Letter

Boundary-driven XYZ chain: Inhomogeneous triangular matrix product ansatz

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We construct an explicit matrix product ansatz for the steady state of a boundary driven XYZ spin-$\frac{1}{2}$ chain for arbitrary local polarizing channels at the ends of the chain. The ansatz, where the Lax operators are written explicitly in terms of infinite-dimensional bidiagonal (triangular) site-dependent matrices, becomes exact either in the (Zeno) limit of infinite dissipation strength or the thermodynamic limit of infinite chain length. The solution is based on an extension of the recently discovered family of separable eigenstates of the model.

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I. INTRODUCTION

Exact and explicit solutions are indispensable for the advancement of our understanding of statistical mechanics of interacting systems. While many such exact solutions in the realm of equilibrium statistical physics have been known for over a half-century [1]—say, Onsager's and Baxter's solutions to two-dimensional classical statistical models at thermal equilibrium or, sometimes equivalent, the Bethe ansatz solutions for quantum models in one dimension—much less is known out of equilibrium [2].

Certain classical stochastic systems which are driven out of equilibrium by boundary dissipation, like simple exclusion processes, could be easily mapped to integrable quantum models in one dimension so that the Bethe ansatz can be used [3]. However, much less is understood for the corresponding boundary-dissipation-driven quantum lattice models [4]. Only a decade ago, an example of such a paradigm, namely, the steady-state density matrix of a boundary-driven Lindblad master equation of the anisotropic Heisenberg (XXZ) spin-$\frac{1}{2}$ chain, has been constructed exactly [5], paralleling analogous results for boundary-driven classical stochastic lattice systems [6,7]. Due to a noncompact (infinite dimensional) representation space of the fundamental algebraic objects needed in the matrix product ansatz (MPA), the solution consequently gave birth to a nonlocal yet quasilocal conserved charge of the model [8], resolving the long debated fundamental question on ballistic spin transport in an easy-plane XXZ chain at high temperature.

Using the dissipative driving to provide a steady-steady excitation of an otherwise conservative system to probe nonequilibrium physics, we will limit our discussion to jump operators which are localized at the boundaries of the system. This boundary-driven paradigm can be considered a quantum analog of a conservative system between two thermodynamic reservoirs [9,10]. So far, exact solutions of steady states of many-body interacting Lindblad equations were limited to particular forms of dissipative boundary driving [5,8,11–14]. Recently, an MPA has been proposed with manifestly spatially inhomogeneous matrices in terms of which one can solve for the steady state of a fully anisotropic Heisenberg (XYZ) spin-$\frac{1}{2}$ chain with arbitrarily oriented boundary polarizing channels in the limit of large coupling (the so-called Zeno regime) [15,16]. However, the Lax matrices forming the MPA, were in general only given in terms of a numerical solution of a nonlinear recurrence in the system size, which turned unstable for long chains.

In this letter, we find an analytic solution of the aforementioned boundary-driven XYZ problem in terms of simple bidiagonal, site-dependent, infinite-dimensional Lax operators whose elements are written explicitly in terms of Jacobi θ functions. Although the solution can be formally considered a leading-order asymptotic in the Zeno regime of strong boundary coupling, it has been shown [17] that it also applies asymptotically in the thermodynamic limit of long chains for fixed boundary coupling. Moreover, the solution provides an exact conserved charge of the XYZ model which, unlike the Hamiltonian and the complete eight-vertex transfer matrix, breaks the spin-reversal symmetry of the model and may have applications beyond the dissipative steady-state paradigm.

II. SEPARABLE EIGENSTATES OF THE XYZ CHAIN

We consider a chain of N spins $\frac{1}{2}$ described by XYZ Hamiltonian H_N acting over 2^N-dimensional Hilbert space $\mathcal{H}_N = \mathbb{C}^{2^N}$:

$$H_N = \sum_{n=1}^{N-1} h_{n,n+1}, \quad h_{n,n+1} = \sum_{\alpha} \sigma_n^{\alpha} J_{\alpha} \sigma_{n+1}^{\alpha}, \tag{1}$$

where σ_n^{α}, $n \in \{1, 2, \ldots N\}$, $\alpha \in \{x, y, z\}$ are Pauli operators embedded in $\mathcal{H}$. It turns out that the natural parameterization of the anisotropy coupling tensor J_{α} is in terms of two complex parameters η, τ and Jacobi θ functions, defining shorthand notation (following Ref. [18]): $\theta_{\alpha}(u) \equiv$

