Letter

Floquet engineering of Lie algebraic quantum systems

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We propose a "Floquet engineering" formalism to systematically design a periodic driving protocol in order to stroboscopically realize the desired system starting from a given static Hamiltonian. The formalism is applicable to interacting and noninteracting quantum systems which have an underlying closed Lie algebraic structure. Unlike previous attempts at Floquet engineering, our method produces the desired Floquet Hamiltonian at any driving frequency and is not restricted to the fast or slow driving regimes. The approach is based on Wei-Norman ansatz, which was originally proposed to construct a time-evolution operator for any arbitrary driving. Here, we apply this ansatz to the micromotion dynamics, defined within one period of the driving, and engineer the functional form and operators of the driving protocol by fixing the gauge of the micromotion. To illustrate our idea, we use a two-band system or the systems consisting of two sublattices as a testbed. Particularly, we focus on engineering the cross-stitched lattice model that has been a paradigmatic flat-band model.

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Introduction. Floquet formalism [1] has been instrumental to study the dynamic evolution of a system subjected to a periodic driving. The dynamics is decomposed into two parts a time-periodic part describing the micromotion of the system within a period, and an effective stroboscopic part governed by a static "Floquet Hamiltonian." The problem of reverse engineering the driving protocol in order to obtain a desired Floquet Hamiltonian stroboscopically from a given *simple* static Hamiltonian is known as *Floquet engineering*. It has garnered a lot of attention over the past several years and has been applied in different experimental paradigms [2–6]. Floquet engineered solid-state materials have been discussed extensively to develop "quantum matter on demand" by controlling post-semiconductor materials [7–9] and several *exotic* properties like unconventional superconductivity [10,11], topologically nontrivial band structures [12], etc. have been realized. Moreover, the effect of periodic driving has been studied on a variety of timely solid-state systems such as Luttinger liquid [13], superconducting circuit [14], bilayer graphene [15], and strongly correlated electrons (Mott materials) [16].

Most of these studies investigated the effect of periodic driving, using a square-wave or sinusoidal protocol, on a given system in either the high [17–21] or low [22,23] driving frequency regime. However, a systematic theory of designing a driving protocol such that the desired Floquet Hamiltonian obtained exactly at *any* driving frequency is still missing in the literature.

In this Letter, we propose to bridge this gap by formulating a theory of Floquet engineering for a class of systems whose Hamiltonians have any closed Lie algebraic structure. Our formalism is based on the Wei-Norman ansatz, which was originally proposed to obtain the dynamics for any time-dependent system [24,25]. Since the form of the long-time evolution part is already known from Floquet theory, we massage the Wei-Norman ansatz to the micromotion part of the dynamics.

The requirement of a Lie algebraic structure is not a drawback, but Hamiltonians of several important classes of solid state systems obey this structure. Hamiltonians for noninteracting two-band systems in any dimension follow the SU(2) algebra [26]. Even interacting one- (two-) dimensional models of unconventional chiral p-wave superconductors at the mean-field level [27,33–35] or Weyl semimetals [36] belong to the SU(2) class. Three-band systems like the kagome and Lieb lattice obey SU(3) algebra [27]. Furthermore, high-T_c superconductors that can be modeled by a half-filled Hubbard Hamiltonian have an underlying SO(4) symmetry. The Bethe ansatz solution of these models show that its elementary excitation can be separated into two fundamental excitations: spinons and holons/antiholons, which reflects that the original SO(4) symmetry can be separated into $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ symmetry [37,38]. Another direct example of the importance of the Lie algebraic structure is a strongly interacting system described by a Tomonaga-Luttinger liquid that obeys SU(1,1) algebraic structure and has been studied under periodic driving [39].

Our formalism can not only be used to control the dynamics of these systems in presence of driving but could also help design the *full* driving protocol to realize these complex Hamiltonians stroboscopically using simple static Hamiltonians [27]. We first outline our general formalism and then illustrate the main idea by designing a driving protocol to realize the *cross-stitched* lattice, an interesting two-band

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system with one band dispersive and another flat, from a static site Hamiltonian [40,41].

