Fractionalization of strongly correlated electrons as a possible route to quantum Hall effect without magnetic field

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We show that the fractionalization of the constrained lattice electrons into charge/spin degrees of freedom driven by strong electron correlation can recover the anomalous quantum Hall effect that is similar to an integer quantum Hall effect in the absence of an external magnetic field.

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I. INTRODUCTION

At sufficiently low temperatures, a system of free charged particles hopping in a square lattice and subjected to a strong perpendicular magnetic field B is characterized by a Landau-band energy spectrum. In case all the bands below a certain gap are completely filled, the system exhibits the lattice integer quantum Hall effect (IQHE) [1]: the Hall conductance is quantized as integer $\times e^2/h$. Those integers are essentially the so-called Chern characters of the $U(1)$ complex line bundle over the base manifold which is a Brillouin torus—a compact closed manifold. In this framework single-electron Bloch wave functions serve as the sections of that bundle. The bundle is twisted giving rise to nonzero Chern characters and then out of that the IQHE becomes manifest.

In a seminal paper [2] Haldane showed that an external magnetic field is not a necessary ingredient to realize the IQHE. This phenomenon is rather driven by the breaking of time-reversal symmetry, which is not necessarily linked to the existence of an external magnetic flux. In a bipartite honeycomb lattice the time-reversal symmetry breaking in the absence of the external magnetic field can instead be accounted for by the presence of phenomenological complex-valued next nearest neighbor (NNN) hopping electron amplitudes. In this case the Hamiltonian is no longer real valued and this by itself breaks time-reversal symmetry. Haldane's theory is constituted in the framework of an electronic band structure. In view of that strong electron correlations are naturally excluded in this scenario.

By contrast, in the present work we consider a system of strongly correlated electrons hopping in a $2d$ lattice as described by the Hubbard model at an infinitely strong on-site repulsion $U=\infty$. In this case, the quantum numbers of the electron break apart, implying that the latter is not a sharply defined quasiparticle and the system can no longer be framed by a conventional electronic band structure [3].

Strong correlations modify the underlying on-site Hilbert space by forbidding doubly occupied sites. The constrained electron operators are isomorphic to the Hubbard operators [4], $X_i^{pq}=|p\rangle\langle q|$, with $p,q=\sigma,0$ and $\sigma=\uparrow,\downarrow$. Those operators appear as the generators of the $su(2|1)$ superalgebra. As a result, charge and spin degrees of freedom are intertwined in this representation under the action of the $SU(2|1)$ supergroup.

Within a conventional framework, Hubbard operators can be split (factorized) into a product of well-defined operators that correspond to spin and charge degrees of freedom. For example, within a standard slave-boson representation one gets $X_i^{\sigma 0}=f_{i\sigma}^\dagger b_i$, where $f_{i\sigma}$ stands for a fermion spinful operator and b_i denotes a boson operator that keeps track of the charge degrees of freedom. However, this representation introduces auxiliary degrees of freedom and it must be accompanied with a local no double occupancy (NDO) constraint $\sum_\sigma f_{i\sigma}^\dagger f_\sigma + b_i^\dagger b_i = 1$. This constraint generates a gauge field with no free Maxwell term that strongly couples the f_i and b_i modes to each other. As a result, the f_i and b_i modes are gauge dependent and they do not represent real physical excitations. As a matter of fact all known slave-particle representations of the Hubbard operators effectively result in strongly coupled compact $U(1)$ lattice gauge theories.

Here we employ instead the Hubbard-operator fractionalization based directly on the $su(2|1)$ superalgebra representation, with the NDO constraint being explicitly resolved prior to any approximations. This provides a description that is free from such auxiliary degrees of freedom. The $su(2|1)$ fractionalization appears then as a dynamical outcome that cannot in general be represented in terms of operator equalities. It is rather formulated in terms of the $su(2|1)$ path-integral action variables. A correspondence with the physical observables can be established in terms of the $su(2|1)$ phase-space correlators.

We first formulate an effective low-energy $su(2|1)$ path-integral action to describe strongly correlated electrons in terms of the factorized spinless charged fermionic ξ_i fields—the holons(dopons)—and the spinful bosonic z_i fields—the spinons. The z_i field describes spin degrees of freedom incorporated in the $su(2|1)$ superalgebra. In fact, this superalgebra can be thought of as just the simplest extension of the spin $su(2)$ algebra to incorporate fermions described by the ξ_i Grassmann numbers. Our second step is to restrict the spin dynamics by fixing a classical spin background to break time-reversal symmetry. This is produced by the addition of a new term that includes a direct Heisenberg spin interaction which determines exclusively the emerging magnetic structure. The resulting magnetic order is assumed to be fixed. In this way, we arrive at a topological band insulator formed by the ξ quasiparticles with no electronic band structure being involved. Our main result is thus this: strong correlations can indeed drive the IQHE without external magnetic field, provided the holons acquire a band structure of their own which, if gapped, can be classified by a Chern character. This is the case provided the other fractionalized degree of freedom—the spinons—allow for a mean-field treatment. Such an approach might be relevant for a description of strongly correlated electrons, due to the presence of a large on-site Coulomb repulsion, provided quantum spin fluctuations can be fully suppressed.

