf{k}')}{\partial k'_s}q_s. \tag{F10}$$

Use of Eq. (15) gives the following expression for $\mathbf{\Psi}_{,s}(\mathbf{k})$:

$$P_{ij,kl}(\mathbf{p})\Psi_{kl,s}(\mathbf{p}) + \Delta\Sigma_{ij,k_1l_1}(\mathbf{p})\Psi_{k_1l_1,s}(\mathbf{p}) - \int_{\mathbf{p}''}\Delta G_{ij,k_2l_2}(\mathbf{p})K_{k_2l_2,k_1l_1}(\mathbf{p},\mathbf{p}'')\Psi_{k_1l_1,s}(\mathbf{p}'') = -\Delta G_{ij,kl}(\mathbf{p})\frac{1}{2i\rho}\frac{\partial L_{kl}(\mathbf{p})}{\partial p_s}. \tag{F11}$$

Using the explicit expression Eq. (29) for the free medium Green's function, as well as Eqs. (7) and (10), we get

$$\Delta G_{ij,k_2l_2}(\mathbf{p}) = (\Delta\Sigma_{ij,n_2m_2}(\mathbf{p}) + P_{ij,n_2m_2}(\mathbf{p}))G_{n_2k_2}(\mathbf{p})G^*_{l_2m_2}(\mathbf{p}). \tag{F12}$$

Next, we define an angular tensor $\mathbf{\Upsilon}$, in analogy to the coefficient that relates the transport mean-free path and extinction length in the diffusion of electromagnetic waves [76,77]:

$$\Psi_{mn,s}(\mathbf{p})q_s = G(\mathbf{p})_{mi}G^*_{nj}(\mathbf{p})\Upsilon_{ij}(\mathbf{p},\mathbf{q}). \tag{F13}$$

It obeys the following integral equation:

$$P_{ij,tt}(\mathbf{p};\mathbf{q}) = \Upsilon_{ij}(\mathbf{p},\mathbf{q}) - \int_{\mathbf{p}''} K_{ij,k_1l_1}(\mathbf{p},\mathbf{p}'')G_{k_1m_1}(\mathbf{p}'')G^*_{n_1l_1}(\mathbf{p}'')\Upsilon_{m_1n_1}(\mathbf{p}'',\mathbf{q}), \tag{F14}$$

so, using Eqs. (F8) and (F13), the following expression for $D^{\mathcal{R}}$ [defined by Eq. (72)] results in:

$$D^{\mathcal{R}} = \frac{\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}} P_{ss,ij}(\mathbf{k};\mathbf{q})G_{im}(\mathbf{k})G^*_{nj}(\mathbf{k})\Upsilon_{mn}(\mathbf{k},\mathbf{q}). \tag{F15}$$

Now, looking at Eq. (F13), we make the ansatz that $\mathbf{\Upsilon}(\mathbf{p},\mathbf{q})$ is proportional to $\mathbf{q}$, and we look for a solution in the form

$$\Upsilon_{mn}(\mathbf{p},\mathbf{q}) = \alpha P_{mn,kk}(\mathbf{p};\mathbf{q}). \tag{F16}$$

Multiplying Eq. (F14) on the left by $P_{ss,k_1l_1}(\mathbf{p};\mathbf{q})G_{k_1i}(\mathbf{p})G^*_{jl_1}(\mathbf{p})$ and integrating over $\mathbf{p}$, we are left with

$$\alpha^{-1} = 1 - \frac{\int_{\mathbf{p}}\int_{\mathbf{p}''} P_{nn,t_1t_2}(\mathbf{p};\mathbf{q})G_{t_1k_1}(\mathbf{p})G^*_{k_2t_2}(\mathbf{p})K_{k_1k_2,m_1n_1}(\mathbf{p},\mathbf{p}'')G_{m_1k_3}(\mathbf{p}'')G^*_{k_4n_1}(\mathbf{p}'')P_{k_3k_4,ll}(\mathbf{p}'',\mathbf{q})}{\int_{\mathbf{k}} P_{ss,k_1l_1}(\mathbf{k};\mathbf{q})G_{k_1i}(\mathbf{k})G^*_{jl_1}(\mathbf{k})P_{ij,tt}(\mathbf{k};\mathbf{q})}. \tag{F17}$$