Formalism. The Hamiltonian of a generic periodically driven quantum system reads

$$H(t) = H_0 + V(t), \quad V(t+T) = V(t), \tag{1}$$

where H_0 and $V(t)$ are the static Hamiltonian and the driving potential with time-period T, respectively. The corresponding time-dependent Schrödinger equation (TDSE) is

$$i\frac{dU(t)}{dt} = H(t)\,U(t), \quad (\hbar = 1). \tag{2}$$

The operator $U(t)$ is the unitary time-evolution operator. According to the Floquet theorem, the solution of the TDSE can always be expressed as

$$U(t) = P(t)\,e^{-iH^{\rm eff}t}, \tag{3}$$

where the *micromotion operator* $P(t+T) = P(t)$ describes the dynamics of the system within one period $[t, t+T]$ and $H^{\rm eff}$ is a static Hamiltonian that governs the long-time dynamics of the system. The initial condition $U(0) = \mathbb{1}$ imposes the condition $P(0) = \mathbb{1}$, whereas the time-periodicity gives $P(nT) = \mathbb{1}$ for every $n \in \mathbb{Z}^+$ (positive integers). Consequently, we have $U(nT) = e^{-iH^{\rm eff}nT} = [e^{-iH^{\rm eff}T}]^n = [U(T)]^n$. If the dynamics of the system is observed stroboscopically at $t = nT$, then it is governed by the effective static Hamiltonian $H^{\rm eff}$.

Obtaining the analytic quantum evolution for any Hamiltonian is highly nontrivial and hence we restrict ourselves to those Hamiltonians whose operators form a closed Lie algebra, i.e.,

$$H_0 = h_0\mathbb{1} + \sum_{\alpha=1}^{N} h_\alpha A_\alpha = h_0\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{A}, \tag{4}$$

$$V(t) = f_0(t)\mathbb{1} + \sum_{\beta=1}^{N} f_\beta(t)A_\beta = f_0(t)\mathbb{1} + \boldsymbol{f}(t)\cdot\boldsymbol{A}, \tag{5}$$

where $\cdot$ denotes the standard scalar product. Above, h_0 and $\boldsymbol{h}$ (column vector with elements h_α and dimension N) are time-independent parameters, whereas $f_0(t)$ and $\boldsymbol{f}(t)$ are time-dependent functions due to the external field. The column vector of the linear operators $\boldsymbol{A} = \{A_\alpha\}$ forms a finite N-dimensional *simple* Lie algebra $\mathcal{L}_N$, which satisfies

$$[A_\alpha, A_\beta] = \sum_{\gamma=1}^{N} \Lambda^{\gamma}_{\alpha\beta} A_\gamma, \tag{6}$$

where Λ's are the structure constants of the algebra $\mathcal{L}_N$. From the Floquet engineering perspective, the underlying Lie algebraic structure will be exploited to design a driving scheme $\{f_0(t), \boldsymbol{f}(t)\}$ to achieve a desired effective Hamiltonian $H^{\rm eff}$ for a given initial static Hamiltonian H_0.

The Wei-Norman ansatz [24,25], i.e., expressing the full evolution operator as a product of exponentials, has been successfully applied to solve the TDSE for a driven quantum system. In our case, since we are particularly interested in Floquet engineering wherein $H^{\rm eff}$ is known, the natural choice is to apply the ansatz to the micromotion operator,

$$P(t) = e^{-im_0(t)}\left(\prod_{\alpha=1}^{N} e^{-im_\alpha(t)A_\alpha}\right). \tag{7}$$

The initial condition and the time-periodic property of $P(t)$ imposes following conditions: $m_0(nT) = 2\nu n\pi$ and $e^{-im_\alpha(nT)A_\alpha} = \mathbb{1}$ for all $\alpha = 1, \ldots, N$, $n = 0, 1, 2, \ldots$, and any integer ν. Besides, we have a gauge freedom to choose any time-dependent functional form of $m_\alpha(t)$. Using the above form of $P(t)$, if we substitute $U(t)$ in the TDSE [Eq. (2)], we get the relations between the driving protocols $\{f_0(t), \boldsymbol{f}(t)\}$ and the functions $\{m_0(t), \boldsymbol{m}(t)\}$ as

$$\begin{aligned}[h_0 + f_0(t)] + [\boldsymbol{h} + \boldsymbol{f}(t)]\cdot\boldsymbol{A} \\ = \dot{m}_0(t) + \boldsymbol{\zeta}(\boldsymbol{m}, \dot{\boldsymbol{m}})\cdot\boldsymbol{A} + P(t)\,H^{\rm eff}\,P^\dagger(t).\end{aligned} \tag{8}$$

