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Letter

Dissipation of moving vortices in thin films

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Moving vortices in thin superconducting films are considered within the time-dependent London description. The dissipation due to out-of-core normal excitations for two vortices moving together turns out to have a minimum for the separation vector $\boldsymbol{a}$ parallel to the velocity and equal to $a_m \approx 2.2\Lambda$, where Λ is the Pearl length. The minimum entropy production suggests that moving vortices should have a tendency to form chains along the velocity with a period of the order a_m.

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Introduction. Problems of vortex dynamics in superconductors have recently come back to the community attention because new and more accurate experimental techniques become available. Vortex velocities well above the speed of sound are now attainable along with more sensitive methods of measuring field distributions [1–3].

Moving vortices, pushed by the Lorentz force due to applied transport current, dissipate energy replenished by the current source. In this situation, the heat transfer should be taken into account [2], to mention one of the complications. One of the facts attracting attention is that moving vortices tend to form chains extended along the velocity [1,4]. The chains have periods $a \gg \xi$, the vortex core size so that the linear London approach may provide useful insights notwithstanding the London inability to treat the vortex core physics.

In this Letter, we study the dissipation $\mathcal{W}$ due to out-of-core quasiparticles in thin films and find that for a pair of vortices $\mathcal{W}(\boldsymbol{a})$ has a minimum at a finite separation $\boldsymbol{a}$ oriented along the pair velocity $\boldsymbol{v}$ (see Ref. [5]). The value of this separation is $a_m \approx 2.2\Lambda$ with the Pearl length $\Lambda = 2\lambda^2/d$ (λ is the penetration depth of the film material, and d is the film thickness). According to the principle of minimum entropy production (or minimum dissipation) in stationary processes [7] the system of moving vortices should have a tendency to form chains along the velocity in which vortices sit at the dissipation minima.

Within the general approach to slow relaxation processes, one relates the time derivative of whatever quantity is relaxing, say Ψ, to the variational derivative of the free-energy functional $\mathcal{F}(\Psi)$, see e.g., Ref. [8],

$$-\chi\frac{\partial\Psi}{\partial t} = \frac{\delta\mathcal{F}}{\delta\Psi}, \tag{1}$$

where χ is the proper relaxation time. The quantity of interest in our case is the vortex field distribution $\boldsymbol{h}(\boldsymbol{r}, t)$ away of the vortex core where the London approach holds and the energy (magnetic + kinetic) is $\mathcal{F} = \int d^2\boldsymbol{r}[h^2 + \lambda^2(\mathrm{curl}\,\boldsymbol{h})^2]/8\pi$ [9],

$$-\chi\frac{\partial\boldsymbol{h}}{\partial t} = \frac{\delta\mathcal{F}}{\delta\boldsymbol{h}}. \tag{2}$$

This yields

$$-\chi\frac{\partial\boldsymbol{h}}{\partial t} = \frac{1}{4\pi}(\boldsymbol{h} - \lambda^2\nabla^2\boldsymbol{h}), \tag{3}$$

which reduces to the common London equation in equilibrium.

The relaxation constant χ is obtained by comparison with the time-dependent London equation [10], which at distances larger relative to the core size is obtained assuming that the current consists of the normal and superconducting parts,

$$\boldsymbol{J} = \sigma\boldsymbol{E} - \frac{2e^2|\Psi|^2}{mc}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\nabla\theta\right), \tag{4}$$

where $\boldsymbol{A}$ is the vector potential, Ψ is the order parameter, θ is the phase, ϕ_0 is the flux quantum, $\boldsymbol{E}$ is the electric field, and σ is the conductivity associated with normal excitations. At these distances, $|\Psi|$ is a constant and acting on Eq. (4) by curl one obtains [10]

$$\boldsymbol{h} - \lambda^2\nabla^2\boldsymbol{h} + \tau\frac{\partial\boldsymbol{h}}{\partial t} = \phi_0\hat{\boldsymbol{z}}\sum_\nu \delta(\boldsymbol{r} - \boldsymbol{r}_\nu), \tag{5}$$

where $\boldsymbol{r}_\nu(t)$ is the position of the νth vortex that may depend on time t, $\hat{\boldsymbol{z}}$ is the direction of vortices. The relaxation time,

$$\tau = 4\pi\sigma\lambda^2/c^2. \tag{6}$$

Comparing this with Eq. (3) one has $\chi = 4\pi\tau$. In fact, the time-dependent Ginzburg-Landau equations can be obtained in a similar manner [8].