The second term on the right-hand side is the analog of the $\langle\cos\theta\rangle$ term in the diffusion of electromagnetic waves [77]. We are left with the following expression:

$$D^{\mathcal{R}} = \frac{\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}} \alpha P_{ii,kl}(\mathbf{k};\mathbf{q})G_{km}(\mathbf{k})G^*_{nl}(\mathbf{k})P_{mn,tt}(\mathbf{k};\mathbf{q}). \tag{F18}$$

The coefficient $\alpha$ is now evaluated: The symmetry properties of the Green tensor, tensor $\mathbf{P}$, and the kernel $\mathbf{K}$ from Eqs. (6), (10), and (47), respectively, yield

$$K_{ij,m_1n_1}(\mathbf{p},-\mathbf{p}'')G_{m_1t_1}(-\mathbf{p}'')G^*_{s_1n_1}(-\mathbf{p}'')P_{t_1s_1,ll}(-\mathbf{p}'',\mathbf{q}) = -K_{ij,m_1n_1}(\mathbf{p},\mathbf{p}'')G_{m_1t_1}(\mathbf{p}'')G^*_{s_1n_1}(\mathbf{p}'')P_{t_1s_1,ll}(\mathbf{p}'',\mathbf{q}).$$

Hence

$$\int_{\mathbf{p}''} K_{ij,m_1n_1}(\mathbf{p},\mathbf{p}'')G_{m_1t_1}(\mathbf{p}'')G^*_{s_1n_1}(\mathbf{p}'')P_{t_1s_1,ll}(\mathbf{p}'',\mathbf{q}) = 0 \tag{F19}$$

and $\alpha = 1$.

Therefore, $D^{\mathcal{R}}$ from the Eq. (F18) is given by

$$D^{\mathcal{R}} = \frac{\mathcal{B}^2}{q^2\omega(1+a)}\int_{\mathbf{k}} P_{ll,mn}(\mathbf{k};\mathbf{q})G_{mi}(\mathbf{k})G^*_{jn}(\mathbf{k})P_{ij,tt}(\mathbf{k};\mathbf{q}). \tag{F20}$$

Finally, using approximation from Eq. (F4) for Eq. (E5) in Eq. (F20), we can write

$$D^{\mathcal{R}} = \frac{-\mathcal{B}^2}{12\pi\rho^2\omega^5(1+a)} \times \left(c_L^4\left(\frac{\mathrm{Re}\left[K_L^2\right]^{7/2}}{\mathrm{Im}\left[K_L^2\right]} + \mathrm{Re}\left[K_L^2\right]\mathrm{Im}\left[K_L^2\right]^{3/2}\right) + 2c_T^4\left(\frac{\mathrm{Re}\left[K_T^2\right]^{7/2}}{\mathrm{Im}\left[K_T^2\right]} + \mathrm{Re}\left[K_T^2\right]^{3/2}\mathrm{Im}\left[K_T^2\right]\right)\right). \tag{F21}$$

Then, the total diffusion constant is

$$\begin{aligned} D &= D^{\mathcal{R}} + D_{\Delta G^{1q}} \\ &= \frac{-\mathcal{B}^2}{12\pi\rho^2\omega^5(1+a)}\left(c_L^4\left(\frac{\mathrm{Re}\left[K_L^2\right]^{7/2}}{\mathrm{Im}\left[K_L^2\right]} + \mathrm{Re}\left[K_L^2\right]\mathrm{Im}\left[K_L^2\right]^{3/2}\right) + 2c_T^4\left(\frac{\mathrm{Re}\left[K_T^2\right]^{7/2}}{\mathrm{Im}\left[K_T^2\right]} + \mathrm{Re}\left[K_T^2\right]^{3/2}\mathrm{Im}\left[K_T^2\right]\right)\right) \\ &\quad + \frac{\mathcal{B}^2\left(c_L^2 - c_T^2\right)\left(\mathrm{Re}\left[K_L^2\right]^{1/2}\mathrm{Im}\left[K_L^2\right] - \mathrm{Re}\left[K_T^2\right]^{1/2}\mathrm{Im}\left[K_T^2\right]\right)}{12\pi\rho^2\omega^3(1+a)}, \end{aligned} \tag{F22}$$

