Soft modes in Fermi liquids at arbitrary temperatures

D. Belitz [1,2] and T. R. Kirkpatrick [3]

[1] *Department of Physics and Institute for Fundamental Science, University of Oregon, Eugene, Oregon 97403, USA*

[2] *Materials Science Institute, University of Oregon, Eugene, Oregon 97403, USA*

[3] *Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742, USA*

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We use kinetic-theory methods to analyze Landau Fermi-liquid theory, and in particular to investigate the number and nature of modes in Fermi liquids that are soft in the long-wavelength limit, both in the hydrodynamic and the collisionless regimes. In the hydrodynamic regime we show that Fermi-liquid theory is consistent with Navier-Stokes hydrodynamics at all temperatures, as expected. The soft modes are the ones familiar from classical hydrodynamics that are controlled by the five conservation laws; namely, two first-sound modes, two shear diffusion modes, and one heat diffusion mode. These modes have a particle-like spectrum and are soft, or scale invariant, at all temperatures. In the collisionless regime we show that the entire single-particle distribution function is soft with a continuous part of the spectrum. This continuous soft mode, which is well known but often not emphasized, has important physical consequences, e.g., for certain quantum phase transitions. In addition, there are the well-known soft zero-sound excitations that describe angular fluctuations of the Fermi surface; their spectra are particle-like. They are unrelated to conservation laws, acquire a mass at any nonzero temperature, and their number depends on the strength of the quasiparticle interaction. We also discuss the fates of these two families of soft modes as the temperature changes. With increasing temperature the size of the collisionless regime shrinks, the damping of the modes grows, and eventually all of the collisionless modes become overdamped. In their stead the five hydrodynamic modes appear in the hydrodynamic regime at asymptotically low frequencies. The two families of soft modes are unrelated and have very different physical origins. In charged Fermi liquids the first-sound modes in the hydrodynamic regime and the $\ell = 0$ zero-sound modes in the collisionless regime get replaced by plasmons, all other modes remain soft.

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I. INTRODUCTION

Fluid mechanics is one of the oldest and most important subfields of physics, going back to Euler's work in the mid-1700s. Classical fluid mechanics [1] demonstrated that by applying the principles of Newtonian mechanics one can successfully describe the behavior of strongly interacting condensed many-particle systems. The advent of quantum mechanics raised the question of how to describe fluids of quantum particles at low temperatures T, in particular fermionic ones such as He-3 or the electron fluid in a metal. Landau's Fermi-liquid (LFL) theory and its generalization to charged systems provided an answer by combining kinetic theory with a phenomenological description of strong interactions and the notion of "quasiparticles"; fictitious entities that allow for a mapping of the spectrum of the strongly interacting system onto that of a noninteracting Fermi gas [2–4].

LFL theory was initially applied to Fermi liquids [5] at asymptotically low temperature in the absence of impurities, where it predicted, *inter alia*, the collective modes known as zero sound. At zero temperature, zero sound is a soft or massless mode, i.e., the frequency goes to zero (in this case linearly) with the wave number, as is the case for first sound. In other words, these excitations are scale invariant. Unlike first sound, their existence has nothing to do with conservation laws. In fact, any nonzero temperature makes the zero-sound modes massive in the sense that the resonance frequency has a damping, or imaginary, part that to leading order is independent of the wave number [6]. By contrast, the masslessness of first sound, which includes the fact that the damping goes to zero as the wave number squared for small wave numbers, is protected by the conservation of particle number and momentum. More generally, one needs to distinguish between the collisionless regime at low temperatures, where the streaming term in a kinetic equation dominates over the collision integral, and the hydrodynamic regime at higher temperatures, where the collisions dominate. The former is delineated by the requirement $\omega\tau \gg 1$ and/or $k\ell \gg 1$, with ω the frequency of an excitation, τ a relevant scattering time, and ℓ the related mean-free path. The latter is delineated by the conditions $\omega\tau \ll 1$ and $k\ell \ll 1$. Since τ and ℓ diverge as $T \to 0$, the collisionless regime extends all the way to zero frequency or wave number only at $T = 0$, see Fig. 1.

These considerations lead to three questions: (1) What is the relation between LFL theory and classical hydrodynamics in the hydrodynamic regime? Since the soft modes in the hydrodynamic regime of both classical fluids and quantum fluids are governed by conservation laws, the structure of the theories should be the same as far as the soft modes are

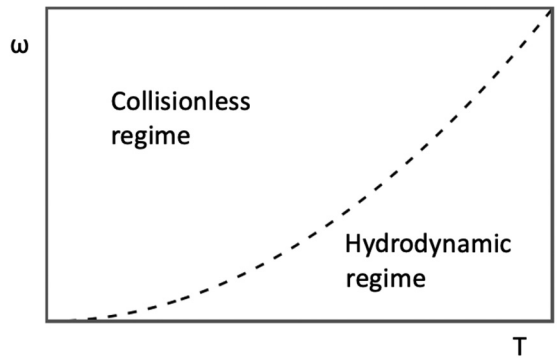

FIG. 1. Schematic locations of the collisionless and hydrodynamic regimes in the frequency-temperature plane. The boundary (dashed line) is not sharp, and its functional form depends on the temperature dependence of the relevant scattering time.

concerned, yet LFL theory is usually assumed to be valid only at low (compared to the Fermi temperature) temperatures. (2) What is the relation, if any, between the soft modes in the hydrodynamic and collisionless regimes? The experimentally observed transition from first sound to zero sound in He-3, with the damping coefficient going through a maximum in between (see, e.g., Fig. 1.4 in Ref. [3]), is often referred to as a "crossover", but the very different origins of the excitations make it implausible that they are continuously connected in any physical sense. This is part of a more general question: What happens to classical soft modes with decreasing temperature, and to quantum soft modes with increasing temperature? (3) What is the relation, if any, between the collective zero-sound modes that are particle-like in the sense that they are characterized by a well-defined frequency with a small damping coefficient, and the continuous excitation known, in a LFL context, as the particle-hole continuum [4], that is NOT particle-like but nonetheless scale invariant with the frequency scaling linearly with the wave number?

The purpose of this paper is to discuss and answer these questions. With respect to question (1), we will show that LFL theory describes the soft modes of a Fermi system qualitatively correctly for all temperatures. In particular, it yields five soft modes in the hydrodynamic regime, namely, two (first) sound modes, two shear diffusion modes, and one heat diffusion mode, that are completely consistent with the linearized Navier-Stokes equations. With respect to question (2), we will discuss that there is no connection between the soft modes in the two regimes. With decreasing temperature the hydrodynamic regime shrinks and the hydrodynamic modes become confined to a smaller and smaller region of parameter space and finally disappear. In their stead, a new family of soft modes emerges in the collisionless regime that is governed by different physics. With respect to question (3), we will show that the particle-like zero-sound excitations and the continuum mode (which we will refer to as the "unparticle" mode, using a term coined in Ref. [7]) are part of one and the same scale invariant spectrum, and hence just different aspects of the same excitation. While historically the zero-sound modes have received much more attention, the continuum unparticle mode has important physical consequences. A particularly striking one is that it is responsible for driving the quantum ferromagnetic transition generically first order by coupling to the magnetic order parameter [8].

Experimental studies of quantum hydrodynamics used to be limited since there was only one known neutral Fermi liquid, He-3, with very limited opportunities for tuning the interaction parameters, while in metals, which represent charged Fermi liquids, the hydrodynamic regime tends to be dominated by impurity scattering, which drastically alters the hydrodynamic behavior from that of clean systems. This has changed in recent years due to the availability of cold-atom systems that allow for the realization of Fermi liquids with widely tunable parameters [9–14]. This is a further motivation for studying quantum hydrodynamics in more detail.

In this paper we will use the usual Boltzmann/Landau kinetic equation [2,15] and construct explicit solutions that identify the soft modes in the system. In a second paper (Ref. [16], to be referred to as Paper II) we add a Langevin force to the quantum kinetic equation, and in particular derive fluctuating Navier-Stokes equations for a fermionic quantum fluid.

This paper is organized as follows. In Sec. II we briefly review key elements of LFL theory and formulate the kinetic equation that governs the time evolution of the single-particle distribution function. In Sec. III we discuss the solution of the linearized kinetic equation in the hydrodynamic regime, and show that the results for the soft modes are the same as in the theory of classical fluids. In Sec. IV we discuss the solution in the collisionless regime, with some emphasis on the continuum unparticle excitation, and highlight the fundamental differences between the soft modes in the two regimes. We conclude in Sec. V with a discussion of various aspects of the theory and its implications. Some technical details regarding thermodynamic relations, continuity equations, and collision operators, are relegated to three appendices. A fourth Appendix highlights the different natures of zero modes and hydrodynamic modes by contrasting transverse zero sound and shear diffusion.

II. KINETIC THEORY FOR FERMIONS

In this section we recall general aspects of Landau Fermi-liquid (LFL) theory [2,3] and discuss the generalized Boltzmann equation that governs the time evolution of the distribution function. For our detailed discussion we will consider systems where the fermions interact only with each other. However, the formalism can easily be generalized to include the interaction of fermions with impurities, or with bosons such as phonons or magnons. Some relevant linearized collision operators are given in Appendix C.

A. Landau's Fermi-liquid theory

Consider a description of a Fermi liquid in terms of Landau quasiparticles (QPs), and let $f(\boldsymbol{p},\boldsymbol{x},t)$ be the single-QP distribution function that depends on the momentum $\boldsymbol{p}$, the real-space position $\boldsymbol{x}$, and the time t. For simplicity we consider spinless fermions; the theory can easily be generalized to include spin. If $e(\boldsymbol{x},t)$ is the energy density of the system as a whole, then the QP energy $\epsilon(\boldsymbol{p},\boldsymbol{x},t)$ is defined via the change

δe of the energy density under a change δf of the distribution function,

$$\delta e(\boldsymbol{x},t) = \frac{1}{V}\sum_{\boldsymbol{p}} \epsilon(\boldsymbol{p},\boldsymbol{x},t)\,\delta f(\boldsymbol{p},\boldsymbol{x},t) \tag{2.1a}$$

with V the system volume [17]. For later reference we also introduce variations of the number density

$$\delta n(\boldsymbol{x},t) = \frac{1}{V}\sum_{\boldsymbol{p}} \delta f(\boldsymbol{p},\boldsymbol{x},t)\,, \tag{2.1b}$$

and of the fluid velocity

$$\delta \boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{nm}\frac{1}{V}\sum_{\boldsymbol{p}} \boldsymbol{p}\,\delta f(\boldsymbol{p},\boldsymbol{x},t)\,. \tag{2.1c}$$

with m the bare fermion mass and n the equilibrium number density, see Eq. (2.5) below.

In equilibrium, $\epsilon(\boldsymbol{p},\boldsymbol{x},t)$ is independent of space and time. For simplicity, we assume an isotropic Fermi liquid, so $\epsilon_{\rm eq}(\boldsymbol{p})$ depends on $p = |\boldsymbol{p}|$ only, and we write $\epsilon_{\rm eq}(\boldsymbol{p}) \equiv \epsilon_p$. The QP velocity is

$$\boldsymbol{v}_p = \boldsymbol{\nabla}_{\boldsymbol{p}}\,\epsilon_p\,. \tag{2.2}$$

Near the Fermi surface one has $\epsilon_p = \mu + v_{\rm F}^*(p - p_{\rm F})$, with μ the chemical potential, $p_{\rm F}$ the Fermi wave number, and $v_{\rm F}^* = p_{\rm F}/m^*$ the Fermi velocity with m^* the QP effective mass. Consequently, for $\boldsymbol{p}$ on the Fermi surface one has $\boldsymbol{v}_p = v_{\rm F}^*\hat{\boldsymbol{p}} = \boldsymbol{p}/m^*$. (We use units such that $\hbar = k_{\rm B} = 1$.) At zero temperature, $\mu(T=0) = p_{\rm F}^2/2m^* \equiv \epsilon_{\rm F}$.

The equilibrium distribution function is given by the Fermi-Dirac distribution

$$f_{\rm eq}(\boldsymbol{p}) = 1/(\exp(\xi_p/T) + 1)\,, \tag{2.3a}$$

with $\xi_p = \epsilon_p - \mu$. For later reference we also define

$$w(\boldsymbol{p}) = -\partial f_{\rm eq}(\boldsymbol{p})/\partial\epsilon_p = \frac{1}{T} f_{\rm eq}(\boldsymbol{p})[1 - f_{\rm eq}(\boldsymbol{p})]$$
$$= \frac{1}{4T\cosh^2(\xi_p/2T)}\,, \tag{2.3b}$$

which plays the role of a weight function. Note that $f_{\rm eq}$ and w depend on the modulus of $\boldsymbol{p}$ only. It is useful to define a scalar product $\langle \ldots | \ldots \rangle$ of $\boldsymbol{p}$-dependent functions

$$\langle g(\boldsymbol{p})|h(\boldsymbol{p})\rangle = \frac{1}{V}\sum_{\boldsymbol{p}} w(\boldsymbol{p})\,g(\boldsymbol{p})\,h(\boldsymbol{p})\,, \tag{2.4a}$$

and an average with respect to the weight function w by

$$\langle g(\boldsymbol{p})\rangle_w = \langle g(\boldsymbol{p})|1\rangle/N_0 \tag{2.4b}$$

where

$$N_0 = \langle 1|1\rangle = \frac{1}{V}\sum_{\boldsymbol{p}} w(\boldsymbol{p}) \tag{2.4c}$$

normalizes the weight function. For $T \to 0$, $N_0 = N_{\rm F}^* + O(T^2)$, where $N_{\rm F}^* = p_{\rm F}m^*/2\pi^2$ is the QP density of states at the Fermi surface.

The equilibrium number density is

$$n = \frac{N}{V} = \frac{1}{V}\sum_{\boldsymbol{p}} f_{\rm eq}(\boldsymbol{p})\,, \tag{2.5}$$

and the physical mass density is $\rho = nm$. At $T = 0$, $n = p_{\rm F}^3/6\pi^2$. The internal energy in equilibrium we will denote by E, and the equilibrium energy density by $e = E/V$.

Within LFL theory, a change δf of the distribution function leads to a change of ϵ given by

$$\delta\epsilon(\boldsymbol{p},\boldsymbol{x},t) = \frac{1}{N_0 V}\sum_{\boldsymbol{p}'} F(\boldsymbol{p},\boldsymbol{p}')\,\delta f(\boldsymbol{p}',\boldsymbol{x},t)\,. \tag{2.6}$$

Here $F(\boldsymbol{p},\boldsymbol{p}') = F(\boldsymbol{p}',\boldsymbol{p})$ is a function that parameterizes the interaction between the QPs, and the factor of N_0 serves to make F dimensionless. The interaction function F is the kernel of the integral equation obeyed by the QP velocity [2],

$$\boldsymbol{v}_p = \frac{1}{m}\boldsymbol{p} - \frac{1}{N_0 V}\sum_{\boldsymbol{p}'} w(\boldsymbol{p}')\,F(\boldsymbol{p},\boldsymbol{p}')\,\boldsymbol{v}_{p'}\,. \tag{2.7}$$

For the discussion in Sec. III we keep only the $\ell = 0$ and $\ell = 1$ terms in an angular-momentum expansion and write

$$F(\boldsymbol{p},\boldsymbol{p}') = F_0 + F_1\,\frac{\boldsymbol{p}\cdot\boldsymbol{p}'}{\langle \boldsymbol{p}^2\rangle_w}\,. \tag{2.8}$$

At $T = 0$ this reduces to the usual definition of the Landau parameter F_1, which determines the relation between the QP effective mass m^* and the bare fermion mass m via [2,3]

$$m^* = m(1 + F_1/3)\,. \tag{2.9}$$

Note that Eq. (2.9), with F_1 as defined in Eqs. (2.8), remains valid if one performs its usual derivation [3] at nonzero temperature. With the model interaction given by Eq. (2.8) the integral equation (2.7) is separable and can be solved. We find

$$\boldsymbol{v}_p = \boldsymbol{p}/m^*\,, \tag{2.10a}$$

and hence

$$\epsilon_p = \frac{1}{2}\,\boldsymbol{p}\cdot\boldsymbol{v}_p = p^2/2m^*\,. \tag{2.10b}$$

Here we have used Eqs. (A12) and (A19b), and we ignore a p-independent contribution to ϵ_p that also depends on the FL interaction [18]. Note that within this model this holds in general, whereas for a general interaction function F it holds only for $\boldsymbol{p}$ on the Fermi surface [19]. The Landau parameter F_0 is a positive number for a neutral Fermi liquid with a short-ranged (SR) repulsive interaction between the QPs. However, for a charged Fermi liquid (such as the conduction-electron system in a metal) F_0 must be augmented by the long-ranged (LR) Coulomb interaction [4,20]:

$$F_0 \to F_0 + 4\pi N_0\,e^2/\boldsymbol{k}^2 \qquad \text{(LR case)}\,, \tag{2.11}$$

where we have anticipated a spatial Fourier transform from the real-space position variable $\boldsymbol{x}$ to a wave vector $\boldsymbol{k}$.