It should be noted that the physics described here does not occur for the "standard" Hubbard model referred to as the Hubbard model at half filling with nearest neighbor tunneling on the square lattice. In the limit of an infinitely strong on-site Coulomb repulsion, the electronic hopping at half filling is strictly forbidden. The fractionalization couples spinons to spinless charged fermions. At finite filling and in the case of a frustrated underlying lattice, integrating out the fermionic degrees of freedom results in an effective spin action that may exhibit spontaneously broken time-reversal symmetry. At a mean-field level, such an action determines a classical spin configuration that exhibits a nonzero spin chirality. However, this is a rather involved technical procedure to be discussed elsewhere. In the following we just postulate the magnetic spin texture in a given ground state. Our aim here is to show that there exists a scenario in which the topological properties of itinerant electrons may arise directly from strong correlation. Two simple examples are considered in Sec. III.

On the other hand, incorporating both quantum spin and fermion dynamics on the same footing might presumably result in a fractional lattice quantum Hall effect in the absence of an external magnetic field [3]. The emergent quasiparticles would obey the fractional statistics and carry fractional charge and the band theory considerations would again fail to account for the underlying physics. In that case the $su(2|1)$ fractionalization could provide a way to calculate the Hall conductance without any reference to the Fermi-liquid band theory approach. However, an explicit theory to deal with that is still under construction and it will be presented elsewhere.

II. THEORETICAL FRAMEWORK

For the reader's convenience we briefly outline a few basic steps to work out a path-integral representation of the partition function for a Hamiltonian given as a polynomial function of the Hubbard operators, $H = H(X)$.

To start with, the fermionic Hubbard operators $X^{0\sigma} = (X^{\sigma 0})^\dagger$ along with the bosonic ones, $X^{\sigma\sigma'}$, X^{00}, are closed under commutation/anticommutation relations into the *superalgebra* $su(2|1)$ [4]. The $su(2|1)$ superalgebra can be thought of as the simplest possible extension of the conventional spin $su(2)$ algebra to incorporate fermionic degrees of freedom. Namely, the bosonic sector of the $su(2|1)$ consists of three bosonic superspin operators,

$$Q^+ = X^{\uparrow\downarrow}, \quad Q^- = X^{\downarrow\uparrow}, \quad Q^z = \tfrac{1}{2}(X^{\uparrow\uparrow} - X^{\downarrow\downarrow}), \tag{1}$$

closed into $su(2)$, and a bosonic operator X^{00} that generates a $u(1)$ factor of the maximal even subalgebra $su(2) \times u(1)$ of $su(2|1)$. The fermionic sector is constructed out of four operators $X^{\sigma 0}, X^{0\sigma}$ that transform in a spinor representation of $su(2)$.

The important ingredient in constructing a corresponding partition function path-integral representation is the $su(2|1)$ coherent state (CS) parametrized by the points of the underlying phase space. Acting with the "lowering" superspin operators $X^{\downarrow\uparrow}$ and $X^{\downarrow 0}$ on the "highest weight" state $|\uparrow\rangle$ we get the normalized $su(2|1)$ coherent state in the $3d$ fundamental representation,

$$\begin{aligned} |z,\xi\rangle &= (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2} \exp\left(zX^{\downarrow\uparrow} + \xi X^{0\uparrow}\right)|\uparrow\rangle \\ &= (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2}(|\uparrow\rangle + z|\downarrow\rangle + \xi|0\rangle), \end{aligned} \tag{2}$$

where z is a complex number and ξ is a complex Grassmann parameter. The Grassmann parameter appears here due to the fact that $X^{\downarrow 0}$ is a fermionic operator in contrast with the operator $X^{\downarrow\uparrow}$. The product $\xi X^{0\uparrow}$ represents therefore a bosonic quantity as required. The set $(z,\xi) \in CP^{1|1}$ parametrizes the underlying phase space, the complex projective superspace $CP^{1|1} = SU(2|1)/SU(2) \times U(1)$. At $\xi = 0$, the $su(2|1)$ CS reduces to the ordinary $su(2)$ CS, $|z,\xi = 0\rangle \equiv |z\rangle$, parametrized by a complex coordinate $z \in \mathrm{CP}^1$. In contrast, at $z = 0$, it represents a pure fermionic CS.

The CS symbols of the X operators, $X_{cs} = \langle z,\xi|X|z,\xi\rangle$, read

$$\begin{aligned} X_{cs}^{0\downarrow} &= -\frac{z\bar{\xi}}{1+|z|^2}, \quad X_{cs}^{\downarrow 0} = -\frac{\bar{z}\xi}{1+|z|^2}, \\ X_{cs}^{0\uparrow} &= -\frac{\bar{\xi}}{1+|z|^2}, \quad X_{cs}^{\uparrow 0} = -\frac{\xi}{1+|z|^2}, \\ Q_{cs}^+ = X_{cs}^{\uparrow\downarrow} &= \frac{z}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right), \\ Q_{cs}^- = X_{cs}^{\downarrow\uparrow} &= \frac{\bar{z}}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right), \\ Q_{cs}^z = \frac{1}{2}(X_{cs}^{\uparrow\uparrow} - X_{cs}^{\downarrow\downarrow}) &= \frac{1}{2}\frac{1-|z|^2}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right). \end{aligned} \tag{3}$$

There is a one-to-one correspondence between the $su(2|1)$ generators and their CS symbols [5].

