Letter

Transitions from Abelian composite fermion to non-Abelian parton fractional quantum Hall states in the zeroth Landau level of bilayer graphene

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The electron-electron interaction in the Landau levels of bilayer graphene is markedly different from that of conventional semiconductors such as GaAs. We show that in the zeroth Landau level of bilayer graphene, in the orbital which is dominated by the nonrelativistic second Landau level wave function, by tuning the magnetic field, a topological quantum phase transition from an Abelian composite fermion to a non-Abelian parton fractional quantum Hall state can be induced at filling factors 1/2, 2/5, and 3/7. The parton states host exotic anyons that can potentially be utilized to store and process quantum information. Intriguingly, some of these transitions may have been observed in a recent experiment (K. Huang *et al.*, arXiv:2105.07058).

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Traditionally, semiconductor quantum wells such as those in GaAs/AlGaAs have been the system of choice to experimentally study fractional quantum Hall effect (FQHE) physics [1,2]. Graphene, with its relativistic dispersion and the presence of multiple components such as spins, valleys, orbitals, and layers adds to the richness of the FQHE phenomenology [3–10]. Aside from these features, in multilayer graphene systems the interactions between electrons can be controlled by parameters such as perpendicular magnetic and electric fields which can assist in stabilizing exotic FQHE states.

Robust even-denominator FQHE states with gaps of the order of a few degrees Kelvin have been observed in the zeroth Landau level (ZLL) of Bernal-stacked bilayer graphene (BLG) [11,12]. When the LL with $\mathcal{N}=0$ orbitals is partially filled, the Jain sequence of odd-denominator Abelian FQHE states described in terms of composite fermions (CFs) [13] is seen. On the other hand, in the LL with $\mathcal{N}=1$ orbitals, only the states at filling factors $\nu=1/3$, $2/3$, and $1/2$ are well established while at some other fractions signatures of FQHE were observed. More recently, Huang *et al.* [14] have observed extensive FQHE in the ZLL of BLG. Furthermore, they showed that transitions between FQHE states at $\nu=2/5$, $3/7$, and $1/2$ can be induced by varying the magnetic field or applying an electric field. The primary result of our work is to show that the transitions they observed are likely from Abelian CF to non-Abelian "parton" states. Encouragingly, these results suggest that BLG could potentially serve as a platform to host Fibonacci anyons which can perhaps form the building blocks of a universal fault-tolerant quantum computer.

Zeroth Landau level of bilayer graphene. The zero-energy manifold in BLG has eight LLs in it with two each coming from the spin ($|\uparrow\rangle, |\downarrow\rangle$), valley ($|+\rangle, |-\rangle$), and orbital ($\mathcal{N}=0,1$) degrees of freedom [15]. The ordering of these single-particle states and their orbital character can be varied by an interlayer electric field and a magnetic field, respectively [14,16–20]. The $\mathcal{N}=0$ LLs are identical to the $n=0$ LL [lowest LL (LLL)] of GaAs (n denotes the LL index of conventional semiconductors while $\mathcal{N}$ refers to the LL index for graphene). However, the $\mathcal{N}=1$ LLs have an admixture of $n=0$ and $n=1$ [second LL (SLL)] orbitals; at small (large) magnetic fields, their orbital nature is more $n=1$ ($n=0$) like.

We model the single-particle spinor wave function for the $\mathcal{N}=1$ LLs as $[\sin(\theta)\phi_1, \cos(\theta)\phi_0]$ [17], where ϕ_n is the wave function of a nonrelativistic electron in the LL indexed by n and θ is a tunable parameter called the mixing angle. The mixing angle is related to the magnetic field B as $\tan(\theta)=t\ell/(\sqrt{2}\hbar v_F)$, where t is the hopping integral (estimated to be ≈350 meV from calculations at zero magnetic field [21]), v_F is the Fermi velocity (typically 10^6 m/s in graphene), and $\ell=\sqrt{\hbar c/(eB)}$ is the magnetic length. There are three special values of θ that are of particular interest: (a) $\theta=0$ corresponds to the LLL, (b) $\theta=\pi/4$ corresponds to the first excited $\mathcal{N}=1$ LL of monolayer graphene (MLG1) [22], and (c) $\theta=\pi/2$ corresponds to the SLL of GaAs. Therefore, for FQHE physics, this simplified model suffices to cover all eight LLs since $\theta=0$ recovers the $\mathcal{N}=0$ LLs.

