Transmission line approach to transport of heat in chiral systems with dissipation

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Measurements of the energy relaxation in the integer quantum Hall edge at filling factor $\nu = 2$ suggest the breakdown of heat current quantization [H. le Sueur *et al.*, Phys. Rev. Lett. **105**, 056803 (2010)]. It was shown in a hydrodynamic model that dissipative neutral modes contributing apparently less than a quantum of heat can be an explanation for the missing heat flux [A. Goremykina *et al.*, arXiv:1908.01213]. This hydrodynamic model relies on the introduction of an artificial high-energy cutoff and lacks a way of *a priori* obtaining the correct definition of the heat flux. In this work we overcome these limitations and present a formalism, effectively modeling dissipation in the quantum Hall edge, proving the quantization of heat flux for all modes. We mapped the quantum Hall edge to a transmission line by analogy and used the Langevin equations and scattering theory to extract the heat current in the presence of dissipation.

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I. INTRODUCTION

The missing heat paradox, reported in the integer quantum Hall (QH) edge at filling factor $\nu = 2$ [1–3], challenges two of the most fundamental phenomena in mesoscopic and nanoscopic physics. On the one hand many experiments confirm that heat flux is quantized, more specifically, the rate at which any type of carrier can transport heat at most in a ballistic channel is proportional to a universal value known as the heat flux quantum quantum $J_q = \frac{\pi k_B^2}{12\hbar} T^2$ [4,5]. Ballistic channels can be found in a variety of different systems including quasi-one-dimensional semiconductor nanostructures [6], carbon nanotubes [7], or QH systems, both at integer [1,2] and fractional fillings [8–10]. On the other hand introducing dissipation may resolve the paradox, since it is suggested to be able to break the aforementioned heat flux quantization [11].

In contrast to charge transport which is often protected by symmetries or topology [12], heat transport in such systems requires a more elaborate theoretical framework. Taking into account the smooth confining potential and screening of the gate electrodes in QH systems, edge states manifest themselves as a charge density profile consisting of alternating compressible and incompressible strips [13,14]. This picture has been experimentally confirmed [15]. The effects of interaction, disorder, or finite temperature effects predict additional nontrivial neutral counterpropagating excitations in the edge [16–24]. Aleiner and Glazman (AG) analyzed the low-energy spectrum of excitations of a compressible electron liquid in a strong magnetic field and showed that the integer QH edge can host neutral copropagating excitation [25]. Theory predicts an infinite number of neutral downstream AG excitations, but they were never detected in experiment. What has been measured is a leakage of the injected energy into the QH edge at different integer fillings [2,3], suggesting the presence of additional degrees of freedoms for energy to be redistributed.

A detailed study of the QH edge at filling factor $\nu = 2$ followed [1]. In the experiment, energy was injected in the form of Joule heat into a QH edge at a constant rate. This creates a nonequilibrium distribution function which eventually relaxes to a "hot" Fermi distribution function, which can be probed by a quantum dot downstream. They found no energy transfer towards the excitation of thermalized states and an efficient energy redistribution between the two channels without particle exchanges. However, after long equilibration lengths $L \geqslant 10\ \mu$m the corresponding temperature of the equilibrium distribution function saturated at a value which was 13% lower than expected for two interacting edge channels. The effective temperature becomes independent of the propagation length indicating that no energy leaked into the bulk of the system or is lost to some external mechanisms not taken into account by the experimental procedure. The authors concluded that the presence of additional degrees of freedom can explain the outcome of the experiment; however, to match the numbers this additional mode would need to carry less than a quantum of heat to explain the loss.

In a previous work we addressed the possibility of a third AG excitation carrying less than a heat flux quantum due to dissipation [11]. The charged mode of the system carries a flux quantum $J = J_q = \frac{\pi}{12\hbar\beta^2}$. The neutral mode subject to a transverse current proportional to the longitudinal conductivity σ_{xx} in the compressible strip will result in this mode having a dissipative term in its low-energy spectrum and thus carrying apparently a reduced amount of heat. The model is limited to the low-energy degrees of freedoms with a wavelength much larger than the inverse size of the compressible strip due to the inhomogeneity of the edge in the transverse direction. Despite the introduction of this artificial cutofff the found reduction of the carried heat is universal, $J = \frac{\sigma_{xx}}{2\pi\sigma_{xy}} J_q$. We also note here that the reported loss of heat might be due to nonlocal relaxation mechanisms [26–29]. However, the results of the present paper show a more complete picture of the low-energy theory of the edge.