The corresponding imaginary time phase-space action takes on the form

$$\mathcal{S}_{su(2|1)} = -\int_0^\beta \langle z,\xi|\frac{d}{dt} + H(X)|z,\xi\rangle dt, \tag{4}$$

with the kinetic term given by

$$\langle z,\xi|\left(-\frac{d}{dt}\right)|z,\xi\rangle = \frac{1}{2}\frac{\dot{\bar{z}}z - \bar{z}\dot{z} + \dot{\bar{\xi}}\xi - \bar{\xi}\dot{\xi}}{1+|z|^2+\bar{\xi}\xi}. \tag{5}$$

Consider now the $U = \infty$ $2d$ Hubbard model:

$$H = -t\sum_{\langle ij\rangle,\sigma} X_i^{\sigma 0}X_j^{0\sigma} + \mu\sum_i X_i^{00}. \tag{6}$$

The chemical potential term is added to fix the total number of dopons. Taking into account Eqs. (3)–(5) and making the change of variables $z_i \to z_i$, $\xi_i \to \xi_i\sqrt{1+|z_i|^2}$, we are led to the partition function in the form of the $su(2|1)$ CS path integral [6]:

$$Z = \int D\mu(z,\xi)\, e^{S}, \tag{7}$$

where the measure

$$D\mu(z,\xi) = \prod_{i,t} \frac{d\bar{z}_i(t) dz_i(t)}{2\pi i(1+|z_i|^2)^2}\, d\bar{\xi}_i(t) d\xi_i(t)$$

splits into the $SU(2)$ invariant spin measure factor and the $U(1)$ fermion measure. Here z_i is a complex number that keeps track of the spin degrees of freedom, while ξ_i is a complex Grassmann parameter that describes the charge degrees of freedom. As $\xi_i^2 = 0$, the NDO constraint is resolved explicitly. In contrast to the slave-particle representations, the $su(2|1)$ dynamical variables are gauge independent: no auxiliary degrees of freedom are introduced.

The effective action

$$S = \sum_i \int_0^\beta \left(ia_i^{(0)} - \bar{\xi}_i\left(\partial_t + ia_i^{(0)}\right)\xi_i\right)dt - \int_0^\beta H\, dt \tag{8}$$

involves the $u(1)$-valued connection one-form of the magnetic monopole bundle [7] that can formally be interpreted as a spin "kinetic" term,

$$ia^{(0)} = -\langle z|\partial_t|z\rangle = \frac{1}{2}\frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{1+|z|^2},$$

with $|z\rangle$ being the $su(2)$ coherent state. This term is also frequently referred to as the Berry connection. The dynamical part of the action takes the form

$$H = -t\sum_{\langle ij\rangle} \bar{\xi}_i \xi_j e^{ia_{ji}} + \text{H.c.} + \mu\sum_i \bar{\xi}_i\xi_i, \tag{9}$$

where

$$a_{ij} = -i\ln\langle z_i|z_j\rangle, \quad \langle z_i|z_j\rangle = \frac{1+\bar{z}_i z_j}{\sqrt{(1+|z_j|^2)(1+|z_i|^2)}}$$

and $z_i(t)$ and $\xi_i(t)$ are the dynamical fields.

The Hamiltonian function (9) cannot be identified with a certain operator expressed in terms of the conventional fermion and spin operators. For instance the CS (or covariant) symbol of the Gutzwiller projected electron operator $\tilde{c}_\uparrow^\dagger = c_\uparrow^\dagger(1-n_\downarrow) = X^{\uparrow 0}$ takes the form

$$X_{cs}^{\uparrow 0} = -\frac{\xi}{\sqrt{1+|z|^2}}.$$

There is no such $su(2)$ spin operator whose covariant symbol is $\sqrt{1+|z|^2}$. In other words the $su(2|1)$ fractionalization of the Hubbard operators into the spin/charge entities occurs in terms of the $su(2|1)$ phase-space coordinates that transform through each other, rather than in terms of the standard spin/fermion operators subjected to the NDO constraint.

Under a *global* $SU(2)$ rotation,

$$z_i \to \frac{uz_i + v}{-\bar{v}z_i + \bar{u}}, \tag{10}$$

we get

$$a_i^{(0)} \to a_i^{(0)} - \partial_t\theta_i, \quad a_{ij} \to a_{ij} + \theta_j - \theta_i, \tag{11}$$

where

$$\theta_i = -\frac{i}{2}\ln\frac{-v\bar{z}_i + u}{-\bar{v}z_i + \bar{u}}, \quad \begin{pmatrix} u & v \\ -\bar{v} & \bar{u}\end{pmatrix} \in \text{SU}(2). \tag{12}$$

Note that the θ_i is an angular variable,

$$\theta_i = -\arg(-v\bar{z}_i + u).$$

The effective action (8) is invariant under the $SU(2)$ rotation given by Eqs. (11) accompanied by the $U(1)$ transformation of the fermionic field,

$$\xi_i \to e^{i\theta_i}\xi_i. \tag{13}$$

A "flux" through a plaquette $\sum_{\text{plaq}} a_{ij}$ generated by the $SU(2)$ transformation remains invariant under (11).