and the leading term in the limit of small $\mathrm{Im}[K_{T,L}^2]$ is

$$D^{\mathrm{lead}} \approx \frac{-\mathcal{B}^2}{12\pi\rho^2\omega^5(1+a)}\left(c_L^4\left(\frac{\mathrm{Re}\left[K_L^2\right]^{7/2}}{\mathrm{Im}\left[K_L^2\right]}\right) + 2c_T^4\left(\frac{\mathrm{Re}\left[K_T^2\right]^{7/2}}{\mathrm{Im}\left[K_T^2\right]}\right)\right) \tag{F23}$$

Explicitly, from Eq. (64), we have

$$-\mathcal{B}^2 = \frac{4\pi\rho^2\omega^2}{\left(2\mathrm{Re}\left[K_T^2\right]^{3/2} + \mathrm{Re}\left[K_L^2\right]^{3/2}\right)}. \tag{F24}$$

Then, using Eqs. (44), (64), (67), and (F23),

$$D^{\mathrm{lead}} \approx \left(1 + \frac{2\mathrm{Re}\left[K_T^2\right]^{3/2}\left(\frac{c_T^2}{\omega^2}\mathrm{Re}\left[K_T^2\right] - 1\right) + \mathrm{Re}\left[K_L^2\right]^{3/2}\left(\frac{c_L^2}{\omega^2}\mathrm{Re}\left[K_L^2\right] - 1\right)}{\left(\frac{\omega_{r1}^2}{\omega^2} - 1\right)\left(2\mathrm{Re}\left[K_T^2\right]^{3/2} + \mathrm{Re}\left[K_L^2\right]^{3/2}\right)}\right)^{-1} \frac{\left(c_L^4\frac{\mathrm{Re}\left[K_L^2\right]^{7/2}}{\mathrm{Im}\left[K_L^2\right]} + 2c_T^4\frac{\mathrm{Re}\left[K_T^2\right]^{7/2}}{\mathrm{Im}\left[K_T^2\right]}\right)}{3\omega^3\left(2\mathrm{Re}\left[K_T^2\right]^{3/2} + \mathrm{Re}\left[K_L^2\right]^{3/2}\right)}, \tag{F25}$$

which is Eq. (74). In the low-frequency limit, it reads as

$$D^{\mathrm{lead}}_{\omega\to 0} \approx \left(\frac{v_T^3 c_L^4}{\left(2v_L^3 + v_T^3\right)v_L^4}\frac{v_L l_L}{3} + \frac{2v_L^3 c_T^4}{\left(2v_L^3 + v_T^3\right)v_T^4}\frac{v_T l_T}{3}\right), \tag{F26}$$

which is Eq. (75).


[1] S.-W. Nam, H.-S. Chung, Y. C. Lo, L. Qi, J. Li, Y. Lu, A. C. Johnson, Y. Jung, P. Nukala, and R. Agarwal, Electrical wind force–driven and dislocation-templated amorphization in phase-change nanowires, Science **336**, 1561 (2012).

[2] M. Höfling, X. Zhou, L. M. Riemer, E. Bruder, B. Liu, L. Zhou, P. B. Groszewicz, F. Zhuo, B.-X. Xu, K. Durst, X. Tan, D. Damjanovic, J. Koruza, and J. Rödel, Control of polarization in bulk ferroelectrics by mechanical dislocation imprint, Science **372**, 961 (2021).