The goal of this paper is to apply a combination of Langevin equations and scattering states to model dissipation

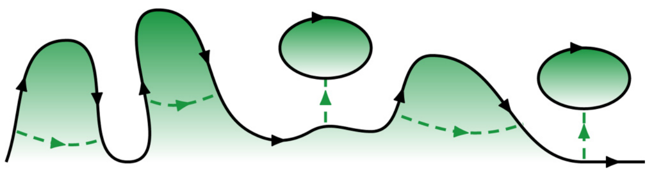
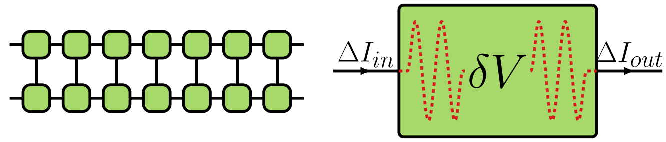


FIG. 1. Possible origin of dissipation in the QH edge: Due to imperfections of the edge, electrons have the possibility to tunnel, resembling a simple reservoir for electrons. This allows us to make an analogy with a transmission line, which has distributed capacitors and resistors. The reservoirs act as a heat bath with a self-consistently determined temperature.



(a) Transmission line (b) Single node

FIG. 2. (a) The transmission line. The two copropagating modes consist of many reservoirs, which can interact longitudinally and transversely. (b) A single node. The incoming current is fully dissipated in the Ohmic reservoir heating it in return. The outgoing current contains a contribution of the collective mode $\delta V(t)/R_q$ and a Langevin source contribution $\delta I(t)$ due to the thermal noise of the resistor.

in chiral systems effectively [30]. We attach a chiral system to a bath modeled as an open system, having the advantage to be able to address equilibrium and nonequilibrium situations, as well as being analytically treatable. These advantages make it a unique approach to modeling dissipation, complementary to the Caldeira-Leggett model, which successfully captured the features of dissipation on a quantum level. We provide an effective theory for chiral dissipative systems which is not restricted to the low-energy degrees of freedom and prove the quantization of heat for these systems, which, to the best of our knowledge, has never been addressed. Furthermore, we present the correct definition of heat flux, which is unobtainable starting from a hydrodynamic point of view, we try to resolve the experimental paradox of the "missing heat," and we discuss the role of AG modes in QH systems.

II. THEORETICAL MODEL

Our goal is to capture the physics of a compressible strip in the presence of disorder. We propose a minimal model and focus on the experimental situation of filling factor $\nu = 2$ and model two copropagating modes with a typically large (spin) resistance between them [31]. Interactions and dissipation can be conveniently introduced using a transmission line (TL) approach. We formally discretize the system into many nodes which interact longitudinally (within the same mode) with a chiral quantum resistor of strength $R_q = \frac{2\pi\hbar}{e^2}$ and transversely (between the modes) with a quantum resistor of, in principle, arbitrary strength $R_\perp$.

One can view this discretization also as the attempt to model inhomogenities of the edge; see Fig. 1. Electrons moving along the edge might be disturbed or stored in these inhomogenities for some time and then are reemitted at a later time. This physically resembles an Ohmic reservoir similar to the one presented in [32]. We consider the situation where the level spacing of the reservoir is smaller than the charging energy, the other limit will be considered elsewhere. Thermal fluctuations incident to the Ohmic contact are absorbed and lead to the creation of voltage $\delta V(t)$ and current fluctuations $\delta I(t)$ being reemitted; see Fig. 2(b). This can be captured within the framework of Langevin equations [33].

$$\frac{d}{dt}\delta Q(t) = \Delta I_{\mathrm{in}}(t) - \Delta I_{\mathrm{out}}(t), \tag{1}$$

$$\Delta I_{\mathrm{out}} = \frac{\delta Q(t)}{R_q C} + \delta I(t), \tag{2}$$

where the first equation is Kirchoff's law guaranteeing current conservation in the Ohmic contact and the second equation is the Langevin equation with C being the capacitance of the node. In our notation ΔI corresponds to the total fluctuation of current containing charge and current fluctuations and δI refers to the current fluctuations in the form of a Langevin source. We take the Langevin equation as a starting point and introduce two chains of N nodes interacting with a transverse resistor. The equation of motion for the voltage fluctuation $\delta V_j^\alpha = \delta Q_j^\alpha / C$ at position j in the upper/lower part of the TL $\alpha = 1, 2$ is given by

