Letter

Kramers' degeneracy for open systems in thermal equilibrium

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Kramers' degeneracy theorem underpins many interesting effects in quantum systems with time-reversal symmetry. We show that the generator of dynamics for Markovian open fermionic systems can exhibit an analogous degeneracy, protected by a combination of time-reversal symmetry *and* the microreversibility (detailed balance) property of systems at thermal equilibrium—the degeneracy is lifted if either condition is not met. We provide simple examples of this phenomenon and show that the degeneracy is reflected in the single-particle Green's functions. Furthermore, we show that certain experimental signatures of topological edge modes in open many-body systems can be protected by microreversibility in the same way. Our results highlight the importance of *detailed balance* in characterizing open topological matter.

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Kramers' theorem is one of the oldest and most celebrated theorems in quantum mechanics. It states that the Hamiltonian of a system with half-integer total spin will have a pairwise degenerate spectrum if time-reversal symmetry (TRS) is preserved [1,2]. This simple claim carries deep implications, ranging from the quantum spin Hall effect [3,4] to the robustness of superconductivity in disordered materials [5].

In this paper we present an analogous degeneracy theorem that pertains to *dissipative* many-body fermionic systems. Dissipative systems propagate irreversibly in time, i.e., arbitrary initial states flow toward a (typically unique) steady state. Therefore it is not *a priori* obvious that TRS should have any bearing on dynamics in this context. However, if the open system is at thermal equilibrium, then TRS manifests itself in terms of microreversibility (also known as quantum detailed balance) [6–14], which has experimental consequences in the solid state [15–18]. Here we show that in Markovian open systems—which are governed by a Lindblad master equation with generator $\mathcal{L}$—the combination of TRS and microreversibility protect degeneracies in the spectrum of $\mathcal{L}$. An analogous nonequilibrium open system (e.g., one coupled to two reservoirs held at different temperatures) will not exhibit this degeneracy; see Fig. 1.

This degeneracy can be inferred in experiments that operate in a regime where coupling to an external environment cannot be ignored, such as noisy quantum simulators. Rather than being manifest in static properties of the steady state, the degeneracy of $\mathcal{L}$ can instead be inferred from time-dependent quantities describing the dynamical response about and/or approach to thermal equilibrium. As an example, we show that fermionic Green's functions and other related correlators [19] show signatures of the degeneracy, making our results directly observable in the solid state [20] and atomic systems [21].

Our theorem helps us identify an important connection between microreversibility and TRS-protected topological phases of matter. For closed systems, certain kinds of gapless edge modes can be attributed to Kramers' degeneracy [3,4,22]. Although the irreversible effects of coupling such systems to an environment can spoil some of their TRS-protected properties [23], here we demonstrate that other properties of the edge modes can persist in the open regime.

These results complement recent work regarding symmetries and topological phenomena in open systems [24–38]. Many of those studies are based on non-Hermitian Hamiltonians [39–46], for which degeneracy theorems akin to Kramers' are known [47–49], and have been central to explain symmetry-protected topological behavior. Our findings are distinct from such "non-Hermitian topological phenomena," which are primarily of relevance to photonic and classical mechanical systems [50]. Indeed, the Lindblad master equation formalism used here is much more widely applicable to interacting, dissipative quantum matter.

Kramers' theorem for Hamiltonians—We review Kramers' theorem in closed fermionic systems. Since fermions have half-integer spin, time reversal is implemented by an antiunitary operator T satisfying

$$T^2 = P, \quad P = (-1)^N, \quad [H, T] = 0, \tag{1}$$

where N is the total fermion number, and P is henceforth referred to as "parity." Note that $[P, T] = 0$. Physical Hamiltonians must conserve parity, $[H, P] = 0$, and so Fock space can be decomposed into even and odd parity sectors $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, wherein H and T are block diagonal

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad T = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} K. \tag{2}$$

