Effective field theories for gapless phases with fractons via a coset construction

Yuji Hirono[1,2,3,*] and Yong-Hui Qi[1,†]
[1]*Asia Pacific Center for Theoretical Physics, Pohang, Gyeongbuk 37673, Korea*
[2]*Department of Physics, Pohang University of Science and Technology, Pohang, Gyeongbuk 37673, Korea*
[3]*RIKEN iTHEMS, RIKEN, Wako 351-0198, Japan*

(Received 3 November 2021; revised 14 March 2022; accepted 27 April 2022; published 10 May 2022)

Fractons are particles with restricted mobility. We give a symmetry-based derivation of effective field theories of gapless phases with fractonic topological defects, such as solids and supersolids, using a coset construction. The resulting theory is identified as the Cosserat elasticity theory, which reproduces the conventional symmetric elasticity theory at low energies. The construction can be viewed as a dynamical realization of the inverse Higgs mechanism. We incorporate topological defects such as dislocations and disclinations, which are nontrivially related by the Bianchi identities of defect gauge fields. The origin of the fractonic nature of defects in those systems can be traced back to the semidirect product structure of translational and rotational groups. The construction is immediately extendable to higher dimensions and systems with broken translational symmetries, such as solids, supersolids, and vortex crystals. We identify Wess-Zumino terms in supersolids, which induce quasiparticle scatterings on topological defects.

DOI: 10.1103/PhysRevB.105.205109

I. INTRODUCTION

Fracton phases are a new class of quantum phases that host excitations with mobility restrictions [1,2]. Those excitations cannot move at all, or their motions are restricted in subdimensional spaces. Such fracton phases were first discussed in exactly solvable lattice models [3–7]. It has been realized that symmetric tensor gauge theories [8–11] can encode mobility restrictions through the conservation laws of multipole moments [12–14]. The elasticity theory of two-dimensional crystals was shown to be dual to a symmetric tensor gauge theory, and disclinations in solids are fractonic [12]. Similar dualities can also be formulated for other systems such as supersolids [15] and vortex crystals [16].

The relation between the immobility of a particle and the conservation of multipole moments can be seen as follows. Suppose that there is a particle number current $j_\mu(x)$, which is conserved, $\partial^\mu j_\mu = 0$. If we define the dipole current by $(J^a)_\mu(x) = x^a j_\mu$, where a is a spatial index, its divergence reads

$$\partial^\mu (J^a)_\mu = j^a. \tag{1}$$

This relation means that the flow of a current should be accompanied by the creation/destruction of dipoles. Therefore, if the dipole current is conserved, $\partial^\mu (J^a)_\mu = 0$, the particle number current vanishes, $j^a = 0$, which means that a particle is immobile. In the case of a solid in $2+1$ dimensions, disclinations and dislocations correspond to j_μ and $(J^a)_\mu$, respectively. When the dipole excitations (dislocations) are gapped, the conservation of dipoles is energetically enforced, and disclinations are immobile. In this way, the relation (1) plays a key role in realizing fractons.

*yuji.hirono@gmail.com
†yonghui.qi@apctp.org

In this paper, we discuss the construction of the effective field theories of gapless phases with fractonic topological defects, such as solids and supersolids, in which the translational symmetry is spontaneously broken. We employ a coset construction [17,18], which is a systematic method of writing down the effective Lagrangian of low-energy theory associated with a spontaneous symmetry breaking (SSB). Topological defects appear as a result of an SSB, and they can be incorporated in the effective theory. In this approach, the relations between the defect currents (1) can be traced back to the underlying structure of the symmetry group and its breaking pattern. The geometric origin of the fractonic feature becomes transparent, and Eq. (1) reflects the fact that the rotational symmetry acts nontrivially on the translational symmetry. Its origin is the same as the Bianchi identities of the torsion and curvature of the Riemann-Cartan spacetime. The coset construction clarifies this connection.

The rest of the paper is organized as follows. In Sec. II, we give the derivation of the effective theories for solids and supersolids. In Sec. III, we describe the properties of dual gauge theories. In Sec. IV, we discuss the scattering processes of quasiparticles off topological defects induced by Wess-Zumino terms. Section V is devoted to the summary.

II. COSET CONSTRUCTION AND THE COSSERAT ELASTICITY

A coset construction is a method for constructing the effective theory of Nambu-Goldstone (NG) modes associated with an SSB. The obtained effective theories are universal, in the sense that their forms are dictated by the symmetry-breaking patterns, and microscopic details are encoded in the values of phenomenological parameters. Although it was invented for the breaking of internal symmetries, it allows for a number of generalizations, such as spacetime symmetries [19–23] and higher-form symmetries [24,25].