$$\frac{d}{dt}\delta Q_j^\alpha = \Delta I_{\mathrm{in},j}^\alpha - \Delta I_{\mathrm{out},j}^\alpha \mp \Delta I_j^\perp, \tag{3}$$

$$\Delta I_j^\perp = \frac{1}{R_\perp}\left(\delta V_j^{(1)} - \delta V_j^{(2)}\right) + \delta I_j^\perp, \tag{4}$$

$$\Delta I_{\mathrm{out},j}^\alpha = \frac{1}{R_q}\delta V_j^\alpha + \delta I_j^\alpha, \tag{5}$$

where the incoming current $\Delta I_{\mathrm{in},j}^\alpha = \Delta I_{\mathrm{out},j-1}^\alpha$ [34] is given by the outgoing current of the previous node, and δI_j^α and $\delta I_j^\perp$ denote a Langevin source and the negative (positive) sign is chosen for $\alpha = 1(2)$. The full transmission line is shown in Fig. 2(a) and the equivalent circuit corresponding to the equation of motion is depicted in Fig. 3.

The outgoing current fluctuations consist of the collective mode contribution and thermal fluctuations. We define the symmetric and antisymmetric combination of all voltages and currents, i.e., the charged and neutral mode $X^{c/n} = \frac{1}{2}(X^{(1)} \pm X^{(2)})$:

$$\frac{d}{dt}\delta Q_j^c = \Delta I_{\mathrm{out},j-1}^c - \Delta I_{\mathrm{out},j}^c, \tag{6}$$

$$\frac{d}{dt}\delta Q_j^n = \Delta I_{\mathrm{out},j-1}^n - \Delta I_{\mathrm{out},j}^n - \frac{2}{R_\perp}\delta V_j^n - \delta I_j^\perp. \tag{7}$$

The Fourier transformations

$$X_j(t) = \sum_{k=0}^{N-1}\int \frac{d\omega}{2\pi} e^{i\frac{2\pi k}{N}j - i\omega t} X_k(\omega), \tag{8}$$

$$X_k(\omega) = \frac{1}{N}\sum_{j=0}^{N-1}\int dt\, e^{-i\frac{2\pi k}{N}j + i\omega t} X_j(t) \tag{9}$$

in time and position allow us to formally solve the equation of motions and express the collective mode contribution $\delta V^{c/n}$ as

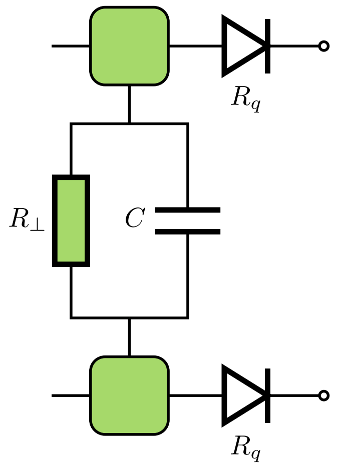


FIG. 3. A transverse cross section of the transmission line depicted in Fig. 2(a). The Ohmic reservoirs longitudinally emit a current according to Ohm's law with the resistance R_q plus thermal fluctuations given by Eq. (5). The upper and lower modes interact via a transverse resistor $R_\perp$ according to Eq. (4). The capacitor represents the self-capacitance C of the Ohmic contacts.

a function of the Langevin sources $\delta I^{c/n}$ and $\delta I^{\perp}$, which have a known correlation function. Furthermore, we can read off the spectra of the charged mode

$$-i\omega_k^c = (R_q C)^{-1}(e^{-2\pi ik/N} - 1) \tag{10}$$

and neutral mode

$$-i\omega_k^n = (R_q C)^{-1}(e^{-2\pi ik/N} - 1 - \Gamma), \tag{11}$$

which contain highly nonlinear terms responsible for dissipation and dispersion of the modes. The neutral mode contains furthermore an additional imaginary part governed by the strength of the perpendicular resistor $\Gamma = 2R_q/R_\perp$.