The kinetic equation that governs the time evolution of the distribution function f is

$$\frac{d}{dt}\,f(\boldsymbol{p},\boldsymbol{x},t) = \left(\frac{\partial f}{\partial t}\right)_{\rm coll}(\boldsymbol{p},\boldsymbol{x},t)\,. \tag{2.12}$$

This holds in complete generality: The total time derivative of f on the left-hand side equals the collision integral on the right-hand side, i.e., the temporal change of f due to collisions

between the QPs. The left-hand side consists of three terms:

$$\frac{d}{dt}f = \partial_t f + \nabla_{\boldsymbol{x}} f \cdot \dot{\boldsymbol{x}} + \nabla_{\boldsymbol{p}} f \cdot \dot{\boldsymbol{p}} \,. \tag{2.13a}$$

The velocity is given by

$$\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{p}}\, \epsilon(\boldsymbol{p}, \boldsymbol{x}, t) \,, \tag{2.13b}$$

and $\dot{\boldsymbol{p}}$ is equal to the force due to the spatially inhomogeneous energy, i.e., the potential energy of the QP interaction,

$$\dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{x}}\, \epsilon(\boldsymbol{p}, \boldsymbol{x}, t) = -\nabla_{\boldsymbol{x}}\, \delta\epsilon(\boldsymbol{p}, \boldsymbol{x}, t) \,. \tag{2.13c}$$

To linear order in small deviations δf from the equilibrium distribution we thus have

$$\frac{d}{dt}f = \partial_t\, \delta f + \boldsymbol{v}_p \cdot \nabla_{\boldsymbol{x}}\, \delta f - \boldsymbol{v}_p \cdot \frac{\partial f_{\rm eq}}{\partial \epsilon_p}\, \nabla_{\boldsymbol{x}}\, \delta\epsilon \,, \tag{2.13d}$$

with $\delta\epsilon$ given by Eq. (2.6). In what follows it will be convenient to write

$$\delta f(\boldsymbol{p}, \boldsymbol{x}, t) = w(\boldsymbol{p})\, \phi(\boldsymbol{p}, \boldsymbol{x}, t) \tag{2.14}$$

with $w(\boldsymbol{p})$ from Eq. (2.3b).

B. The linearized kinetic equation

We write the collision integral as a linear collision operator Λ acting on δf:

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll}(\boldsymbol{p}, \boldsymbol{x}, t) = \Lambda(\boldsymbol{p})\, \delta f(\boldsymbol{p}, \boldsymbol{x}, t). \tag{2.15}$$

After a temporal Laplace transform with z as the complex frequency [21] and a spatial Fourier transform with $\boldsymbol{k}$ as the wave vector the kinetic equation (2.12) takes the form

$$[-iz + L_{\boldsymbol{k}}(\boldsymbol{p})]\phi(\boldsymbol{p}, \boldsymbol{k}, z) = \phi(\boldsymbol{p}, \boldsymbol{k}, t = 0) \,. \tag{2.16a}$$

Here $L_{\boldsymbol{k}}(\boldsymbol{p})$ is a linearized kinetic operator given by

$$L_{\boldsymbol{k}}(\boldsymbol{p}) = i\boldsymbol{k} \cdot \boldsymbol{v}_p \left[1 + \frac{1}{N_0 V} \sum_{\boldsymbol{p}'} w(\boldsymbol{p}')\, F(\boldsymbol{p}', \boldsymbol{p})\, R_{\boldsymbol{p}\to\boldsymbol{p}'}\right] - \Lambda(\boldsymbol{p}) \,, \tag{2.16b}$$

with the replacement operator R defined by $R_{\boldsymbol{p}\to\boldsymbol{p}'}\, g(\boldsymbol{p}) = g(\boldsymbol{p}')$ for any $\boldsymbol{p}$-dependent function g. The first contribution to $L_{\boldsymbol{k}}(\boldsymbol{p})$ represents the streaming term and the QP interaction term in Eq. (2.13d). The collision operator Λ must respect the five collision invariants that are unaffected by QP collisions, viz., the density, the three components of the momentum, and the energy [22].

Using the scalar product defined in Eq. (2.4a) we can write the density, velocity, and energy fluctuations from Eqs. (2.1) as

$$\delta n(\boldsymbol{x}, t) = \langle 1 | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.17a}$$

$$\delta \boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{nm}\langle \boldsymbol{p} | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.17b}$$

$$\delta e(\boldsymbol{x}, t) = \langle \epsilon_p | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,. \tag{2.17c}$$

We separate the momentum into a longitudinal (with respect to the wave vector $\boldsymbol{k}$) and two transverse components:

$$\boldsymbol{p} = (\hat{\boldsymbol{k}} \cdot \boldsymbol{p})\hat{\boldsymbol{k}} + \sum_{i=1,2} (\hat{\boldsymbol{k}}_\perp^{(i)} \cdot \boldsymbol{p})\hat{\boldsymbol{k}}_\perp^{(i)} \tag{2.18}$$

with $\hat{\boldsymbol{k}}$ the unit vector in $\boldsymbol{k}$ direction, and two unit vectors $\hat{\boldsymbol{k}}_\perp^{(1,2)}$ that are perpendicular to $\boldsymbol{k}$ and to each other. This separates the velocity fluctuations into one longitudinal component δu_L and two transverse components $\delta u_\perp^{(1,2)}$ and we write the five fluctuations from Eqs. (2.17) in the form

$$\delta n(\boldsymbol{x}, t) = \langle a_1(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.19a}$$

$$\delta u_L(\boldsymbol{x}, t) = \frac{1}{nm}\langle a_2(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.19b}$$

$$\delta u_\perp^{(1)}(\boldsymbol{x}, t) = \frac{1}{nm}\langle a_3(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.19c}$$

$$\delta u_\perp^{(2)}(\boldsymbol{x}, t) = \frac{1}{nm}\langle a_4(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.19d}$$

$$\delta e(\boldsymbol{x}, t) = \langle a_5(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle + \langle \epsilon_p \rangle_w\, \delta n(\boldsymbol{x}, t) \,. \tag{2.19e}$$

Also of interest are the temperature fluctuations

$$\delta T(\boldsymbol{x}, t) = \frac{1}{c_V}\langle a_5(\boldsymbol{p}) | \phi(\boldsymbol{p}, \boldsymbol{x}, t)\rangle \,, \tag{2.19f}$$

the fluctuations of the entropy density $s = S/V$,

$$T\, \delta s(\boldsymbol{x}, t) = \delta e(\boldsymbol{x}, t) - \mu\, \delta n(\boldsymbol{x}, t) \tag{2.19g}$$

and the pressure fluctuations

$$\delta p(\boldsymbol{x}, t) = \left(\frac{\partial p}{\partial T}\right)_{N,V} \delta T(\boldsymbol{x}, t) + \left(\frac{\partial p}{\partial n}\right)_{T,V} \delta n(\boldsymbol{x}, t) \,. \tag{2.19h}$$

Here we have defined

$$a_1(\boldsymbol{p}) = 1 \,, \tag{2.20a}$$

$$a_2(\boldsymbol{p}) = \hat{\boldsymbol{k}} \cdot \boldsymbol{p} \,, \tag{2.20b}$$

$$a_3(\boldsymbol{p}) = \hat{\boldsymbol{k}}_\perp^{(1)} \cdot \boldsymbol{p} \,, \tag{2.20c}$$

$$a_4(\boldsymbol{p}) = \hat{\boldsymbol{k}}_\perp^{(2)} \cdot \boldsymbol{p} \,, \tag{2.20d}$$

$$a_5(\boldsymbol{p}) = \epsilon_p - \langle \epsilon_p \rangle_w \,, \tag{2.20e}$$

and we have split the energy fluctuation into its overlap with the density fluctuation plus a part $\langle a_5 | \phi\rangle$ that is orthogonal to the density. We will also need the normalizations of these vectors. We define

$$A_\alpha^{-2} = \langle a_\alpha(\boldsymbol{p}) | a_\alpha(\boldsymbol{p})\rangle \tag{2.21a}$$

and find

$$A_1^{-2} = \langle 1 | 1\rangle = (1 + F_0)(\partial n/\partial\mu)_T \,, \tag{2.21b}$$

$$A_2^{-2} = A_3^{-2} = A_4^{-2} = \frac{1}{3}\langle \boldsymbol{p} | \boldsymbol{p}\rangle = nm^* \,, \tag{2.21c}$$

$$A_5^{-2} = \langle a_5(\boldsymbol{p}) | a_5(\boldsymbol{p})\rangle = T c_V \,, \tag{2.21d}$$

with c_V the specific heat at constant volume. See Appendix A 1 for the final equalities in Eqs. (2.21), and Appendix A 2 for the origin of Eq. (2.19f). Equation (2.19g) follows from Eq. (A11).

Now consider the five conservation laws. Particle number conservation implies

$$0 = \frac{d}{dt}\,\delta n(\boldsymbol{x},t) = \frac{1}{V}\sum_{p} w(\boldsymbol{p})\,\frac{d}{dt}\,\phi(\boldsymbol{p},\boldsymbol{x},t)$$
$$= \frac{1}{V}\sum_{p} w(\boldsymbol{p})\,\Lambda(\boldsymbol{p})\,\phi(\boldsymbol{p},\boldsymbol{x},t) = \langle 1|\Lambda(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{x},t)\rangle. \tag{2.22}$$

Consequently, $\Lambda(\boldsymbol{p})$ has a zero eigenvalue with eigenvector $a_1(\boldsymbol{p})$. Analogously, the remaining four conservation laws are reflected by four additional zero eigenvalues. We will refer to the space spanned by the $a_\alpha(\boldsymbol{p})$ listed in Eqs. (2.20) as the "zero-eigenvector space" and denote it by $\mathcal{L}_0$. The space orthogonal to $\mathcal{L}_0$ we will refer to as the "orthogonal space" and denote it by $\mathcal{L}_\perp$. For a discussion of the continuity equations related to the five conservations laws see Appendix B.

These properties of the collision operator suffice for determining the soft modes in the hydrodynamic regime. In the collisionless regime, the collision operator can be neglected to leading order. If desirable, one can construct explicit model collision operators that have the required five zero eigenvalues, see Sec. IV and Appendix C.

The solution of the kinetic equation (2.16a) is qualitatively different depending on whether the collision operator or the streaming and interaction terms in the kinetic operator $L_{\boldsymbol{k}}(\boldsymbol{p})$ dominate. The corresponding regions of parameter space are known as the "hydrodynamic regime" and the "collisionless regime", respectively, see Fig. 1. We will discuss the hydrodynamic regime first, and the collisionless regime second.

III. SOLUTIONS OF THE KINETIC EQUATION I: HYDRODYNAMIC REGIME

The system is always in the hydrodynamic regime for fixed nonzero temperature at asymptotically small wave numbers (or frequencies), or for fixed wave number or frequency at sufficiently high temperature. This is because of the linear wave-number dependence of the streaming/interaction contribution to $L_{\boldsymbol{k}}(\boldsymbol{p})$ and the fact that collisions become less frequent with decreasing temperature.

A. Short-ranged case

In the hydrodynamic regime, the collision operator dominates over the first term on the right-hand side of Eq. (2.16b). As a result, all modes are massive (i.e., they have a nonzero frequency at $\boldsymbol{k}=0$) except for the five hydrodynamic modes tied to the five conservation laws. The corresponding hydrodynamic frequencies are given by the eigenvalues of the kinetic operator $L_{\boldsymbol{k}}(\boldsymbol{p})$ defined in Eq. (2.16b), and the hydrodynamic modes are given by the corresponding left eigenfunctions, see Eqs. (2.19). Both the eigenvalues and the eigenfunctions can be determined by studying the perturbed zero eigenvalues [23]. We thus consider the left eigenproblem

$$\langle\psi^{\mathrm{L}}_\alpha(\boldsymbol{k},\boldsymbol{p})|\,L_{\boldsymbol{k}}(\boldsymbol{p}) = \langle\psi^{\mathrm{L}}_\alpha(\boldsymbol{k},\boldsymbol{p})|\,\omega_\alpha(\boldsymbol{k})\,. \tag{3.1}$$

We are interested in the small-$\boldsymbol{k}$ behavior of the five eigenvalues that are zero at $\boldsymbol{k}=0$. Accordingly, we perform an expansion in powers of $\boldsymbol{k}$:

$$\omega_\alpha(\boldsymbol{k}) = \omega^{(1)}_\alpha(\boldsymbol{k}) + \omega^{(2)}_\alpha(\boldsymbol{k}) + O(k^3) \qquad (\alpha = 1,2,3,4,5) \tag{3.2a}$$

where $\omega^{(n)}_\alpha = O(k^n)$. Analogously, we expand the eigenfunctions

$$\langle\psi^{\mathrm{L}}_\alpha(\boldsymbol{k},\boldsymbol{p})| = \langle\psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{k},\boldsymbol{p})| + \langle\psi^{\mathrm{L}(1)}_\alpha(\boldsymbol{k},\boldsymbol{p})| + \langle\psi^{\mathrm{L}(2)}_\alpha(\boldsymbol{k},\boldsymbol{p})| + O(k^3)\,. \tag{3.2b}$$

$\psi^{\mathrm{L}(0)}_\alpha$ can depend on $\boldsymbol{k}$ at most via $\hat{\boldsymbol{k}}$, $\psi^{\mathrm{L}(1)}_\alpha$ is linear in k, etc. For notational simplicity we will not show this $\boldsymbol{k}$ dependence explicitly from here on and write $\psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{k},\boldsymbol{p}) \equiv \psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{p})$, etc. In the same expansion, the linearized kinetic operator is [see Eq. (2.16b)

$$L_{\boldsymbol{k}}(\boldsymbol{p}) = -\Lambda(\boldsymbol{p}) + L^{(1)}_{\boldsymbol{k}}(\boldsymbol{p})\,. \tag{3.3}$$

The operator $L^{(1)}_{\boldsymbol{k}}(\boldsymbol{p})$ that is linear in $\boldsymbol{k}$ has two contributions:

$$L^{(1)}_{\boldsymbol{k}}(\boldsymbol{p}) = L^{(1,1)}_{\boldsymbol{k}}(\boldsymbol{p}) + L^{(1,2)}_{\boldsymbol{k}}(\boldsymbol{p}) \tag{3.4a}$$

where

$$L^{(1,1)}_{\boldsymbol{k}}(\boldsymbol{p}) = i\,\boldsymbol{k}\cdot\boldsymbol{v}_p \tag{3.4b}$$

reflects the streaming term for noninteracting QPs with mass m^*, and

$$L^{(1,2)}_{\boldsymbol{k}}(\boldsymbol{p}) = i\,\boldsymbol{k}\cdot\boldsymbol{v}_p\,\frac{1}{N_0 V}\sum_{\boldsymbol{p}'} w(\boldsymbol{p}')\,F(\boldsymbol{p}',\boldsymbol{p})\,R_{\boldsymbol{p}\to\boldsymbol{p}'} \tag{3.4c}$$

reflects the QP interaction. If we use the LFL model interaction from Eq. (2.8), then by utilizing the scalar product notation this can be written as

$$L^{(1,2)}_{\boldsymbol{k}}(\boldsymbol{p}) = \frac{F_0}{\langle 1|1\rangle}\,|i\boldsymbol{k}\cdot\boldsymbol{v}_p\rangle\langle 1| + \frac{F_1}{\langle\boldsymbol{p}|\boldsymbol{p}\rangle}\,|(i\boldsymbol{k}\cdot\boldsymbol{v}_p)\boldsymbol{p}\rangle\cdot\langle\boldsymbol{p}|\,. \tag{3.4d}$$

As we pointed out in Sec. II, this model for the interaction implies the simple form (2.10a) for the QP velocity. That is, within this model $\boldsymbol{v}_p \in \mathcal{L}_0$. We will nonetheless usually write $\boldsymbol{v}_p$ rather than $\boldsymbol{p}/m^*$ in order to point out several aspects of the theory that will emerge if one considers a more general QP interaction function so that $\boldsymbol{v}_p$ has a component in the orthogonal space $\mathcal{L}_\perp$.

To zeroth order in this wave-number expansion we have

$$\langle\psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{p})|\,\Lambda(\boldsymbol{p}) = 0\,, \tag{3.5}$$

which implies that the $\psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{p})$ are linear combinations of the $a_\alpha(\boldsymbol{p})$ from Eqs. (2.20):

$$\langle\psi^{\mathrm{L}(0)}_\alpha(\boldsymbol{p})| = \sum_{\beta=1}^{5} c^{(\alpha)}_\beta(\hat{\boldsymbol{k}})\,\langle a_\beta(\boldsymbol{p})|\,. \tag{3.6}$$

By systematically going to higher order in k we can now determine the hydrodynamic modes and their eigenvalues. In particular, in order to determine the coefficients $c^{(\alpha)}_\beta$ in Eq. (3.6) one needs to go to $O(k)$.