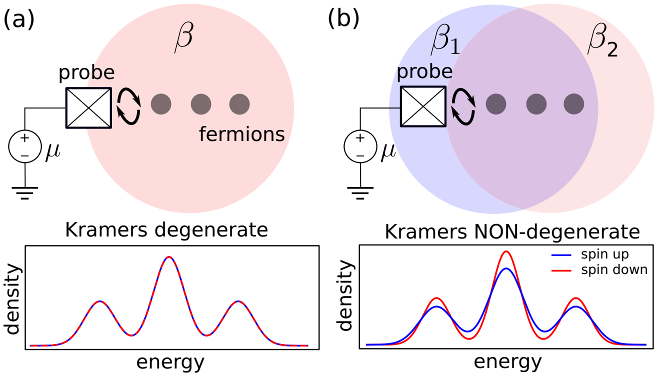

FIG. 1. (a) A fermionic system (three small circles) coupled to an environment (big circle) in thermal equilibrium at temperature β^{-1} has a Kramers' degeneracy. (b) The same system (three small circles) coupled to two baths (two big circles) at different temperatures β_1^{-1}, β_2^{-1} will not be Kramers' degenerate. This leads to a spin splitting of the Green's function (13), which can be experimentally detected using, e.g., spin-resolved tunneling spectroscopy (bottom panels). For both (a) and (b), the system-bath Hamiltonian obeys TRS: $[H_{SB}, T] = 0$.

Here, K takes the complex conjugate of any scalar to its right, and $U_\pm U_\pm^* = \pm\mathbb{I}$.

Kramers' theorem states that for a TRS-invariant Hamiltonian (i.e., when $THT^{-1} = H$), the eigenvalues of H_- must be twofold degenerate. Specifically, any eigenstate $|\psi_-\rangle$ with odd parity has a time-reversed partner $T|\psi_-\rangle$ which is also an eigenstate of the same energy and is orthogonal to $|\psi_-\rangle$. In contrast, H_+ is generically nondegenerate.

Symmetries of open quantum systems—For the purposes of this Letter, we will focus on Markovian open systems described by a density matrix ρ, whose dynamics is governed by a Lindblad master equation [51,52]

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_i (2L_i\rho L_i^\dagger - \{L_i^\dagger L_i, \rho\}). \quad (3)$$

Here the Hamiltonian part H describes the coherent evolution of the system, and the dissipators L_i arise from coupling to an environment. The generating superoperator $\mathcal{L}$ is often referred to as the "Lindbladian."

We define time-reversal and parity superoperators ($\mathcal{T}$ and $\mathcal{P}$, respectively) which specify how these symmetries transform the state $\rho(t)$. These act as $\mathcal{T}[\rho] = T\rho T^{-1}$ and $\mathcal{P}[\rho] = P\rho P^{-1}$. The space of operators can be split into even and odd superparity sectors $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_+ \oplus \mathcal{B}(\mathcal{H})_-$, where $\mathcal{B}(\mathcal{H})_\pm$ contains operators satisfying $\mathcal{P}(A) = \pm A$ (where A is an arbitrary operator). Specifically, $\mathcal{B}(\mathcal{H})_+$ contains operators of the form $|\text{even}\rangle\langle\text{even}|$ or $|\text{odd}\rangle\langle\text{odd}|$, and operators in $\mathcal{B}(\mathcal{H})_-$ are $|\text{even}\rangle\langle\text{odd}|$ or $|\text{odd}\rangle\langle\text{even}|$, which are traceless. (We use the convention $P|\text{even}\rangle = +|\text{even}\rangle$, $P|\text{odd}\rangle = -|\text{odd}\rangle$.) Since coherent superpositions of states with opposite fermion parity are not physical, the system density matrix must belong to $\mathcal{B}(\mathcal{H})_+$ (note that this does not prohibit classical mixtures of wave functions with opposite parity). Accordingly, physical generators must satisfy $[\mathcal{L}, \mathcal{P}] = 0$. By analogy to Eq. (2), the matrix representations of $\mathcal{L}$ and $\mathcal{T}$ then become block diagonal:

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \mathcal{U}_+ & 0 \\ 0 & \mathcal{U}_- \end{pmatrix} K. \quad (4)$$

From Eq. (1) we have $\mathcal{T}^2 = \mathcal{P}$, and hence $\mathcal{U}_\pm \mathcal{U}_\pm^* = \pm\mathbb{I}$. Evidently the superoperator $\mathcal{T}^2$ leaves operators in $\mathcal{B}(\mathcal{H})_-$ invariant only up to a (-1) phase. This contrasts with systems made up of bosonic or spin degrees of freedom, where $T^2 = \pm 1$, and hence $\mathcal{T}^2 = \mathbb{I}$, even for half-integer spins.