A. Computation of the heat flux

The advantage of the TL formulation is that we can compute the heat flux locally by considering a cross section between nodes. Each node is connected to perfect chiral channels, a Hamiltonian system, for which a continuity equation for the energy density $\partial_t \hat{h} + \partial_x \hat{J} = 0$ can be derived. Starting from the Hamiltonian density of free chiral bosons $\hat{h} = \frac{\hbar v_F}{4\pi}[\partial_x \phi(x,t)]^2$, applying the equation of motion $\partial_t \phi(x,t) + v_F \partial_x \phi(x,t) = 0$ and using the definition for the bosonic charge density $\hat{\rho}(x,t) = \frac{e}{2\pi}\partial_x \phi(x,t)$ and bosonic current $\hat{j}(x,t) = -\frac{e}{2\pi}\partial_t \phi(x,t)$. We arrive at the following expression for the average heat flux [36]:

$$J = \langle \hat{J} \rangle = \frac{\pi \hbar}{e^2}\langle \hat{j}(x,t)^2 \rangle. \tag{12}$$

By means of Fourier transformation we obtain the current-current correlation function $\langle \Delta I_{\text{out},k}^{c/n}(\omega) \Delta I_{\text{out},q}^{c/n}(\omega')\rangle$ from the equation of motion, Eqs. (6) and (7). Using the inverse Fourier transform of Eq. (12) we can directly substitute the correlation function and obtain the heat carried by the charged and neutral mode. The details of how to solve the equations of motion and how to compute the resulting integrals can be found in the Supplemental Material [37].

B. Noise power from FDT

Equation (12) allows us to express the heat flux through the correlation function of the Langevin sources appearing in Eqs. (6) and (7). Due to locality in space and time of the Langevin sources the following simplification arises:

$$\left\langle \delta I_k^x(\omega) \delta I_q^{x'}(\omega')\right\rangle = 2\pi\delta(\omega + \omega')\delta_{x,x'}\frac{\delta_{k,-q}}{N}S_x(\omega), \tag{13}$$

where $x, x' \in \{c, n, \perp\}$ labels the sources belonging to the charged, neutral, or perpendicular source. Additionally, we have a Kronecker δ function for the momenta and a Dirac δ function for the frequencies with their proper normalization. The noise spectral density S_x is evaluated assuming a local thermal equilibrium in the channel, thus the fluctuation dissipation theorem (FDT) applies,

$$S_x(\omega) = G_x\left[\frac{\hbar\omega}{1 - e^{-\beta\hbar\omega}} - S_{\text{vac}}(\omega)\right], \tag{14}$$

where the vacuum noise $S_{\text{vac}}(\omega) = \hbar\omega\theta(\omega)$ is subtracted, with the inverse temperature $\beta = (k_B T)^{-1}$, $G_{c/n} = 1/(2R_q)$, and $G_\perp = \Gamma/R_q$. Note the difference of factor 2 between the longitudinal and transverse resistor, which is due to the chirality of the longitudinal resistors [30]. With these ingredients the average heat flux can be computed. Note that for equilibrium noise, one finds the aforementioned heat flux quantum

$$J = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi}\left[\frac{\hbar\omega}{1 - e^{-\beta\hbar\omega}} - \hbar\omega\theta(\omega)\right] = \frac{\pi}{12\hbar\beta^2} = J_q. \tag{15}$$

III. RESULTS

The heat flux through the upper and lower arm can be expressed as the contribution carried by the charged mode and the neutral mode

$$\frac{J}{R_q} = \frac{1}{2}\sum_{\alpha=1,2}\left\langle\left(\Delta I_{\text{out},j}^{\alpha}\right)^2\right\rangle = \sum_{\alpha'=c,n}\left\langle\left(\Delta I_{\text{out},j}^{\alpha}\right)^2\right\rangle. \tag{16}$$

We will compute the charged J^c and neutral mode J^n contributions separately. Note that the charged mode contribution can be computed simply by taking the limit $J^c = \lim_{\Gamma\to 0} J^n$. We proceed with the following steps: (1) take the limit of infinitely many nodes $N \to \infty$, which allows us to replace the discrete k summation by a k integration, but keep the nodes separated by the distance ξ. (2) Compute the k integral by mapping it onto the unit circle $z \to e^{ik\xi}$ and use the residue theorem with the poles enclosed by the unit circle contour. (3) Compute the ω integral.