1. Speed of first sound

To first order in the k expansion we have

$$\langle\psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})|\, L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) - \langle\psi_\alpha^{\mathrm{L}(1)}(\boldsymbol{p})|\, \Lambda(\boldsymbol{p}) = \omega_\alpha^{(1)}(\boldsymbol{k})\,\langle\psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})|\,. \tag{3.7}$$

Multiplying from the right with $w(\boldsymbol{p})\,a_\gamma(\boldsymbol{p})$ and summing over $\boldsymbol{p}$ we obtain

$$\sum_\beta c_\beta^{(\alpha)}(\hat{\boldsymbol{k}})\big(L_{\beta\gamma}^{(1)}(\boldsymbol{k},\boldsymbol{p}) - \delta_{\beta\gamma}\,\omega_\alpha^{(1)}(\boldsymbol{k})\big) = 0 \tag{3.8a}$$

with

$$L_{\beta\gamma}^{(1)}(\boldsymbol{k},\boldsymbol{p}) = \langle a_\beta(\boldsymbol{p})|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|a_\gamma(\boldsymbol{p})\rangle A_\gamma^2 \tag{3.8b}$$

the elements of a matrix $L^{(1)}$. Due to the angular integration in the elements of $L^{(1)}$ the 5×5 system decouples into two scalar equations for the transverse velocity, or shear, modes and a 3×3 system for the density, the longitudinal velocity, and the energy or heat mode. Furthermore, the two shear eigenvalues are the same by symmetry, $\omega_3=\omega_4=\omega_\perp$, and the angular integration makes them vanish at this order,

$$\omega_\perp^{(1)} = 0\,. \tag{3.9}$$

One of the eigenvalues of the 3×3 submatrix $L_L^{(1)}$ for the longitudinal modes ($\alpha,\beta=1,2,5$) also vanishes at this order,

$$\omega_5^{(1)} = 0\,. \tag{3.10}$$

This eigenvalue corresponds to the heat mode, see below. The other two, which correspond to first sound, have the form

$$\omega_{1,2}^{(1)} = \pm i c_1 k\,, \tag{3.11}$$

In order to determine the speed of first sound c_1, we need the matrix $L_L^{(1)}$ explicitly. Evaluating the matrix elements in Eq. (3.8b) we find

$$L_L^{(1)} = \frac{ik}{3}\begin{pmatrix} 0 & \frac{\langle v_p|p\rangle}{nm} & 0 \\ \frac{\langle v_p|p\rangle}{(\partial n/\partial\mu)_{T,V}} & 0 & \frac{\langle v_p|p\,a_5(p)\rangle}{c_V T} \\ 0 & \frac{\langle v_p|p\,a_5(p)\rangle}{nm} & 0 \end{pmatrix} \tag{3.12a}$$

$$= ik\begin{pmatrix} 0 & 1/m & 0 \\ \left(\frac{\partial p}{\partial n}\right)_{T,V} & 0 & \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V} \\ 0 & \frac{T}{nm}\left(\frac{\partial p}{\partial T}\right)_{N,V} & 0 \end{pmatrix}; \tag{3.12b}$$

see Appendix A 3 for the determination of the matrix elements in Eq. (3.12b). We note that Eq. (3.12b) represents the linearized Euler equations for an inviscid fluid [1,24]. This illustrates that the theory so far is consistent with general hydrodynamics; we will see below that this remains true if one takes into account dissipation. Using Eq. (3.12b) in (3.8a) we find

$$c_1^2 = \frac{1}{mn\chi_T}\left[1 + \frac{T\chi_T}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V}^2\right] = 1/mn\chi_S\,, \tag{3.13}$$

where $\chi_T = (-1/V)(\partial V/\partial p)_{T,N}$ is the isothermal compressibility, and $\chi_S = \chi_T c_V/c_p$ is the adiabatic one. We have used Eqs. (A20b) and (A27b) to cast the speed of sound in this form. Note that it is the coupling to the heat mode that changes the isothermal compressibility into the adiabatic one. This is the correct result for the speed of sound; it is identical with the expressions obtained in the theory of classical fluids [25], and in a phenomenological treatment of a Fermi gas [18]. However, we stress that it is not obvious *a priori* that LFL theory is consistent with general hydrodynamics in complete generality, not just at low temperatures. At $T=0$ we recover the well-known result [2,3]

$$c_1^2 = \tfrac{1}{3}\,(v_{\mathrm{F}}^*)^2(1+F_0)(1+F_1/3)\,. \tag{3.14}$$

2. Hydrodynamic modes

From Eqs. (3.6) and (3.8a) we see that the left eigenvectors of $L^{(1)}$ determine the eigenmodes $\psi_\alpha^{\mathrm{L}(0)}$ that correspond to the eigenvalues $\omega_\alpha^{(1)}$. The shear modes decouple from the longitudinal modes and from each other. The remaining three eigenmodes are determined by the left eigenvectors of the matrix $L_L^{(1)}$. Note that the matrix is not symmetric, so we need to distinguish between left and right eigenvectors.

(a) Shear modes. Since the shear modes decouple there is no difference between left and right, and we have

$$\psi_{3,4}^{\mathrm{L}(0)}(\boldsymbol{p}) = \psi_{3,4}^{\mathrm{R}(0)}(\boldsymbol{p}) \equiv \psi_{3,4}^{(0)}(\boldsymbol{p}) = a_{3,4}(\boldsymbol{p}) = \hat{\boldsymbol{k}}_\perp^{1,2}\cdot\boldsymbol{p}\,. \tag{3.15a}$$

The normalization is

$$\langle\psi_{3,4}^{(0)}(\boldsymbol{p})|\psi_{3,4}^{(0)}(\boldsymbol{p})\rangle = A_2^{-2} = nm^*\,. \tag{3.15b}$$

(b) Heat mode. The left and right eigenvectors associated with the zero eigenvalue $\omega_1^{(5)}=0$ are not the same. For the left eigenvector we find

$$\begin{aligned}\psi_5^{\mathrm{L}(0)}(\boldsymbol{p}) &= a_5(\boldsymbol{p}) - \frac{T}{n}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_1(\boldsymbol{p}) \\ &= \epsilon_p - (Ts/n+\mu) \\ &= \epsilon_p - \frac{e+p}{n}\,,\end{aligned} \tag{3.16}$$

where $s=S/V$ is the entropy density. Here we have used Eqs. (A7) and (2.8) to go from the first line to the second one; see also Eq. (2.19g). The third line follows from the general identity $Ts = e+p-n\mu$, with e the energy density. We recognize $(e+p)/n$ as the enthalpy per particle.

Equation (3.16) shows that the thermodynamic quantity that represents the heat mode is

$$q(\boldsymbol{x},t) = e(\boldsymbol{x},t) - \frac{e+p}{n}\,n(\boldsymbol{x},t)\,, \tag{3.17}$$

which we recognize as the same quantity that represents the heat mode in a classical fluid [24,25]. This is as it should be, since the hydrodynamic arguments that lead to this mode are completely general. One can discern the physical meaning of $q(\boldsymbol{x},t)$ by considering the fluctuation of the entropy per

particle. From the expression (A11) for the entropy density of a Fermi liquid we obtain

$$\begin{aligned}\delta(s/n)(\boldsymbol{x},t) &= \frac{1}{n}\,\delta s(\boldsymbol{x},t) - \frac{s}{n^2}\,\delta n(\boldsymbol{x},t)\\ &= \frac{1}{Tn}\frac{1}{V}\sum_{\boldsymbol{p}}(\epsilon_p - \mu - sT/n)\delta f(\boldsymbol{p},\boldsymbol{x},t)\\ &= \frac{1}{Tn}\frac{1}{V}\sum_{\boldsymbol{p}}(\epsilon_p - (e+p)/n)\delta f(\boldsymbol{p},\boldsymbol{x},t).\end{aligned} \tag{3.18}$$

We thus have

$$\left\langle \psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{x},t)\right\rangle = Tn\,\delta(s/n)(\boldsymbol{x},t)\,, \tag{3.19}$$

which identifies the heat mode as the fluctuation of the entropy per particle.

To determine the right eigenvector we consider the right eigenproblem that corresponds to Eqs. (3.8). The relevant matrix is

$$\tilde{L}^{(1)}_{\beta\gamma}(\boldsymbol{k},\boldsymbol{p}) = A^2_\beta \langle a_\beta(\boldsymbol{p})|L^{(1)}_{\boldsymbol{k}}(\boldsymbol{p})|a_\gamma(\boldsymbol{p})\rangle \tag{3.20}$$

and the longitudinal submatrix is

$$\tilde{L}^{(1)}_L = ik\begin{pmatrix} 0 & \frac{A_1^2}{A_2^2}\frac{1}{m} & 0\\ \frac{A_2^2}{A_1^2}\left(\frac{\partial p}{\partial n}\right)_{T,V} & 0 & \frac{A_2^2}{A_5^2}\frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V}\\ 0 & \frac{A_5^2}{A_2^2}\frac{T}{nm}\left(\frac{\partial p}{\partial T}\right)_{N,V} & 0\end{pmatrix}. \tag{3.21}$$

The right eigenvector that corresponds to the eigenvalue $\omega_5^{(1)} = 0$ is

$$\begin{aligned}\psi_5^{\mathrm{R}(0)}(\boldsymbol{p}) &= a_5(\boldsymbol{p}) - \frac{T}{\langle 1|1\rangle}\frac{(\partial p/\partial T)_{N,V}}{(\partial p/\partial n)_{T,V}}\,a_1(\boldsymbol{p})\\ &= a_5(\boldsymbol{p}) + \frac{T}{\langle 1|1\rangle}\left(\frac{\partial n}{\partial T}\right)_{p,V} a_1(\boldsymbol{p}) \qquad (3.22a)\\ &= a_5(\boldsymbol{p}) - \frac{T}{n}\left(\frac{\partial p}{\partial T}\right)_{N,V}\frac{1}{1+F_0}\,a_1(\boldsymbol{p})\,. \qquad (3.22b)\end{aligned}$$

Here we have used Eqs. (A.20) to arrive at Eq. (3.22b). For the normalization of the heat mode this yields

$$\left\langle \psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = Tc_p \tag{3.23}$$

where we have used Eqs. (A20b) and (A27b).

(c) Sound modes. For the left eigenvectors associated with the eigenvalues $\omega_{1,2}^{(1)}$ we find, from Eq. (3.12b),

$$\begin{aligned}\psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p}) &= \pm c_1 a_2(\boldsymbol{p}) + \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_5(\boldsymbol{p})\\ &\quad + \left(\frac{\partial p}{\partial n}\right)_{T,V} a_1(\boldsymbol{p}) \qquad (3.24a)\\ &= \pm c_1 a_2(\boldsymbol{p}) + \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V}\psi_5^{\mathrm{L}(0)}(\boldsymbol{p}) + mc_1^2 a_1(\boldsymbol{p})\,. \qquad (3.24b)\end{aligned}$$

Here we have used Eqs. (3.16) and (3.14) to arrive at the second line. Combining Eqs. (2.19f), (2.19h), and (3.24a) we see that the sound modes are linear combinations of longitudinal velocity fluctuations, Eq. (2.19b), and pressure fluctuations δp:

$$\left\langle \psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{x},t)\right\rangle = \pm c_1 m\,\delta u_{\mathrm{L}}(\boldsymbol{x},t) + \delta p(\boldsymbol{x},t)\,. \tag{3.25}$$

The corresponding right eigenvector is obtained from Eq. (3.21) as

$$\begin{aligned}\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p}) &= \pm c_1 a_2(\boldsymbol{p}) + \frac{m^*}{c_V m}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_5(\boldsymbol{p})\\ &\quad + \frac{nm^*}{m}\frac{1}{\langle 1|1\rangle}\,a_1(\boldsymbol{p})\,, \qquad (3.26a)\end{aligned}$$

and for the normalization we obtain

$$\left\langle \psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = 2nm^* c_1^2 \tag{3.26b}$$

where we have used Eqs. (A20b) and (3.13).

All of these results are consistent with the corresponding ones for a classical fluid [23,26]. Note that the hydrodynamic modes are all orthogonal to one another, as they must be:

$$\left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\middle|\psi_\beta^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = \delta_{\alpha\beta}\left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\middle|\psi_\alpha^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle. \tag{3.27}$$

With the zeroth order eigenmodes $\psi^{(0)}$ in place, Eq. (3.7) now constitutes a well-defined integral equation that determines the modes at $O(k)$. The solution of this equation is not unique since one can add an arbitrary linear combination of the a_α to $\psi_\alpha^{L(1)}$ and still satisfy the equation. Uniqueness is achieved by the requirement that $\psi_\alpha^{L(1)}$ be an element of the orthogonal space $\mathcal{L}_\perp$. We can formally write this unique solution as

$$\left\langle \psi_\alpha^{\mathrm{L}(1)}(\boldsymbol{p})\right| = \left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\right|\left[L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) - \omega_\alpha^{(1)}(\boldsymbol{k})\right]\Lambda^{-1}(\boldsymbol{p})\,\mathcal{P}_\perp\,. \tag{3.28a}$$

Here the projection operator $\mathcal{P}_\perp$ projects onto the orthogonal space $\mathcal{L}_\perp$, and the inverse collision operator formally exists since it acts on a vector in $\mathcal{L}_\perp$. The formal expression (3.28a) should be interpreted as the solution of the integral Eq. (3.7) made unique by the orthogonality requirement that is enforced by $\mathcal{P}_\perp$. The corresponding right eigenvector at this order is

$$\left|\psi_\alpha^{\mathrm{R}(1)}(\boldsymbol{p})\right\rangle = \mathcal{P}_\perp\Lambda^{-1}(\boldsymbol{p})\left[L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p} - \omega_\alpha^{(1)}(\boldsymbol{k})\right]\left|\psi_\alpha^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle. \tag{3.28b}$$

3. *Shear diffusion, heat diffusion, and sound attenuation coefficients*

We now consider the eigenproblem, Eq. (3.1), at second order in k. At this order we have

$$\begin{aligned}&-\left\langle \psi_\alpha^{\mathrm{L}(2)}(\boldsymbol{p})\right|\Lambda(\boldsymbol{p}) + \left\langle \psi_\alpha^{\mathrm{L}(1)}(\boldsymbol{p})\right|\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) - \omega_\alpha^{(1)}(\boldsymbol{k})\right)\\ &\quad = \left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\right|\omega_\alpha^{(2)}(\boldsymbol{k},\boldsymbol{p})\,.\end{aligned} \tag{3.29}$$

If we multiply with $|\psi_\alpha^{\mathrm{R}(0)}\rangle$ from the right and use Eq. (3.28a) this becomes

$$\begin{aligned}\omega_\alpha^{(2)}(\boldsymbol{k}) &= \frac{1}{\left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\middle|\psi_\alpha^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle}\left\langle \psi_\alpha^{\mathrm{L}(0)}(\boldsymbol{p})\right|\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\right.\\ &\quad \left. - \omega_\alpha^{(1)}(\boldsymbol{k})\right)\Lambda^{-1}(\boldsymbol{p})\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) - \omega_\alpha^{(1)}(\boldsymbol{k})\right)\left|\psi_\alpha^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle.\end{aligned} \tag{3.30}$$

a. Shear diffusion.. We again first consider the shear modes, $\alpha = 3, 4$. In this case $\omega_{3,4}^{(1)} \equiv \omega_\perp^{(1)} = 0$, and we obtain

for the shear eigenvalue at second order

$$\omega_\perp^{(2)} = \frac{1}{nm^*}\left\langle \psi_\perp^{(0)}(\boldsymbol{p})|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\Lambda^{-1}(\boldsymbol{p})L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_\perp^{(0)}(\boldsymbol{p})\right\rangle \tag{3.31a}$$

$$= \nu k^2 \,. \tag{3.31b}$$

Here $\psi_\perp^{(0)}$ stands for either $\psi_3^{(0)}$ or $\psi_4^{(0)}$, and ν is the kinematic viscosity, which is given by the matrix element in Eq. (3.31a). It is related to the shear viscosity η via the physical mass density:

$$\nu = \eta/nm \,. \tag{3.32}$$

This is an Einstein relation that relates the kinematic viscosity ν, which is the shear diffusion coefficient, and the transport coefficient η via the static transverse momentum susceptibility, which is given by nm.