Here, the components of the column vector $\boldsymbol{\zeta}$ are linear functions of $\dot{\boldsymbol{m}}(t) = \{dm_\alpha(t)/dt\}$ and nonlinear functions of $\boldsymbol{m}(t)$. Therefore, we can always express $\boldsymbol{\zeta}(\boldsymbol{m}, \dot{\boldsymbol{m}}) = \boldsymbol{\mathcal{M}}_1(t)\cdot\dot{\boldsymbol{m}}$, where $\boldsymbol{\mathcal{M}}_1(t)$ is a $N \times N$ matrix whose elements are nonlinear functions of $\boldsymbol{m}$. This nonlinearity is decided by the underlying Lie algebra. Consider the general form $H^{\rm eff} = h_0^{\rm eff}\mathbb{1} + \boldsymbol{h}^{\rm eff}\cdot\boldsymbol{A}$, the last term on the right hand side of Eq. (8) can also be represented in terms of the operators $\boldsymbol{A}$ as

$$P(t)\,H^{\rm eff}\,P^\dagger(t) = h_0^{\rm eff}\mathbb{1} + \boldsymbol{\xi}(\boldsymbol{m}, \boldsymbol{h}^{\rm eff})\cdot\boldsymbol{A}. \tag{9}$$

The vector $\boldsymbol{\xi}(\boldsymbol{m}, \boldsymbol{h}^{\rm eff})$ is a linear function of $\boldsymbol{h}^{\rm eff}$, but a nonlinear function of $\boldsymbol{m}(t)$, i.e., $\boldsymbol{\xi}(\boldsymbol{m}, \boldsymbol{h}^{\rm eff}) = \boldsymbol{\mathcal{M}}_2(t)\cdot\boldsymbol{h}^{\rm eff}$ where $\boldsymbol{\mathcal{M}}_2(t)$, similar to $\boldsymbol{\mathcal{M}}_1(t)$, is a matrix whose elements are nonlinear functions of $\boldsymbol{m}$. Using Eqs. (8) and (9) and equating the coefficients of the operators, we get

$$\begin{aligned} h_0 + f_0(t) &= \dot{m}_0(t) + h_0^{\rm eff}, \\ \boldsymbol{h} + \boldsymbol{f}(t) &= \boldsymbol{\mathcal{M}}_1(t)\cdot\dot{\boldsymbol{m}}(t) + \boldsymbol{\mathcal{M}}_2(t)\cdot\boldsymbol{h}^{\rm eff}, \end{aligned} \tag{10}$$

where $\boldsymbol{\mathcal{M}}_1(nT) = \boldsymbol{\mathcal{M}}_2(nT) = \mathbb{1}$ for $n = 0, 1, 2, \ldots$. The gauge freedom in the micromotion operator makes $\boldsymbol{\mathcal{M}}_1(t)$ and $\boldsymbol{\mathcal{M}}_2(t)$ nonunique, but it can be fixed at any arbitrary time $t \neq nT$ by choosing an appropriate gauge. According to Wei-Norman, if $\mathcal{L}_N$ is not a solvable algebra, the transformation matrices $\boldsymbol{\mathcal{M}}_1(t)$ and $\boldsymbol{\mathcal{M}}_2(t)$ could be ill-defined for an arbitrary representation. Therefore, unless we find a representation which is globally well-defined, we cannot apply the Wei-Norman ansatz to design the driving protocol.

Our Lie algebraic Floquet engineering protocol can be applied to any system having an underlying finite dimensional closed algebra. We now apply this formalism to an arbitrary two-bands system that naturally follows the SU(2) algebra. Here we are particularly focusing on this algebra because a large class of interacting and noninteracting systems obey this symmetry [26,27,33–35,42–44]. In principle, this formalism can also be applied to multiband systems, but the complexity of the problem increases with the number of bands (see supplementary [27] for the three-band case).