Thin films. Let the film of thickness d be on the xy plane. Integration of Eq. (5) over the film thickness gives for the z component of the field for a Pearl vortex moving with velocity $\boldsymbol{v}$,

$$\frac{2\pi\Lambda}{c}\text{curl}_z\boldsymbol{g} + h_z + \tau\frac{\partial h_z}{\partial t} = \phi_0\delta(\boldsymbol{r} - \boldsymbol{v}t). \tag{7}$$

Here, $\boldsymbol{g}$ is the sheet current density related to the tangential field components at the upper film face by $2\pi\boldsymbol{g}/c = \hat{z}\times\boldsymbol{h}$; $\Lambda = 2\lambda^2/d$ is the Pearl length. With the help of $\text{div}\,\boldsymbol{h} = 0$ this equation is transformed to

$$h_z - \Lambda\frac{\partial h_z}{\partial z} + \tau\frac{\partial h_z}{\partial t} = \phi_0\delta(\boldsymbol{r} - \boldsymbol{v}t). \tag{8}$$

As was stressed by de Gennes [9] and Pearl [11], the problem of a vortex in a thin film is reduced to that of the stray field distribution in free space subject to the boundary condition (8) at the film surface. Since outside the film $\text{curl}\,\boldsymbol{h} = \text{div}\,\boldsymbol{h} = 0$, one can introduce a scalar potential for the *outside* field,

$$\boldsymbol{h} = \nabla\varphi, \qquad \nabla^2\varphi = 0. \tag{9}$$

The general form of the potential satisfying the Laplace equation and vanishing at $z \to \infty$ is

$$\varphi(\boldsymbol{r}, z) = \int\frac{d^2\boldsymbol{k}}{4\pi^2}\varphi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz}, \tag{10}$$

that is checked by direct differentiation. Here, $\boldsymbol{k} = (k_x, k_y)$, $\boldsymbol{r} = (x, y)$, and $\varphi(\boldsymbol{k})$ is the two-dimensional (2D) Fourier transform of $\varphi(\boldsymbol{r}, z = 0)$.

As performed in Ref. [10], one applies the 2D Fourier transform to Eq. (8) to obtain a linear differential equation for $h_{z\boldsymbol{k}}(t)$, the solution of which is

$$h_{z\boldsymbol{k}} = -k\varphi_{\boldsymbol{k}} = \frac{\phi_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{1 + \Lambda k - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}. \tag{11}$$

For two vortices separated by $\boldsymbol{a}$, the right-hand side of Eqs. (7) and (8) is

$$\phi_0[\delta(\boldsymbol{r} - \boldsymbol{v}t) + \delta(\boldsymbol{r} - \boldsymbol{a} - \boldsymbol{v}t)], \tag{12}$$

so that we obtain for the field,

$$h_{z\boldsymbol{k}} = \frac{\phi_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}(1 + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}})}{1 + \Lambda k - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}. \tag{13}$$

Electric field and dissipation for slow motion. This field is found from quasistationary Maxwell equations $\text{curl}\,\boldsymbol{E} = -\partial_t\boldsymbol{h}/c$ and $\text{div}\,\boldsymbol{E} = 0$ [8,12], which yield in 2D Fourier space,

$$E_{x\boldsymbol{k}} = -\frac{k_y}{k_x}E_{y\boldsymbol{k}} = -\frac{ik_y}{ck^2}\frac{\partial h_{z\boldsymbol{k}}}{\partial t}. \tag{14}$$

For a pair of vortices separated by $\boldsymbol{a}$, we have

$$\frac{\partial h_{z\boldsymbol{k}}}{\partial t} = -i\frac{\phi_0(\boldsymbol{k}\cdot\boldsymbol{v})(1 + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}})}{1 + \Lambda k - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}. \tag{15}$$

We are interested in motion with constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ so that we can evaluate the fields at $t = 0$, i.e., the factor $e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$ can be omitted. Then, Eqs. (14) and (15) yield

$$E_{x\boldsymbol{k}} = \frac{\phi_0 v}{c}\frac{k_y k_x(1 + e^{-i\boldsymbol{k}\boldsymbol{a}})}{k^2(1 + \Lambda k)},$$
$$E_{y\boldsymbol{k}} = -\frac{\phi_0 v}{c}\frac{k_x^2(1 + e^{-i\boldsymbol{k}\boldsymbol{a}})}{k^2(1 + \Lambda k)}. \tag{16}$$

Since the prefactor here contains v, in linear approximation in velocity, the term $i\boldsymbol{k}\cdot\boldsymbol{v}\tau$ in denominators can be discarded for slow motion.