Using the $L_{\boldsymbol{k}}^{(1,2)}$ part of the kinetic operator in the form of Eq. (3.4d) we find

$$L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\left|\psi_\perp^{(0)}(\boldsymbol{p})\right\rangle = (1+F_1/3)\left|i(\boldsymbol{k}\cdot\boldsymbol{v}_p)(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{p})\right\rangle \,, \tag{3.33a}$$

$$\left\langle\psi_\perp^{(0)}(\boldsymbol{p})\right|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) = \left\langle i(\boldsymbol{k}\cdot\boldsymbol{v}_p)(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{p})\right| \,, \tag{3.33b}$$

where $\hat{\boldsymbol{k}}_\perp$ stands for either $\hat{\boldsymbol{k}}_\perp^1$ or $\hat{\boldsymbol{k}}_\perp^2$. Note that both of these vectors are orthogonal to all five vectors $a_\alpha(\boldsymbol{p})$ that span the zero-eigenvector space $\mathcal{L}_0$, as they must be, so the matrix element in Eq. (3.31a) exists. We thus have

$$\eta = -\left\langle(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\right|\Lambda^{-1}(\boldsymbol{p})\left|(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{p})\right\rangle \,. \tag{3.34}$$

That is, the shear viscosity is given by a transverse stress correlation, as expected from classical kinetic theory at the level of the Boltzmann equation [23].

If we replace the collision operator by the BKG model collision operator from Appendix C, then $\Lambda^{-1}(\boldsymbol{p})$ in Eq. (3.34) effectively becomes $-\tau$, with τ the collision time. At $T=0$ we then recover the expression for the kinematic viscosity and the shear viscosity familiar from LFL theory [3]:

$$\nu = \tfrac{1}{5}\,(v_{\rm F}^*)^2\tau(1+F_1/3) \,, \tag{3.35a}$$

$$\eta = \frac{nm}{5}\,(v_{\rm F}^*)^2\tau(1+F_1/3) \,. \tag{3.35b}$$

(b) Heat diffusion. Equation (3.30) for $\alpha = 5$ yields

$$\omega_5^{(2)} = \frac{1}{Tc_p}\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\Lambda^{-1}(\boldsymbol{p})L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle \tag{3.36a}$$

$$= D_T k^2 \tag{3.36b}$$

where D_T is the heat diffusion coefficient. It is related to the thermal or heat conductivity κ via

$$D_T = \kappa/c_p \,. \tag{3.37}$$

Using Eq. (3.4d) to calculate the vectors in the matrix element in Eq. (3.36b) we find

$$\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) = ik\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\right| \,, \tag{3.38a}$$

$$L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\left|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = ik\left|(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right\rangle \,. \tag{3.38b}$$

Note that the kinetic operator $L_{\boldsymbol{k}}^{(1)}$ turns the right eigenfunction, Eq. (3.22) into a current related to the left eigenfunction, given by Eq. (3.16). We have used Eqs. (A4) and (A.20) to arrive at this result. For the thermal conductivity this yields

$$\kappa = \frac{-1}{T}\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\right|\Lambda^{-1}(\boldsymbol{p})\left|(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right\rangle \,. \tag{3.39}$$

Again, this is consistent with the corresponding result in classical kinetic theory [23], and Eq. (3.37) is an Einstein relation that relates the heat diffusion coefficient D_T to the transport coefficient $T\kappa$ via the heat susceptibility Tc_p.

If we replace $\Lambda(\boldsymbol{p})$ by the BGK model collision operator from Appendix C and evaluate Eq. (3.39) to lowest order in the temperature, we find

$$\kappa = c_p\,\tfrac{1}{3}(v_{\rm F}^*)^2\tau \,, \tag{3.40a}$$

$$D_T = \tfrac{1}{3}(v_{\rm F}^*)^2\tau \,, \tag{3.40b}$$

which is the result familiar from LFL theory [3]. Note that at this order there is no difference between c_p and c_V.

(c) Sound attenuation. We finally consider Eq. (3.30) for $\alpha = 1, 2$. We have

$$\omega_{1,2}^{(2)} = \frac{1}{2nm^*c_1^2}\left\langle\psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p})\right|\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) \mp ic_1k\right)\Lambda^{-1}(\boldsymbol{p}) \times \left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) \mp ic_1\right)\left|\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle \tag{3.41a}$$

$$= \tfrac{1}{2}\,\Gamma\,k^2 \tag{3.41b}$$

with Γ the sound attenuation coefficient.

Using various identities from Appendix A, as well as Eq. (2.10a), we can write the relevant left vector in the form

$$\left\langle\psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p})\right|\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) \mp ic_1k\right) = ik\Bigg[\pm c_1\,\hat{k}_i\hat{k}_j\langle\sigma^{ij}(\boldsymbol{p})| + \frac{1}{c_V} \times \left(\frac{\partial p}{\partial T}\right)_{N,V}\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)| + mc_1^2\langle\hat{\boldsymbol{k}}\cdot(\boldsymbol{v}_p - \boldsymbol{p}/m^*)|\Bigg] \,, \tag{3.42a}$$

where we have used the sound mode in the form (3.24b). The second term in Eq. (3.42a) we recognize as proportional to the heat current density from Eqs. (3.38) and Appendix B 2. For the corresponding right vector we find

$$\left(L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) \mp ic_1k\right)\left|\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = ik\,\frac{m^*}{m}\Bigg[\pm c_1\,\hat{k}_i\hat{k}_j|\sigma^{ij}(\boldsymbol{p})\rangle + \frac{1}{c_V} \times \left(\frac{\partial p}{\partial T}\right)_{N,V}|\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\rangle + mc_1^2|\hat{\boldsymbol{k}}\cdot(\boldsymbol{v}_p - \boldsymbol{p}/m^*)\rangle\Bigg] \,. \tag{3.42b}$$

The last term on the right-hand side of both Eq. (3.42a) and (3.42b) is the component of the longitudinal QP velocity in the orthogonal space $\mathcal{L}_\perp$. It thus vanishes for the model interaction given by Eq. (3.4d), and we drop it. The stress tensor σ^{ij} is given by

$$\sigma^{ij}(\boldsymbol{p}) = \sigma_1^{ij}(\boldsymbol{p}) - \delta^{ij}\,\sigma_2(\boldsymbol{p}) \tag{3.43a}$$

where

$$\sigma_1^{ij}(\boldsymbol{p}) = p^i v_p^j - \delta^{ij}\frac{1}{3}\,\boldsymbol{p}\cdot\boldsymbol{v}_p\,, \tag{3.43b}$$

and

$$\sigma_2(\boldsymbol{p}) = \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V}\epsilon_p - \frac{1}{3}\,\boldsymbol{p}\cdot\boldsymbol{v}_p + \frac{n}{\langle 1|1\rangle} - \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V}\langle\epsilon_p\rangle_w\,. \tag{3.43c}$$

Note that σ_1 and σ_2 are separately orthogonal to the zero-eigenvector space. To calculate the eigenfrequencies, Eqs. (3.41), we use the fact that the collision operator is isotropic in momentum space. As a result, the contributions from σ_1, σ_2, and the heat current do not mix, and the contribution from σ_1 can be related to the shear viscosity η, Eq. (3.34). After a calculation that makes extensive use of the identities in Appendix A, in particular Eqs. (A.20) and (A.27), we obtain

$$\Gamma = \frac{1}{nm}\left(\frac{4}{3}\,\eta + \zeta\right) + D_T\left(\frac{c_p}{c_V} - 1\right). \tag{3.44a}$$

Here D_T is the heat diffusion coefficient, Eqs. (3.37) and (3.39), η is the shear viscosity, Eq. (3.34), and

$$\zeta = -\langle\sigma_2(\boldsymbol{p})|\Lambda^{-1}(\boldsymbol{p})|\sigma_2(\boldsymbol{p})\rangle \tag{3.44b}$$

is a contribution to the bulk viscosity [27]. The latter vanishes in a Fermi gas, where $p = 2e/3$ and hence $\sigma_2(\boldsymbol{p}) = 0$. It also vanishes in the model where the QP interaction is given by Eq. (3.4d). This can be seen by realizing that $-3|\sigma_2(\boldsymbol{p})\rangle\rangle = \mathcal{P}_\perp|\boldsymbol{p}\cdot\boldsymbol{v}_p\rangle\rangle$ is the projection of $\boldsymbol{p}\cdot\boldsymbol{v}_p$ onto the orthogonal space $\mathcal{L}_\perp$. But from Eq. (2.10b) it follows that $\boldsymbol{p}\cdot\boldsymbol{v}_p \in \mathcal{L}_0$, and hence $\sigma_2(\boldsymbol{p}) = 0$. However, in the case of a more general QP interaction it will be nonzero and of $O(T^4)$ relative to the shear viscosity, see the remarks in Ref. [19]. These properties of ζ are consistent with the results of Ref. [28], and the expression for the sound attenuation, Eq. (3.44a), is the same as the one for a classical fluid [25].

The complete result for the eigenvalues $\omega_{1,2}$ that describe first sound is

$$\omega_{1,2}(\boldsymbol{k}) = \pm i c_1 k + \tfrac{1}{2}\,\Gamma\,k^2 + O(k^3)\,, \tag{3.45}$$

with c_1 the speed of sound from Eq. (3.13), and Γ the sound attenuation coefficient from Eq. (3.44a).

B. Long-ranged case

As mentioned in the context of Eq. (2.11), in a charged Fermi liquid the interaction parameter F_0 acquires an additional contribution that is due to the Coulomb interaction and diverges for small wave numbers as $1/k^2$. The main effect of this term is to turn the soft sound mode into the massive (in $d = 3$) plasmon mode. It also affects the heat diffusion coefficient, but does not change the diffusive nature of the heat transport.

We define the plasma frequency ω_p

$$\omega_p^2 = 4\pi n e^2/m\,, \tag{3.46a}$$

and the Thomas-Fermi screening wave number $p_{\rm TF}$,

$$p_{\rm TF}^2 = nm\chi_T\,\omega_p^2\,. \tag{3.46b}$$

Here $\chi_T = (\partial n/\partial\mu)_{N,V}/n^2 = N_0/n^2(1+F_0)$ is the isothermal compressibility in the SR case. The isothermal compressibility of the Coulomb system takes the form

$$\chi_T^{\rm LR} = \chi_T\,\frac{k^2}{k^2 + p_{\rm TF}^2}\,. \tag{3.47}$$

Note that $\chi_T^{\rm LR}$ is wave-number dependent and vanishes at $k = 0$, i.e., the system is incompressible with respect to uniform deformations.

1. Hydrodynamic modes

The linearized kinetic operator is now given by

$$L_{\boldsymbol{k}}(\boldsymbol{p}) = L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p}) - \Lambda(\boldsymbol{p}) + L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\,, \tag{3.48}$$

with $L_{\boldsymbol{k}}^{(1)}$ from Eqs. (3.4) and

$$L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p}) = \frac{4\pi e^2}{k^2}\,\big|i\boldsymbol{k}\cdot\boldsymbol{v}_p\rangle\rangle\langle\langle 1\big| \tag{3.49}$$

Accordingly, the left eigenproblem from Eqs. (3.8) needs to be augmented by a matrix

$$L_{\beta\gamma}^{(-1)}(\boldsymbol{k},\boldsymbol{p}) = \langle a_\beta(\boldsymbol{p})|L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p})|a_\gamma(\boldsymbol{p})\rangle A_\gamma^2 \tag{3.50}$$

whose only nonzero element is $L_{21}^{(-1)}(\boldsymbol{k},\boldsymbol{p}) = i\,\omega_p^2\,m/k$. As a result, the matrix L_L for the longitudinal modes acquires a contribution of $O(1/k)$, and instead of Eq. (3.12b) we have

$$L_L^{(-1)} + L_L^{(1)} = ik\begin{pmatrix} 0 & 1/m & 0 \\ \left(\frac{\partial p}{\partial n}\right)_{T,V} + \frac{m\omega_p^2}{k^2} & 0 & \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V} \\ 0 & \frac{T}{nm}\left(\frac{\partial p}{\partial T}\right)_{N,V} & 0 \end{pmatrix}. \tag{3.51}$$

This again represents the linearized Euler equations, albeit in the presence of a long-ranged interaction. The eigenvalue ω_5, which corresponds to the heat mode, is still zero at this level. However, the eigenvalues $\omega_{1,2}$ are now given by

$$\omega_{1,2}^2 = -\left(\omega_p^2 + c_1^2 k^2\right) \tag{3.52a}$$

with c_1 the speed of first sound in the SR case, Eq. (3.13). They thus have a contribution at $O(k^0)$,

$$\omega_{1,2}^{(0)} = \pm i\omega_p\,, \tag{3.52b}$$

and one at $O(k^2)$,

$$\omega_{1,2}^{(2)} = \pm i\,k^2\,c_1^2/2\omega_p\,, \tag{3.52c}$$

as well as contributions at higher order. This reflects the fact that the soft first-sound modes have been turned into massive plasmon modes by the long-ranged interaction [29].

(a) Shear modes. The shear modes are unaffected by the long-ranged interaction and are still given by Eqs. (3.15).

(b) Heat mode. Similarly, the heat mode, i.e., the left eigenvector associated with the eigenvalue $\omega_5 = 0$, is unaffected and still given by Eq. (3.16). However, the corresponding right eigenvector, which we obtain from the modification of Eq. (3.21) that corresponds to Eq. (3.51), now has one contribution at $O(k^0)$ and another at $O(k^2)$, as can be seen from

Eq. (3.22b) in conjunction with Eq. (2.11). The components of the eigenvectors in the zero eigenvector space $\mathcal{L}_0$ thus are

$$\psi_5^{\mathrm{L}(0)}(\boldsymbol{p}) = a_5(\boldsymbol{p}) - \frac{T}{n}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_1(\boldsymbol{p})\,, \tag{3.53a}$$

$$\psi_5^{\mathrm{R}(0)}(\boldsymbol{p}) = a_5(\boldsymbol{p})\,, \tag{3.53b}$$

$$\psi_5^{\mathrm{R}(2)}(\boldsymbol{p}) = -k^2\,\frac{T(\partial p/\partial T)_{N,V}}{m\,\omega_p^2\,\langle 1|1\rangle}\,a_1(\boldsymbol{p}). \tag{3.53c}$$

Accordingly, the normalization of the heat mode to lowest order in k is given by c_V rather then c_p:

$$\left\langle \psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = T c_V\,. \tag{3.54}$$

(c) Plasmon modes. The plasmon modes are given by the left eigenvectors of the matrix in Eq. (3.51) for the eigenvalues $\omega_{1,2}$. If we normalize the modes such that the generalization of Eq. (3.26b) is finite at $k = 0$, we have, up to $O(k^2)$,

$$\psi_{1,2}^{\mathrm{L}(-1)}(\boldsymbol{p}) = \frac{1}{k}\,m\,\omega_p^2\,a_1(\boldsymbol{p}), \tag{3.55a}$$

$$\psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p}) = \pm\omega_p\,a_2(\boldsymbol{p}), \tag{3.55b}$$

$$\psi_{1,2}^{\mathrm{L}(1)}(\boldsymbol{p}) = k\left[\frac{1}{n\chi_T}\,a_1(\boldsymbol{p}) + \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_5(\boldsymbol{p})\right]. \tag{3.55c}$$

$$\psi_{1,2}^{\mathrm{L}(2)}(\boldsymbol{p}) = \pm k^2\,\frac{c_1^2}{2\omega_p}\,a_2(\boldsymbol{p})\,. \tag{3.55d}$$

For the corresponding right eigenvector one finds

$$\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p}) = \pm\,\omega_p\,a_2(\boldsymbol{p}), \tag{3.56a}$$

$$\psi_{1,2}^{\mathrm{R}(1)}(\boldsymbol{p}) = k\,\frac{m^*}{m}\left[\frac{n}{\langle 1|1\rangle}\,a_1(\boldsymbol{p}) + \frac{1}{c_V}\left(\frac{\partial p}{\partial T}\right)_{N,V} a_5(\boldsymbol{p})\right], \tag{3.56b}$$

$$\psi_{1,2}^{\mathrm{R}(2)}(\boldsymbol{p}) = \pm k^2\,\frac{c_1^2}{2\omega_p}\,a_2(\boldsymbol{p})\,. \tag{3.56c}$$

The normalization now is

$$\left\langle \psi_{1,2}^{\mathrm{L}}(\boldsymbol{p})|\psi_{1,2}^{\mathrm{R}}(\boldsymbol{p})\right\rangle = 2nm^*\omega_p^2 + O(k^2)\,. \tag{3.57}$$

This completes the determination of the components of the hydrodynamic modes that are elements of $\mathcal{L}_0$. In contrast to the SR case they do not all occur at the same order in the k expansion. Note that all of the results so far can be obtained from those for the SR case, Secs. III A 1 and III A 2, by using the substitution (2.11) for the Fermi-liquid parameter F_0.

2. *Transport coefficients, and plasmon damping and dispersion*

(a) Shear diffusion. As we have seen, the shear modes are unaffected by the Coulomb interaction. Accordingly, the shear viscosity and the shear diffusion coefficient are still given by Eqs. (3.34) and (3.32), respectively.