Since $\rho(t)$ belongs to $\mathcal{B}(\mathcal{H})_+$, it is often stated that the odd-superparity part of the Lindbladian $\mathcal{L}_-$ is unphysical. However, this is not true if we ask about the joint state of two fermionic systems. If system S and some probe R evolve independently under Lindbladians $\mathcal{L}_{S,R}$, then their joint state $\rho_{SR}(t)$ evolves as $\partial_t \rho_{SR} = (\mathcal{L}_S \otimes \text{id}_R + \text{id}_S \otimes \mathcal{L}_R)[\rho_{SR}]$. When the two are coupled, the overall superparity $\mathcal{P}_{\text{tot}} = \mathcal{P}_S \mathcal{P}_R$ must still be even ($\mathcal{P}_S$ and $\mathcal{P}_R$ are superparities for S and R, respectively), however the state $\rho_{SR}(t)$ may contain components which are odd under $\mathcal{P}_S$ and $\mathcal{P}_R$ separately, e.g., if a fermion is in a superposition between S and R. These components will evolve under $\mathcal{L}_{S,-}$ and $\mathcal{L}_{R,-}$. We will later identify specific physical observables that can be used to infer properties of $\mathcal{L}_-$. We now establish a degeneracy theorem analogous to Kramers' which applies to the spectrum of $\mathcal{L}_-$.

Kramers' degeneracy for Lindbladians—Because open quantum systems evolve irreversibly in time, TRS cannot be expected to play the same role as in closed systems. Indeed, if the Hamiltonian H is TRS invariant in the sense of Eq. (1), then we have $\mathcal{T}^{-1}\mathcal{L}\mathcal{T} = -\mathcal{L}$. This relation is incompatible with a nontrivial dissipative part of Eq. (3), which is negative semi-definite [53]. One way in which an antiunitary symmetry can be imposed on open systems without such an inconsistency is to demand that the Hamiltonian be *odd* under TRS, i.e., $THT^{-1} = -H$, in which case $\mathcal{T}^{-1}\mathcal{L}\mathcal{T} = \mathcal{L}$ can be satisfied [54]. Although such a symmetry is mathematically well defined, it does not physically correspond to time reversal in the closed system limit.

Instead, for systems in thermal equilibrium, time-reversal symmetry of the system and environment degrees of freedom naturally gives rise to a "microreversibility" property, otherwise known as detailed balance. This condition relates the rate of each possible physical process to the rate of its time-reversed process. Mathematically, a Lindbladian that respects detailed balance satisfies a superoperator equation [6,7,9]

$$\mathcal{L}^\dagger = \mathcal{Q}^{-1}\mathcal{T}^{-1}\mathcal{L}\mathcal{T}\mathcal{Q}, \quad (5)$$

where $\mathcal{Q}$ acts as $\mathcal{Q}[A] = qA$, and $q = \exp[-\beta H]/Z$ is the density matrix in the Gibbs ensemble at inverse temperature β with respect to the system Hamiltonian H (Z is the partition function). We have defined

$$\mathcal{L}^\dagger[\rho] = +i[H,\rho] + \sum_i (2L_i^\dagger \rho L_i - \{L_i^\dagger L_i, \rho\}), \quad (6)$$

which is the usual adjoint of $\mathcal{L}$ with respect to the Hilbert-Schmidt inner product $\langle A, B\rangle := \text{Tr}[A^\dagger B]/\text{Tr}\mathbb{I}$.

A simple example of dissipators satisfying (5) is a pair $L_1 = \sqrt{\gamma_1}V$, $L_2 = \sqrt{\gamma_2}V^\dagger$, where V is TRS invariant up to a phase, $\mathcal{T}[V] = e^{i\theta}V$ [55], and acts as a lowering operator $[H, V] = -\omega V$. The temperature is implicitly determined by $\gamma_1/\gamma_2 = e^{\beta\omega}$. In words, L_1 lowers the system energy by ω at a given rate, and L_2 does the opposite at a rate that differs by a Boltzmann factor. This is the essence of the detailed balance condition. More generally, Eq. (5) is naturally satisfied when a time-reversal symmetric system is coupled to a TRS-respecting environment at thermal equilibrium with temperature β^{-1} [56].