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$\vartheta_\alpha(\pi u, e^{2i\pi\tau})$, $\bar{\theta}_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{i\pi\tau})$:

$$\frac{J_x}{J} = \frac{\bar{\theta}_4(\eta)}{\bar{\theta}_4(0)}, \quad \frac{J_y}{J} = \frac{\bar{\theta}_3(\eta)}{\bar{\theta}_3(0)}, \quad \frac{J_z}{J} = \frac{\bar{\theta}_2(\eta)}{\bar{\theta}_2(0)}. \tag{2}$$

Fixing the energy scale, say, $J = 1$, the remaining two independent coupling constants J_α are uniquely parameterized—up to permutation of the axes—by taking $\eta, i\tau \in \mathbb{R}$. However, all the results of this letter remain valid for arbitrary choice $J, \eta, \tau \in \mathbb{C}$ parameterizing a general complex coupling tensor J_α.

Our analysis starts by the following remarkable observation. Defining a one-parameter family of spinors:

$$|\psi_n\rangle \equiv |\psi(u+n\eta)\rangle = \begin{pmatrix} \theta_1(u+n\eta) \\ -\theta_4(u+n\eta) \end{pmatrix}, \tag{3}$$

where $u \in \mathbb{C}$ is a free parameter, we find a family of spatially inhomogeneous separable eigenstates of the XYZ model with boundary fields:

$$(H_N - a_1\sigma_1^z + a_N\sigma_N^z)\,|\Psi\rangle = E\,|\Psi\rangle\,, \tag{4}$$

$$|\Psi(u)\rangle = \bigotimes_{n=1}^{N} |\psi_n\rangle\,,$$

$$E = \sum_{n=1}^{N-1} d_n. \tag{5}$$

The eigenvalue condition straightforwardly follows from telescoping the following divergence condition:

$$h\,|\psi_n\rangle \otimes |\psi_{n+1}\rangle = (a_n\sigma^z \otimes \mathbb{I}_2 - a_{n+1}\mathbb{I}_2 \otimes \sigma^z + d_n\mathbb{I}_4)\,|\psi_n\rangle \otimes |\psi_{n+1}\rangle\,, \tag{6}$$

where $h = \sum_\alpha J_\alpha \sigma^\alpha \otimes \sigma^\alpha$ is a 4×4 Hamiltonian density operator. Consistency of Eq. (6) requires that coefficients $a_n, d_n \in \mathbb{C}$ satisfy a set of recurrence relations which can be explicitly solved [19]:

$$a_n \equiv a(u+n\eta),$$
$$d_n = f(\eta) + f(u+n\eta) - f(u+(n+1)\eta),$$
$$a(u) = \frac{\bar{\theta}_1(\eta)\bar{\theta}_2(u)}{\bar{\theta}_2(0)\bar{\theta}_1(u)}, \quad f(u) = \frac{\bar{\theta}_1(\eta)\bar{\theta}_1'(u)}{\bar{\theta}_1'(0)\bar{\theta}_1(u)}. \tag{7}$$

Note that this fixes the magnitude of the boundary fields a_1, a_N, while their direction (chosen here along the z axis) is arbitrary, so the result can be generalized to arbitrarily oriented boundary fields which need not be collinear.

We note that the separable eigenstates, in the special case of the XXZ model—the so-called spin-helix states—have been proven experimentally useful [20,21]. While such separable eigenstates have finite lifetime within existing experimental protocols, we will show how to make them stable by boundary dissipation, which in turn can be implemented using a repeated interaction protocol [22] using fully polarized spins, see Ref. [19].

III. INHOMOGENEOUS BIDIAGONAL LAX OPERATORS

However, the choice in Eq. (4) serves our purpose, which is to promote Eq. (6) to a divergence relation for local Lax operators [15,16]:

$$[h_{n,n+1}, \mathbf{L}_n\mathbf{L}_{n+1}] = 2i(I\mathbf{L}_{n+1} - \mathbf{L}_n I). \tag{8}$$