(b) Heat diffusion. Consider the left eigenproblem, Eq. (3.1), for $\alpha = 5$ with the kinetic operator given by Eq. (3.48). The parts of the eigenfunctions that are elements of the zero eigenvector space are given by Eqs. (3.53). Expanding in powers of k as in Sec. III A 1 we have at lowest order

$$\left\langle \psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right| L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p}) = 0\,, \tag{3.58a}$$

which holds since the angular integration vanishes. At $O(k^0)$ we have

$$-\left\langle \psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right|\Lambda(\boldsymbol{p}) + \left\langle \psi_5^{\mathrm{L}(1)}(\boldsymbol{p})\right| L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p}) = 0\,. \tag{3.58b}$$

Here the first term on the left-hand side vanishes due to the conservation laws. The vector $\langle\psi_5^{\mathrm{L}(1)}(\boldsymbol{p})|$ is orthogonal to the zero eigenvector space, whereas the vector $|\boldsymbol{k}\cdot\boldsymbol{v}_p\rangle$ in the operator $L_{\boldsymbol{k}}^{(-1)}$ is proportional to the zero eigenvector $|a_2(\boldsymbol{p})\rangle$ due to Eq. (2.10a). Hence this condition is also fulfilled. Anticipating that the first nonzero contribution to the eigenvalue appears only at $O(k^2)$, at $O(k)$ we have

$$\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\,L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) - \langle\psi_5^{\mathrm{L}(1)}(\boldsymbol{p})|\,\Lambda(\boldsymbol{p}) + \langle\psi_5^{\mathrm{L}(2)}(\boldsymbol{p})|\,L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p}) = 0\,. \tag{3.58c}$$

The last term on the left-hand side again vanishes since $\psi_5^{\mathrm{L}(2)}$ is orthogonal to the zero eigenvector space, and we obtain a formal expression for $\psi_5^{\mathrm{L}(1)}$ in analogy to Eq. (3.28a):

$$\begin{aligned}\left\langle \psi_5^{\mathrm{L}(1)}(\boldsymbol{p})\right| &= \langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\,L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\,\Lambda^{-1}(\boldsymbol{p})\,\mathcal{P}_\perp \\ &= ik\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\right|\Lambda^{-1}(\boldsymbol{p})\,\mathcal{P}_\perp\,.\end{aligned} \tag{3.59a}$$

Analogous arguments for the right eigenproblem yield

$$\begin{aligned}\left|\psi_5^{\mathrm{R}(1)}(\boldsymbol{p})\right\rangle = \mathcal{P}_\perp\Lambda^{-1}\big(L_{\boldsymbol{k}}^{(-1)}(\boldsymbol{p})\left|\psi_5^{\mathrm{R}(2)}(\boldsymbol{p})\right\rangle + L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p}) \\ \times\,\left|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle\big) = ik\,\mathcal{P}_\perp\Lambda^{-1}(\boldsymbol{p})\left|(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\right\rangle.\end{aligned} \tag{3.59b}$$

Note that linear combination of vectors on the right-hand side of Eq. (3.59b) produces the same result for $\psi_5^{\mathrm{R}(1)}$ as in the SR case, see Eqs. (3.38).

In order to determine the eigenvalue at $O(k^2)$ we multiply Eq. (3.1) from the right with $|\psi_5^{\mathrm{R}}(\boldsymbol{p})\rangle$ and expand order by order in powers of k. The terms at $O(k^{-1})$ and $O(k^0)$ vanish due to a combination of $L_{\boldsymbol{k}}^{(-1)}$ acting on the orthogonal space and the conservation laws. At $O(k)$ we find

$$\omega_5^{(1)}(\boldsymbol{k})\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle = \left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle, \tag{3.60}$$

which vanishes due to the angular integration, so

$$\omega_5^{(1)}(\boldsymbol{k}) = 0 \tag{3.61}$$

as expected. At $O(k^2)$ we find

$$\begin{aligned}\omega_5^{(2)}(\boldsymbol{k})\left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\right\rangle &= \left\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(1)}(\boldsymbol{p})\right\rangle \\ &- \left\langle\psi_5^{\mathrm{L}(1)}(\boldsymbol{p})|\Lambda(\boldsymbol{p})|\psi_5^{\mathrm{R}(1)}(\boldsymbol{p})\right\rangle + \langle\psi_5^{\mathrm{L}(1)}(\boldsymbol{p}) \\ &\times\,|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\rangle\,.\end{aligned} \tag{3.62a}$$

The first two terms on the right-hand side cancel by Eq. (3.59a). To evaluate the third term we note that $L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\rangle = ik|(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})a_5(\boldsymbol{p})\rangle$ has a component in the zero eigenvector space that is eliminated by the projection

operator in Eq. (3.59a). This yields

$$\omega_5^{(2)}(\boldsymbol{k})\langle\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\psi_5^{\mathrm{R}(0)}(\boldsymbol{p})\rangle = -k^2\langle(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})|\Lambda^{-1}(\boldsymbol{p}) \times |(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})\psi_5^{\mathrm{L}(0)}(\boldsymbol{p})\rangle\,. \tag{3.62b}$$

Using the normalization (3.54) we finally obtain

$$\omega_5^{(2)}(\boldsymbol{k}) = D_T k^2\,, \tag{3.62c}$$

with the heat diffusion coefficient

$$D_T = \kappa/c_V \tag{3.63}$$

The heat conductivity κ is given by the same expression as in the SR case, Eq. (3.39), but the susceptibility in the Einstein relation (3.63) is now c_V instead of c_p.

(c) Plasmon damping and dispersion. An analogous analysis for the $\alpha = 1, 2$ channels confirms Eq. (3.52b) for the eigenvalues at $O(k^0)$. The eigenvalues at this order combine with the collision operator, and it is convenient to define

$$\tilde{\Lambda}_{1,2} = \Lambda + \omega_{1,2}^{(0)}\,. \tag{3.64}$$

At $O(k)$ we find contributions $\delta\psi_{1,2}^{\mathrm{L,R}(1)}$ in the orthogonal space $\mathcal{L}_\perp$ that need to be added to Eqs. (3.55c) and (3.56b). These are

$$\langle\delta\psi_{1,2}^{\mathrm{L}(1)}(\boldsymbol{p})| = \langle\psi_{1,2}^{\mathrm{L}(0)}(\boldsymbol{p})|\,L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\,\tilde{\Lambda}_{1,2}^{-1}\,\mathcal{P}_\perp\,, \tag{3.65a}$$

$$|\delta\psi_{1,2}^{\mathrm{R}(1)}(\boldsymbol{p})\rangle = \mathcal{P}_\perp\tilde{\Lambda}^{-1}L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})\,|\psi_{1,2}^{\mathrm{R}(0)}(\boldsymbol{p})\rangle\,. \tag{3.65b}$$

These expressions need to be interpreted as the solutions of the underlying integral equations, with uniqueness enforced by the projection operator $\mathcal{P}_\perp$, see the comments in the context of Eq. (3.28a). There is no contribution to the eigenvalues at this order, but at $O(k^2)$ one finds a contribution in addition to Eq. (3.52c):

$$\delta\omega_{1,2}^{(2)} = \frac{-k^2\,\omega_p^2}{2nm}\langle\hat{k}_i\hat{k}_j\sigma^{ij}(\boldsymbol{p})|\tilde{\Lambda}_{1,2}^{-1}|\hat{k}_l\hat{k}_m\sigma^{lm}(\boldsymbol{p})\rangle \tag{3.66a}$$

$$= \frac{-k^2\,\omega_p^2}{2nm}\,\frac{4}{3}\langle(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p)|\tilde{\Lambda}_{1,2}^{-1}|(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p)\rangle \tag{3.66b}$$

with $\sigma^{ij}(\boldsymbol{p})$ from Eq. (3.43a). In writing Eq. (3.66b) we have used the fact that the contribution σ_2 to the stress tensor, Eq. (3.43c), vanishes for the model QP interaction we are using, see the comment after Eq. (3.44b).

This contribution to the eigenfrequency has the structure of a transport coefficient; it can be interpreted as a high-frequency shear viscosity, see Appendix E in Paper II. It has both a real part and an imaginary part, so it contributes both to the plasmon dispersion and the plasmon damping. We emphasize that these results can *not* be obtained from the SR case by means of a simple substitution. In a low-temperature expansion one can take advantage of the fact that the collision operator scales as some positive power of the temperature and separate the real and imaginary parts of $\delta\omega_{1,2}^{(2)}$,

$$\omega_{1,2}(\boldsymbol{k}) = \pm i\Omega_p(\boldsymbol{k}) + \tfrac{1}{2}\,\Gamma_p\,k^2 + O(k^4)\,. \tag{3.67a}$$

Here

$$\Omega_p(\boldsymbol{k}) = \omega_p\sqrt{1 + c_1^2k^2/\omega_p^2} + \frac{2}{3}\,\frac{1}{nm^*\omega_p}\langle(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p) |1 + O(\Lambda^2)|(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p)\rangle\,k^2 + O(k^4) \tag{3.67b}$$

is the wave-number dependent plasmon frequency, and

$$\Gamma_p = \frac{-1}{nm^*\omega_p^2}\,\frac{4}{3}\langle(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p)|\Lambda + O(\Lambda^3)|(\hat{\boldsymbol{k}}\cdot\boldsymbol{p})(\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{v}_p)\rangle \tag{3.67c}$$

is the plasmon damping coefficient. Note that the leading contribution to the plasmon damping is linear in $\Lambda \sim 1/\tau(T)$, with $\tau(T)$ a relaxation time, and thus decreases with decreasing temperature, whereas the first-sound damping is proportional to τ [see Eqs. (3.44) and (3.34), and thus increases with decreasing temperature. A related observation is that the plasmon damping coefficient in the low-temperature limit is no longer given by the solution of an integral equation, in contrast to the SR case, see Eqs. (3.41). If we use the model BGK operator from Appendix C we find, to $O(k^2)$ and to lowest order in the temperature,

$$\Omega_p(\boldsymbol{k}) = \omega_p + \frac{1}{2\omega_p}\left[c_1^2 + \frac{4}{15}\,(v_{\mathrm{F}}^*)^2\right]k^2 \tag{3.68a}$$

$$\Gamma_p = \frac{4}{15}\left(\frac{v_{\mathrm{F}}^*}{\omega_p}\right)^2\frac{1}{\tau}\,. \tag{3.68b}$$

C. The fate of the hydrodynamic modes in the low-temperature limit

As we have seen, the hydrodynamic modes in a degenerate Fermi liquid are essentially the same as those in a classical fluid. With decreasing temperature the hydrodynamic regime shrinks; it is confined to wave numbers smaller than

$$q^* \approx 1/v_{\mathrm{F}}\tau(T)\,, \tag{3.69}$$

where $1/\tau(T)$ is a generic relaxation rate that vanishes as $T \to 0$. At low temperatures for fixed wave number, or at larger wave numbers for fixed temperature, the system enters a different regime where collisions between QPs are no longer dominating the physics and the soft modes are of a very different nature. It is sometimes said that the hydrodynamic modes "cross over" to the collisionless ones. This is misleading for various reasons. First, the hydrodynamic modes to not evolve into something else; rather, they disappear since the regime that supports their existence shrinks to zero. Second, there is no one-to-one correspondence between soft modes in the two regimes. The number of soft modes in the collisionless regime is not limited by the number of conservation laws, and in some sense are infinitely many of them, as we will discuss next.

The above discussion pertains to the five hydrodynamic modes in the SR case, and to the modes that remain soft in

the LR case. The plasmon is qualitatively different: It is not a hydrodynamic mode, but rather a consequence of gauge invariance [30]. Therefore, it exists, and actually becomes better defined, even in the absence of collisions. The plasmon excitation in the hydrodynamic regime is thus identical to the one in the collisionless regime, as we will see in Sec. IV B below.

IV. SOLUTIONS OF THE KINETIC EQUATIONS II: COLLISIONLESS REGIME

So far we have discussed the hydrodynamic regime, where the collision operator Λ dominates the kinetic operator $L_{\boldsymbol{k}}$. As we have seen, the soft modes in that case are controlled by the five conservation laws. The soft modes are the zero eigenfunctions that correspond to the five zero eigenvalues of Λ, and their long-wavelength properties are given by the operator $L^{(1)}_{\boldsymbol{k}}$, Eq. (3.4a), which perturbs the zero eigenvalues. We now turn to the collisionless regime, which is defined by the linearized kinetic operator $L^{(1)}_{\boldsymbol{k}}$ dominating over the collision operator. This realized for wave numbers larger than q^* in Eq. (3.69). At $T = 0$, where there are no collisions, the collisionless regime extends all the way to zero wave number. This regime has been discussed extensively in the context of He-3 [3,31], and we will focus on aspects that are either less well known or important for other applications. In Appendix D we briefly discuss the relation, or rather lack thereof, between the zero modes in the collisionless regime and the hydrodynamic modes, in order to demonstrate the different physical origins of these respective excitations.

A. Short-ranged case

Consider the kinetic equation (2.16a) at $T = 0$, where $\Lambda(\boldsymbol{p}) = 0$. Since the kinetic operator $L_{\boldsymbol{k}}$ is now linear in $\boldsymbol{k}$, the entire distribution function ϕ is soft. Consequently, all of its moments with respect to the momentum $\boldsymbol{p}$ are soft, and in this sense we have an infinite number of soft modes; see Sec. V for an elaboration. For simplicity, let us first consider the case where the only nonzero Landau parameter is F_0. Then Eq. (2.16a) yields

$$\phi(\boldsymbol{p},\boldsymbol{k},z) = \frac{F_0}{N_0}\,\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}{\zeta-\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}\,\delta n(\boldsymbol{k},z) + \frac{i\,\phi(\boldsymbol{p},\boldsymbol{k},t=0)}{z-v_{\mathrm{F}}^*k\,(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}})}\,, \tag{4.1}$$

where $\zeta = z/v_{\mathrm{F}}^*k$. By summing over $\boldsymbol{p}$ we obtain a linear equation for the density fluctuation δn:

$$\delta n(\boldsymbol{k},z)[1-F_0I_1(\zeta)] = \frac{1}{V}\sum_{\boldsymbol{p}} w(\boldsymbol{p})\,\frac{i\,\phi(\boldsymbol{p},\boldsymbol{k},t=0)}{z-v_{\mathrm{F}}^*k\,(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}})} \tag{4.2a}$$

where

$$I_1(\zeta) = \frac{1}{2}\int_{-1}^{1} d\eta\,\frac{\eta}{\zeta-\eta} = -1-\frac{\zeta}{2}\,\log\left(\frac{\zeta-1}{\zeta+1}\right). \tag{4.2b}$$

Substituting this expression back into Eq. (4.1) yields an explicit expression for ϕ. We see that all of the soft modes are characterized by a propagator

$$P(\boldsymbol{k},z) = P(\zeta) = \frac{1}{1-F_0I_1(\zeta)} - 1\,. \tag{4.3}$$

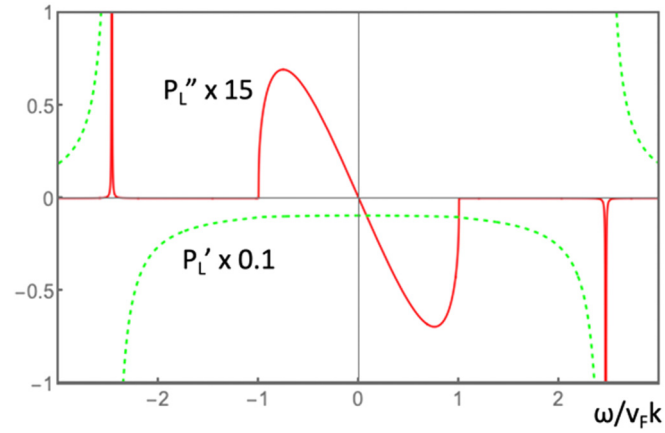

FIG. 2. Spectrum P_L'' (solid red) and reactive part P_L' (dashed green) of the longitudinal propagator, Eq. (4.6a), for $F_0 = F_1 = 5$. The spectrum consists of the continuous unparticle contribution in the center and the two zero-sound delta-function contributions. The transverse propagator, Eq. (4.6b), is qualitatively the same except that the zero-sound contributions to the spectrum exist only if $F_1 > 6$. For either propagator the entire spectrum is scale invariant.

Here the constant subtraction term serves to make P a proper causal function that vanishes for $\zeta \to \infty$. The spectrum of P, $P''(\boldsymbol{k},\omega) = \mathrm{Im}\,P(\boldsymbol{k},\omega+i0)$, consists of a continuous part and, in addition, delta-function contributions that correspond to zeros of the denominator. Two such zeros exist for any $F_0 > 0$; they are the well known zero-sound modes [2]. The resonance frequencies (which correspond to $-i$ times the perturbed zero eigenvalues ω in Sec. III) are

$$z = \pm c_0 k\,, \tag{4.4a}$$

where the speed of zero sound is given by

$$c_0 = \sigma_0 v_{\mathrm{F}}^*\,, \tag{4.4b}$$

with σ_0 the solution of

$$I_1(\sigma_0) = 1/F_0\,, \tag{4.4c}$$

which exists and is unique for all $F_0 > 0$.