[3] K. K. Adepalli, J. Yang, J. Maier, H. L. Tuller, and B. Yildiz, Tunable oxygen diffusion and electronic conduction in $SrTiO_3$ by dislocation-induced space charge fields, Adv. Funct. Mater. **27**, 1700243 (2017).

[4] L. Porz, T. Frömling, A. Nakamura, N. Li, R. Maruyama, K. Matsunaga, P. Gao, H. Simons, C. Dietz, M. Rohnke, J. Janek, and J. Rödel, Conceptual framework for dislocation-modified conductivity in oxide ceramics deconvoluting mesoscopic structure, core, and space charge exemplified for $SrTiO_3$, ACS Nano **15**, 9355 (2021).

[5] F. C.-P. Massabuau, P. Chen, M. K. Horton, S. L. Rhode, C. X. Ren, T. J. O'Hanlon, A. Kovács, M. J. Kappers, C. J. Humphreys, R. E. Dunin-Borkowski, and R. A. Oliver, Carrier localization in the vicinity of dislocations in InGaN, J. Appl. Phys. **121**, 013104 (2017).

[6] D. Kotchetkov, J. Zou, A. A. Balandin, D. I. Florescu, and F. H. Pollak, Effect of dislocations on thermal conductivity of GaN layers, Appl. Phys. Lett. **79**, 4316 (2001).

[7] M. Kamatagi, N. Sankeshwar, and B. Mulimani, Thermal conductivity of GaN, Diam. Relat. Mater. **16**, 98 (2007).

[8] J. Ma, X. Wang, B. Huang, and X. Luo, Effects of point defects and dislocations on spectral phonon transport properties of wurtzite GaN, J. Appl. Phys. **114**, 074311 (2013).

[9] M. Kamatagi, R. Vaidya, N. Sankeshwar, and B. Mulimani, Low-temperature lattice thermal conductivity in free-standing GaN thin films, Int. J. Heat Mass Transfer **52**, 2885 (2009).

[10] B. K. Singh, V. J. Menon, and K. C. Sood, Phonon conductivity of plastically deformed crystals: Role of stacking faults and dislocations, Phys. Rev. B **74**, 184302 (2006).

[11] J. Shuai, H. Geng, Y. Lan, Z. Zhu, C. Wang, Z. Liu, J. Bao, C.-W. Chu, J. Sui, and Z. Ren, Higher thermoelectric performance of Zintl phases $(Eu_{0.5}\ Yb_{0.5})_{1-x}\ Ca_x\ Mg_2\ Bi_2$ by band engineering and strain fluctuation, Proc. Natl. Acad. Sci. **113**, E4125 (2016).

[12] Y. Wu, Z. Chen, P. Nan, F. Xiong, S. Lin, X. Zhang, Y. Chen, L. Chen, B. Ge, and Y. Pei, Lattice strain advances thermoelectrics, Joule **3**, 1276 (2019).

[13] L. You, Y. Liu, X. Li, P. Nan, B. Ge, Y. Jiang, P. Luo, S. Pan, Y. Pei, W. Zhang, G. J. Snyder, J. Yang, J. Zhang, and J. Luo, Boosting the thermoelectric performance of PbSe through dynamic doping and hierarchical phonon scattering, Energy Environ. Sci. **11**, 1848 (2018).

[14] J. Xin, H. Wu, X. Liu, T. Zhu, G. Yu, and X. Zhao, Mg vacancy and dislocation strains as strong phonon scatterers in $Mg_2\ Si_{1-x}\ Sb_x$ thermoelectric materials, Nano Energy **34**, 428 (2017).

[15] C. Zhou, Y. K. Lee, J. Cha, B. Yoo, S.-P. Cho, T. Hyeon, and I. Chung, Defect engineering for high-performance n-Type PbSe thermoelectrics, J. Am. Chem. Soc. **140**, 9282 (2018).

[16] Y. Yu, S. Zhang, A. M. Mio, B. Gault, A. Sheskin, C. Scheu, D. Raabe, F. Zu, M. Wuttig, Y. Amouyal, and O. Cojocaru-Mirédin, Ag-segregation to dislocations in PbTe-based thermoelectric materials, ACS Appl. Mater. Interfaces **10**, 3609 (2018).