Two-bands systems. In the momentum space ($\boldsymbol{k}$-space), the Hamiltonian of the periodically driven two-band systems can be written in terms of the Nambu spinors $\Psi_{\boldsymbol{k}} = (a_{\boldsymbol{k}}, b_{\boldsymbol{k}})^{\rm T}$ as

$$H(t) = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} H_{\boldsymbol{k}}(t) \Psi_{\boldsymbol{k}}, \tag{11}$$

where $H_k(t) = H_{k0} + V_k(t)$ and $V_k(t+T) = V_k(t)$. The components of the Nambu spinors $a_k(a_k^\dagger)$ and $b_k(b_k^\dagger)$ are respectively representing the annihilation (creation) operators corresponding to the valence and conduction bands. The time-independent part H_{k0} and the time-periodic $V_k(t)$ can be expressed in general as

$$\begin{aligned} H_{k0} &= h_{k0}\mathbb{1} + \boldsymbol{h}_k \cdot \boldsymbol{S}, \\ V_k(t) &= f_{k0}(t)\mathbb{1} + \boldsymbol{f}_k(t) \cdot \boldsymbol{S}. \end{aligned} \tag{12}$$

The operators $2\boldsymbol{S} = \boldsymbol{\sigma}$ follow SU(2) algebra, where the components of $\boldsymbol{\sigma}$ are the Pauli matrices. This finite dimensional algebra facilitates the application of the Wei-Norman formalism to study the dynamics of two-band systems.

The unsolvable SU(2) algebra has two well-known representations: XYZ representation with $\boldsymbol{S}_{\mathrm{XYZ}} = (S_x, S_y, S_z)^{\mathrm{T}}$ and $\pm Z$ representation with $\boldsymbol{S}_{\pm Z} = (S_+, S_-, S_z)^{\mathrm{T}}$, where $S_\pm = (S_x \pm iS_y)$. For an arbitrary choice of $\boldsymbol{A}$, e.g., $\boldsymbol{A} = \boldsymbol{S}_{\mathrm{XYZ}}$, it is not guaranteed that the time-dependent functions $m_\alpha(t)$ appearing in the micromotion operator, Eq. (7), are smooth continuous functions for all time t [24,25]. However, following Ref. [45], we later show that the $(\pm Z)$ representation gives a globally well-defined $\boldsymbol{\mathcal{M}}_1(t)$ matrix. Therefore, for the two-bands case, Wei-Norman ansatz along with a proper choice of representation $\boldsymbol{S} \equiv \boldsymbol{S}_{\pm Z}$ can be applied to design a driving scheme, with arbitrary driving frequency, to achieve the desired effective Hamiltonian from a static Hamiltonian.

Floquet engineering protocol. We now provide the basic steps to Floquet engineer a two-band system, where the desired stroboscopic Hamiltonian is $H_k^{\mathrm{eff}} = h_{k0}^{\mathrm{eff}}\mathbb{1} + \boldsymbol{h}_k^{\mathrm{eff}} \cdot \boldsymbol{S}$. The protocol is divided into three essential steps.

(1) Wei-Norman ansatz. Use $U_k(t) = P_k(t)e^{-iH_k^{\mathrm{eff}}t}$ via Floquet theorem, and apply the Wei-Norman ansatz to construct the micromotion operator

$$P_k(t) = e^{-im_{k0}}e^{-im_{k+}S_+}e^{-im_{k-}S_-}e^{-im_{kz}S_z}. \tag{13}$$

The function m_{k0} is real, but the other functions $(m_{k\pm}, m_{kz})$ are complex. The explicit time-dependence of the functions m has been suppressed for notational simplicity. Also note that, the last three terms of the above expression are not individually unitary, but their product is unitary which imposes

$$\begin{aligned} \mathrm{Im}[m_{kz}] &= \ln(1 + |m_{k+}|^2), \\ m_{k-} &= \frac{m_{k+}^*}{1 + |m_{k+}|^2}. \end{aligned} \tag{14}$$