We now state the main claim of this work: Suppose a fermionic Lindbladian satisfies microreversibility via Eq. (5); then the odd-superparity superoperator $\mathcal{L}_-$ [see Eq. (4)] is guaranteed to have a twofold degenerate spectrum. Our proof proceeds as follows. The odd parity part of the Lindbladian satisfies

$$\mathcal{L}_-^\dagger = \mathcal{Q}_-^{-1}\mathcal{T}_-^{-1}\mathcal{L}_-\mathcal{T}_-\mathcal{Q}_-, \tag{7}$$

where $\mathcal{T}_-^2 = -\mathbb{I}$, $\mathcal{T}_- = \mathcal{U}_- K$. We drop the "−" index. Define right and left eigenoperators

$$\mathcal{L}(r_i) = \Lambda_i r_i, \quad \mathcal{L}^\dagger(l_i) = \Lambda_i^* l_i, \tag{8}$$

with $\mathrm{Tr}[l_i^\dagger r_j] = \delta_{ij}$. Substituting the expression for microreversibility into the left eigenoperator equation leads to

$$\mathcal{Q}^{-1}\mathcal{T}^{-1}\mathcal{L}\mathcal{T}\mathcal{Q}(l_i) = \Lambda_i^* l_i \Rightarrow \mathcal{L}\mathcal{T}\mathcal{Q}(l_i) = \Lambda_i \mathcal{T}\mathcal{Q}(l_i). \tag{9}$$

We find that r_i and $\mathcal{T}\mathcal{Q}(l_i)$ are both right eigenoperators of $\mathcal{L}$ with eigenvalue Λ_i. However, one can show that $\mathrm{Tr}[l_i^\dagger \mathcal{T}\mathcal{Q}(l_i)] = 0$ (see the Supplemental Material (SM) [56]). Since $\mathrm{Tr}[l_i^\dagger r_i] = 1$, we find that r_i and $\mathcal{T}\mathcal{Q}(l_i)$ are linearly independent eigenoperators, and hence the complex Lindblad spectrum in the odd superparity sector must be twofold degenerate.

In contrast to the Kramers' theorem for closed systems, our result explicitly relies on the presence of thermal equilibrium. Intuition can be gained from microscopic considerations: Linear response about a thermal state can be formulated in terms of the (Kramers' degenerate) eigenstates of the system-bath Hamiltonian. A system's response in thermal equilibrium should thus be sensitive to the TRS of the microscopic Hamiltonian. Our work captures this behavior from the perspective of the system's master equation.

Our analysis also suggests that a similar Kramers' degeneracy is present in the spectrum of a thermal quantum channel superoperator, which can describe the evolution of a system coupled to a non-Markovian bath [56].

Example: random quadratic Hamiltonian—We confirm the generalized Kramers' theorem via an example. Consider a system of spin-1/2 fermions with N twofold degenerate single-particle orbitals. The most general particle-conserving quadratic Hamiltonian is

$$H = \sum_{ij;\sigma,\sigma'} H_{ij,\sigma\sigma'} f_{i,\sigma}^\dagger f_{j,\sigma'} = \sum_{k=1}^{N}\sum_{\tau=\pm} \epsilon_k c_{k,\tau}^\dagger c_{k,\tau}, \tag{10}$$

with $i, j \in [1, \dots, N]$ and $\sigma, \sigma' \in \pm$. We impose a TRS on the Hamiltonian: $[H, T] = 0$, such that the single-particle spectrum is twofold degenerate, with $T f_{i,\sigma} T^{-1} = \sigma f_{i,-\sigma}$, $T c_{k,\tau} T^{-1} = \tau c_{k,-\tau}$. Let us define the following dissipators:

$$L_{1,pq} = \sqrt{\gamma_{1,pq}} \sum_{\tau=\pm} (c_{q,\tau}^\dagger c_{p,\tau} + \tau c_{q,\tau}^\dagger c_{p,-\tau}), \tag{11}$$

$$L_{2,pq} = \sqrt{\gamma_{2,pq}} \sum_{\tau=\pm} (c_{p,\tau}^\dagger c_{q,\tau} + \tau c_{p,-\tau}^\dagger c_{q,\tau}), \tag{12}$$