For further convenience let us consider separately the real and imaginary parts of a_{ij}:

$$a_{ji} = \phi_{ji} + i\chi_{ji}, \quad \bar{\phi}_{ji} = \phi_{ji}, \quad \bar{\chi}_{ji} = \chi_{ji}.$$

We get

$$\begin{aligned}\phi_{ji} &= \frac{i}{2}\ln\frac{1+\bar{z}_i z_j}{1+\bar{z}_j z_i} \\ &= \frac{i}{2}\ln\frac{(\frac{1}{2}+S_i^z)(\frac{1}{2}+S_j^z)+S_i^- S_j^+}{(\frac{1}{2}+S_i^z)(\frac{1}{2}+S_j^z)+S_j^- S_i^+}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\chi_{ji} &= -\frac{1}{2}\ln\frac{(1+\bar{z}_i z_j)(1+\bar{z}_j z_i)}{(1+|z_i|^2)(1+|z_j|^2)} \\ &= -\frac{1}{2}\ln\left(2\vec{S}_i\cdot\vec{S}_j + \frac{1}{2}\right),\end{aligned} \tag{15}$$

where $\vec{S}_i$ stand for the coherent-state symbols of the $su(2)$ generators, $\vec{S}_i = \vec{Q}_i\,|_{\bar{\xi}_i,\xi_i=0}$. It can be checked that the ϕ_{ji} potential transforms under (10) in the same way as the a_{ji} does, i.e.,

$$\phi_{ji} \to \phi_{ji} + \theta_i - \theta_j, \tag{16}$$

whereas, in contrast, χ_{ji} remains intact. This transformation appears as a gauge fixing by choosing a specific rotationally covariant frame. The dynamical fluxes do not depend on that choice.

The low-energy physics is governed by the fluctuations of the phase variable,

$$\phi_{ji} = -\arg(1+\bar{z}_i z_j) = \arg(1+\bar{z}_j z_i),$$

defined modulo (2π). It is clear that $\phi_{ji} = -\phi_{ij}$. The potentials $\phi_i^{(0)} := ia_i^{(0)}$ and ϕ_{ji} formally recall those gauge fields that define a compact $U(1)$ lattice gauge theory. This is due to the fact that both theories are formulated as $U(1)$ complex line bundles. The gauge potentials (local connections in these bundles) in both theories transform formally in the same way under a change in the local trivialization [8]. In our case, such a change is caused by a rotation of the underlying base space—a canonical transformation of the phase space.

Topology

Substituting representations (14) and (15) into Eq. (9), the Hamiltonian function takes on the form

$$H = -t\sum_{\langle ij\rangle}\bar{\xi}_i\xi_j e^{i\phi_{ji}}\sqrt{2\vec{S}_i\cdot\vec{S}_j+\frac{1}{2}} + \mu\sum_i\bar{\xi}_i\xi_i. \quad (17)$$

This Hamiltonian describes the interaction between a spin texture and the itinerant spinless fermions, with the local NDO constraint being automatically incorporated since $\xi_i^2=0$. The model may display nontrivial topology—a nonzero Chern number—only provided that there are at least two available bands. This is because the Chern number of a full set of bands is zero—the corresponding vector bundle being always trivial [9].

The simplest insulator possesses two bands, the empty one above the gap and the filled one (or valence band) below the gap, with the chemical potential lying inside the gap. We consider examples of such a two-band model that may display a topologically insulating phase. To characterize the topology of the valence band bundle we only focus on the one-particle eigenstates of (17). In this case the chemical potential term just shifts the corresponding eigenvalues, which has no effect on topological properties, provided the system remains insulating. As we focus only on the topological behavior of the filled band, we from now on ignore the explicit μ dependence of the Hamiltonian.

We restrict ourselves to the two-band case with two opposite Chern numbers. To this end, let us consider a bipartite $2d$ lattice L, which is a direct sum of two sublattices A and B, i.e., $L=A\oplus B$. We then make the following change of variables on the sublattice B:

$$z_i\to-\frac{1}{\bar{z}_i},\quad \xi_i\to\xi_i e^{i\theta_i^{(0)}},\quad i\in B. \quad (18)$$

Here $\theta_i^{(0)}=\theta_i|_{u=0,v=1}$. Under this transformation, $\phi_{ji}\to\phi_{ij}+\theta_j^{(0)}-\theta_i^{(0)}$. The CS image of the on-site electron spin operator changes sign, $\vec{S}_i\to-\vec{S}_i$ [6].

An important remark is in order at this stage. Haldane's theory is based on electronic band structure for fully polarized electrons. The spin degrees of freedom are simply ignored. Freezing the spin degrees of freedom within the mean-field theory is somehow equivalent to ignoring spins in the problem or fixing it to provide the correct Chern number, as in the case of noncollinear spin textures [10]. We just want to note that in our approach we deal with the fractionalization into the effective holon/spinon degrees of freedom. The freezing of the spinon degrees of freedom results in the band structure displayed by the holons rather than by the spin polarized physical electrons.

To proceed further we note that the phase ϕ_{ji} enters the hopping terms on the A and B sublattices with opposite signs. This in turn means that the dynamical fluxes through elementary plaquettes in the A and B within a unit cell are opposite to each other. The Hall conductance σ_{xy} by definition changes sign under the time-reversal transformation. A nonzero value can only occur if time-reversal invariance is broken. To realize a finite quantum Hall response, it is therefore necessary to break time-reversal symmetry. In the Haldane model for graphene, this is done by inserting local fluxes which sum up to zero over a unit cell. These fluxes can be described by introducing homogeneous c-valued phase factors in the second neighbor hopping amplitude, $t_2\to t_2e^{\pm\phi}$. In our dynamical model (17) this can be achieved by fixing the underlying spin background to enforce the breaking of the time-reversal symmetry.