The above condition reduces the seven independent parameters (real m_{k0} and real and imaginary parts of $m_{k\pm,z}$) to four. We choose $(m_{k0}, m_{k+}, m_{k+}^*, m_{kz}^{\mathrm{R}})$, where $m_{kz}^{\mathrm{R}} = \mathrm{Re}[m_{kz}]$, as the independent variables and the micromotion operator reads

$$P_k(t) = \frac{e^{-im_{k0}}}{\sqrt{1 + |m_{k+}|^2}} \begin{pmatrix} e^{-\frac{i}{2}m_{kz}^{\mathrm{R}}} & -im_{k+}e^{\frac{i}{2}m_{kz}^{\mathrm{R}}} \\ -im_{k+}^* e^{-\frac{i}{2}m_{kz}^{\mathrm{R}}} & e^{\frac{i}{2}m_{kz}^{\mathrm{R}}} \end{pmatrix}.$$

(2) Transformation matrices. Consider $\boldsymbol{h}_k^{\mathrm{eff}} = \{h_{k-}^{\mathrm{eff}}, h_{k+}^{\mathrm{eff}}, h_{kz}^{\mathrm{eff}}\}$ in $(\pm Z)$ representation. Substituting $U_k(t)$ in the TDSE and using Eq. (10), we obtain $\boldsymbol{\mathcal{M}}_1$ and $\boldsymbol{\mathcal{M}}_2$ for a given k as

$$\boldsymbol{\mathcal{M}}_{k1} = \frac{1}{1 + |m_{k+}|^2} \begin{pmatrix} 1 & 0 & im_{k+} \\ 0 & 1 & -im_{k+}^* \\ im_{k+}^* & -im_{k+} & 1 - |m_{k+}|^2 \end{pmatrix}, \tag{15}$$

$$\boldsymbol{\mathcal{M}}_{k2} = \frac{1}{1 + |m_{k+}|^2} \begin{pmatrix} e^{-im_{kz}^{\mathrm{R}}} & -iq_k m_{k+} & im_{k+} \\ iq_k^* m_{k+}^* & e^{im_{kz}^{\mathrm{R}}} & -im_{k+}^* \\ -2q_k^* & -2q_k & 1 - |m_{k+}|^2 \end{pmatrix},$$

with $q_k = im_{k+}e^{im_{kz}^{\mathrm{R}}}$. The above form ensures that these matrices are identity at $t = nT$ for $n = 0, 1, 2, \ldots$.

(3) Driving protocol. For the Floquet engineering protocol, a bare minimal Hamiltonian is considered as the initial static Hamiltonian H_0. For example, here we set $\boldsymbol{h} = 0$, i.e. $H_0 = h_0\mathbb{1}$. Therefore, using the previous two steps, the driving functions are

$$\begin{aligned} h_0 + f_{k0}(t) &= \dot{m}_{k0} + \boldsymbol{h}_{k0}^{\mathrm{eff}}, \\ \boldsymbol{f}_k(t) &= \boldsymbol{\mathcal{M}}_{k1} \cdot \dot{\boldsymbol{m}}_k + \boldsymbol{\mathcal{M}}_{k2} \cdot \boldsymbol{h}_k^{\mathrm{eff}}. \end{aligned} \tag{16}$$

The transformation matrices are well-defined at all times t (implicit dependence in m_k) and not just stroboscopically [see from Eq. (15)]. Moreover, the globally well-defined $\boldsymbol{\mathcal{M}}_k\ \forall \boldsymbol{k}$ ensures the driving protocol is well-defined for all times. It is worth emphasizing our Floquet engineering protocol is *exact* in the driving frequency ω and does not require frequency-based perturbative expansions that lead to nonconvergent series [17,18,20,21].