Here, $\mathbf{L}_n = \sum_\alpha L_n^\alpha \sigma_n^\alpha$ are the so-called Lax operators with components $L_n^\alpha \in \mathrm{End}(\mathcal{H}_\mathrm{a})$ as well as $I \in \mathrm{End}(\mathcal{H}_\mathrm{a})$ acting as linear operators over a suitable *auxiliary space* $\mathcal{H}_\mathrm{a}$. Note that I acts trivially over the physical space $\mathcal{H}_N$. We first show that the solution in Eq. (6), together with a solution of an equivalent relation for a dual, bi-orthogonal spinor [19]:

$$\langle\psi_n^\perp| = (\theta_4(u+n\eta),\ \theta_1(u+n\eta)),$$
$$\langle\psi_n^\perp| \otimes \langle\psi_{n+1}^\perp|\,h = \langle\psi_n^\perp| \otimes \langle\psi_{n+1}^\perp| \times(-a_n\sigma^z \otimes \mathbb{I}_2 + a_{n+1}\mathbb{I}_2 \otimes \sigma^z + d_n\mathbb{I}_4), \tag{9}$$

provides a solution to Eq. (8) for one-dimensional auxiliary space $\mathcal{H}_\mathrm{a} = \mathbb{C}$:

$$\mathbf{L}_n = \frac{1}{\kappa(u+n\eta)}\,|\psi_n\rangle\,\langle\psi_n^\perp|\,, \quad I = 1, \tag{10}$$

where $\kappa(u) = -i\theta_1(u)\theta_4(u)a(u)$. The proof follows from inserting Eq. (10) into Eq. (8), while facilitating divergence conditions Eqs. (6) and (9) and a trivially verifiable identity $\sigma^z\,|\psi_n\rangle\,\langle\psi_n^\perp| + |\psi_n\rangle\,\langle\psi_n^\perp|\,\sigma^z = 2\theta_1(u+n\eta)\theta_4(u+n\eta)\mathbb{I}_2$.

Now we are in position to state our main result:

Theorem: The operator divergence condition in Eq. (8) is generally solved, for any auxiliary space $\mathcal{H}_\mathrm{a} = \mathbb{C}^M$, with the following inhomogeneous bidiagonal ansatz:

$$L_n^\alpha = \sum_{j=1}^{M} s_{n-2(j-1)}^\alpha\,|j\rangle\,\langle j| + \sum_{j=1}^{M-1} s_{n-2(j-1)}^\alpha\,|j\rangle\,\langle j+1|\,,$$
$$I = \sum_{j=1}^{M} |j\rangle\,\langle j| - \sum_{j=1}^{M-1} |j\rangle\,\langle j+1|\,, \tag{11}$$

where $s_n^\alpha \equiv s^\alpha(u+n\eta)$ and $s^x(u) = \frac{i}{2a(u)}[\frac{\theta_1(u)}{\theta_4(u)} - \frac{\theta_4(u)}{\theta_1(u)}]$, $s^y(u) = -\frac{1}{2a(u)}[\frac{\theta_1(u)}{\theta_4(u)} + \frac{\theta_4(u)}{\theta_1(u)}]$, $s^z(u) = \frac{i}{a(u)}$.

Proof: It is straightforward to check that Eq. (11) is equivalent to Eq. (10) for $M = 1$. For the general proof of Eq. (11), we make the following observation: diagonal elements of triangular matrices (bidiagonal ones being special cases thereof) form a commutative subalgebra $\mathbb{C}$; hence, the diagonal elements of Lax operators $\langle j|\,L_n^\alpha\,|j\rangle$ must all have the same functional form (independent of j) apart from a possible shift in the variable u (which may depend on j). Within the ansatz in Eq. (11), the matrix elements $\langle j|\,L_n^\alpha\,|j'\rangle$ for $j' \geqslant j+2$ all identically vanish. Hence, we only need to check the case $j' = j+1$, which is equivalent to studying the 2×2 problem (in auxiliary space) with

$$\mathbf{L}_n = \begin{pmatrix} \mathbf{u}_n & \mathbf{u}_n \\ 0 & \mathbf{v}_n \end{pmatrix}, \quad I = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $\mathbf{u}_n = \frac{1}{\kappa(u+n\eta)}\,|\psi(u+n\eta)\rangle\,\langle\psi^\perp(u+n\eta)|$, and $\mathbf{v}_n = \frac{1}{\kappa(v+n\eta)}\,|\psi(v+n\eta)\rangle\,\langle\psi^\perp(v+n\eta)|$. Inserting this ansatz into $\langle 1|\,\mathrm{Eq.}\ (8)\,|2\rangle$ and using the established identities, e.g., Eqs. (6) and (9), the only nontrivial condition that remains connects u and v, i.e., $v = u - 2\eta$, which proves Eq. (11) for any u, M.