The dissipation power follows:

$$\mathcal{W} = \sigma d\int d^2\boldsymbol{r}\,E^2 = \sigma d\int\frac{d^2\boldsymbol{k}}{4\pi^2}(|E_{x\boldsymbol{k}}|^2 + |E_{y\boldsymbol{k}}|^2)$$
$$= \frac{\phi_0^2 v^2\sigma d}{2\pi^2 c^2}\int\frac{d^2\boldsymbol{k}\,k_x^2(1 + \cos\boldsymbol{k}\boldsymbol{a})}{k^2(1 + k\Lambda)^2}. \tag{17}$$

We now go to dimensionless $\boldsymbol{q} = \Lambda\boldsymbol{k}$,

$$\frac{\mathcal{W}}{\mathcal{W}_0} = \int\frac{d^2\boldsymbol{q}\,q_x^2(1 + \cos\boldsymbol{q}\boldsymbol{R})}{q^2(1 + q)^2} = W_1 + W_2, \tag{18}$$

where $\mathcal{W}_0 = \phi_0^2 v^2\sigma d/2\pi^2 c^2\Lambda^2$ and $\boldsymbol{R} = \boldsymbol{a}/\Lambda$. The first contribution,

$$W_1 = \int\frac{d^2\boldsymbol{q}q_x^2}{q^2(1 + q)^2} = \pi\,\ln\frac{1}{e\xi}, \tag{19}$$

where the upper limit of the divergent integral over q is taken as $1/\xi$ to avoid the vortex core (ξ here is the dimensionless core size). The second contribution is

$$W_2 = \int\frac{d^2\boldsymbol{q}q_x^2\cos\boldsymbol{q}\boldsymbol{R}}{q^2(1 + q)^2}$$
$$= \int_0^\infty\frac{dq\,q}{(1 + q)^2}\int_0^{2\pi}d\phi\,\cos^2\phi\,\cos[qR\,\cos(\phi - \alpha)], \tag{20}$$

with ϕ being the azimuth of $\boldsymbol{q}$ and α is the angle between $\boldsymbol{R} = \boldsymbol{a}/\Lambda$ and X. After substitution $\beta = \phi - \alpha$, the angular integral takes the form

$$\int_0^{2\pi}d\beta\,\cos^2(\beta + \alpha)\cos(qR\,\cos\beta)$$
$$= 2\pi\left(\frac{J_1(qR)}{qR} - J_2(qR)\cos^2\alpha\right), \tag{21}$$

where $J_{1,2}$ are Bessel functions of the first kind. The integration over q can be performed analytically resulting in a cumbersome combination of Bessel and hypergeometric functions. We avoid this by performng this integration numerically. The contours of $\mathcal{W}_2(X, Y) = \text{const}$ are shown in Fig. 1; contours of the total dissipation $\mathcal{W} = \text{const}$ are, in fact, the same because W_1 is a coordinate-independent constant.

A surprising feature of this plot is the two minima at the X axis situated symmetrically relative to the origin ($\boldsymbol{v}$ is along X). One of these minima is shown in Fig. 2 where $W_2(X, 0)$ is plotted to indicate the minimum position at $X_m \approx 2.2$. To see a clear picture of the dissipation $\mathcal{W}(\boldsymbol{a}) = W_2(\boldsymbol{a}) + \text{const}$, we also show the three-dimensional (3D) version of the same result in Fig. 3.