[17] S. Giaremis, J. Kioseoglou, P. Desmarchelier, A. Tanguy, M. Isaiev, I. Belabbas, P. Komninou, and K. Termentzidis, Decorated dislocations against phonon propagation for thermal management, ACS Appl. Energy Mater. **3**, 2682 (2020).

[18] A. J. Minnich, M. S. Dresselhaus, Z. F. Ren, and G. Chen, Bulk nanostructured thermoelectric materials: Current research and future prospects, Energy Environ. Sci. **2**, 466 (2009).

[19] L. Lindsay, D. A. Broido, and T. L. Reinecke, Ab initio thermal transport in compound semiconductors, Phys. Rev. B **87**, 165201 (2013).

[20] S. Lee, K. Esfarjani, J. Mendoza, M. S. Dresselhaus, and G. Chen, Lattice thermal conductivity of Bi, Sb, and Bi-Sb alloy from first principles, Phys. Rev. B **89**, 085206 (2014).

[21] Z. Tian, S. Lee, and G. Chen, Comprehensive review of heat transfer in thermoelectric materials and devices, Ann. Rev. Heat Transfer **17**, 425 (2014).

[22] P. Jund and R. Jullien, Molecular-dynamics calculation of the thermal conductivity of vitreous silica, Phys. Rev. B **59**, 13707 (1999).

[23] F. Müller-Plathe, A simple nonequilibrium molecular dynamics method for calculating the thermal conductivity, J. Chem. Phys. **106**, 6082 (1997).

[24] O. N. Bedoya-Martínez, J.-L. Barrat, and D. Rodney, Computation of the thermal conductivity using methods based on classical and quantum molecular dynamics, Phys. Rev. B **89**, 014303 (2014).

[25] P. G. Klemens, The scattering of low-frequency lattice waves by static imperfections, Proc. Phys. Soc. Sect. A **68**, 1113 (1955).

[26] P. Carruthers, Scattering of phonons by elastic strain fields and the thermal resistance of dislocations, Phys. Rev. **114**, 995 (1959).

[27] F. Lund and B. Scheihing H., Scattering of phonons by quantum-dislocation segments in an elastic continuum, Phys. Rev. B **99**, 214102 (2019).

[28] F. Lund and B. Scheihing-Hitschfeld, The scattering of phonons by infinitely long quantum dislocations segments and the generation of thermal transport anisotropy in a solid threaded by many parallel dislocations, Nanomaterials **10**, 1711 (2020).

[29] A. Granato and K. Lücke, Theory of mechanical damping due to dislocations, J. Appl. Phys. **27**, 583 (1956).

[30] A. Granato and K. Lücke, Application of dislocation theory to internal friction phenomena at high frequencies, J. Appl. Phys. **27**, 789 (1956).

[31] K. Lücke and A. V. Granato, Simplified theory of dislocation damping including point-defect drag. I. Theory of drag by equidistant point defects, Phys. Rev. B **24**, 6991 (1981).

[32] G. A. Kneezel and A. V. Granato, Effect of independent and coupled vibrations of dislocations on low-temperature thermal conductivity in alkali halides, Phys. Rev. B **25**, 2851 (1982).

[33] D. Shilo and E. Zolotoyabko, X-ray Imaging of Phonon Interaction with Dislocations, in *Dislocations in Solids*, edited by F. R. N. Nabarro and J. P. Hirth (Elsevier, Amsterdam, 2007), Vol. 13, Chap. 80, pp. 603–639.

[34] A. Maurel, V. Pagneux, F. Barra, and F. Lund, Interaction between an elastic wave and a single pinned dislocation, Phys. Rev. B **72**, 174110 (2005).

[35] N. Mujica, M. T. Cerda, R. Espinoza, J. Lisoni, and F. Lund, Ultrasound as a probe of dislocation density in aluminum, Acta Mater. **60**, 5828 (2012).