A. Heat flux of the charged and neutral mode

In this section we give the results for the fluxes computed as described above. For the charged mode we find

$$J^c = \int_{-\infty}^{\infty}\int_{-\pi/\xi}^{\pi/\xi}\frac{d\omega\, dk}{8\pi^2}\frac{\xi\omega^2 S_0(\omega)}{[\omega - \omega^c(k)][\omega + \omega^c(-k)]} = J_q, \tag{17}$$

where the dispersion relation of the collective mode is given by Eq. (10), and we abbreviate the noise power with $S_0(\omega) = \hbar\omega(1 - e^{-\beta\hbar\omega})^{-1} - S_{\text{vac}}(\omega)$. The charged mode carries a full flux quantum as expected. For the neutral mode we surprisingly find

$$J^n = \int \frac{d\omega\, dk}{4\pi^2} \frac{\xi\left(R_q^2 C^2 \omega^2 + \Gamma + \frac{\Gamma^2}{2}\right) S_0(\omega)}{R_q^2 C^2 [\omega - \omega^n(k)][\omega + \omega^n(-k)]} = J_q, \tag{18}$$

with the limits of integration being the same as the ones for the charged mode and the spectrum of the neutral modes given by Eq. (11). The heat carried by the neutral mode is given by a full flux quantum and completely independent of $R_\perp$. This can be seen after computing the momentum integral, which makes it equivalent to Eq. (15). See the Supplemental Material [37] for details on the calculation. This is unexpected since the dispersion relation of the neutral mode is subject to arbitrary strong dissipation, e.g., in the strongly damped limit $\Gamma \to \infty$. This is part of our proof, showing that heat is universally quantized even in the strongly overdamped regime.

B. The sum rule

The second part of our proof can be understood as a generalized sum rule [32] originating from each individual node of the transmission line. This is the direct manifestation of the unitarity of the scattering matrix at each node. This is one strength compared to the Caldeira-Leggett model our formalism offers. One can straightforwardly solve (3) to (5) for the outgoing current $\Delta I^\alpha_{\text{out},j}$ as a function of the incoming current $\Delta I^\alpha_{\text{in},j}$ and the Langevin sources. This gives for the current fluctuations of the $\alpha \in \{1, 2\}$ channel

$$\Delta I^\alpha_{\text{out},j} = \sum_{\beta=1,2} \mathcal{T}^\alpha_{\text{in},\beta} \Delta I^\beta_{\text{in},j} + \mathcal{T}^\alpha_{\parallel,\beta} \delta I^\beta_j + \mathcal{T}^\alpha_\perp \delta I^\perp_j. \tag{19}$$

The corresponding coefficients are given by

$$\mathcal{T}^1_{\text{in},1} = \mathcal{T}^2_{\text{in},2} = \tfrac{1}{2}[A(\omega) + B(\omega)],$$
$$\mathcal{T}^1_{\text{in},2} = \mathcal{T}^2_{\text{in},1} = -\mathcal{T}^1_{\parallel,2} = -\mathcal{T}^2_{\parallel,1} = \tfrac{1}{2}[A(\omega) - B(\omega)],$$
$$\mathcal{T}^1_{\parallel,1} = \mathcal{T}^2_{\parallel,2} = 1 - \mathcal{T}^1_{\text{in},1},$$
$$\mathcal{T}^1_\perp = -\mathcal{T}^2_\perp = -B(\omega),$$

with $A(\omega) = (1 - iR_q C\omega)^{-1}$ and $B(\omega) = R_\perp(R_\perp + 2R_q - iR_\perp R_q C\omega)^{-1}$. If one computes the noise power in the upper/lower channel one finds

$$\left\langle \Delta I^\alpha_{\text{out},j}(\omega) \Delta I^\alpha_{\text{out},j}(-\omega)\right\rangle = \sum_{\beta=1,2} \left|\mathcal{T}^\alpha_{\text{in},\beta}\right|^2 \left\langle \Delta I^\beta_{\text{in},j}(\omega) \Delta I^\beta_{\text{in},j}(-\omega)\right\rangle + \left|\mathcal{T}^\alpha_{\parallel,\beta}\right|^2 S_{\parallel,\beta} + |\mathcal{T}^\alpha_\perp|^2 S_\perp. \tag{20}$$

The sum rule states, that if the current incident to a node is equilibrium, i.e., it has a noise power given by Eq. (14) with $G_{\text{in}} = \frac{1}{R_q}$ we find that

$$\sum_{\beta=1,2} \left|\mathcal{T}^\alpha_{\text{in},\beta}\right|^2 G_{\text{in}} + \left|\mathcal{T}^\alpha_{\parallel,\beta}\right|^2 G_\parallel + |\mathcal{T}^\alpha_\perp|^2 G_\perp = \frac{1}{R_q}, \tag{21}$$

with $G_{\text{in}} = G_\parallel = \frac{1}{R_q}$ and $G_\perp = \frac{2}{R_\perp}$. This immediately explains the earlier findings since every outgoing current from a node will be in equilibrium if the incident currents are in equilibrium.