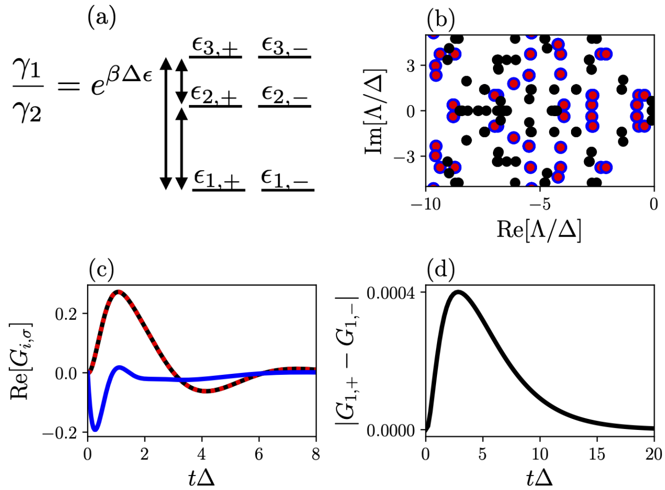

FIG. 2. (a) Three doubly degenerate energy levels ($N = 3$) labeled by $\epsilon_{i,\pm}$. TRS-respecting dissipators cause jumps between energy levels at a rate consistent with detailed balance in thermal equilibrium at temperature β^{-1}. (b) Lindblad spectrum for the model described in the main text (for states with 0, 1, or 2 occupied fermions) at a temperature $\beta\Delta = 1$, with $N = 3$, $\Delta_d/\Delta = 0.1$, and $g/\Delta = 1$. The odd-superparity sector (blue and red dots) is twofold degenerate, while the even sector (black dots) is not. (c) Real part of the retarded Green's function for same parameters as in (b). The Kramers' degeneracy ensures that $G_{i,\sigma} = G_{i,-\sigma}$ in the expectation value of the thermal steady state with one occupied fermionic mode: black line and red dashed line correspond to Re[$G_{1,+}$] and Re[$G_{1,-}$], respectively. In general, Green's function pairs other than $G_{i,\pm}$ are nondegenerate: blue line corresponds to Re[$G_{2,+}$]. (d) A system that is coupled to *two* thermal baths $g_1/\Delta = 1$, $g_2/\Delta = 0.4$ at different temperatures: $\beta_1\Delta = 1$, $\beta_2\Delta = 10$ [otherwise same parameters as in (b)]. Microreversibility is broken, hence the Green's functions for $i, +$ and $i, -$ split.

for $\gamma_{1,pq} = g[n_\beta(\epsilon_p - \epsilon_q) + 1]$, $\gamma_{2,pq} = g n_\beta(\epsilon_p - \epsilon_q)$, $n_\beta(\omega) = (e^{\beta\omega} - 1)^{-1}$ (the Bose function), and $\epsilon_p > \epsilon_q$. Physically, such terms can appear due to coupling to a bosonic bath (since the dissipators are quadratic in fermions). L_1 represents a process that lowers the energy of the system; L_2 raises the energy. The Ls satisfy: $[H, L_{1,pq}] = (\epsilon_q - \epsilon_p)L_{1,pq}$, $L_{2,pq} \sim L_{1,pq}^\dagger$, $\gamma_{1,pq}/\gamma_{2,pq} = e^{\beta(\epsilon_p - \epsilon_q)}$, and $[L_{1/2,pq}, T] = 0$, thus respecting microreversibility [see Fig. 2(a)]. We include these dissipators between each pair of energy levels in the system, and also consider uniform dephasing in the energy basis: $L_{d,i,\pm} = \sqrt{\Delta_d} c_{i,\pm}^\dagger c_{i,\pm}$ (preserves microreversibility). For a fixed number of fermions in the system, the thermal state is the unique steady state.

Figure 2(b) plots the Lindblad spectrum associated with a random TRS-respecting Hamiltonian ($|H_{ij}| \in [0, \Delta]$; Δ sets the energy scale) coupled to a thermal bath for states with up to two occupied fermionic modes. The spectrum of the odd superparity sector (blue and red dots) is indeed twofold degenerate, while the spectrum of the even sector (black dots) is not. Note that Kramers' theorem does not imply a degeneracy of the steady state.