Throughout this Letter, we shall neglect the effects of screening by gates, rotation between the layers, valley-symmetry breaking, and disorder. We also neglect the effects of LL mixing and thus states related by particle-hole symmetry are considered on an equal footing. Furthermore, we shall restrict ourselves to only one-component states. In the case of two components, where the components can be considered as spins residing in the $n=0$ LL [therefore the interaction is SU(2) invariant], the spin-phase diagram of many FQHE states has been studied in detail in the past [22–29]. Recently, a detailed phase diagram of two-component states in the $n=0$ LL of double-layer graphene [two graphene layers separated by an insulator, such as hexagonal boron nitride (hBN), which breaks the SU(2) symmetry of the interaction] has been worked out both experimentally [30,31] and theoretically [32]. Under appropriate settings, these two-component states could also be stabilized in BLG. We leave out an exploration of multicomponent FQHE states in BLG for the future. Our attention will be solely focused on the single-component FQHE states that could arise in any of the LLs with $\mathcal{N}=1$ (denoted by the pseudospins $|1,\uparrow,\pm\rangle$ and $|1,\downarrow,\pm\rangle$). We refer to any of these LLs with $\mathcal{N}=1$ as the ZLL of BLG.

Parton states. The parton theory [33] was introduced by Jain as a generalization of his CF theory. In the parton theory, one imagines dividing the electron into q fictitious entities called partons. To construct a gapped state of the electrons, each of the partons is placed in an integer quantum Hall effect (IQHE) state at filling n_α, where $\alpha=1,2,\dots,q$ labels the various species of the partons. The resulting electronic state, denoted as "$n_1n_2n_3\cdots$," is described by the wave function

$$\Psi_\nu^{n_1n_2n_3\cdots}=\mathcal{P}_{\rm LLL}\prod_{\alpha=1}^{q}\Phi_{n_\alpha}(\{z_j\}), \tag{1}$$

where the coordinate of the jth electron is given by the complex number $z_j=x_j-iy_j$, Φ_n is the Slater determinant wave function for n-filled LLs of nonrelativistic electrons, and $\mathcal{P}_{\rm LLL}$ denotes projection into the LLL. We allow the parton fillings to be negative, which we denote by $\bar{n}$, with $\Phi_{\bar{n}}=\Phi_{-n}=\Phi_n^*$. In these states, the partons see a magnetic field that is antiparallel to that seen by the electrons. The parton theory can also capture compressible states. In particular, when $n\to\infty$, the wave function Φ_n describes the gapless Fermi sea (FS).

As the partons are unphysical objects they have to be glued back together to recover the physical electrons. This gluing procedure is already implemented in the wave function given in Eq. (1) since the different parton species coordinates z_j^α are all set equal to the electron coordinate z_j, i.e., $z_j^\alpha=z_j$ for all α. Each Φ_{n_α} in Eq. (1) is made up of *all* the electronic coordinates $\{z_j\}$. The density of each parton species is the same as the electronic density and all the partons see the same magnetic field that the electrons experience. Thus, the charge of the α parton species $e_\alpha=-e\nu/n_\alpha$, where $-e$ is the electronic charge. The parton charges add up to that of the electron, which results in the constraint $\nu=[\sum_{\alpha=1}^{q}n_\alpha^{-1}]^{-1}$. A parton state with a repeated factor of n, with $|n|\geqslant 2$, hosts excitations that carry non-Abelian braiding statistics [34].

The $\nu=1/r$ Laughlin state [35], described by the wave function $\Psi_{\nu=1/r}^{\rm Laughlin}=\Phi_1^r$, can be reinterpreted as the r-parton state where all the partons form a $\nu=1$ IQHE state. The $\nu=s/(2ps\pm1)$ Jain/CF state, described by the wave function $\Psi_{\nu=s/(2ps\pm1)}^{\rm Jain}=\mathcal{P}_{\rm LLL}\Phi_1^{2p}\Phi_{\pm s}$, can be viewed as a $(2p+1)$-parton state where one parton forms a $\nu=\pm s$ IQHE state and rest of the $2p$ partons form a $\nu=1$ IQHE state. The Rezayi-Read [36] wave function for the CF Fermi sea (CFFS) at $\nu=1/2$ can be interpreted as a "FS11" state, where one parton forms a Fermi sea and two partons form a $\nu=1$ IQHE state. Several parton states, beyond the Abelian Laughlin and Jain states, have been proposed as feasible candidates to describe FQHE plateaus that arise in the LLL [37–39], SLL [40–46], LLL of wide quantum wells [47], and in the LLs of graphene [10,44,46,48,49]. Furthermore, recently some very high-energy excited states have also been described in terms of partons [50].