For an arbitrary velocity, one has to keep the term $ik_x v\tau$ in denominators of electric-field components (16). One then obtains

$$\frac{\mathcal{W}}{\mathcal{W}_0} = \int\frac{d^2\boldsymbol{q}\,q_x^2(1 + \cos\boldsymbol{q}\boldsymbol{R})}{q^2[(1 + q)^2 + q_x^2 S^2]}. \tag{22}$$

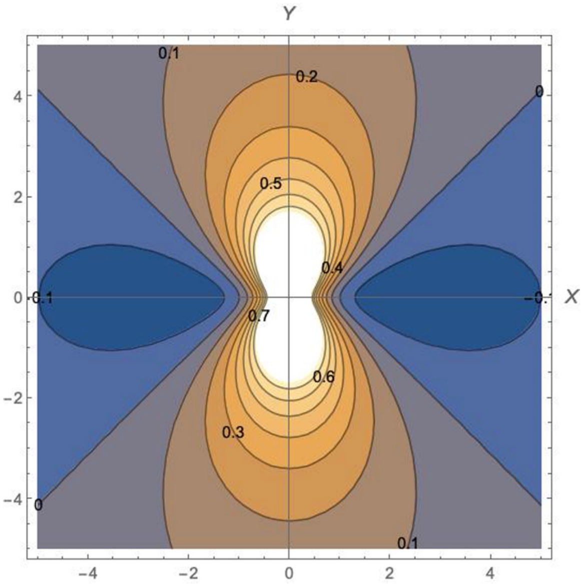

FIG. 1. Contours of constant dissipation $W_2(X, Y)$ for a pair of vortices; one at the origin and the other at $(X, Y) = (a_x, a_y)/\Lambda$ moving with the same velocity along the X axis.

The dimensionless parameter,

$$S = v\,\frac{2\pi\sigma d}{c^2} \tag{23}$$

is small even for vortex velocities exceeding the speed of sound presently attainable [1–3] if one takes for the estimate the conductivity σ of normal quasiparticles as equal to the normal-state conductivity. Unfortunately, there is not much experimental information about the T dependence of σ. Theoretically, this question is still debated, e.g., Ref. [13] discusses possible strong enhancement of σ due to inelastic scattering.

We employ the fast Fourier transform (FFT) to evaluate the integral (22). The position X_m of the minimum of $\mathcal{W}(X, 0)$ for each S was obtained from the contour plot similar to Fig. 1, which was sliced out of the 2D map obtained from the cosine term of Eq. (22) via 2D FFT. The result is shown in Fig. 4. Hence, for $S \lesssim 0.2$, which is the domain of our interest, the minimum is practically in the same place at $X_m = x_m/\Lambda \approx 2.2$ [5].

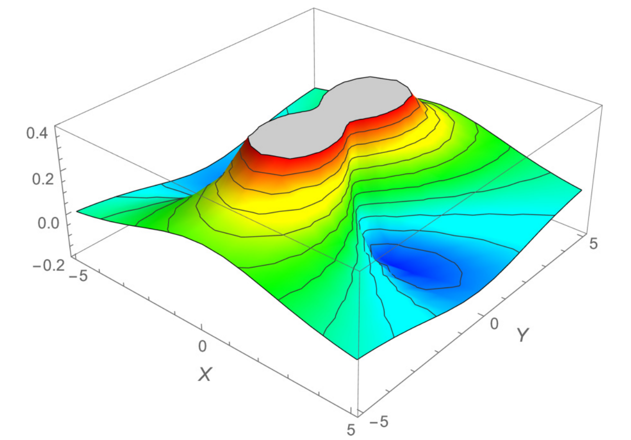

FIG. 3. The 3D plot of $W_2(X, Y)$ for the velocity along the X axis. $(X, Y) = (a_x, a_y)/\Lambda$.

Discussion. Hence, the dissipation $\mathcal{W}$ of two vortices separated by $\boldsymbol{R} = (X, Y)$ depends on the pair orientation relative to the velocity $\hat{\boldsymbol{v}}$ and on the pair size R. The numerically evaluated dissipation $\mathcal{W}(X, Y)$ is shown in Fig. 3. The dissipation power has a minimum if the pair is oriented parallel to $\boldsymbol{v}$ and the vortices are separated by $a_m \approx 2.2\Lambda$.

The physical reason for this minimum can be traced to the magnetic structure of a single moving vortex. It was shown in Refs. [6,10] that the magnetic field is depleted in front of the moving vortex and enhanced behind it due to induced currents of normal excitations. If two vortices move so that one follows the other and $\boldsymbol{a} \parallel \boldsymbol{v}$, in the space between them the depletion of the second is compensated by the enhancement due to the leader. The resulting magnetic-field variation in this space is weaker than for a single vortex. Then the electric field induced in this intervortex region $\boldsymbol{E} \propto \partial_t \boldsymbol{h} \propto (\boldsymbol{v}\cdot\nabla\boldsymbol{h})$ is suppressed along with the dissipation. Clearly, this simple argument does not work if the pair orientation differs from $\boldsymbol{a} \parallel \boldsymbol{v}$.