IV. STEADY STATE OF THE BOUNDARY-DRIVEN CHAIN

We wish to construct the nonequilibrium steady state (NESS) density matrix ρ of the Lindblad equation:

$$\frac{d}{dt}\rho = -i[H_{N+2},\rho] + \Gamma\,\mathcal{D}_\mathrm{l}[\rho] + \Gamma\,\mathcal{D}_\mathrm{r}[\rho] = 0, \tag{13}$$

at large dissipation strength Γ, where $\mathcal{D}_\mu[\rho]$, $\mu \in \{\mathrm{l},\mathrm{r}\}$, denote the dissipators at the left and right ends of the chain of $N+2$ sites, which we label by 0 and $N+1$, respectively. They are of the form $\mathcal{D}_\mu[\rho] = 2k_\mu\rho k_\mu^\dagger - \{k_\mu^\dagger k_\mu,\rho\}$ with jump operators $k_{\mathrm{l/r}} = (\mathbf{n}'_{\mathrm{l/r}} + i\mathbf{n}''_{\mathrm{l/r}})\cdot\boldsymbol{\sigma}_{0/N+1}$ targeting polarizations $\mathbf{n}_\mu = \mathbf{n}(\theta_\mu,\phi_\mu)$, where $\mathbf{n}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Here, $\mathbf{n}'_\mu = \mathbf{n}(\frac{\pi}{2}-\theta_\mu, \pi+\phi_\mu)$ and $\mathbf{n}''_\mu = \mathbf{n}(\frac{\pi}{2}, \phi_\mu - \frac{\pi}{2})$, which together with $\mathbf{n}_\mu$ form an orthonormal basis of $\mathbb{R}^3$. The targeted states of the dissipators are single-site pure states ρ_μ, such that $\mathcal{D}_\mu[\rho_\mu] = 0$.

In our previous work [15–17], we have shown that, in the regime of either large Γ or large N, NESS can be written in the leading order as $\rho = \rho_\mathrm{l}\otimes\rho_N\otimes\rho_\mathrm{r} + \mathcal{O}[(N\Gamma)^{-1}]$, where $\rho_N = \Omega\Omega^\dagger/\mathrm{Tr}(\Omega\Omega^\dagger)$ is completely fixed with the condition:

$$\left[H_N + \sum_\alpha (J_\alpha n_\mathrm{l}^\alpha \sigma_1^\alpha + J_\alpha n_\mathrm{r}^\alpha \sigma_N^\alpha),\, \rho_N\right] = 0, \tag{14}$$

and the MPA:

$$\Omega = \langle w_\mathrm{l}|\,\mathbf{L}_1\mathbf{L}_2\cdots\mathbf{L}_N\,|w_\mathrm{r}\rangle, \tag{15}$$

where $\mathbf{L}_n$ obey the divergence condition in Eq. (8). The boundary vectors $|w_\mu\rangle \in \mathcal{H}_\mathrm{a}$ are fixed by projecting the commutativity conditions to the boundary sites, yielding

$$\langle w_\mathrm{l}|\,V_\mathrm{l} = 0, \tag{16}$$

$$V_\mathrm{r}\,|w_\mathrm{r}\rangle = \varepsilon(1,-1,1,-1,\ldots)^T, \tag{17}$$
$$V_\mathrm{l} = \sum_\alpha J_\alpha n_\mathrm{l}^\alpha L_1^\alpha + iI,$$
$$V_\mathrm{r} = \sum_\alpha J_\alpha n_\mathrm{r}^\alpha L_N^\alpha - iI,$$

while the commutativity in Eq. (14) in the bulk follows from Eq. (8). Parameter ε is arbitrary, and in generic case where the matrix V_r is nonsingular, we may fix it to $\varepsilon = 1$ without loss of generality, while a special homogeneous case $\varepsilon = 0$ should be treated separately.