Motivated by a recent experiment [14] we consider FQHE at $\nu=2/5$, 3/7, and 1/2 in the ZLL of BLG. The parton states that are relevant to these fillings are as follows: (a) $\nu=2/5$: (a1) 211, and (a2) $\bar{2}^31^4$ [42], which lies in the same universality class as the particle-hole conjugate of the three-cluster Read-Rezayi state [51] which supports Fibonacci anyons; (b) $\nu=3/7$: (b1) 311 and (b2) $\bar{3}^21^3$ [49], whose excitations, such as those of the $\bar{2}^31^4$, are also parafermionic; and (c) $\nu=1/2$: (c1) FS11 and (c2) $\bar{2}^21^3$ [40], which lies in the same topological phase as the anti-Pfaffian state [52,53]. The $\bar{2}^21^3$ state can be interpreted as a topological p-wave superconductor of CFs [40,54]. The aforementioned noninteracting CF states are known to be stabilized in the LLL [25] and MLG1 [9,10] while the non-Abelian parton states likely prevail in the SLL [32,40,42]. In the SLL of GaAs, FQHE has been observed at 2/5 and 1/2 [2,55–59] and some signatures of it have been seen at 3/7 [57].

Numerical results. All our calculations are carried out on the Haldane sphere [60]. In this geometry, N electrons move on the spherical surface in the presence of a radial magnetic flux of $2Qhc/e$ ($2Q$ is an integer) which is generated by a magnetic monopole placed at the center of the sphere. In the LL indexed by $\mathcal{N}$, the total number of single-particle orbitals is $2l+1$, where $l=Q+\mathcal{N}$ is the shell-angular momentum. FQHE ground states on the sphere occur when $2l=N/\nu-\mathcal{S}$, where $\mathcal{S}$ is a rational number called the shift [61]. The shift can often differentiate between candidate states occurring at the same filling. The shift of the parton state described by the wave function of Eq. (1) is $\mathcal{S}^{n_1n_2\cdots n_q}=\sum_{\alpha=1}^{q}n_\alpha$. Due to the rotational symmetry, the total orbital angular momentum L and its z component are good quantum numbers on the sphere. FQHE ground states are uniform, i.e., they have $L=0$ while excitations generically have $L>0$. Although the sphere is not the best geometry to study the gapless CFFS, in this work we will consider filled-shell CF states on the sphere that have previously been shown to serve as representatives of the uniform CFFS [29,36,62,63].

An important feature of an FQHE state is that it is incompressible, i.e., it has a nonzero gap to charge and neutral excitations. The charge gap, which can be accessed in transport experiments, gives the energy cost to create a far-separated pair of fundamental (smallest magnitude charge) quasiparticle and quasihole. From exact diagonalization (ED), the charge gap for a system in which the ground state of N electrons occurs at shell-angular momentum $2l$ can be obtained as $\Delta_c=[\mathcal{E}(2l-1)+\mathcal{E}(2l+1)-2\mathcal{E}(2l)]/n_q$, where $\mathcal{E}(2l)=E_0(2l)-N^2\mathcal{C}(2l)/2$. Here, $E_0(2l)$ is the Coulomb energy of the ground state of N electrons at $2l$, $\mathcal{C}(2l)$ is the average charging energy at $2l$ which accounts for the contribution of the background [43], and n_q is the number of fundamental quasiparticles (quasiholes) created upon the removal (insertion) of a flux quantum in the ground state. The neutral gap $\Delta_n=E_1(2l)-E_0(2l)$ is the difference in energies of the two lowest-lying states at the N and $2l$ corresponding to the ground state. All the gaps are quoted in units of $e^2/(\epsilon\ell)$, where ϵ is the dielectric constant of the host. We map the FQHE problem in the ZLL of BLG to a problem of electrons in the LLL interacting with a set of pseudopotentials $\{V_m\}$ [60], where V_m is the energy penalty for placing two electrons in a relative angular momentum m state in the ZLL of BLG. To allow for some variation in the interaction we shall carry out ED using the spherical