Guiding principle to fix the gauge of the micromotion operator. The gauge for the micromotion is fixed by choosing $\{m_{k0}, \boldsymbol{m}_k\}$ satisfying the boundary conditions at $t = nT$: $e^{-im_{k0}\mathbb{1}} = e^{-im_{k+}S_\pm} = e^{-im_{kz}S_z} = \mathbb{1}, \forall n$. This can be achieved in various ways and here we illustrate a physically motivated gauge choice. We first consider the natural choice of a separable form in which each $m_k(t)$ is a product of momentum and time-dependent functions, such that $\{m_{k0}(t), \boldsymbol{m}_k(t)\} = \{\phi_{k0}\mu_0(t), \phi_{k+}\mu_+(t), \phi_{k+}^*\mu_+^*(t), \phi_{kz}^{\mathrm{R}}\mu_z^{\mathrm{R}}(t)\}$. Furthermore, we set $\phi_{kz}^{\mathrm{R}} = 1$ and $\phi_{k+} = e^{ik}$ suggesting that intra sublattice hopping is suppressed during the micromotion and only inter sublattice hopping is allowed. Consequently, Eq. (16) simplifies as

$$\begin{aligned} f_{k0}(t) &= \phi_{k0}\dot{\mu}_0(t) + \boldsymbol{h}_{k0}^{\mathrm{eff}}, \\ \boldsymbol{f}_k(t) &= \tilde{\boldsymbol{\mathcal{M}}}_{k1} \cdot \dot{\boldsymbol{\mu}}_k(t) + \tilde{\boldsymbol{\mathcal{M}}}_{k2} \cdot \boldsymbol{h}_k^{\mathrm{eff}}, \end{aligned} \tag{17}$$

where

$$\tilde{\boldsymbol{\mathcal{M}}}_{k1} = \frac{1}{1 + |\mu_+(t)|^2} \begin{pmatrix} e^{ik} & 0 & i\mu_+(t)e^{ik} \\ 0 & e^{-ik} & i\mu_+^*(t)e^{-ik} \\ i\mu_+^*(t) & -i\mu_+(t) & 1 - |\mu_+(t)|^2 \end{pmatrix},$$

and $\boldsymbol{\mathcal{M}}_{k2} \to \tilde{\boldsymbol{\mathcal{M}}}_{k2}$, defined in Eq. (15), with $m_{kz}^{\mathrm{R}} \to \mu_z^{\mathrm{R}}(t)$ and $m_{k+} \to \mu_+(t)e^{ik}$. We now set $\mu_0(t) = a_0 \sin(\omega t)$, $\mu_+(t) = a_+e^{i\theta}\sin(\omega t)$, and $\mu_z^{\mathrm{R}}(t) = p\omega t$ where p is any integer. This choice respects the boundary conditions and ensures the frequency of all the time-dependent functions equals $\omega = 2\pi/T$. The real amplitudes $\{a_0, a_+\}$, the phase factor $e^{i\theta}$, and the integer p are arbitrary that depend on the physical system as shown below with a specific example. For any arbitrary values of the parameters a_0, a_+, θ, and p, the matrices $\tilde{\boldsymbol{\mathcal{M}}}_{k1}$

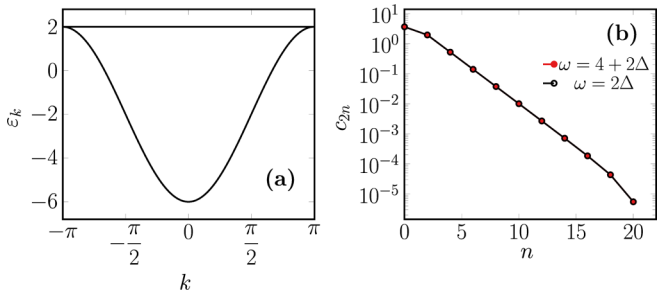

FIG. 1. (a) Band diagram of the cross-stitch lattice for $\alpha = 1.0$ and $\Delta = 2.0$. (b) Fourier coefficients of the envelope function for $a_+^2 = 2.0$.

and $\tilde{\mathcal{M}}_{k2}$ are globally well-defined which ensures the validity of the Wei-Norman ansatz for all k and t.