If we add the Landau parameter F_1, we obtain an expression for ϕ in terms of the density fluctuation δn and the velocity fluctuation $\delta\boldsymbol{u}$ that is a generalization of Eq. (4.1):

$$\phi(\boldsymbol{p},\boldsymbol{k},z) = \frac{F_0}{N_0}\,\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}{\zeta-\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}\,\delta n(\boldsymbol{k},z) + \frac{F_1}{N_0}\,\frac{n}{v_{\mathrm{F}}^*} \times \frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}{\zeta-\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}\,\hat{\boldsymbol{p}}\cdot\delta\boldsymbol{u}(\boldsymbol{k},z) + \frac{i\,\phi(\boldsymbol{p},\boldsymbol{k},t=0)}{z-v_{\mathrm{F}}^*k\,\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}}\,. \tag{4.5}$$

Using this expression to calculate δn and $\delta\boldsymbol{u}$ via Eqs. (2.17a) and (2.17b) we obtain a 2×2 system for δn and δu_L whose determinant generalizes the propagator P to

$$P_L(\zeta) = \frac{1}{1+F_1/3-[F_0(1+F_1/3)+F_1\zeta^2]I_1(\zeta)} - 1\,. \tag{4.6a}$$

The spectrum is qualitatively the same as for $F_1 = 0$. In addition, the equation for the transverse velocity yields a

transverse propagator

$$P_T(\zeta) = \frac{1}{1 - \frac{1}{2}F_1\left[\frac{1}{3} - (\zeta^2 - 1)I_1(\zeta)\right]}. \tag{4.6b}$$

The spectrum of P_T also has a continuous part, and for $F_1 > 6$ it contains, in addition, transverse zero-sound modes. For a comparison between the latter and the hydrodynamic shear diffusion modes, see Appendix D.

These results demonstrate that the soft modes in the collisionless regime are qualitatively different from those in the hydrodynamic regime: There always is a continuous part of the spectrum, which represents a continuum of soft modes that obey a linear scaling of the frequency with the wave number. This continuous part is not particle-like and has no analog in the hydrodynamic regime; in a particle-physics context such continuous scale invariant excitations have been dubbed "unparticles" [7]. In addition, there are particle-like zero-sound modes that are unrelated to conservation laws; how many of these there are depends on the QP interaction. Figure 2 demonstrates these features. We emphasize that the continuum and the zero-sound poles are part of the same spectrum, but the continuum is the more fundamental part in the sense that it does not depend on the values of the Landau parameters and exists even in a noninteracting Fermi system.

So far we considered only the Landau parameters F_0 and F_1. If one keeps higher terms in the expansion of $F(\boldsymbol{p}, \boldsymbol{p}')$ in Legendre polynomials, Eq. (2.8), additional zero-sound modes can appear. To demonstrate this, we consider a model that keeps the Landau parameter F_2 and repeat the above analysis. The longitudinal propagator then becomes

$$P_L(\zeta) = \frac{1}{\left(1 + \frac{1}{3}F_1\right)\left[1 - \frac{1}{20}(1 + 5F_0)F_2\right] + \frac{3}{4}F_2\zeta^2 - d(\zeta)I_1(\zeta)} - 1\,, \tag{4.7a}$$

where

$$d(\zeta) = \left(1 + \frac{1}{3}F_1\right)\left[F_0 + \frac{1}{4}(1 + \frac{9}{5}F_0)F_2\right] + \left[F_1 - \frac{3}{2}\left(1 + \frac{1}{2}F_0(1 + \frac{1}{3}F_1) + \frac{1}{30}F_1\right)F_2\right]\zeta^2 + \frac{9}{4}F_2\zeta^4\,. \tag{4.7b}$$

An inspection shows that the denominator of P_L has one zero that is continuously related to the zero-sound pole in Eq. (4.6a). It has a second zero, closer to the continuous part of the spectrum, provided the following conditions are fulfilled:

$$F_0 > \frac{10}{3}\,\frac{1 + F_1/30}{1 + F_1/3} \tag{4.8a}$$

and

$$F_2 > \frac{30F_0\left(1 + \frac{1}{3}F_1\right)}{9F_0\left(1 + \frac{1}{3}F_1\right) - F_1 - 30}\,. \tag{4.8b}$$

While the existence of the additional collective mode requires a rather large positive value of F_2, it illustrates again that the zero-sound modes are governed by the QP interaction rather than by conservation laws.

At a low nonzero temperature the collision operator gives all of the soft modes in the collisionless regime a mass. For the zero-sound modes this takes the form of a damping term that broadens the δ function in the spectrum of the propagator. For an explicit example we use the BGK model collision operator from Appendix C, with $\tau(T)$ the temperature-dependent relaxation time. The infinitely many soft modes now couple, and the kinetic equation is no longer exactly soluble even if one keeps only the Landau parameter F_0. In an approximation that keeps only the density and the longitudinal momentum, and ignores the coupling to the modes at higher angular momenta, we find for the resonance frequencies that generalize Eq. (4.4a) to linear order in $1/\tau$

$$z = \pm c_0 k - i\gamma\,, \tag{4.9a}$$

with c_0 from Eq. (4.4b) and

$$\gamma = \frac{1}{\tau(T)}\left[1 - \frac{1 + F_0 + 3\sigma_0^2}{F_0^2\sigma_0|I_1'(\sigma_0)|}\right], \tag{4.9b}$$

which is positive for all $F_0 > 0$. Here I_1' is the derivative of the function I_1 from Eq. (4.2b). Note that the damping coefficient γ is independent of the wave number, i.e., the mode is massive for all $T > 0$. γ vanishes at $T = 0$, and increases with increasing temperature. This is qualitatively different from the damping of the first-sound mode in the hydrodynamic regime, Eq. (3.41a), which is proportional to k^2 and increases with decreasing temperature.

The damping of other modes can be analyzed analogously. In particular, the continuum unparticle excitation acquires a mass that is proportional to $1/\tau$.

B. Long-ranged case

In the collisionless regime we can deduce the spectrum in the LR case by substituting Eq. (2.11) into the results for the SR case. We illustrate the result by performing this substitution in the propagator P from Eq. (4.3), which now reads

$$P(\boldsymbol{k}, z) = \frac{1}{1 - [F_0 + p_{\rm TF}^2(1 + F_0)/k^2]I_1(\zeta)} - 1\,. \tag{4.10}$$

The continuous part of the spectrum is qualitatively the same as in the SR case, but instead of the zero-sound pole we now have a plasmon pole at frequency

$$z = \pm\omega_p + O(k^2). \tag{4.11}$$

The continuous excitation still displays scale invariance, but the plasmon does not. This is illustrated in Fig. 3, which shows the spectrum for two difference wave numbers.

The dispersion of the plasmon and its damping can be obtained by keeping terms to $O(k^2)$. A calculation analogous

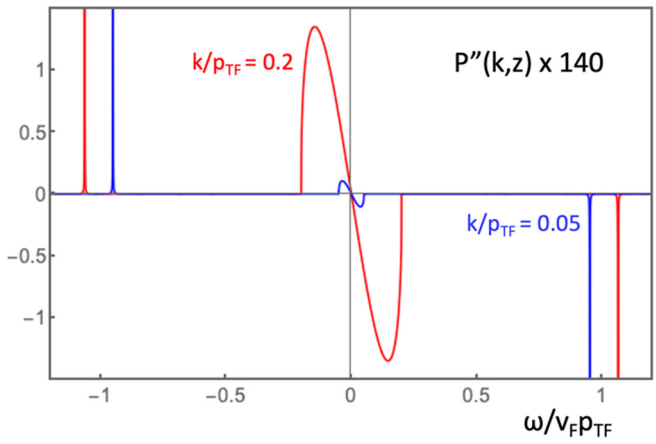

FIG. 3. Spectrum of the propagator in Eq. (4.10) for $F_0 = 5$ and $k = 0.2\, p_{\rm TF}$ (red) and $k = 0.05\, p_{\rm TF}$ (blue), respectively, showing the continuous unparticle contribution and the plasmon poles. The continuum is still scale invariant, with the frequency scaling linearly with the wave number, whereas the plasmon poles are not and depend only weakly on the wave number.

to the one that leads to Eq. (4.9b) yields

$$z = \pm\omega_p\left[1 + \frac{1}{2}\left(\frac{3}{5} + \frac{1}{3}F_0\right)\left(\frac{v_{\rm F}^*}{\omega_p}\right)^2 k^2\right] - \frac{i}{2}\Gamma_p k^2 + O(k^4)\,, \tag{4.12a}$$

with

$$\Gamma_p = \frac{4}{15}\,(v_{\rm F}^*/\omega_p)^2\,\frac{1}{\tau(T)} \tag{4.12b}$$

the plasmon damping coefficient. This is the same result as in the hydrodynamic regime, see Eqs. (3.68).

We emphasize that the only zero-sound mode that becomes massive in the LR case is the one that is present even with F_0 the only nonzero Landau parameter. For instance, the additional soft mode discussed in connection with Eqs. (4.7) remains soft, so does the transverse zero-sound mode described by Eq. (4.6b), and so does the continuous unparticle excitation.

C. The fate of the collisionless modes with increasing temperature

The collisionless regime is complementary to the hydrodynamic one, it is confined to wave numbers larger than q^* defined in Eq. (3.69). The collisionless soft modes are truly soft only at $T = 0$; for any nonzero temperature they acquire a damping term that does not vanish in the limit of zero wave number, but still is small for low temperatures. With increasing temperature the relaxation time decreases, and the collisionless regime gets pushed to larger wave numbers and frequencies, while the hydrodynamic regime grows. At the same time, the damping of the modes increases and they eventually become overdamped, while the hydrodynamic modes emerge at low frequencies and wave numbers and their damping decreases.

As already mentioned in Sec. III C, these observations do not apply to the plasmon, which is governed by gauge invariance and therefore is the same in both the hydrodynamic and the collisionless regimes, see Eqs. (3.68) and (4.12).

V. DISCUSSION AND CONCLUSION

We conclude with a summary and discussion of some of the salient points of the paper. We also add some remarks regarding points that were mentioned only briefly, or not at all, in the main text.

(1) We have addressed two very general questions regarding Fermi liquids: First, we have shown that LFL theory, which is often thought of as being valid only at low temperatures, is fully consistent with Navier-Stokes hydrodynamics irrespective of the temperature. We have done so by using kinetic theory to explicitly solve the kinetic equation for the hydrodynamic modes, by means of the method of perturbed zero eigenvalues of the collision operator. Alternatively, one can derive the Navier-Stokes equations for the LFL. This program will be carried out in Paper II. For our explicit solution we have used the model interaction given by Eq. (2.8), which implies (2.10a). However, we have used Eq. (2.10a) only twice: Once to eliminate the additional term on the right-hand-side of Eq. (3.42a), and once to ascertain that the model kinetic operator $L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})$ is consistent with particle number conservation. This, and the general structure of the theory, strongly suggests that an analogous analysis is possible for a completely general QP interaction. Carrying out this program will lead to a qualitatively new effect, namely, a component of the QP velocity in the orthogonal space $\mathcal{L}_\perp$. That is, physical particles and quasiparticles will have the same density, but different currents. The resulting hydrodynamic theory will have a structure that is different from that of a simple classical fluid and share some (but not all) aspects with a classical binary mixture. Among the physical consequences will be a nonzero bulk viscosity, and an additional contribution to the sound attenuation.

It should be mentioned that the fact that LFL theory is internally consistent, and consistent with general hydrodynamics, at all T does not imply that it is exact. For instance, nonlocalities in the collision operator (i.e., different single-particle distributions occurring at different points in real space) will lead to contributions at $O(T^2)$ that are not included in LFL theory.

Second, we have discussed the absence of a relation between the soft modes in the hydrodynamic and collision regimes, respectively. With decreasing temperature the damping of the hydrodynamic modes increases and the hydrodynamic regime shrinks until it disappears at $T = 0$. At the same time, a completely unrelated family of soft modes emerges in the collisionless regime. Their number is governed by the QP interaction rather than by conservation laws, their damping decreases with decreasing temperature, and they are truly soft only at $T = 0$. We have demonstrated this explicitly by means of a model calculation of the shear modes in Appendix D. Tables I and II summarizes the soft modes, as well as the plasmon modes, in both regimes.

The soft modes in the collisionless regime fall into three distinct classes. The first class consists of the modes that are related to conserved quantities. These are the two longitudinal $\ell = 0, 1$ zero-sound modes, and the two transverse zero-sound

TABLE I. Modes in the hydrodynamic regime.

	Short ranged	Long ranged
Shear modes (2)	Diffusive $z \sim -iD_\perp k^2$	Diffusive $z \sim -iD_\perp k^2$
Heat mode (1)	Diffusive $z \sim -iD_T k^2$	Diffusive $z \sim -iD_T k^2$
First sound (2)	Propagating $z \sim c_1 k - i v_F k^2 \tau$	None
Plasmon (2)	None	Propagating $z \sim \omega_p - i(\frac{v_F k}{\omega_p})^2 \frac{1}{\tau}$

modes. Their hydrodynamic counterparts are the two first-sound modes and the two shear diffusion modes, respectively. (The heat diffusion mode has no analog at $T = 0$.) The second class consists of propagating zero modes for higher angular momenta. They have no analogs in the hydrodynamic regime, and their existence depends on the strength of the QP interaction. The third class is represented by the continuum or unparticle mode, which also has no hydrodynamic analog. It has important consequences for a variety of quantum phase transitions, see Ref. [8] and the last paragraph of point (2) below, and therefore is in some sense physically more important than the propagating modes.

We have also discussed the special role played by the plasmon in a charged Fermi liquid, which is not a hydrodynamic mode and extends through both the collisionless and the hydrodynamic regimes for reasons related to gauge invariance [30]. We have discussed 3-d systems where the plasmon is massive and its damping is independent of the wave number in the homogeneous limit, see Eqs. (3.68) and (4.12). This changes in 2-d systems, where both the plasmon frequency and the damping go to zero as $k \to 0$; the former as $\sqrt{k}$ and the latter as k^2 [32].

We also note that "Fermi liquid", in our context, can be interpreted rather broadly: We have not specified the temperature dependence of the relaxation rate $1/\tau$, and we have not made use of the concept of "well-defined quasiparticles". For instance, our analysis of the hydrodynamic regime applies to what is known as a marginal Fermi liquid [33].

TABLE II. Modes in the collisionless regime.

	Short ranged	Long ranged
Unparticle (1)	Continuous function of $z/v_F k$	Continuous function of $z/v_F k$
$\ell = 0, 1$ longitudinal zero sound (2)	Propagating $z \sim c_0 k - i/\tau$	None
Plasmon (2)	None	Propagating $z \sim \omega_p - i(\frac{v_F k}{\omega_p})^2 \frac{1}{\tau}$
$\ell = 1$ transverse zero sound (2)	Propagating $z \sim c_0 k - i/\tau$ [a]	Propagating $z \sim c_0 k - i/\tau$ [a]
$\ell \geqslant 2$ zero sound (many)	Propagating $z \sim c_0 k - i/\tau$ [a]	Propagating $z \sim c_0 k - i/\tau$ [a]

[a]The zero sound velocities and the damping coefficients are different for different modes. However, they all are proportional to those of the longitudinal zero-sound mode given in Eqs. (4.4b) and (4.9b), respectively.

(2) The number of soft modes in the collisionless regime (or, strictly speaking, at $T = 0$), is to some extent a matter of interpretation. In the $\boldsymbol{p} - \boldsymbol{k}$ momentum space, sometimes referred to as μ-space in kinetic theory [23], there is only one soft mode, viz., the fluctuation ϕ of the single-particle distribution function, whose denominator is given by $z - \boldsymbol{p} \cdot \boldsymbol{k}$, see Eq. (4.1). However, as a result of this denominator all of the moments of ϕ with respect to $\boldsymbol{p}$ are soft, and in this sense there is an infinite number of soft modes. This is true in a clean Fermi system; in the presence of quenched disorder only the zeroth moment with respect to $\boldsymbol{p}$ is soft, see Refs. [34–38], and Ref. [39] for a review.

It should be emphasized that the spectrum of ϕ has a continuous part and, in general, δ-function contributions describing zero-sound modes that are both part of the same spectrum (this supports the single-soft mode interpretation). Our results are consistent with a quantum-field-theoretic analysis in Ref. [40], which kept only the equivalent of the Landau parameter F_0. It is interesting that kinetic theory, which uses quantum mechanics only in the form of the equilibrium Fermi distribution, and the field theory are equivalent. Following Ref. [7], we have referred to the continuous part of the spectrum as the unparticle excitation, whereas the zero-sound poles represent particle-like excitations. We note that the unparticle part of the spectrum, while an exotic idea in a high-energy context, has been known since the earliest days of the quantum theory of condensed matter, where it is usually referred to as the particle-hole continuum. For instance, it gives the Lindhard function [41] its characteristic scale-invariant structure. Neither the continuum nor the zero-sound poles are related to conservation laws, and all of them acquire a mass at any nonzero temperature. Some remarks to the contrary related to zero sound in Ref. [40] were incorrect.