We not only have a fully explicit form of the Lax operators in Eq. (11), but we can also solve the boundary equations explicitly [Eqs. (16) and (17)] and determine the free complex variable u. Namely, we shall parameterize targeted boundary polarizations $n_\mathrm{l}, n_\mathrm{r}$ via two complex numbers:

$$u_\mu = x_\mu + iy_\mu, \quad \mu\in\{\mathrm{l},\mathrm{r}\}, \tag{18}$$

as (see Ref. [19])

$$n_\mu^x = -\frac{\bar\theta_2(iy_\mu)}{\bar\theta_3(iy_\mu)}\,\frac{\bar\theta_1(x_\mu)}{\bar\theta_4(x_\mu)},$$
$$n_\mu^y = -i\frac{\bar\theta_1(iy_\mu)}{\bar\theta_3(iy_\mu)}\,\frac{\bar\theta_2(x_\mu)}{\bar\theta_4(x_\mu)},$$
$$n_\mu^z = -\frac{\bar\theta_4(iy_\mu)}{\bar\theta_3(iy_\mu)}\,\frac{\bar\theta_3(x_\mu)}{\bar\theta_4(x_\mu)}. \tag{19}$$

We can now prove, see Ref. [19] for details, that Eq. (14) is satisfied with the choice:

$$u = u_\mathrm{l}, \quad M = N+1 \tag{20}$$

in Eq. (11), and Eq. (16) is solved by

$$\langle w_\mathrm{l}| = \langle 1,1,0,\ldots,0|\,. \tag{21}$$

The solution of Eq. (17) for $|w_\mathrm{r}\rangle = (r_1, r_2, \ldots, r_{N+1})^T$ for $\epsilon = 1$ is given by recurrence:

$$r_{k-1} = \frac{(-1)^k - r_k\,(V_\mathrm{r})_{k-1,k}}{(V_\mathrm{r})_{k-1,k-1}}, \quad k = 1,\ldots,N+1,$$
$$r_{N+1} = \frac{(-1)^N}{(V_\mathrm{r})_{N+1,N+1}}, \tag{22}$$

which is valid if operator V_r in Eq. (17) does not have zero eigenvalues, i.e., $\prod_n (V_\mathrm{r})_{nn} \neq 0$.

We stress that the vector on right-hand side of Eq. (17) should be allowed since it is essentially in the joint kernel (null space) of all L_n^α, disregarding the last component, namely, $L_n^\alpha(1,-1,1,-1,\ldots)^T = (0,0,\ldots,0,0,*)^T$, where $*$ denotes any nonzero element. The next action of L_m^β creates another nonzero element $L_m^\beta(0,0,\ldots,0,0,*)^T = (0,0,\ldots,0,0,*,*)^T$, and so on. The property in Eq. (14) is thus guaranteed by $\langle w_\mathrm{l}|\,L_1^{\alpha_1}\ldots L_{N-1}^{\alpha_{N-1}}V_\mathrm{r}\,|w_\mathrm{r}\rangle = (1,1,0,0,\ldots)(0,0,*,*,\ldots,*)^T = 0$. In the previous study [15,16], on the other hand, Lax operators had a trivial joint kernel; hence, only $\varepsilon = 0$ applied there.

However, for a submanifold of fine-tuned driving/coupling parameters, the matrix V_r can be singular $\det[V_\mathrm{r}] = \prod_k (V_\mathrm{r})_{kk} = 0$ [23]. This situation corresponds to a NESS with large modulations of the local magnetization, see Fig. 1. In this case, Eq. (17) for the right auxiliary vector $|w_\mathrm{r}\rangle$ must be solved with $\epsilon = 0$, while the recurrence in Eq. (22) breaks down.

The most prominent NESS of this singular type is obtained if just the first diagonal term of V_r vanishes, $(V_\mathrm{r})_{11} = 0$, yielding the unique solution of Eq. (17) with $\epsilon = 0$: $|w_\mathrm{r}\rangle = (1,0,0,\ldots,0)^T$. The respective right boundary polarization $\mathbf{n}_\mathrm{r}$ is given by Eq. (19) with $y_\mathrm{r} = y_\mathrm{l},\ x_\mathrm{r} = x_\mathrm{l} + (N+1)\eta$, see Ref. [19]. Due to the upper trigonal structure of all L_n^α, every expression of the form $\langle w_\mathrm{l}|\,L_1^{\alpha_1}L_2^{\alpha_2}\cdots L_N^{\alpha_N}\,|w_\mathrm{r}\rangle$ will contain only one nonzero term, rendering the steady state site factorized:

$$\rho_N = (\mathbf{L}_1\mathbf{L}_1^\dagger)\otimes(\mathbf{L}_2\mathbf{L}_2^\dagger)\otimes\ldots\otimes(\mathbf{L}_N\mathbf{L}_N^\dagger), \tag{23}$$