C. Exact cancellation of Joule heating and backaction in equilibrium

The structure of the outgoing current is always of the form $\Delta I_{\text{out}} = R_q^{-1} \delta V(t) + \delta I(t)$, hence the total current-current correlation function in Eq. (16) consists of three parts. The first contribution contains the autocorrelation function of the collective mode $\mathcal{C}_{VV}$, the second the cross-correlation function between the collective mode contribution and the Langevin sources $\mathcal{C}_{IV}$, and finally the autocorrelation function of the Langevin sources $\mathcal{C}_{II}$. We find the remarkable result

$$\frac{1}{R_q} \int \frac{dk}{2\pi} \mathcal{C}_{VV} = -\int \frac{dk}{2\pi} (\mathcal{C}_{IV} + \mathcal{C}_{VI}),$$
$$\mathcal{C}_{XY} = \langle \delta X^{\alpha'}(k,\omega) \delta Y^{\alpha'}(-k,-\omega)\rangle,$$

which holds for the charged and neutral mode, $\alpha' = c, n$ and $X, Y \in \{V, I\}$. The exact cancellation after taking the momentum integral implies that the Joule heating on the nodes, given by $\mathcal{C}_{VV}$, is fully compensated by the backaction of the sources on the nodes, $(\mathcal{C}_{IV} + \mathcal{C}_{VI})$. The correlation function of the source $\mathcal{C}_{II}$ is the only remaining part of Eq. (16), which explains why there is always a flux quantum for the charged and neutral modes. The sources are thus not just auxiliary but real physical entities of the system. This principle of exact cancellation due to the backaction of thermal noise was first mentioned in [38] by Nyquist who stated that two resistors in thermal equilibrium connected by ideal wires excite thermal fluctuations, which in principle leads to a heat flux from one resistor to another, but is compensated by the fluctuations of the second resistor. This also holds true in every frequency window, since one would be able to extract energy by placing a frequency filter in the system. The same is true here. The cancellation holds before the integration over frequencies is done. The simple reason to explain the exact cancellation is the second law of thermodynamics, which forbids the extraction of heat, i.e., dissipation of heat in the present case, if the system is in thermal equilibrium. This is a key difference with [11], where the energy flux was defined as potential energy flux proportional to $\mathcal{C}_{VV}$, a contribution which is now canceled by the backaction effect of the sources. The implications of this will be discussed in the next section.

IV. LOW-ENERGY THEORY OF AN EDGE WITH INTRINSIC DISSIPATION

In this section we want to address the differences between the hydrodynamic model [11] and the transmission line approach presented in this paper and why the former yields different results, despite correctly applying fluctuation-dissipation relations, a standard procedure, to obtain equilibrium correlation functions.

The general idea is now to obtain a low-energy theory from the discrete transmission line model, to compare it to the one obtained in the previous paper and to comment on its universality. It is clear that in the low-energy limit, e.g., for small temperatures, not all possible modes of the nonlinear spectrum of the charged and neutral mode will be excited. This justifies linearizing the spectrum, if possible, to the point where the heat flux integrals Eqs. (17) and (18) converge and yield the same heat flux quantum. This is equivalent to finding the low-energy field theory, which correctly describes chiral heat transport in the QH edge in the presence of dissipation. We will discuss the charged and neutral mode separately.

A. Low-energy field theory for the charged mode

The dispersion relation of the charged mode is given by Eq. (10). It contains a "hidden" type of dispersion coming from the discreteness of the transmission line. This can be seen by expanding the dispersion relation in small ξ:

$$-i\omega^c(k) \approx -ikv_q - \frac{k^2\xi v_q}{2} + i\frac{k^3\xi^2 v_q}{6}, \quad (22)$$

where we rescaled to intensive quantities by introducing the velocity $v_q = \xi/R_qC$. The second term plays the role of a dissipative term coming from the retardation of the collective mode and vanishes in the true continuum limit $\xi \to 0$. Notice that this crossover is nontrivial; physically this means that the decay length of the collective mode becomes much larger than the distance between the nodes, so no dissipation is happening and thus no heating and backaction effect, as discussed in the previous section. In this limit the system is susceptible to its boundary conditions, and the potential energy flux described in [11] correctly predicts a flux quantum. If one keeps dissipation in the system, the definition of the heat flux in terms of the current current correlation function correctly gives a heat flux quantum, if one restricts the energies to be small; see the Supplemental Material [37] for details on the calculation. This allows us to write the equation of motion in a coarse grained fashion, i.e., depending on a continuous variable, rather than a discrete node index. This can be understood as the minimal expansion of the discrete difference operator in Kirchhoff's law in order to capture the feature of dissipation correctly at low energies plus a Langevin equation. In real space the equation of motion for the coarse-grained collective mode $V^c(x,t)$ and source $\delta I^c(x,t)$ read