III. EXAMPLES

In a system of lattice electrons, the topological Hall effect may be viewed as arising from the electron hopping in a nontrivial classical spin background that exhibits a nonzero scalar spin chirality. The corresponding topological spin quantum numbers can be related to the anomalous Hall conductivity in the absence of any externally applied magnetic field. The simplest examples come from a mean-field treatment of the Kondo-lattice interaction (the double exchange model) of the localized spins and itinerant electrons yielding noncoplanar spin textures [11–14]. In this way the underlying topological spin structures are composed of multiple spin density waves with a spin scalar chirality defined by the triple product of three neighboring sites, $\langle\vec{S}_i\cdot\vec{S}_j\times\vec{S}_k\rangle\neq0$, in the ground states. Typical postulated spin textures are chiral stripes, spiral spin configurations, and skyrmion spin structures [15]. In particular, the experimentally observed skyrmion lattice can be viewed as a lattice of topologically stable knots in the underlying spin structure [16,17].

In the present work we propose an alternative way to address the topological properties of the hopping electrons in the nontrivial spin background. We show that the entanglement between the itinerant fermion degrees of freedom and that of the localized spins is actually built in a very definition of the Hubbard operators. Instead of the double exchange model to describe electron dynamics to account for the emergence of nonzero topological quantum numbers, the Hubbard model in the strong coupling limit is employed. The established charge/spin fractionalization plays, in this limit, an essential role in driving the system into a topologically nontrivial phase. The emergent band topology may thus be viewed as arising from strong correlation.

In this section we consider two examples to illustrate our approach. Both of them are based on the classical treatment of the underlying spin textures. In view of that, the ground state in the spin sector is always classical. If magnetic frustration (geometric or dynamic due to doping) is sufficiently strong, a spin system may evade spontaneous symmetry breaking at low temperatures and instead form a highly entangled state where the spins fluctuate in a cooperative manner. Below we consider strongly correlated electrons on top of different classical spin backgrounds. This results in both topologically trivial and nontrivial models. We consider the simplest possible models that however can be discriminated from each other by their topological properties. The first example is based on considering a coplanar spin structure, whereas the second one deals with a noncoplanar spin texture.

A. Spin spirals

Let us consider first the classical coplanar spin spirals of the form [18]

$$\vec{S}_i = S(\cos\vec{q}\cdot\vec{r}_i, \sin\vec{q}\cdot\vec{r}_i, 0)|_{S=1/2} = \tfrac{1}{2}(\cos\vec{q}\cdot\vec{r}_i, \sin\vec{q}\cdot\vec{r}_i, 0). \quad (19)$$

The emergence of the spin-spiral periodic structure with vector $\vec{q}$ that differs from the BZ sizes can be related to magnetic structures in antiferromagnetic metals [19]. An issue of the stability of the spiral phase against thermal and quantum fluctuation was considered in [20].

We get

$$2\vec{S}_i\cdot\vec{S}_j + \frac{1}{2} = \cos^2\left(\frac{\vec{q}\cdot\vec{a}_{ij}}{2}\right), \quad \vec{a}_{ij} = \vec{r}_i - \vec{r}_j,\ i, j \in A.$$

On the other hand,

$$2\vec{S}_i\cdot\vec{S}_j + \frac{1}{2} = \sin^2\left(\frac{\vec{q}\cdot\vec{a}_{ij}}{2}\right), \quad i \in A,\ j \in B.$$

On the sublattice A the phase factor ϕ_{ij} takes on the form

$$\phi_{ji} = \frac{i}{2}\ln\frac{1+e^{-i\vec{q}\cdot\vec{a}_{ij}}}{1+e^{i\vec{q}\cdot\vec{a}_{ij}}} = \arctan\frac{\sin\vec{q}\cdot\vec{a}_{ij}}{1+\cos\vec{q}\cdot\vec{a}_{ij}} = \arctan\left(\tan\frac{\vec{q}\cdot\vec{a}_{ij}}{2}\right) = \frac{\vec{q}\cdot\vec{a}_{ij}}{2}, \quad i, j \in A, \quad (20)$$

whereas

$$\phi_{ji} = -\frac{\pi}{2} + \frac{\vec{q}\cdot\vec{a}_{ij}}{2}, \quad i \in A,\ j \in B.$$

The Hamiltonian function (17) then becomes

$$H = it_1\sum_{\langle ij\rangle}\bar{\xi}_i\xi_j|\sin\phi_{ij}|e^{-i\phi_{ij}} - t_2\sum_{\langle ij\rangle\in A}\bar{\xi}_i\xi_j|\cos\phi_{ij}|e^{-i\phi_{ji}} + (A\to B) + \text{H.c.}, \quad \phi_{ij} = \frac{\vec{q}\cdot\vec{a}_{ij}}{2}. \quad (21)$$

In contrast to a general representation where the phase factors ϕ_{ji} enter sublattices A and B with the opposite signs, in the present case these factors are the same. This is due to the fact that $\phi_{ij} + \theta_j^{(0)} - \theta_i^{(0)} = \phi_{ji}$ provided $S_i^z = 0$. Because of this the fluxes are generated by the nonzero z components of the total on-site spins. At $S_i^z = 0$ those phases can be eliminated through appropriate unitary transformations of the ξ_i fields. Finally we arrive at the operator Hamiltonian

$$H = t_1\sum_{\langle ij\rangle}f_i^\dagger f_j|\sin\phi_{ij}| - t_2\sum_{\langle ij\rangle\in A}f^\dagger f_j|\cos\phi_{ij}| + (A\to B) + \text{H.c.} \quad (22)$$

It is clear that $H^* = H$ so that the time-reversal symmetry remains intact: in $2d$, spatial inversion is contained in the connected part of the spatial rotation group. This in turn means that this model does not exhibit a Hall response and it is topologically trivial. This fully agrees with the fact that the spin chirality is absent in a coplanar spin configuration.