How can an open system violate microreversibility (and hence break its degeneracy)? One obvious way involves a system-environment coupling that directly violates TRS.

However, a more subtle way to break degeneracy involves coupling the system to a nonequilibrium environment. Consider a system that is connected to *two* thermal baths, each at a different temperature. Even if all system-environment couplings respect TRS, the system will host a *nonequilibrium* (nonthermal) steady state, and hence microreversibility will be violated. We have numerically verified that the degeneracy of $\mathcal{L}_-$ for the spin-1/2 system is broken when coupled to baths at different temperatures. (We note that such nonequilibrium splitting requires quartic terms (quadratic dissipators) in the master equation, see the SM [56].)

Physical observables—As argued above, $\mathcal{L}_-$ only governs dynamics in scenarios where fermions can move between the system (S) and some external "probe" (R). Therefore, rather than looking at expectation values of the system density matrix $\mathrm{Tr}[O_S\rho_S(t)]$ (which are determined by $\mathcal{L}_+$), we suggest that the Kramers'-like degeneracy of $\mathcal{L}_-$ can be detected in the retarded Green's function of the steady state ρ_{SS}:

$$G^R_{i,\sigma}(t) = -i\Theta(t)\mathrm{Tr}[\{f_{i,\sigma}(t), f^\dagger_{i,\sigma}\}\rho_{\mathrm{SS}}]. \tag{13}$$

Here $\Theta(t)$ is the Heaviside step function, and we work in the Heisenberg picture where operators evolve as $A(t) = e^{\mathcal{L}^\dagger t}[A]$ (although this expression should be slightly modified for open systems coupled to fermionic baths; see the Supplemental Material [56]).

The Green's function (13) can be measured in solid state systems using, e.g., photoemission or tunneling spectroscopy, which indeed involve fermions moving in/out of the system [20]. Probing single-particle Green's functions in ultracold atoms is more challenging, but protocols involving stimulated Raman spectroscopy have been developed [21]. More concretely, $G^R_{i,\sigma}(t)$ is sensitive to the Kramers' degeneracy of $\mathcal{L}_-$ because $f_{i,\sigma} \in \mathcal{B}(\mathcal{H})_-$ is superparity odd, and so the time evolution of (13) is governed by $\mathcal{L}_-$. The generalized Kramers' theorem ensures the relation $G^R_{i,\sigma}(t) = G^R_{i,-\sigma}(t)$ [56], which we confirm numerically in Fig. 2(c). However, when the system is coupled to two baths at different temperatures, microreversibility is broken and the Green's functions differ for opposite spins [Fig. 2(d)]. We note that the Fourier transform of the temporal Green's function is directly probed in solid-state electron-tunneling experiments (see below, and the SM [56]).

It is rather natural that microreversibility has implications for response functions such as (13); indeed, the very definition of microreversibility is sometimes framed in terms of fluctuation-dissipation relations for steady-state correlators [6,18]. The above demonstrates that the correlation functions of superparity-odd operators have a particular structure associated with the Kramers' degeneracy of $\mathcal{L}_-$.

Degenerate zero-bias peak from microreversibility—Kramers' degeneracy plays an important role in determining the stability of symmetry-protected topological edge modes of Hamiltonians [3,57], e.g., the presence of spinful TRS ensures that a pair of Majorana zero modes in 1D cannot gap. For closed systems, degenerate Majorana modes are detectable via degenerate spin-resolved tunneling spectroscopy at the edge of the superconductor [19]. Here we show that the spin-resolved tunneling spectra remain unspilt if the superconductor is coupled to a thermal bath with TRS-respecting terms, while a splitting can arise if microreversibility is violated.

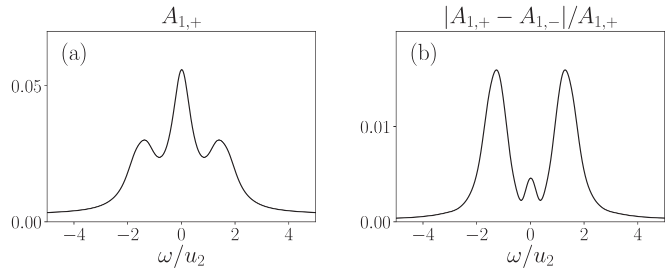

FIG. 3. Steady-state, spin-resolved spectral function at the boundary of the Majorana chain (14) coupled to two baths found via exact diagonalization. (a) Zero-bias peak due to the Majorana modes. The width of the peaks is set by the temperature. (b) Splitting between spin-up and spin-down spectral functions due to breaking of microreversibility via a nonequilibrium environment. Parameters: $N = 3, u_1/v_1 = 0.25, u_2/v_1 = 0.25, v_2/v_1 = 0.5, \beta v_1 = 3.3, g/v_1 = 0.05, g_{\mathrm{neq}}/v_1 = 2$.