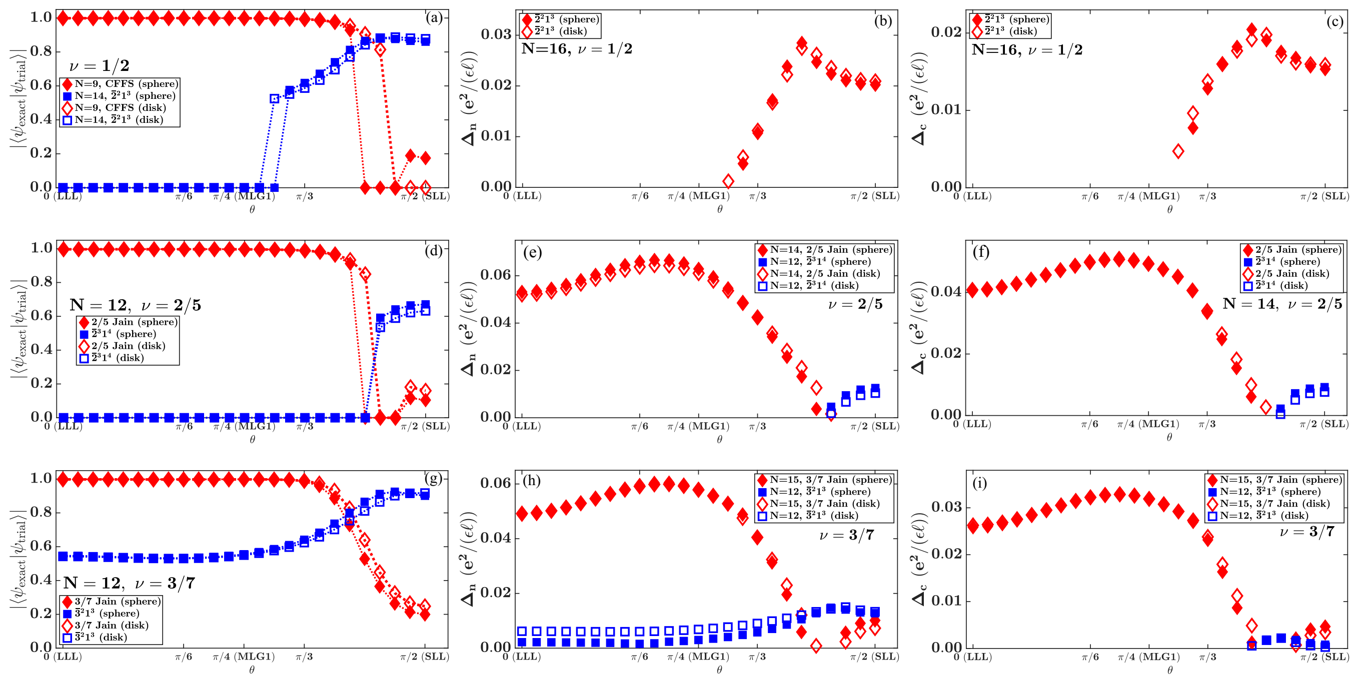

FIG. 1. Overlaps with the exact Coulomb ground state in the zeroth Landau level of bilayer graphene [(a), (d), and (g)], neutral gaps [(b), (e), and (h)], and charge gaps [(c), (f), and (i)] as a function of the mixing angle θ that parametrized the magnetic field for candidate states at $\nu = 1/2$ [(a)–(c)], 2/5 [(d)–(f)], and 3/7 [(g)–(i)] evaluated using the spherical (solid symbols) and disk (open) pseudopotentials for N electrons residing on the surface of a sphere. The gaps are only shown when they are positive and the corresponding ground state is uniform.

and planar disk pseudopotentials [see Supplemental Material (SM) [64]].