B. Spin precession: Honeycomb lattice

Let us now consider a classical noncoplanar spin spiral configuration:

$$\vec{S}_i = (\epsilon\cos f_i, \epsilon\sin f_i, \sqrt{S^2-\epsilon^2}), \quad \epsilon \ll S. \quad (23)$$

It describes a precession of the spin $\vec{S}_i$ with an amplitude ϵ around the z axis. Here $f_i(t) = \omega t - 2\vec{q}\cdot\vec{r}_i$. At $S_i = 1/2$ we get

$$2\vec{S}_i\cdot\vec{S}_j + \tfrac{1}{2} = 1 - 4\epsilon^2\sin^2 f_{ij}, \quad i, j \in A$$

and

$$e^{-\chi_{ij}} = 1 - 2\epsilon^2\sin^2 f_{ij} + O(\epsilon^4).$$

Here $f_{ij} = \vec{q}\cdot(\vec{r}_i - \vec{r}_j)$. The phase factor ϕ_{ji} on the sublattice A becomes

$$\phi_{ji} = \frac{i}{2}\ln\frac{1-2\epsilon^2+\epsilon^2e^{-2if_{ij}}}{1-2\epsilon^2+\epsilon^2e^{+2if_{ij}}} = \arctan\frac{\epsilon^2\sin 2f_{ij}}{1+\epsilon^2\cos 2f_{ij}} = \epsilon^2\sin 2f_{ij} + O(\epsilon^4). \quad (24)$$

On sublattice B, $\vec{S}_i \to -\vec{S}_i$. Since

$$\frac{(\frac{1}{2}-S_i^z)(\frac{1}{2}-S_j^z)+S_i^-S_j^+}{(\frac{1}{2}-S_i^z)(\frac{1}{2}-S_j^z)+S_j^-S_i^+} = \frac{(\frac{1}{2}+S_i^z)(\frac{1}{2}+S_j^z)+S_j^-S_i^+}{(\frac{1}{2}+S_i^z)(\frac{1}{2}-S_j^z)+S_i^-S_j^+}\cdot\frac{S_j^+S_i^-}{S_i^+S_j^-}, \quad (25)$$

we get $\phi_{ji} \to -\phi_{ji} + 2f_{ij}$. Explicitly

$$\phi_{ji} = -\epsilon^2\sin 2f_{ij} + 2f_{ij} + O(\epsilon^4), \quad i, j \in B, \quad (26)$$

and χ_{ij} remains intact.

In case $i \in A$ and $j \in B$ one gets

$$2\vec{S}_i\cdot\vec{S}_j + \tfrac{1}{2} = 4\epsilon^2\sin^2 f_{ij},$$

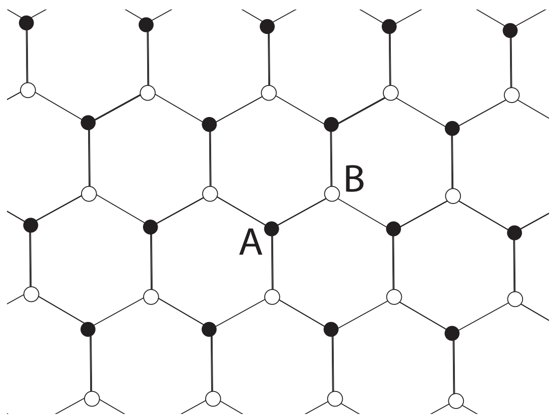
so that

$$e^{-\chi_{ij}} = 2\epsilon|\sin f_{ij}|, \quad \phi_{ji} = -\pi/2 + f_{ij}.$$

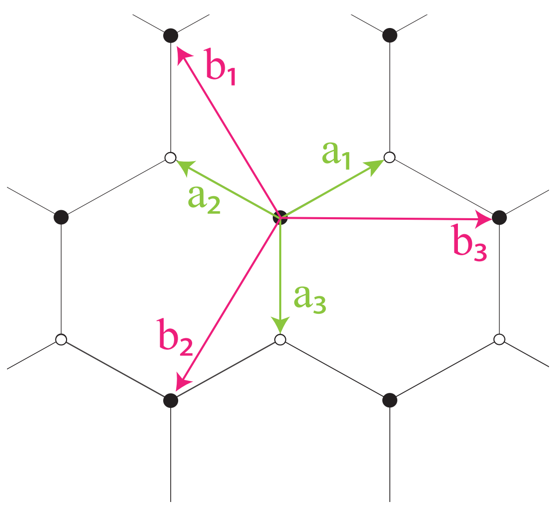
Combining all the factors together we finally get the Hamiltonian

$$H = 2t_1\epsilon\sum_{i\in A, j\in B}f_i^\dagger f_j|\sin f_{ij}| + t_2\sum_{i,j\in A}f_i^\dagger f_j(1+2i\epsilon^2\sin f_{ij}e^{if_{ij}})e^{-if_{ij}} + t_2\sum_{i,j\in B}f_i^\dagger f_j(1-2i\epsilon^2\sin f_{ij}e^{-if_{ij}})e^{if_{ij}} + \text{H.c.} \quad (27)$$

The NN hoping amplitude $\sim t_1$ is a real quantity, whereas the NNN ones $\sim t_2$ are complex numbers. An inversion symmetry breaking on-site energy $+M$ on A sites and $-M$ on sites can also be added to this Hamiltonian, $M\sum_i(-1)^iX_i^{00}$, with its CS symbol being $M\sum_i(-1)^i\bar{\xi}_i\xi_i$. Here $(-1)^i = \pm 1$ on A and B sublattices, respectively. Representation (27) implies that $(t_{ij\in A})^* = t_{ij\in B}$, whereas Im $t_{i\in A, j\in B} = 0$. Here i, j denote



(a) Honeycomb Lattice



(b) Displacement vectors.