[36] F. Barra, R. Espinoza-González, H. Fernández, F. Lund, A. Maurel, and V. Pagneux, The use of ultrasound to measure dislocation density, Journal of The Minerals, Metals & Materials Society **67**, 1856 (2015).

[37] V. Salinas, C. Aguilar, R. Espinoza-González, F. Lund, and N. Mujica, In situ monitoring of dislocation proliferation during plastic deformation using ultrasound, Int. J. Plast. **97**, 178 (2017).

[38] C. Espinoza, D. Feliú, C. Aguilar, R. Espinoza-González, F. Lund, V. Salinas, and N. Mujica, Linear versus nonlinear

acoustic probing of plasticity in metals: A quantitative assessment, Materials **11**, 2217 (2018).

[39] A. Maurel, V. Pagneux, F. Barra, and F. Lund, Wave propagation through a random array of pinned dislocations: Velocity change and attenuation in a generalized Granato and Lücke theory, Phys. Rev. B **72**, 174111 (2005).

[40] D. Churochkin, F. Barra, F. Lund, A. Maurel, and V. Pagneux, Multiple scattering of elastic waves by pinned dislocation segments in a continuum, Wave Motion **60**, 220 (2016).

[41] P. Sheng, *Introduction to Wave Scattering, Localization and Mesoscopic Phenomena* (Academic, New York, 2006).

[42] D. Vollhardt and P. Wölfle, Diagrammatic, self-consistent treatment of the Anderson localization problem in $d \leqslant 2$ dimensions, Phys. Rev. B **22**, 4666 (1980).

[43] P. Wölfle and R. N. Bhatt, Electron localization in anisotropic systems, Phys. Rev. B **30**, 3542 (1984).

[44] R. N. Bhatt, P. Wölfle, and T. V. Ramakrishnan, Localization and interaction effects in anisotropic disordered electronic systems, Phys. Rev. B **32**, 569 (1985).

[45] T. R. Kirkpatrick, Localization of acoustic waves, Phys. Rev. B **31**, 5746 (1985).

[46] Y. Barabanenkov and V. Ozrin, Asymptotic solution of the Bethe-Salpeter equation and the Green-Kubo formula for the diffusion constant for wave propagation in random media, Phys. Lett. A **154**, 38 (1991).

[47] Y. Barabanenkov and V. Ozrin, Diffusion asymptotics of the Bethe-Salpeter equation for electromagnetic waves in discrete random media, Phys. Lett. A **206**, 116 (1995).

[48] H. Stark and T. C. Lubensky, Multiple light scattering in anisotropic random media, Phys. Rev. E **55**, 514 (1997).

[49] R. Weaver, Diffusivity of ultrasound in polycrystals, J. Mech. Phys. Solids **38**, 55 (1990).

[50] B. A. van Tiggelen, L. Margerin, and M. Campillo, Coherent backscattering of elastic waves: Specific role of source, polarization, and near field, J. Acoust. Soc. Am. **110**, 1291 (2001).

[51] N. P. Trégourès and B. A. van Tiggelen, Generalized diffusion equation for multiple scattered elastic waves, Waves in Random Media **12**, 21 (2002).

[52] N. P. Trégourès and B. A. van Tiggelen, Quasi-two-dimensional transfer of elastic waves, Phys. Rev. E **66**, 036601 (2002).

[53] L. Trujillo, F. Peniche, and L. D. G. Sigalotti, Derivation of a Schrödinger-like equation for elastic waves in granular media, Granular Matter **12**, 417 (2010).

[54] L. A. Cobus, W. K. Hildebrand, S. E. Skipetrov, B. A. van Tiggelen, and J. H. Page, Transverse confinement of ultrasound through the Anderson transition in three-dimensional mesoglasses, Phys. Rev. B **98**, 214201 (2018).

[55] A. Goïcoechea, S. E. Skipetrov, and J. H. Page, Suppression of transport anisotropy at the Anderson localization transition in three-dimensional anisotropic media, Phys. Rev. B **102**, 220201(R) (2020).