$$\Delta I^c_{\text{out}}(x,t) = \frac{1}{R_q}\delta V^c(x,t) + \delta I^c(x,t), \quad (23)$$

$$\partial_t \delta V^c(x,t) + v_q R_q \mathcal{D}\Delta I^c_{\text{out}}(x,t) = 0, \quad (24)$$

where $\mathcal{D} = \partial_x - \frac{\xi}{2}\partial_x^2 + \frac{\xi^2}{6}\partial_x^3$. With these modified hydrodynamic equations one can proceed as follows: (1) Solve Eqs. (23) and (24) for the collective mode contribution δV^c. (2) The heat flux carried by the chiral system is given by $J = R_q\langle[\Delta I^c_{\text{out}}(x,t)]^2\rangle$. (3) Take into account only low energies and momenta, i.e., expand the cubic and quartic terms in the momenta around $k \to \omega/v_q$, which will result in a heat flux quantum, shown explicitly in the Supplemental Material [37]. This completes the low-energy hydrodynamic theory for chiral dissipative systems.

B. Low-energy field theory for the neutral mode

The dispersion relation of the neutral mode is given by Eq. (11) and contains two types of dissipation. The dissipation due to the discreteness of the transmission line, similarly to the charged mode and the dissipation introduced by the transverse resistor. Focusing on the transverse dissipation only, we keep the leading order terms

$$-i\omega^n(k) \approx -iv_qk - \frac{v_q\Gamma}{\xi}. \quad (25)$$

As we show explicitly in the Supplemental Material [37], if $\Gamma \approx k\xi \ll 1$ we find that the heat integral gives a flux quantum. If the above condition is not met, the integral does not come from low momenta, but from the bandwidth of the system, and one needs to integrate over all nonlinearities of the spectrum. Physically this means that the neutral collective mode decays over distances shorter than the nodes for stronger dissipations and is essentially pinned. There is no low-energy description of the strongly overdamped neutral mode. The thermal fluctuations, however, still carry a flux quantum. We are able to write a coarse-grained equation of motion for the case that the low-energy theory exists [39]:

$$\Delta I^n_{\text{out}}(x,t) = \frac{1}{R_q}\delta V^n(x,t) + \delta I^n(x,t), \quad (26)$$

$$\Delta I^{\perp}(x,t) = \frac{2}{R_{\perp}}\delta V^n(x,t) + \delta I^{\perp}(x,t), \quad (27)$$

$$\partial_t \delta V^n(x,t) + v_q R_q \partial_x \Delta I^n_{\text{out}}(x,t) - \Delta I^{\perp}(x,t) = 0. \quad (28)$$

We are now able to discuss the role of the artificial cutoff ξ_a introduced in [11] for the momentum integration, roughly given by the transverse size of the compressible strip. This was the restriction of the low-energy approach to the edge. We can see that if the integral comes from small energies and momenta, replacing the artificial cutoff by the true cutoff can change only the numerical prefactor, but not the parametrical suppression found in the heat flux of the neutral mode. We want to emphasize once more that the definition of heat flux as potential energy flux was the cause for this result, but the mistake is rather conceptual. It can be seen that the Langevin sources introduced in the present formalism are not just a mathematical trick, but a real part of the system leading to a cancellation of the autocorrelator of the collective mode. The Joule heating is canceled by the backaction of the sources on the collective mode in thermal equilibrium. We thus conclude that a Langevin equation is the more complete approach to dissipative systems, and we naturally give a more correct definition of energy flux in these systems, which is unobtainable starting from the hydrodynamic point of view.

V. THE ROLE OF ALEINER-GLAZMAN MODES

So far we have been able to write an effective model for a system with either two copropagating modes, i.e., filling factor $\nu = 2$ with a transverse spin resistance or, equivalently after rescaling $R_q \to R_q/2$, the hydrodynamic model consisting of one charged and one half-filled Aleiner-Glazman neutral mode, stemming from approximating the density profile obtained by the electrostatic analysis of the edge [25].