Consider the following spin-1/2 Kitaev chain:

$$H = -i\sum_{j=1,\sigma=\pm}^{j=N}[u_1 a_{j,\sigma}b_{j,\sigma} + u_2 a_{j,\sigma}b_{j,-\sigma}] - i\sum_{j=1,\sigma=\pm}^{j=N-1}[v_1 b_{j,\sigma}a_{j+1,\sigma} + v_2 b_{j,\sigma}a_{j+1,-\sigma}], \tag{14}$$

where $a_{j,\sigma}, b_{j,\sigma}$ are Majoranas corresponding to site j with spin σ. This model can be diagonalized as $H = \sum_{j=1}^N\sum_{\tau=\pm}\epsilon_j d^\dagger_{j,\tau}d_{j,\tau}$, where $Td_{j,\tau}T^{-1} = \tau d_{j,-\tau}$. We include the same thermal dissipators as before [see Eq. (11)] between each pair of energy levels p, q in the system at inverse temperature β. While such dissipators are manifestly nonlocal, they can arise from local system-bath coupling [53]. The weak-coupling Markovian approximation is commonly used in studying topological matter connected to a thermal bath [58,59]. To break microreversibility, we consider a nonequilibrium bath that removes pairs of fermions on each site: $L_j = \sqrt{g_{\mathrm{neq}}}\psi_{j,+}\psi_{j,-}$, where we define the complex fermion $\psi_{j,\sigma} = a_{j,\sigma} + ib_{j,\sigma}$. These dissipators obey $[L_j, T] = 0$ but do not evolve the system toward a thermal state.

The spectral degeneracy (or its splitting) can be experimentally detected using spin-resolved tunneling spectroscopy [20] (see Fig. 1). For a zero-temperature probe, the current-voltage characteristic of the tunnel junction is $\partial I_\sigma/\partial\mu \propto A_{1,\sigma}(-\mu)$, where $A_{i,\sigma}(\omega) = -\mathrm{Im}[G^R_{i,\sigma}(\omega)]$ is the spectral function, I_σ is spin-σ current, and μ is the chemical potential (see the SM [56]). Figure 3 plots the spin-up spectral function for the probe attached to the boundary, and the relative difference between the spin-up and spin-down spectral functions for the chain coupled to two baths. The zero-bias peak in the topological phase corresponds to the boundary-localized Majoranas. The splitting that emerges between the spin-up and spin-down spectral functions is due to the nonequilibrium setup. This splitting vanishes if the nonequilibrium bath is turned off and micoreversibility is restored (not shown). While isolating a Majorana chain from its larger environment is a difficult task, our result suggests that signatures of the TRS-protected Majorana modes should still be observable in experiment provided that the environment is *thermal*.

Conclusions and outlook—We have proved a generalization of Kramers' theorem for open quantum systems, and shown that it has implications for symmetry-protected topological phases.

Future work should further investigate the role that detailed balance plays in protecting topological signatures and phases of open quantum systems. For example, it is known that non-Hermitian generalizations of time-reversal symmetry can prevent the non-Hermitian skin effect [39], i.e., extreme spectral sensitivity to boundary conditions [43,60]. Can microreversibility guarantee the absence of the skin effect in Lindbladians?

Imposing TRS on a Hamiltonian allows us to identify topologically distinct ground states which cannot smoothly evolve into one another without closing the energy gap [22]. It is currently unclear whether microreversibility has similar implications for the topological properties of the steady-state density matrix [37].

Finally, we note that microreversibility may also have interesting implications for the random-matrix theory of open systems [61–64], e.g., the non-Hermitian random matrix generalization of class AII might constrain the spectral statistics of the odd-superparity sector of microreversible Lindbladians [65].

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