We have discussed only spinless Fermi liquids for simplicity's sake. It is well known how to incorporate spin in LFL theory [2,3], and the generalization in the current context is straightforward. The spin channel again supports the unparticle continuum, and in addition spin-zero-sound modes whose existence and number depends on the values of the Landau parameters.

The importance of the unparticle continuum is often downplayed in favor of the particle-like collective zero-sound excitations. This ignores the fact that it has dramatic physical consequences. For instance, in the spin channel (which we have not explicitly discussed) it is responsible for a non-analytic wave-number dependence of the spin susceptibility [42], and for the ferromagnetic quantum phase transition to be generically a first-order transition [8,43].

(3) The origin and interpretation of the soft modes in the collisionless regime has been the subject of several studies. The scale invariant unparticle continuum mode has been interpreted as the Goldstone mode of a spontaneously broken rotational symmetry in Matsubara frequency space; roughly speaking, a broken symmetry between retarded and advanced degrees of freedom [40]. This is in analogy to Wegner's interpretation of the diffusive soft mode in disordered Fermi systems known as the "diffusion" [34]. Reference [44] has

interpreted it as a Goldstone mode related to a spontaneously broken Lorentz boost invariance. The relation between these two interpretations is not clear.

(4) We have discussed clean fermion systems, but impurity scattering can easily be taken into account; see Appendix C for the relevant collision integral. It qualitatively changes the hydrodynamic modes: fermion momentum is no longer conserved, and the density response is diffusive. In bulk metals the clean hydrodynamic behavior we have discussed is very difficult to realize, since impurity scattering tends to dominate even in the cleanest samples. In two-dimensional systems the ultraclean hydrodynamic regime, where momentum is conserved, is easier to realize [32,45]. Also, it recently has become possible to realize clean Fermi liquids in cold-atom systems [9–14].

(5) We have discussed LFL theory at a level analogous to the linearized Navier-Stokes equations of classical hydrodynamics. An interesting problem is the generalization of this treatment by adding a fluctuating Langevin force. This will be analogous to the fluctuating hydrodynamics description of classical fluids [1] and allow for the calculation of time-correlation functions in both equilibrium and nonequilibrium situations. This problem is considered in Paper II [16].

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APPENDIX A: THERMODYNAMIC RELATIONS

Here we list and explain various thermodynamic relations that were used in Secs. II and III. We start with the normalization factors in Eqs. (2.21).

1. Normalizations of the zero eigenvectors

To determine A_1, consider the variation $\delta f_{\rm eq}(\boldsymbol{p})$ of the equilibrium distribution function due to a homogeneous variation $\delta\mu$ of the chemical potential at fixed T and V:

$$\delta f_{\rm eq}(\boldsymbol{p})\Big|_{T,V} = -w(\boldsymbol{p})(\delta\epsilon_p - \delta\mu)\,. \tag{A1}$$

$\delta\epsilon_p$ is related to $\delta f_{\rm eq}$ by Eq. (2.6), and by symmetry only the Landau parameter F_0 contributes,

$$\delta\epsilon_p = \frac{1}{N_0}\,F_0\,\delta n\,. \tag{A2}$$

For the variation of the number density, Eq. (2.1b), we thus have

$$\delta n = \frac{N_0}{1+F_0}\,\delta\mu \tag{A3}$$

with $N_0 = \langle 1|1\rangle$ from Eq. (2.4c). Hence [3,4],

$$\left(\frac{\partial n}{\partial\mu}\right)_{T,V} = \frac{\langle 1|1\rangle}{1+F_0}\,, \tag{A4}$$

which is the second equality in Eq. (2.21b). In the limit of low temperature,

$$\langle 1|1\rangle = N_{\rm F}^* + O(T^2)\,. \tag{A5}$$

The same line of reasoning for a variation of $f_{\rm eq}$ under a variation δT at fixed μ and V yields

$$\left(\frac{\partial n}{\partial T}\right)_{\mu,V} = \frac{1}{1+F_0}\,\frac{1}{T}\,\langle\xi_p|1\rangle\,. \tag{A6}$$

Combining this with Eq. (A4) we obtain

$$\langle\xi_p\rangle_w = \langle\epsilon_p\rangle_w - \mu = -T(\partial\mu/\partial T)_{N,V}\,. \tag{A7}$$

From Eq. (A7) we can obtain the normalization A_3 as follows. Consider a variation of the energy density, Eq. (2.1a). For fixed number density n we have $\delta\epsilon_p = 0$, and hence

$$\delta f_{\rm eq}(\boldsymbol{p})|_{N,V} = w(\boldsymbol{p})\left(\delta\mu + \frac{1}{T}\,\xi_p\,\delta T\right), \tag{A8a}$$

and therefore, from Eq. (2.1a),

$$\delta e = \langle\epsilon_p|1\rangle\delta\mu + \frac{1}{T}\,\langle\epsilon_p|\xi_p\rangle\delta T\,. \tag{A8b}$$

For the specific heat at constant volume this yields

$$T c_V = T\left(\frac{\partial e}{\partial T}\right)_{V,N} = T\langle\epsilon_p|1\rangle\left(\frac{\partial\mu}{\partial T}\right)_{V,N} + \langle\epsilon_p|\xi_p\rangle\,. \tag{A9}$$

By using Eq. (A7) we obtain

$$T c_V = \langle\epsilon_p|\epsilon_p\rangle - \langle\epsilon_p\rangle_w^2\langle 1|1\rangle = \langle a_5(\boldsymbol{p})|a_5(\boldsymbol{p})\rangle \tag{A10}$$

with a_5 from Eq. (2.20e). This is the second equality in Eq. (2.21d). Alternatively, we obtain the same result by starting with the entropy density of a Fermi liquid in the form [2,3]

$$s = \frac{-1}{V}\sum_{\boldsymbol{p}}\big[f_{\rm eq}\ln f_{\rm eq} + (1-f_{\rm eq})\ln(1-f_{\rm eq})\big]\,, \tag{A11}$$

using Eq. (A8a), and calculating the specific heat as $c_v = T(\partial s/\partial T)_{V,N}$.

For A_2, we need the same expression as for $\langle 1|1\rangle$ with an additional $\boldsymbol{p}^2$ in the integrand. At $T=0$, the $\boldsymbol{p}^2$ gets replaced by $p_{\rm F}^2$, and by using Eq. (A5) we have $\langle\boldsymbol{p}|\boldsymbol{p}\rangle = p_{\rm F}^2 N_{\rm F}^* + O(T^2)$. Due to the f-sum rule this remains valid in general, i.e., there are no explicit temperature corrections [4], and we have

$$\langle\boldsymbol{p}|\boldsymbol{p}\rangle = p_{\rm F}^2 N_{\rm F}^* = 3m^* n\,. \tag{A12}$$

We finally list some of the above quantities explicitly in the low-temperature limit. The normalizations are, in addition to Eq. (A12), which is valid at all T,

$$\langle 1|1\rangle = N_{\rm F}^* + O(T^2)\,, \tag{A13a}$$

$$\langle a_5(\boldsymbol{p})|a_5(\boldsymbol{p})\rangle = c_v T = sT + O(T^4) = \frac{\pi^2}{3}N_{\rm F}^*\,T^2 + O(T^4)\,. \tag{A13b}$$

In Eq. (A13b) we have used the LFL theory expression for the entropy at low temperature [3]. Also of interest is $\langle\epsilon_p\rangle_w$. The chemical potential is [3]

$$\mu(T) = \epsilon_{\rm F} - \frac{\pi^2}{12}\,\frac{T^2}{\mu(T=0)} + O(T^4). \tag{A14}$$

Equation (A7) thus yields

$$\langle \epsilon_p \rangle_w = \mu + \frac{\pi^2}{6}\frac{T^2}{\mu} + O(T^4) = \epsilon_F + \frac{\pi^2}{12}\frac{T^2}{\mu} + O(T^4)\,. \tag{A15}$$

2. Temperature fluctuations

In order to derive Eq. (2.19f) we start with Eqs. (A1) and (A2) to write energy density fluctuations at constant T and V as

$$\delta e|_{T,V} = \frac{-1}{V}\sum_{\boldsymbol{p}} \epsilon_p w(\boldsymbol{p})(\delta\epsilon_p - \delta\mu) = \left(\delta\mu - \frac{F_0}{\langle 1|1\rangle}\delta n\right)\langle \epsilon_p|1\rangle. \tag{A16}$$

With the help of Eq. (A4) this yields

$$(\partial e/\partial n)_{T,V} = -c_V(\partial T/\partial n)_{E,V} = \langle \epsilon_p \rangle_w\,. \tag{A17}$$

The first equality is generally valid, the second one is valid within LFL theory. Now consider

$$c_V \delta T = c_V\left(\frac{\partial T}{\partial e}\right)_{N,V}\delta e + c_V\left(\frac{\partial T}{\partial n}\right)_{E,V}\delta n = \delta e - \langle \epsilon_p \rangle_w \delta n = \langle a_5|\phi\rangle\,, \tag{A18}$$

which is Eq. (2.19f). Here we have used $c_V = (\partial e/\partial T)_{V,N}$ and Eq. (A17) to go from the first line to the second one.

3. The matrix $L_L^{(1)}$

Equation (3.12b) can be obtained from Eq. (3.12a) by means of the following manipulations.

In order to calculate $\langle \boldsymbol{v}_p|\boldsymbol{p}\rangle$, consider

$$\langle v_p^i|p_j\rangle = \frac{1}{V}\sum_{\boldsymbol{p}} w(\boldsymbol{p})\, p_j \frac{\partial}{\partial p_i}\epsilon_p = \frac{-1}{V}\sum_{\boldsymbol{p}} p_j \frac{\partial f_{\rm eq}}{\partial p_i} = \delta_{ij}\frac{1}{V}\sum_{\boldsymbol{p}} f_{\rm eq}(\boldsymbol{p}) = \delta_{ij}\, n\,, \tag{A19a}$$

where the second line is obtained from the first one by partial integration. We thus have

$$\langle \boldsymbol{v}_p|\boldsymbol{p}\rangle = 3n\,. \tag{A19b}$$

This yields the (1,2) matrix element in Eq. (3.12b).

$\partial n/\partial\mu$ is related to the compressibility χ:

$$(\partial n/\partial\mu)_{T,V} = n^2\chi_T\,, \tag{A20a}$$

where

$$\chi_T = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = \frac{1}{n}\left(\frac{\partial n}{\partial p}\right)_{T,V} \tag{A20b}$$

is the isothermal compressibility, with p the pressure. These identities follow from standard Jacobian manipulations [46] combined with the fact that the particle number is an extensive quantity, and hence $(\partial N/\partial V)_{p,T} = N/V = n$. Combining Eqs. (A20b) and (A19b) we obtain the (2,1) matrix element in Eq. (3.12b).

To express $\langle \boldsymbol{v}_p|\boldsymbol{p}\, a_5(\boldsymbol{p})\rangle$ in terms of thermodynamic quantities, we start with

$$\langle v_p^i|p_j\,\epsilon_p\rangle = \delta_{ij}\frac{1}{V}\sum_{\boldsymbol{p}}\epsilon_p f_{\rm eq}(\boldsymbol{p}) + \frac{1}{V}\sum_{\boldsymbol{p}} f_{\rm eq}(\boldsymbol{p})\, p_j\, v_p^i = \delta_{ij}\frac{1}{V}\sum_{\boldsymbol{p}}\epsilon_p f_{\rm eq}(\boldsymbol{p}) + \delta_{ij}\frac{T}{V}\sum_{\boldsymbol{p}}\ln(1-f_{\rm eq})\,. \tag{A21}$$

For the first line we have integrated by parts as in Eq. (A19a), and to obtain the second line we have used the identity from the first line of Eq. (2.3b). The entropy, Eq. (A11), can be rewritten as

$$s = \frac{-1}{V}\sum_{\boldsymbol{p}}\ln(1-f_{\rm eq}) + \frac{1}{TV}\sum_{\boldsymbol{p}}\epsilon_p f_{\rm eq}(\boldsymbol{p}) - n\mu/T\,. \tag{A22}$$

Combining this with Eq. (A21) yields

$$\langle v_p^i|p_j\,\epsilon_p\rangle = \delta_{ij}(Ts + n\mu) \tag{A23a}$$

and hence

$$\langle \boldsymbol{v}_p|\boldsymbol{p}\,\epsilon_p\rangle = 3(Ts + n\mu)\,. \tag{A23b}$$

Now the Duhem-Gibbs relation, $\mu = G/N$ with G the Gibbs free energy, yields the general identity

$$\left(\frac{\partial\mu}{\partial T}\right)_{N,V} = \frac{-s}{n} + \frac{1}{n}\left(\frac{\partial p}{\partial T}\right)_{N,V}\,. \tag{A24}$$

This allows us to write Eq. (2.6) as

$$\langle v_p^i|p_j\,\epsilon_p\rangle = \delta_{ij} n\left[\mu - T\left(\frac{\partial\mu}{\partial T}\right)_{N,V} + \frac{T}{n}\left(\frac{\partial p}{\partial T}\right)_{N,V}\right]. \tag{A25}$$

Combining this with Eq. (A7) yields

$$\langle v_p^i|p_j\, a_5(\boldsymbol{p})\rangle = \delta_{ij} T(\partial p/\partial T)_{N,V} \tag{A26a}$$

and hence

$$\langle \boldsymbol{v}_p|\boldsymbol{p}\, a_5(\boldsymbol{p})\rangle = 3T(\partial p/\partial T)_{N,V}\,. \tag{A26b}$$

From this result we obtain the (2,3) and (3,2) matrix elements in Eq. (3.12b).

We note that Eqs. (A19a) and (A26b) are valid for an arbitrary QP velocity $\boldsymbol{v}_p$, i.e., and arbitrary interaction function $F(\boldsymbol{p}, \boldsymbol{p}')$, not just for the particular form (2.10a).

4. The speed of first sound

To derive Eq. (3.13) we start with the following expression for the ratio c_p/c_V [46]:

$$\frac{c_p}{c_V} = 1 - \frac{T}{V}\frac{[(\partial V/\partial T)_{p,N}]^2}{c_V(\partial V/\partial p)_{T,N}}\,. \tag{A27a}$$

By means of general Jacobian identities we can rewrite this as

$$\frac{c_p}{c_V} = 1 - \frac{T}{c_V}\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}\left[\left(\frac{\partial p}{\partial T}\right)_{V,N}\right]^2 = 1 + \frac{T\chi_T}{c_V}\left[\left(\frac{\partial p}{\partial T}\right)_{V,N}\right]^2 , \tag{A27b}$$

with χ_T the isothermal compressibility from Eq. (A20b). This shows that the second equality in Eq. (3.13) follows form the first one.

APPENDIX B: CONTINUITY EQUATIONS

In this Appendix we discuss the continuity equations associated with the conservation of the particle number, the momentum, and the energy, respectively, in the context of our kinetic theory.

1. Particle number conservation

Consider the kinetic equation as written in Eqs. (2.16). Multiplying from the left with the constant function $\langle a_1(\boldsymbol{p})| = \langle 1|$ we obtain

$$-iz\,\delta n(\boldsymbol{k},z) + i\boldsymbol{k}\cdot\boldsymbol{j}_n(\boldsymbol{k},z) = \delta n(\boldsymbol{k},t=0)\,, \tag{B1a}$$

where the divergence of the number-current density fluctuation is given by

$$i\boldsymbol{k}\cdot\boldsymbol{j}_n(\boldsymbol{k},z) = \langle 1|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\rangle. \tag{B1b}$$

Since the number current density is given by $n\,\delta\boldsymbol{u}(\boldsymbol{k},z)$, with $\delta\boldsymbol{u}$ from Eq. (2.17b), this implies

$$\frac{ik}{m}\langle a_2(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\rangle = \langle 1|L_{\boldsymbol{k}}^{(1)}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\rangle\,. \tag{B2}$$

Since this holds for arbitrary functions ϕ, we have

$$\frac{1}{m}\hat{\boldsymbol{k}}\cdot\boldsymbol{p} = \hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p + \frac{1}{N_0V}\sum_{\boldsymbol{p}'} w(\boldsymbol{p}')\,F(\boldsymbol{p},\boldsymbol{p}')\,(\hat{\boldsymbol{k}}\cdot\boldsymbol{p}')\,, \tag{B3}$$

which is the longitudinal part of Eq. (2.7). Equation (2.7) is valid for any interaction function F. With the specific model interaction given by Eq. (2.8) we obtain the longitudinal part of Eq. (2.10a). We see that Eq. (2.10a) is necessary for particle-number conservation to hold.