where $\mathbf{L}_n$ is given by Eq. (10). It is easy to verify that the state is pure and is fully characterized by the corresponding magnetization profile, given by Jacobi elliptic functions:

$$\langle\sigma_n^x\rangle = A_x\,\mathrm{sn}[2K_k(\eta n + x_\mathrm{l}), k],$$
$$\langle\sigma_n^y\rangle = A_y\,\mathrm{cn}[2K_k(\eta n + x_\mathrm{l}), k],$$
$$\langle\sigma_n^z\rangle = A_z\,\mathrm{dn}[2K_k(\eta n + x_\mathrm{l}), k], \tag{24}$$

(explicit A_α given in Ref. [19]), where $k = [\frac{\bar\theta_2(0)}{\bar\theta_3(0)}]^2$, $K_k = \frac{1}{2}\pi[\bar\theta_3(0)]^2$, with periods $2/\eta$ $(1/\eta)$ for x, y (z) components,

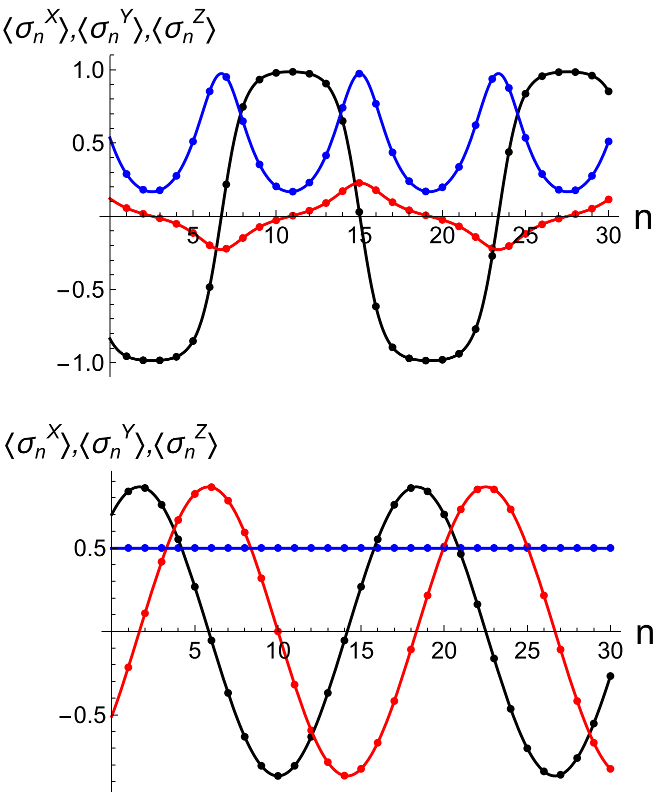

FIG. 1. Local magnetization profiles for a factorized pure nonequilibrium steady state (NESS) in the elliptic XYZ case (upper panel) and for the XXZ case (lower panel). x-, y-, and z-spin projections are indicated with black, red, and blue points, respectively. Interpolating curves for x-, y-, and z-spin projections are given by Jacobi elliptic functions in Eq. (24) for the XYZ model and trigonometric functions for the XXZ model, and n is the site number. For both cases, boundary polarizations are chosen to render $(V_r)_{11} = 0$ in Eq. (17) and $N = 30$, so that Eq. (17) is solved with $\epsilon = 0$. Parameters: $\eta = 0.12$, $\tau = i/2$, $u = 0.0477 + 0.123361i$ (upper panel), $\gamma = 0.12$, $\theta_l = \pi/3$, $u = \phi_l = -0.2\pi$ (lower panel).

see upper panel of Fig. 1. The state in Eq. (23) is an elliptic counterpart of the spin-helix state (see lower panel of Fig. 1) appearing in models with uniaxial spin anisotropy (XXZ) [24,25].

While in special cases the NESS can be obtained fully analytically, see Eq. (24) using our MPA, for generic parameters, simplicity and sparse structure of our MPA allows efficient numerical calculus of arbitrary NESS observables for large chains, as exemplified in Fig. 2. A crucial advantage of our representation with respect to earlier results [15,16] is a full control of the auxiliary vector $|w_r\rangle$ via a recurrence in Eq. (22) and explicit Lax operator expression in Eq. (11) for the fully anisotropic XYZ case.