We obtain the Jain and CFFS states for small systems using a brute-force projection to the LLL. The parton states are constructed by evaluating *all* the $L = 0$ states for the corresponding system and expanding the parton wave function on the basis of all $L = 0$ states [44,64,65]. In Fig. 1 we show the overlaps of the exact Coulomb ground state in the ZLL of BLG with different candidate states and the charge and neutral gaps at $\nu = 2/5$, 3/7, and 1/2. From the high overlaps, as well as the nonzero charge gaps, we see that at low magnetic fields, i.e., in the vicinity of the SLL point, the non-Abelian parton states $\bar{2}^3 1^4$, $\bar{3}^2 1^3$, and $\bar{2}^2 1^3$ could be stabilized. On the other hand, at higher magnetic fields the CF states prevail. The charge and neutral gaps of the parton state at 3/7 are quite small, indicating that it is quite fragile (gaps also decrease with decreasing B). Strong finite-size effects are seen near the SLL point as can be deduced from the fact that we find $\Delta_c < \Delta_n$ while in the thermodynamic limit we expect $\Delta_c \geqslant \Delta_n$. In summary, at $\nu = 2/5$, 3/7, and 1/2 in the ZLL of BLG, in the vicinity of the SLL point the ground state is likely a non-Abelian parton state while in the rest of the parameter space, which includes LLL and MLG1 points, the ground state is a CF state.

Discussion. In a recent experiment [14] strong signatures of FQHE states at $\nu = 1/2$, 2/5, and 3/7 have been reported in the $\mathcal{N} = 1$ LLs of ultrahigh-quality BLG devices. Owing to the FQHE observed at half filling and a concomitant absence of it at many of the Jain fractions near $\nu = 1/2$, we propose that these plateaus could be described by the parton states considered in this work. From Fig. 1 we estimate the critical mixing angle at which the transition from an Abelian CF state to a non-Abelian parton state occurs at all three fillings to be in the vicinity of $\theta_c = 5\pi/12$. For typical parameters of graphene, this critical mixing angle corresponds to a magnetic field of $B_c \approx 7$ T. Since we have considered a simplified model and made several assumptions, this value of the critical field should only be viewed as a ballpark estimate.

TABLE I. The table gives some experimentally measurable properties of the various states that can arise at filling factors $\nu = 1/2$, 3/7, and 2/5 in the zeroth Landau level (ZLL) of bilayer graphene (BLG) as the magnetic field B is varied. The states are labeled using the notation given in Eq. (1). Using a simplified model for the interaction in the ZLL of BLG we estimate the critical value of the magnetic field B_c at which a transition from an Abelian composite fermion (CF) to a non-Abelian parton state is $B_c \approx 7$ T at all three fillings. The shift $\mathcal{S}$ on the sphere is related to the Hall viscosity $\eta_H = \hbar\nu\mathcal{S}/(8\pi\ell^2)$, κ_{xy} is the thermal Hall conductance in units of $[\pi^2 k_B^2/(3h)]T$ (filled LLs provide an additional integral contribution), and $\mathcal{Q}_{\rm qp}$ is the charge of the fundamental (smallest charged in magnitude) quasiparticle in units of the electron charge. The thermal Hall conductance of the CF Fermi sea (CFFS) is not expected to be quantized to a universal value since its bulk is gapless.

B field	ν	State	Nature of state	$\mathcal{S}$	κ_{xy}	$\mathcal{Q}_{\rm qp}$
$[0, B_c)$	2/5	$\bar{2}^3 1^4$	Non-Abelian	-2	$-4/5$	1/5
(B_c, ∞)	2/5	$211 \equiv 2/5$ Jain	Abelian	4	2	1/5
$[0, B_c)$	3/7	$\bar{3}^2 1^3$	Non-Abelian	-3	$-11/5$	1/7
(B_c, ∞)	3/7	$311 \equiv 3/7$ Jain	Abelian	5	3	1/7
$[0, B_c)$	1/2	$\bar{2}^2 1^3$	Non-Abelian	-1	$-1/2$	1/4
(B_c, ∞)	1/2	FS11, CFFS	Abelian	2		0

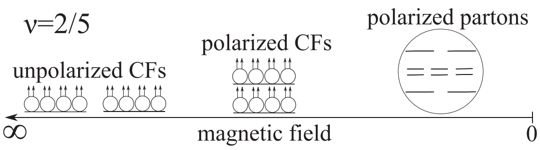

FIG. 2. Schematic representation of the candidate fractional quantum Hall phases that can arise at filling $\nu = 2/5$ in the zeroth Landau level of bilayer graphene as a function of the magnetic field. The circle and arrows together denote a composite fermion (CF) which is a bound state of an electron (circle) and two vortices (arrows). The partonic substructure is shown by the various partons filling different numbers of Landau levels (lines) inside the electron. In this work, we consider a single-component system and thereby focus solely on the transition between the polarized CF and parton states.