2. Energy conservation

The continuity equation related to energy conservation is conveniently expressed in terms of the heat mode, Eq. (3.16), which is a linear combination of the energy and the density. Multiplying Eq. (2.16a) from the left with $\langle\psi_5^{(0)L}(\boldsymbol{p})|$ we find

$$-iz\big\langle\psi_5^{(0)L}(\boldsymbol{p})\big|\phi(\boldsymbol{p},\boldsymbol{k},z)\big\rangle + i\boldsymbol{k}\cdot\boldsymbol{j}_T(\boldsymbol{k},z) = \big\langle\psi_5^{(0)L}(\boldsymbol{p})\big|\phi(\boldsymbol{p},\boldsymbol{k},t=0)\big\rangle\,. \tag{B4a}$$

The heat current density is given by

$$\boldsymbol{j}_T(\boldsymbol{k},z) = \big\langle\boldsymbol{v}_p\psi_5^{(0)L}(\boldsymbol{p})\big|\phi(\boldsymbol{p},\boldsymbol{k},z)\big\rangle\,; \tag{B4b}$$

it determines the heat conductivity via Eq. (3.39). As we discussed in Sec. III A 2 b, the physical interpretation of the heat mode is an entropy density. Accordingly, $\boldsymbol{j}_T$ is an entropy current density. We note that this identification of the entropy density as a conserved quantity does not violate the Second Law since the description of irreversibility requires going past the linearized kinetic theory considered in this paper.

3. Momentum conservation

To obtain the continuity equation related to momentum conservation we multiply Eq. (2.16a) from the left with $\langle\boldsymbol{p}|$. This yields

$$-iz\,nm\,\delta\boldsymbol{u}(\boldsymbol{k},z) + ik\langle\boldsymbol{p}(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p)|\phi(\boldsymbol{p},\boldsymbol{k},z)\rangle + ik\,\frac{F_0}{\langle 1|1\rangle}\,\langle\boldsymbol{p}|\hat{\boldsymbol{k}}\cdot\boldsymbol{v}_p\rangle\,\delta n(\boldsymbol{k},z) = nm\,\delta\boldsymbol{u}(\boldsymbol{k},t=0)\,. \tag{B5}$$

Here we have used the model interaction from Eq. (2.8); this can be generalized if desirable. By using various identities from Appendix A, as well as the identity $-3|\sigma_2(\boldsymbol{p})\rangle = \mathcal{P}_\perp|\boldsymbol{p}\cdot\boldsymbol{v}_p\rangle$ noted after Eq. (3.44b), this can be written

$$-iz\,nm\,\delta u^i(\boldsymbol{k},z) + ik_j\tau^{ij}(\boldsymbol{k},z) = nm\,\delta u^i(\boldsymbol{k},t=0)\,. \tag{B6a}$$

Here

$$\tau^{ij}(\boldsymbol{k},z) = \delta^{ij}\delta p(\boldsymbol{k},z) + \big\langle\sigma^{ij}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\big\rangle \tag{B6b}$$

is the stress tensor. The first term is the reactive part, which is given by the pressure fluctuation δp, Eq. (2.19h). The second term is the dissipative part, with $\sigma^{ij}(\boldsymbol{p})$ from Eqs. (3.43).

If we break up the momentum into its longitudinal and transverse component, respectively, as in Eqs. (2.18) and (2.19), this can be written

$$-iz\,nm\,\delta u_\perp(\boldsymbol{k},z) + ik\,j_{u\perp}(\boldsymbol{k},z) = nm\,\delta u_\perp(\boldsymbol{k},t=0)\,, \tag{B7a}$$

$$-iz\,nm\,\delta u_{\rm L}(\boldsymbol{k},z) + ik\,j_{u{\rm L}}(\boldsymbol{k},z) = nm\,\delta u_{\rm L}(\boldsymbol{k},t=0), \tag{B7b}$$

for the transverse and longitudinal components, respectively. The transverse momentum current

$$j_{u\perp}(\boldsymbol{k},z) = \hat{k}_\perp^i\hat{k}^j\langle(\sigma_1)_{ij}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\rangle \tag{B7c}$$

agrees with Eq. (3.33b) and determines the shear viscosity via Eq. (3.34). The longitudinal one,

$$j_{u{\rm L}}(\boldsymbol{k},z) = \delta p(\boldsymbol{k},z) + \hat{k}_i\hat{k}_j\big\langle\sigma^{ij}(\boldsymbol{p})|\phi(\boldsymbol{p},\boldsymbol{k},z)\big\rangle \tag{B7d}$$

combines with the remaining contributions to the sound mode to form Eq. (3.42a). $\delta u_\perp$ and $\hat{\boldsymbol{k}}_\perp$ in Eqs. (B7a) and (2.5) represent either of the two transverse directions.

APPENDIX C: LINEARIZED COLLISION OPERATORS

The linearized collision operator $\Lambda(\boldsymbol{p})$ is in general a sum of several terms that represent fermion-fermion interactions, fermion-impurity interactions, and fermion-boson interactions. The latter include, e.g., electron-phonon interactions or electron-magnon interactions in metals. We will denote these three contributions by $\Lambda_{\text{f-f}}$, $\Lambda_{\text{f-b}}$, and $\Lambda_{\text{f-i}}$, respectively. They

can be obtained by linearizing the more general expressions given in Ref. [15].

The linearized fermion-fermion collision operator can be written

$$\Lambda_{\text{f-f}}(\boldsymbol{p})\,\phi(\boldsymbol{p}) = \frac{1}{1-f_{\text{eq}}(\boldsymbol{p})}\,\frac{1}{V^3}\sum_{\boldsymbol{p}',\boldsymbol{p}_1,\boldsymbol{p}_1'} W(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}_1') \times \delta(\epsilon_p+\epsilon_{p_1}-\epsilon_{p'}-\epsilon_{p_1'})\,\delta(\boldsymbol{p}+\boldsymbol{p}_1-\boldsymbol{p}'-\boldsymbol{p}_1') \times f_{\text{eq}}(\boldsymbol{p}_1)[1-f_{\text{eq}}(\boldsymbol{p}')][1-f_{\text{eq}}(\boldsymbol{p}_1')]\,[\phi(\boldsymbol{p}') + \phi(\boldsymbol{p}_1')-\phi(\boldsymbol{p})-\phi(\boldsymbol{p}_1)]\,. \tag{C1a}$$

Here W is the probability for two fermions in momentum states $\boldsymbol{p}$ and $\boldsymbol{p}_1$ to be scattered into momentum states $\boldsymbol{p}'$ and $\boldsymbol{p}_1'$. Time reversal symmetry implies

$$W(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}_1') = W(\boldsymbol{p}',\boldsymbol{p}_1';\boldsymbol{p},\boldsymbol{p}_1)\,. \tag{C1b}$$

In order to obtain Eq. (2.7) from the general expression given in Ref. [15] we have repeatedly used the energy-conservation expressed by the first δ function as well as the explicit form of the equilibrium distribution function, Eq. (2.3a). Equation (2.7) makes explicit the five conservation laws: $\Lambda_{\text{f-f}}(\boldsymbol{p})\,\phi(\boldsymbol{p}) = 0$ if $\phi(\boldsymbol{p}) = a_\alpha(\boldsymbol{p})$ with a_α any of the five functions defined in Eqs. (2.20).

The fermion-impurity collision operator is [47,48]

$$\Lambda_{\text{f-i}}(\boldsymbol{p})\,\phi(\boldsymbol{p}) = \frac{1}{V}\sum_{\boldsymbol{p}'} W(\boldsymbol{p}',\boldsymbol{p})\,\delta(\epsilon_{p'}-\epsilon_p)[\phi(\boldsymbol{p}')-\phi(\boldsymbol{p})]\,. \tag{C2}$$

Here the fermion particle number and energy are conserved, but the momentum is not.

Finally, the fermion-boson collision operator can be written

$$\Lambda_{\text{f-b}}(\boldsymbol{p})\,\phi(\boldsymbol{p}) = \frac{1}{f_{\text{eq}}(\boldsymbol{p})(1-f_{\text{eq}}(\boldsymbol{p}))}\,\frac{1}{V^2}\sum_{\boldsymbol{p}',\boldsymbol{k}} \delta(\boldsymbol{p}'-\boldsymbol{p}-\boldsymbol{k})\,n_{\text{eq}}(\boldsymbol{k})[W(\boldsymbol{p}';\boldsymbol{p},\boldsymbol{k})\,f_{\text{eq}}(\boldsymbol{p})(1-f_{\text{eq}}(\boldsymbol{p}'))\,\delta(\epsilon_{p'}-\epsilon_p-\omega_k) + W(\boldsymbol{p}',-\boldsymbol{k};\boldsymbol{p})\,f_{\text{eq}}(\boldsymbol{p}')(1-f_{\text{eq}}(\boldsymbol{p}))\,\delta(\epsilon_{p'}-\epsilon_p+\omega_k)][\phi(\boldsymbol{p}')-\phi(\boldsymbol{p})]\,. \tag{C3}$$

Here $n_{\text{eq}}(\boldsymbol{k}) = 1/(\exp(\omega_k/T)-1)$ is the equilibrium Bose-Einstein distribution function, and ω_k is the energy of a boson with wave number k. W is again a transition probability. If the bosons are phonons, then umklapp processes can be taken into account by adding a reciprocal lattice vector to the argument of the momentum-conserving δ function. From Eq. (C3) we see that the fermion particle number is still conserved, but the fermion momentum and energy are not.

For explicit calculations a model fermion-fermion collision operator that is the quantum version of the Bhatnagar-Gross-Krook collision operator in classical kinetic theory [49] is also useful. It uses a simple relaxation-time approximation with a momentum-independent collision rate $1/\tau$ that is temperature dependent. For an ordinary Fermi liquid it is given, up to a prefactor of $O(1)$, by

$$1/\tau \approx T^2/\epsilon_{\text{F}} \tag{C4}$$

with ϵ_{F} the Fermi temperature. The five conservation laws are taken into account by projecting on the orthogonal space $\mathcal{L}_\perp$:

$$\Lambda_{\text{f-f}}^{\text{BGK}}(\boldsymbol{p}) = \frac{-1}{\tau}\,\mathcal{P}_\perp\,. \tag{C5a}$$

Here

$$\mathcal{P}_\perp = \mathbb{1} - \sum_{\alpha=1}^{5}\frac{|a_\alpha(\boldsymbol{p})\rangle\langle a_\alpha(\boldsymbol{p})|}{\langle a_\alpha(\boldsymbol{p})|a_\alpha(\boldsymbol{p})\rangle}\,, \tag{C5b}$$

with $\mathbb{1}$ the unit operator and the $a_\alpha(\boldsymbol{p})$ from Eqs. (2.20), is the projection operator onto $\mathcal{L}_\perp$ that we have used, e.g., in Eq. (3.28a). Acting with $\Lambda_{\text{f-f}}^{\text{BGK}}(\boldsymbol{p})$ on ϕ yields

$$\Lambda_{\text{f-f}}^{\text{BGK}}(\boldsymbol{p})\,\phi(\boldsymbol{p},\boldsymbol{x},t) = \frac{-1}{\tau}\Big[\phi(\boldsymbol{p},\boldsymbol{x},t) - \frac{1}{N_0}\,\delta n(\boldsymbol{x},t) - \frac{m}{m^*}\,\boldsymbol{p}\cdot\delta\boldsymbol{u}(\boldsymbol{x},t) - \frac{1}{Tc_V}[\delta e(\boldsymbol{x},t) - \langle\epsilon_p\rangle_w\delta n(\boldsymbol{x},t)]a_5(\boldsymbol{p})\Big]. \tag{C6}$$

Substituting Eq. (C6) in (2.16b), and solving the resulting kinetic equation (2.16a) in the hydrodynamic regime, one obtains the same results as in Sec. III (as must be the case), but with explicit expressions for the transport coefficients and the speed of sound in terms of the parameter τ. Some of the results have been quoted in the main text.

APPENDIX D: SHEAR MODES IN THE COLLISIONLESS AND HYDRODYNAMIC REGIMES

Here we illustrate the different natures and origins of the zero modes and the hydrodynamics modes, respectively, by means of a discussion of the shear modes in the two regimes. For simplicity and transparency we use the BGK collision operator, Eq. (C6), for this purpose.

Consider the kinetic equation (2.16) with the collision operator given by Eq. (C6), and focus on the transverse velocity fluctuations, Eqs. (2.19c) and (2.19d). The integral over the azimuthal angle then vanishes for the terms involving F_0, δn, and a_5, and it suffices to consider

$$\Big[-iz + i\boldsymbol{k}\cdot\boldsymbol{v}_p + \frac{1}{\tau}\Big]\phi(\boldsymbol{p},\boldsymbol{k},z) + i\boldsymbol{k}\cdot\boldsymbol{v}_p\,\frac{F_1}{N_0V} \times \sum_{\boldsymbol{p}'} w(\boldsymbol{p}')\,\frac{\boldsymbol{p}\cdot\boldsymbol{p}'}{\langle\boldsymbol{p}^2\rangle_w}\,\phi(\boldsymbol{p}',\boldsymbol{k},z) - \frac{1}{\tau}\,\frac{m}{m^*}\,\boldsymbol{p}\cdot\delta\boldsymbol{u}(\boldsymbol{k},z) = \phi(\boldsymbol{p},\boldsymbol{k},t=0)\,. \tag{D1}$$

Here the relaxation time τ is understood to be T dependent. The third term on the left-hand side reflects the conservation law for the momentum, and the second term reflects the QP interaction. If we divide by the term in angular brackets that multiplies ϕ in the first term, multiply by $\hat{\boldsymbol{k}}_\perp\cdot\boldsymbol{p}/nm$, and sum over $\boldsymbol{p}$, we obtain an equation for $\delta u_\perp$ of the form

$$\delta u_\perp(\boldsymbol{k},z)[1-f_\perp(\boldsymbol{k},z)] = (IC)_\perp(\boldsymbol{k},z)\,. \tag{D2a}$$

Here $\delta u_\perp$ stands for either of the two transverse velocity fluctuation, and $(IC)_\perp$ is the initial-condition term involving $\phi(t=0)$ whose precise form is irrelevant for our current purposes. The function $f_\perp$ is given by

$$f_\perp(\boldsymbol{k},z) = F_1\,\frac{1}{4}\int_{-1}^{1} d\eta\,\frac{\eta(1-\eta^2)}{z/v_F^*k-\eta+i/v_F^*k\tau} + \frac{i}{v_F^*k\tau}\,\frac{3}{4}\int_{-1}^{1} d\eta\,\frac{1-\eta^2}{z/v_F^*k-\eta+i/v_F^*k\tau}\,. \quad \text{(D2b)}$$

The two terms on the right-hand side correspond to the second and third term, respectively, on the left-hand side of Eq. (2.5).

Now we analyze the term $1-f_\perp$, which determines the resonance frequency. In the hydrodynamic regime, $v_F^*k\tau \ll 1$, we find for the resonance frequency

$$z = -i\nu k^2 + O(k^3)\,, \quad \text{(D3)}$$

with ν the kinematic viscosity from Eq. (3.32). In this limit the shear mode is diffusive, in agreement with Sec. III A 3 a. Note that the soft nature of this mode is due to the structure of the third term on the right-hand side of Eq. (2.7), which reflects the conservation law. The Landau parameter F_1 just provides a contribution to the diffusion coefficient.

In the collisionless regime, $v_F^*k\tau \gg 1$, the resonance frequency is

$$z = \pm c_{0\perp}k - i\gamma + O(1/v_F^*k\tau)\,. \quad \text{(D4a)}$$

Here $c_{0\perp} = \sigma_{0\perp}v_F$ is the speed of transverse zero sound. It is given by the solution of the transcendental equation

$$I_1(\sigma_{0\perp}) = \frac{F_1-6}{3F_1\left(\sigma_{0\perp}^2-1\right)}\,, \quad \text{(D4b)}$$

Here I_1 is the integral defined in Eq. (4.2b), and the solution exists and is unique for all $F_1 > 6$. The existence of this soft mode is due to the first term on the right-hand side of Eq. (2.7). It thus is a consequence of the QP interaction rather than the conservation law. This result agrees with Eq. (4.6b), and with (9.24) in Ref. [31]. It has observable consequences in both He-3 [57] and in electronic Fermi liquids [58,59]. For the damping coefficient one finds

$$\gamma = \frac{1}{\tau}\left[1+\frac{1}{1+F_1I_1(\sigma_{0\perp})}\left(\frac{1}{2}\,\sigma_{0\perp}+\frac{6-F_1}{12F_1}\right)\right], \quad \text{(D4c)}$$

which is positive for all $F_1 > 6$.

This simple examples illustrates various points: (1) The hydrodynamic shear modes are a result of the conservation law for the transverse momentum, whereas the transverse zero-sound mode is due to the QP interaction and has nothing to do with the conservation law. (2) The leading contribution to the zero-sound damping is independent of the wave number, in contrast to the behavior of hydrodynamic sound modes. (3) The nature of the excitation changes qualitatively from one regime to the other: it is diffusive in the hydrodynamic regime, but propagating in the collisionless regime.

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