As remarked in the Introduction, the dissipation of two moving vortices has also been considered in our earlier work [6], however, the minimum of it, the main result of the current Letter, had not been found. A formal reason for this omission

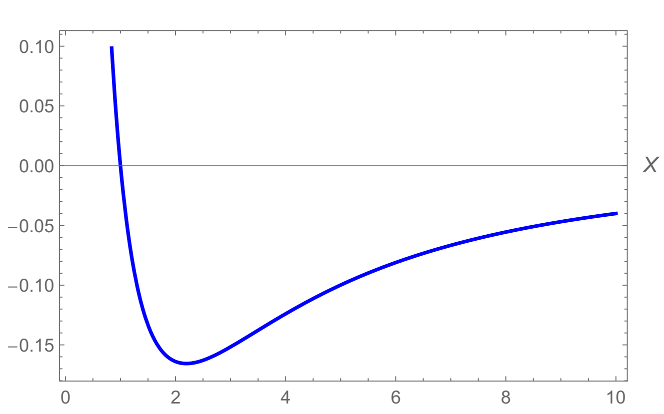

FIG. 2. $W_2(X, 0)$ vs X for the velocity along the X axis. $S = 0.1$, and X is in units of Λ.

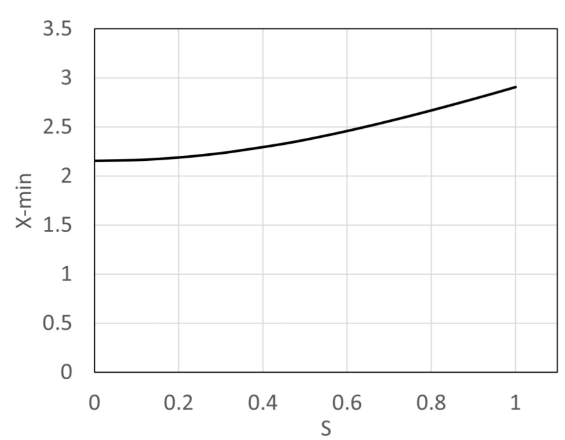

FIG. 4. The minimum position X_m vs S.

was that for the minimum to be visible, the divergent part (19) had to be subtracted in advance. Instead, we have chosen to evaluate the divergent double integral (17) "brute-force" numerically, an uncontrollable procedure. Hence, the part of Ref. [6] related to dissipation of two moving vortices is, in fact, incorrect.

Moving vortices in Pb films were studied in Ref. [1]. The penetration depth of bulk Pb is $\lambda \approx 96$ nm and the film thickness $d = 75$ nm so that the Pearl length $\Lambda \approx 250$ nm. Vortices driven across the thin-film bridge by a transport current are reported to form chains along the velocity with spacing a depending on the distance from the bridge edge. Since the driving current decreases with the distance x from the edge, the vortex velocity depends on x as well. The team [1] was able to estimate both $v(x)$ and $a(x)$.

According to our model, the pair of moving vortices dissipates the least if it is oriented along the velocity and separated by $a_m \approx 2.2\Lambda$. One can expect the chain of vortices to have a period of the order a_m. Taking the experimental estimate of Λ we obtain $a_m \approx 540$ nm. In the experiment [1] the chain period varies from $\approx$1500 nm near the bridge edge to $\approx$600 nm (for the set of data with the transport current 18.9 mA). Hence, the order of magnitude provided by our model is correct. In other words, the idea that the chain period is dictated by the minimum of dissipation agrees qualitatively with observations.

From the data [1], close to the bridge edge the chain period $a \approx 1.5$ μm and the velocity $v \approx 16$ km/s, i.e., the ratio $a/v \approx 10^{-10}$ s. On the other hand,

$$\frac{a_m}{v} = 2.2\frac{\Lambda}{v} = \frac{4\pi\sigma\lambda^2}{c^2 S}, \tag{24}$$

where we replaced the velocity with S of Eq. (23). Taking for a_m/v the experimental ratio $a/v \approx 10^{-10}$ s and $\lambda \approx 96$ nm, we estimate the conductivity of normal excitations $\sigma \approx (3 \times 10^{19} S)$ s^{-1}. With $S \sim 10^{-2}$ this gives the Pb conductivity that again suggests a qualitative relevance of our model.

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