Application. We now apply our Floquet engineering protocol to realize the *cross-stitch lattice* Hamiltonian H_k^{eff}, which is a two-band system whose one band is dispersionless (flat) and the other is dispersive [40,41]. In the momentum space, our target Hamiltonian is $H_k^{\text{eff}} = h_{k0}^{\text{eff}}\mathbb{1} + h_k^{\text{eff}}(S_+ + S_-)$, where $h_{k0}^{\text{eff}} = -2\alpha\cos(k)$ and $h_k^{\text{eff}} = -(2\alpha\cos(k) + \Delta)$. The energy of the flat band is Δ and the dispersive band is $-4\alpha\cos(k) - \Delta$ [see Fig. 1(a)].

We choose our initial bare static Hamiltonian $H_{k0} = h_{k0}\mathbb{1}$, where $h_{k0} = -2\alpha\cos(k)$, which describes two uncoupled sublattices with each sublattice being a 1D chain with zero onsite energy. The parameter α determines the nearest neighbor hopping strength in each of the sublattices. The choice of the simple static Hamiltonian reduces the complexity of the expressions and we obtain $h_{k0}^{\text{eff}} = 0$ implying $f_{k0}(t) = \phi_{k0}\dot{\mu}_0(t)$ using Eq. (17). As mentioned earlier, the gauge can be fully set with a physical model and hence in this case we have a freedom to set $\phi_0(k) = 0$ and $\theta = 0$. Thus we have $f_{k0}(t) = 0\ \forall t$ and the function $\mu_+(t)$ becomes real.

Using Eq. (17) and the relations $f_{kx}(t) = 2\,\text{Re}[f_{k-}(t)]$ and $f_{ky}(t) = -2\,\text{Im}[f_{k-}(t)]$, we obtain the driving functions in the XYZ representation as

$$\begin{aligned}
f_{kx}(t) &= f_e(t)\big[a_+\omega\,\mathrm{C}_{\omega t}\mathrm{C}_k + h_k^{\text{eff}}\,\mathrm{C}_{p\omega t} - a_+ p\omega\,\mathrm{S}_{\omega t}\mathrm{S}_k \\
&\quad + a_+^2 h_k^{\text{eff}}\,\mathrm{C}_{2k+p\omega t}\mathrm{S}_{\omega t}^2\big], \\
f_{ky}(t) &= -f_e(t)\big[a_+\omega\,\mathrm{C}_{\omega t}\mathrm{S}_k - h_k^{\text{eff}}\,\mathrm{S}_{p\omega t} + a_+ p\omega\,\mathrm{S}_{\omega t}\mathrm{C}_k \\
&\quad + a_+^2 h_k^{\text{eff}}\,\mathrm{S}_{2k+p\omega t}\mathrm{S}_{\omega t}^2\big], \\
f_{kz}(t) &= f_e(t)\left[p\omega\left(1 - \frac{a_+^2}{2}\right) + \frac{p\omega a_+^2}{2}\mathrm{C}_{2\omega t}\right. \\
&\quad \left. + 4a_+ h_k^{\text{eff}}\,\mathrm{S}_{k+p\omega t}\mathrm{S}_{\omega t}\right],
\end{aligned} \tag{18}$$

where $f_e(t) = (1 + a_+^2\,\mathrm{S}_{\omega t}^2)^{-1}$, $\mathrm{C}_w = \cos(w)$ and $\mathrm{S}_w = \sin(w)$. In the above expression, we have two free parameters: a real parameter a_+ and an integer p. We set these two parameters such that each of the driving function does not have any static part. First, we set $a_+^2 = 2$ which removes the first term of the driving function $f_{kz}(t)$. Next we set $p = 3$, which is the minimal integer that ensures absence of any static term in the driving protocols [46]. The above protocol in lattice space