FIG. 1. Honeycomb lattice and associated displacement vectors.

the NN sites. This is a necessary condition to enforce time-reversal symmetry breaking. In the momentum space,

$$H = \sum_{\vec{k}\in BZ} \psi^\dagger_{\vec{k}} \mathcal{H}_{\vec{k}} \psi_{\vec{k}}. \tag{28}$$

Here, BZ stands for the Brillouin zone and $\psi^\dagger_{\vec{k}} = (f^\dagger_{\vec{k},A}, f^\dagger_{\vec{k},B})$. On a honeycomb lattice the Hamiltonian becomes

$$\begin{aligned}\mathcal{H}_{\vec{k}} &= t_2 \sum_{i=1}^{3} \cos(\vec{q}\cdot\vec{b}_i)\cos(\vec{k}\cdot\vec{b}_i)\cdot I \\ &+ t_1\epsilon \sum_{i=1}^{3} |\sin\vec{q}\cdot\vec{a}_i|\cos(\vec{k}\cdot\vec{a}_i)\cdot\sigma_1 \\ &- t_2(1-2\epsilon^2)\sum_{i=1}^{3}\sin(\vec{q}\cdot\vec{b}_i)\sin(\vec{k}\cdot\vec{b}_i)\cdot\sigma_3. \end{aligned} \tag{29}$$

A honeycomb lattice is defined (see Fig. 1) by a set of the NN displacement vectors, $\vec{a}_1 = (\sqrt{3}/2, 1/2)$, $\vec{a}_2 = (-\sqrt{3}/2, 1/2)$, $\vec{a}_3 = (0, -1,)$, $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = 0$, and a set of the NNN vectors, $\vec{b}_1 = \vec{a}_2 - \vec{a}_3 = (-\sqrt{3}/2, 3/2)$, $\vec{b}_2 = \vec{a}_3 - \vec{a}_1 = (-\sqrt{3}/2, -3/2)$, $\vec{b}_3 = \vec{a}_1 - \vec{a}_2 = (\sqrt{3}, 0)$, $\vec{b}_1 + \vec{b}_2 + \vec{b}_3 = 0$. As usual, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denote the Pauli matrices. The term $\sim\sigma_3$, being odd in $\vec{k}$, breaks time reversal symmetry, provided $\epsilon \neq 0$ and $\vec{q} \neq 0$.

At $t_2 = 0$ and $\epsilon \neq 0$, the system is an insulator if the term $\sim\sigma_1$ is nonzero. In case it is zero the two bands are degenerate at the corresponding points of the BZ. This degeneracy can be lifted by adding the NNN interaction. The emergent gap $\sim t_2$ protected by time-reversal symmetry breaking is enforced by the σ_3 dependent contribution. In case we choose $q_x = q/\sqrt{3}$ and $q_y = q$ this occurs at the BZ points $\vec{K} = (\pi/2\sqrt{3}, \pi/2)$ and $-\vec{K}$. As a result,

$$c_1 = \mathrm{sgn}(\sin q), \tag{30}$$

where c_1 is the first Chern character of the corresponding $U(1)$ Bloch bundle. At $\epsilon = 0$ we get instead

$$\mathcal{H}_{\vec{k}} \sim \mathrm{diag}\left[\sum_i \cos[\vec{b}_i\cdot(\vec{q}+\vec{k})], \sum_i \cos[\vec{b}_i\cdot(\vec{q}-\vec{k})]\right]. \tag{31}$$

By an appropriate change $\vec{k} \to \vec{k} \pm \vec{q}$ on the A and B sublattices we end up with

$$\mathcal{H}_{\vec{k}} \sim \cos k_x \cos k_y \cdot I.$$

This is the time-reversal invariant Hamiltonian that exhibits no topological properties and $c_1 = 0$.

We thus see that a noncoplanar spin configuration on a frustrated (honeycomb) lattice results in the nontrivial band topology. However, it is still unclear whether these conditions are both necessary or/and sufficient.

IV. DISCUSSION

In this section, we would like to clarify some limitations of our work and to touch upon some issues to be discussed in a future work.

We start with the $U = \infty$ Hubbard model (6). The Hubbard operators X_i incorporate both the charge (spinless fermion) as well as the $su(2)$ spin degrees of freedom. If we vary the doping regime, different magnetic ground-state structures can emerge from representation (6). Those structures are generated by the so-called kinetic magnetism since there is no direct exchange magnetic term in the $U = \infty$ Hubbard model. Although an itinerant electron-driven chiral magnetic ordering can be derived from (6) on a triangular lattice [10], the spontaneously formed noncoplanar topological magnetic textures used in our paper and in other related publications cannot be accounted for by just such kinetic magnetism. Such structures necessarily require the presence of a direct exchange spin interaction as well. For this reason we add to (6) a new spin exchange term,