[56] F. Lund, Normal modes and acoustic properties of an elastic solid with line defects, Phys. Rev. B **91**, 094102 (2015).

[57] E. Bianchi, V. M. Giordano, and F. Lund, Elastic anomalies in glasses: Elastic string theory understanding of the cases of glycerol and silica, Phys. Rev. B **101**, 174311 (2020).

[58] Y. M. Beltukov, D. A. Parshin, V. M. Giordano, and A. Tanguy, Propagative and diffusive regimes of acoustic damping in bulk amorphous material, Phys. Rev. E **98**, 023005 (2018).

[59] L. Ryzhik, G. Papanicolaou, and J. B. Keller, Transport equations for elastic and other waves in random media, Wave Motion **24**, 327 (1996).

[60] F. Lund, Response of a stringlike dislocation loop to an external stress, J. Mater. Res. **3**, 280 (1988).

[61] D. Churochkin and F. Lund, Diffusion of elastic waves in a two dimensional continuum with a random distribution of screw dislocations, Wave Motion **69**, 16 (2017).

[62] D. Churochkin and F. Lund, Coherent propagation and incoherent diffusion of elastic waves in a two dimensional continuum with a random distribution of edge dislocations, Wave Motion **105**, 102768 (2021).

[63] H. T. Nieh, L. Chen, and P. Sheng, Ward identities for transport of classical waves in disordered media, Phys. Rev. E **57**, 1145 (1998).

[64] Y. N. Barabanenkov and V. D. Ozrin, Comment on "Ward identities for transport of classical waves in disordered media," Phys. Rev. E **64**, 018601 (2001).

[65] H. T. Nieh, L. Chen, and P. Sheng, Reply to "Comment on 'Ward identities for transport of classical waves in disordered media,' " Phys. Rev. E **64**, 018602 (2001).

[66] D. H. Berman, Diffusion of waves in a layer with a rough interface, Phys. Rev. E **62**, 7365 (2000).

[67] B. A. van Tiggelen and A. Lagendijk, Rigorous treatment of the speed of diffusing classical waves, Europhys. Lett. **23**, 311 (1993).

[68] D. Livdan and A. A. Lisyansky, Transport properties of waves in absorbing random media with microstructure, Phys. Rev. B **53**, 14843 (1996).

[69] R. L. Weaver, On diffuse waves in solid media, J. Acoust. Soc. Am. **71**, 1608 (1982).

[70] A. Lubatsch, J. Kroha, and K. Busch, Theory of light diffusion in disordered media with linear absorption or gain, Phys. Rev. B **71**, 184201 (2005).

[71] R. L. Weaver, J. Y. Yoritomo, and J. P. Coleman, Diffuse elastic waves in a nearly axisymmetric body: Energy distribution, transport and dynamical localization, Eur. Phys. J.: Spec. Top. **226**, 1371 (2017).

[72] L. D. Sotelo, H. Hadidi, C. S. Pratt, M. P. Sealy, and J. A. Turner, Ultrasonic mapping of hybrid additively manufactured 420 stainless steel, Ultrasonics **110**, 106269 (2021).

[73] J.-H. Ee, D.-W. Jung, U.-R. Kim, and J. Lee, Combinatorics in tensor-integral reduction, Eur. J. Phys. **38**, 025801 (2017).

[74] G. D. Mahan, *Many Particle Physics,* 3rd ed. (Plenum, New York, 2000).

[75] Y. N. Barabanenkov and V. D. Ozrin, Problem of Light Diffusion in Strongly Scattering Media, Phys. Rev. Lett. **69**, 1364 (1992).

[76] P. Sheng, *Scattering and Localization of Classical Waves in Random Media* (World Scientific, Singapore, 1990).

[77] B. van Tiggelen and H. Stark, Nematic liquid crystals as a new challenge for radiative transfer, Rev. Mod. Phys. **72**, 1017 (2000).