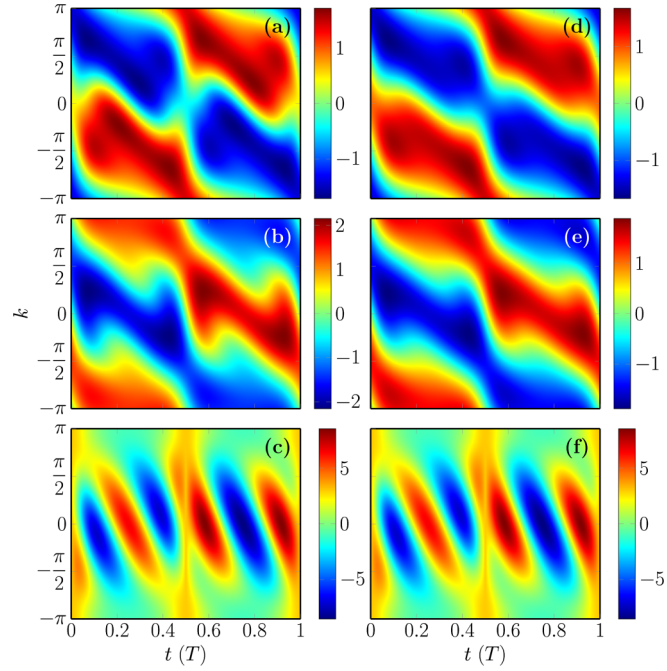

FIG. 2. Density plot of the driving functions $f_{kx}(t)$ [(a) and (d)], $f_{ky}(t)$ [(b) and (e)], and $f_{kz}(t)$ [(c) and (f)] are plotted as a function of momentum k and time t for $\omega = 4 + 2\Delta = 8$ [(a)–(c)] and $\omega = 2\Delta = 4$ [(d)–(f)].

turns out to be *local* involving only the next-to-next nearest neighbors, ensuring experimental feasibility (see Ref. [27]).

We consider two moderate (same order of the band gap) cases of the driving frequency: $\omega = 4 + 2\Delta = 8$ and $\omega = 2\Delta = 4$. For these two cases, the Fourier coefficients are shown in Fig. 1(b) as a function of the coefficient indices. For both frequencies, the odd coefficients c_{2n+1} are zero and the even coefficients c_{2n} fall exponentially with n. Therefore the envelope can be realized with high accuracy considering only a few even harmonics. Figure 2 illustrates all the driving functions, given by Eq. (18), for driving frequency $\omega = 4 + 2\Delta = 8$ [panels (a)–(c)] and $\omega = 2\Delta = 4$ [panels (d)–(f)]. Clearly, these are not simple functions having sine or cosine periodicity in time t as typically considered in the literature.

Conclusion. We have introduced a Floquet engineering protocol applicable to systems whose Hamiltonians have an underlying Lie algebraic structure. A large number of physically relevant interacting and noninteracting models in any dimension fall into this class of systems. In our formalism, we have applied the Wei-Norman ansatz [25] to the micromotion part of the Floquet dynamics, and from that, we have prescribed how to design a driving protocol to reach the desired system starting from a given simple static Hamiltonian. We have explicitly solved the case of two-band systems that obey the unsolvable SU(2) algebra and described a guiding principle to fix the gauge of the micromotion operator. We then illustrate our idea by stroboscopically realizing the cross-stitched model's flat and dispersive band diagram [40,41].

Our formulation does not rely on any perturbative expansions and is exact. The main idea is to provide a recipe

to design a driving protocol on a simple static Hamiltonian such that a desired stroboscopic Hamiltonian is obtained for *any* driving frequency. Unlike previous works that rely on a specific form of the driving (sine or cosine), the exactness of our approach allows us to engineer the functional form of the drive. Even though we tackled generic condensed matter setup of systems having two-energy bands, our formalism can be easily adapted for any two-level system with a driving protocol that is a generalization to those studied in Refs. [47–49].

In principle, the formalism presented here can be applied to multiband systems (see Ref. [27] for a discussion on three-bands systems) to Floquet engineer technologically relevant materials like higher-order topological insulator (HOTI) [50–52] or reproduce $\mathbb{Z}_2$ lattice gauge theory in cold atom setup [53].

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[27] See Supplemental Material http://link.aps.org/supplemental/10.1103/PhysRevB.105.L020301 which includes Refs. [28–32]. It provides the mathematical details for our general formulation and explicitly works out the two-band problem. It also provides a real-space representation of the driving protocol used to engineer the cross-stitch lattice. Finally, the supplementary material deals with interacting and non-interacting systems focusing on superconductors. Also, it works out a three-band system

exemplifying that our method can be used beyond archetypal two-band systems.

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