$$H_{cl} = H_{cl}(\vec{S}_i, \vec{S}_j), \tag{32}$$

that includes the *classical* Heisenberg (anisotropic) exchange interactions. The emergent magnetic phases are determined exclusively by H_{cl}. In our work we make a further assumption that the magnetic degrees of freedom are much slower than the degrees of freedom of the itinerant electrons. In view of that H and H_{cl} can be decomposed without affecting the low-energy physics. The resulting magnetic order is treated as being essentially static. The connection one-form in (9) is then replaced with its classical counterpart given by Eqs. (14) and (15). As a result, the energy bands are renormalized by the underlying spin texture and the Hamiltonian (6) reduces to an effective tight-binding model given by Eq. (17). The underlying spin background has no feedback from the fermionic dynamics and remains fixed as initially postulated. If the spin ordering opens a full gap in the charge excitation spectrum the conditions are given for the topological Hall effect to be fully manifest. The doping is then fixed to allow the insertion of the Fermi level in the interband gap and the resulting state is a topological insulator driven by the local no double occupancy constraint.

Of course, such an approach is not self-consistent since it ignores quantum spin fluctuations. However, provided the theory is stable against such fluctuations our approach demonstrates explicitly in what way strong correlation can directly affect topology, which is one of the goals of the present work.

In a full dynamical theory, the possible magnetic phases described in the paper could presumably be accounted for by the development of doping-dependent magnetic states. To derive this explicitly, the fermionic fields in path integral (7) must be integrated out in order for us to work out an effective spin action which could determine the average spin values as functions of doping.

Here we postulate instead a given fixed classical spin background. The relevant magnetic structures under analysis have been widely seen in itinerant magnets on various lattices (e.g., chiral stripes, spin spirals, skyrmion textures, etc.) [15]. Since the localized classical spins are taken to be fixed and bear no dynamics of their own they do not exhibit a doping dependence. The textures being dealt with affect instead the connection one-form a_{ij}. In a sense this amounts to threading an effective "flux" through a unit cell as discussed in the present work.

Our work leaves open the possibility of a more integrated approach which can describe the emergence of quantum state of matter with a nonzero first Chern number as a result of the intertwining of fully dynamical magnetic structures and strongly correlated electrons. We intend to explore this route in the near future.

V. CONCLUSION

In conclusion, we show that the anomalous quantum Hall effect can be placed in the context of phenomena associated with strongly correlated electron systems. This can be achieved via the dynamical fractionalization of strongly correlated electrons into spin/charge degrees of freedom driven by the $su(2|1)$ superalgebra representation of the strongly coupled Hubbard model. The necessary ingredients for that to happen are (i) the fermionic gapped bundle structure related to holons and (ii) the underlying $su(2)$ spin texture that explicitly enforces time-reversal symmetry breaking. While it is not clear whether the particular model presented here can be directly physically realizable, the classical spin spiral/precession structures highlighted above can arise in some models that describe localized spin configurations coupled with conduction electrons [19]. Among possible topologically nontrivial spin textures that might essentially affect the topological properties of itinerant electrons of a special interest are those driven by dynamical models that display experimentally observed skyrmion [21] and hedgehog [22] spin structures. This problem will be discussed elsewhere.

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[1] T. H. Hansson, M. Hermanns, S. H. Simon, and S. F. Viefers, Rev. Mod. Phys. **89**, 025005 (2017).
[2] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[3] J. Maciejko and G. A. Fiete, Nat. Phys. **11**, 385 (2015).
[4] P. B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988).
[5] F. A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[6] E. Kochetov and V. Yarunin, Phys. Rev. B **56**, 2703 (1997); A. Ferraz, E. Kochetov, and M. Mierzejewski, *ibid.* **73**, 064516 (2006); A. Ferraz and E. Kochetov, Nucl. Phys. B **853**, 710 (2011).
[7] M. Stone, Nucl. Phys. B **314**, 557 (1989).
[8] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (Taylor and Francis, London, 2003).
[9] G. Panati, Ann. Henri Poincaré **8**, 995 (2007).
[10] I. Martin and C. D. Batista, Phys. Rev. Lett. **101**, 156402 (2008).
[11] J. Ye, Y. B. Kim, A. J. Millis, B. I. Shraiman, P. Majumdar, and Z. Tesanovic, Phys. Rev. Lett. **83**, 3737 (1999).
[12] S. H. Chun, M. B. Salamon, Y. Lyanda-Geller, P. M. Goldbart, and P. D. Han, Phys. Rev. Lett. **84**, 757 (2000).
[13] K. Ohgushi, S. Murakami, and N. Nagaosa, Phys. Rev. B **62**, R6065 (2000).
[14] S. D. Yi, S. Onoda, N. Nagaosa, and J. H. Han, Phys. Rev. B **80**, 054416 (2009).
[15] S. Hayami and Y. Motome, Phys. Rev. B **104**, 144404 (2021).
[16] A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni, Phys. Rev. Lett. **102**, 186602 (2009).
[17] J. H. Han, J. Zang, Z. Yang, J-H. Park, and N. Nagaosa, Phys. Rev. B **82**, 094429 (2010).
[18] T. A. Kaplan, Phys. Rev. **116**, 888 (1959).
[19] I. E. Dzyaloshinskii, Sov. Phys. JETP **19**, 960 (1964); **20**, 223 (1965).
[20] Y. Iqbal, T. Müller, H. O. Jeschke, R. Thomale, and J. Reuther, Phys. Rev. B **98**, 064427 (2018).
[21] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Science **323**, 915 (2009).
[22] N. Kanazawa *et al.*, Phys. Rev. B **96**, 220414(R) (2017).