V. SPECIAL CASE OF THE XXZ CHAIN

In the partially anisotropic case $J_x/J = J_y/J = 1, J_z/J = \Delta = \cos\gamma$ (γ either real or imaginary), the divergence condition in Eq. (6) is satisfied with $|\psi_n\rangle = [\cos\frac{\theta}{2}\exp(-iu_n/2), \sin\frac{\theta}{2}\exp(iu_n/2)]^T$, $u_n = u + n\gamma$, where $a_{n+1} = a_n = -i\sin\gamma$, $d_n = \Delta$, see Ref. [19].

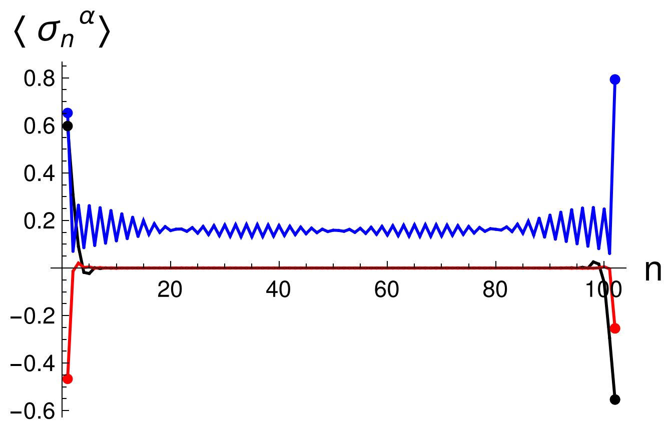

FIG. 2. Local magnetization profiles for the elliptic XYZ case for a generic choice of parameters, when Eq. (17) is solved with $\epsilon = 1$, i.e., by recurrence in Eq. (22). x-, y-, and z-spin projections are indicated, respectively, with black, red, and blue points connected by lines for clarity. Boundary-targeted magnetizations (site numbers $n = 0$, $n = N + 1$) are indicated by bullet symbols at the ends. Parameters: $N = 100$, $\eta = 0.4511$, $\tau = i/2$, $u_l = -0.89 + 0.4\,i$, $u_r = 0.1 + 0.55\,i$. The corresponding anisotropy tensor eigenvalues are $\{J_x, J_y, J_z\} = \{2.37994, 0.427449, 0.128303\}$.

Following the same line of argument as for the XYZ case, we obtain explicit Lax operators in the bidiagonal form in Eq. (11):

$$s_n^x(u) \pm i s_n^y(u) = \mp\frac{\left(\tan\frac{\theta}{2}\right)^{\pm 1}}{\sin\gamma} e^{\pm i(u+n\gamma)},$$
$$s^z(u) = -\frac{1}{\sin\gamma}. \tag{25}$$

Equation (16) is satisfied with the same left boundary vectors as in Eq. (21), provided that the parameters u, θ in Eq. (25) relate to the spherical coordinates ϕ_l, θ_l of the left boundary polarization n_l via $\theta = \theta_l, u = \phi_l$. The right boundary vector is calculated using Eq. (17) with either $\epsilon = 1$ or 0, as discussed above.

VI. DISCUSSION

We have proposed an analytic method of constructing an inhomogeneous MPA on the basis of the local divergence condition in Eq. (6), by means of which we solve the driven dissipative problem in the Zeno regime for a quantum spin chain, with boundary spins kept in fixed arbitrary quantum states. Based on our results, we identified parameters allowing us to generate remarkably simple pure steady states with local magnetization described via Jacobi elliptic functions in Eq. (24) depicted in Fig. 1. These states are elliptic counterparts of spin-helix states [24–26], discussed recently in connection with cold atom experiments [20,27] from one side and in connection with remarkable underlying algebraic structure (phantom Bethe roots) [28–30] from the other side. In the XYZ model context, we can show that highly atypical quantum states of the type in Eq. (24) result from emergence of low-dimensional invariant subspaces in the spectrum of an open XYZ spin chain, under special choice of boundary fields [31].

Our results enable efficient study of steady-state properties of driven dissipative spin chains, reducing complexity from exponential degree 2^{2N} to polynomial degree N^2, thus allowing access to hydrodynamic scales. From the theoretical viewpoint, our construction is easily generalizable to other models satisfying the property in Eq. (6), see Ref. [24], e.g., for the Izergin-Korepin model.

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