However, writing an effective model for dissipation in these systems, we see that it is not permitted to write a low-energy theory for the neutral mode at all in the presence of dissipation. We are not able to properly construct the continuum limit of the TL based on the edge reconstruction of the AG modes. Furthermore, the dissipation in the charged mode of the system depends on length scales,

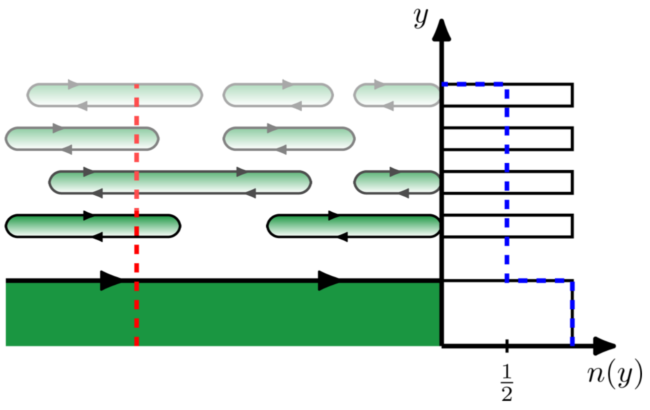


FIG. 4. An effective description of edge defects as quantum Hall puddles and the corresponding density profile in black, cutting through the red dashed line. The blue dashed line shows the "average" hydrodynamic density, which suggests the presence of a half-filled neutral mode. The model presented in this paper suggests the presence of the disorder length scale in the hydrodynamic equations of motions. This implies that viewing the nonhomogenous density profile at the edge due to quantum puddles as an on average "half-filled" neutral mode is wrong, or, if it exists, the neutral mode will decay on length scales comparable to the disorder length scale.

e.g., the correlation length of disorder, which are not captured in the low-energy approach. We thus conclude that the electrostatic/hydrodynamic picture is not applicable to describe dissipative chiral transport in QH systems and furthermore neglects the experimental truth in the QH edge. The picture of QH puddles (see Fig. 4) is not compatible with the prediction of AG modes. The half-filling is an artifact of coarse graining many puddles which have co- and counterpropagating modes. This is different from a half-filled copropagating mode.

VI. CONCLUSION

We have shown that the heat flux quantization in chiral systems in thermal equilibrium is robust and reflects fundamental thermodynamic laws. We analyzed chiral quantum systems with the "transmission line approach," an extension of the Langevin formalism in combination with scattering states developed in earlier works [30,32]. The formalism allows to take into account dissipation on an effective level. Similarly to the Caldeira-Leggett model, we attach a bath consisting of many oscillators to a chiral system. In contrast to the Caldeira-Leggett model, we consider the baths to be an open system, which allows for the analytical treatment of equilibrium and nonequilibrium situations. We were not only able to address the question of the apparent reduction of heat flux carried by the strongly overdamped neutral mode in [11], but have proven the quantization of heat flux for chiral systems in the presence of different types of dissipation. In a next step we considered chiral systems in the presence of additional diffusive modes, making the system essentially nonchiral.

For all of the systems above we could prove that heat flux is quantized if the system is in thermal equilibrium and could relate this to the local unitarity condition of the scattering matrix and the backaction principle of the Langevin sources on the collective mode. We have shown the correct definition of heat flux for chiral dissipative systems and discussed the crossover to the potential energy flux used in [11]. This question of the definition of energy flux reveals the nature of the Langevin sources as true physical entities which are able to give back heat to the system rather than just being a mathematical aid. We continued formulating the low-energy theories which can be deduced from the transmission line model and capture the universal aspects of the edge. We discussed the minimal expansion of Kirchhoff's law, which completes the low-energy theory.

The results of this paper are not able to answer the missing heat paradox [1], but expand a field containing many interesting ideas. An explanation for the experiment could be a potential discrepancy between the actual and measured heat flux. To proceed further one needs to look into the physics of probing the edge states. This involves the construction of a electronic operator which is not just the bosonized vertex operator but goes beyond the bosonization formalism due to correlations through the collective mode if a tunnel probe is connected. The results of this paper also show that the strongly overdamped AG modes cannot be described in a low-energy theory and if they are present cannot explain the experiment. So far their experimental detection remains an open question. Finally, the formalism developed in this paper can be easily applied to different edge reconstructions and to fractional quantum Hall systems.

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