Now we discuss various experimentally measurable properties that can distinguish the CF and parton states. The non-Abelian nature of the $\bar{3}^2 1^3$ and $\bar{2}^3 1^4$ states does not cause a further fractionalization of their quasiparticle charge. This should be contrasted with the $\bar{2}^2 1^3$ state which does support a quasiparticle of charge $(-e)/4$ at $\nu = 1/2$. Therefore, at $\nu = 2/5$ and $3/7$ the fundamental quasiparticles of the parton and CF states both carry the same charge of $(-e)/5$ and $(-e)/7$, respectively.

Due to the presence of the $\bar{2}$ and $\bar{3}$ factors the parton states are expected to host upstream edge modes which can be detected experimentally [66–68]. In contrast, the CF states only carry downstream edge modes. Assuming a full equilibration of the edge states, the thermal Hall conductance κ_{xy} of an FQHE state at temperatures T much smaller than the gap is expected to be quantized as $\kappa_{xy} = c_{-}[\pi^2 k_B^2/(3h)]T$, where c_{-} is the chiral central charge [69]. The chiral central charge of all the CF states is integral while those of the parton states we considered fractional (see Table I). Recently, thermal Hall measurements have been carried out at many filling factors in GaAs [70,71] and graphene [72,73]. An extension of these experiments to the ZLL of BLG could help detect the partonic topological order.

The Hall viscosity, which measures the stress response of the FQHE state to perturbations of the underlying metric, is also anticipated to be quantized as [74] $\eta_H = \hbar\rho\mathcal{S}/4$, where $\rho = \nu/(2\pi\ell^2)$ is the electronic density and $\mathcal{S}$ is the shift [61]. Since the parton and CF states have different shifts, they carry different Hall viscosities. In Table I we have summarized these plausibly experimentally accessible properties of the CF and parton states at $\nu = 1/2$, $2/5$, and $3/7$. We note here that these phase transitions can potentially be studied using field theoretic techniques [75].

We mention here that in the spinful LLL, the ground state at $\nu = 2/5$ is a spin-singlet Jain state [28,29]. On the other hand, even in the spinful $\mathcal{N} = 1$ LL of MLG, the ground state is fully polarized, i.e., the interactions in the first excited LL of MLG are such that the CFs spontaneously polarize [22]. Therefore, in a two-component system in the ZLL of BLG, as the magnetic field is lowered, the FQHE state at $\nu = 2/5$ transitions from a spin-singlet CF state to a fully polarized one and eventually at low magnetic fields goes to a parton state. These transitions are schematically depicted in Fig. 2. Similarly, at $\nu = 3/7$ in the ZLL of BLG, at large magnetic fields, the ground state would be a partially polarized Jain state. Likewise, at $\nu = 1/2$, as the interaction is continuously tuned from the LLL to the SLL points in the half-filled ZLL of BLG, the unpolarized CFs first polarize, and then the polarized CFs pair up to form a *p*-wave superconducting state [20].

We have not considered $\nu = 1/3$ here since at all three special points, namely LLL, MLG1, and SLL, the ground state at one-third filling is believed to be Abelian [45,76–78]. Thus it is unlikely that a non-Abelian state is stabilized in the ZLL of BLG at $\nu = 1/3$. In the SM [64], we have considered transitions between different *Abelian* states at $\nu = 1/3$ in the ZLL of BLG. The $s/(4s \pm 1)$ states, such as at $\nu = 1/5$, $2/7$, and $2/9$, reside in the same topological phase as the corresponding Jain state at all three special points [10,44,78,79]. Thus, we expect the topological nature of the ground state at these fillings does not change as we transition from the very high to very low magnetic field limits in the ZLL of BLG.

In conjunction with previous works, our results show that for all the experimentally observed plateaus promising candidate parton wave functions can be constructed. Furthermore, it appears that the parton theory is sufficiently rich to capture all FQHE orders. More generally, it would be interesting to explore the possibility that structures inspired by the parton construction could aid in understanding other strongly correlated systems.

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