We here perform a coset construction for solids and supersolids in $D = d + 1$ spacetime dimensions.[1] We consider nonrelativistic systems and the underlying spacetime symmetry is the Galilean group. This symmetry is spontaneously broken because of the formation of solids or supersolids. In the case of supersolids [27], we have an additional $U(1)$ symmetry, which is also spontaneously broken. In a coset construction, we first parametrize the coset space, and for each broken generator we have an NG field, which is a degree of freedom of the low-energy theory. A subtlety for the broken spacetime symmetry is that, unlike the case of internal symmetries, there is no one-to-one correspondence between a broken symmetry generator and a gapless NG mode [20]. When a broken symmetry generator does not commute with the translation, there can be a nontrivial coupling of NG modes through the covariant derivative and that results in the reduction of gapless modes compared to the number of broken generators. For the details of the coset construction for broken spacetime symmetries, see Ref. [21].

Below, d denotes the spatial dimension, and $D = d + 1$ is the spacetime dimension. We use the mostly-plus convention for the Minkowski metric, $\eta_{\mu\nu} = (-1, +1, \dots, +1)$. The symmetrization/antisymmetrization of indices are denoted by brackets, $(\cdots)$ and $[\cdots]$, respectively. For example, $A_{(ab)} \equiv (A_{ab} + A_{ba})/2$ and $A_{[ab]} \equiv (A_{ab} - A_{ba})/2$.

A. Symmetry and its breaking pattern

A Galilean group is the spacetime symmetry group of nonrelativistic systems. The generators of the Galilean group in d spatial dimensions, Gal(d), satisfy the Galilean algebra $\mathfrak{Gal}(d)$, whose nonvanishing commutation relations are

$$\begin{aligned} &[J_{ab}, J_{cd}] = -4i\eta_{[a[c}J_{d]b]}, \quad [J_{ab}, P_c] = 2i\eta_{c[a}P_{b]}, \\ &[J_{ab}, B_c] = 2i\eta_{c[a}B_{b]}, \quad [B_a, H] = iP_a, \end{aligned} \tag{2}$$

where J_{ab}, P^a, and B^a are the generators of rotation, spatial translation, and Galilean boost, respectively, and H corresponds to the non-mass energy. The Bargmann algebra $\mathfrak{B}(d)$ is a central extension of the Galilean algebra $\mathfrak{Gal}(d)$ with the following relation,

$$[B_a, P_b] = -i\eta_{ab}Q_0, \tag{3}$$

where Q_0 represents the total mass, which can be written as $Q_0 = -mQ$, where m is the mass of a particle, and Q is the particle number. Since Q is a $U(1)$ charge and commutes with every other generator, this is a central extension. The Bargmann algebra can be obtained as the nonrelativistic limit of the Poincaré algebra [28]. For $d \geqslant 3$, this is the only central extension. In the case of two spatial dimensions, the Galilean group, Gal(2), admits another central extension. In addition to Eq. (3), we can have the following nonvanishing commutation relations,

$$[B_a, B_b] = -i\epsilon_{ab}\kappa\, Q_0. \tag{4}$$

In addition to m, there is another parameter κ, which can be interpreted as the spin (per mass) of a particle [29].

[1]See Ref. [26] for a related construction.

We consider the following symmetry-breaking pattern to realize solid and supersolid phases,

$$\text{Gal}(d) \times U(1) \to (\mathbb{Z}^d \rtimes \mathcal{G}) \times \mathbb{R}, \tag{5}$$

where $\rtimes$ is the semidirect product and $\mathcal{G} \subset SO(d)$ is a discrete subgroup of $SO(d)$, $\mathbb{R}$ corresponds to the time translation, and $\mathbb{Z}^d$ represents discrete spatial translations. The low-energy behavior of solids can be described by a d-dimensional field $\phi^a(t, \boldsymbol{x})$. The field $\phi^a(t, \boldsymbol{x})$ can be regarded as the comoving coordinate of the material. The actions of internal rotational and translational symmetries are given by

$$\begin{aligned} &\phi^a \mapsto \xi^a_b\phi^b, \quad \xi^a_b \in SO(d), \\ &\phi^a \mapsto \phi^a + c^a, \quad c^a \in \mathbb{R}^d. \end{aligned} \tag{6}$$

For supersolids, to describe the $U(1)$-symmetry breaking, we introduce an additional scalar field $\phi^0(t, \boldsymbol{x})$, which is shifted by the $U(1)$ symmetry,

$$\phi^0 \mapsto \phi^0 + c^0, \quad c^0 \in \mathbb{R}/2\pi\mathbb{Z} \simeq U(1). \tag{7}$$

The action of the $U(1)$ symmetry and the translations can be combined into a four-vector notation,

$$\phi^A \mapsto \phi^A + c^A, \tag{8}$$

where c^A are constants with index $A = (0, a)$. Correspondingly, we introduce a four-vector notation $Q_\mu = (Q_0, Q_a)$. When the supersolid (or solid) is in the ground state, we can choose the comoving coordinates ϕ^A to coincide with the spacetime coordinates $x^A = (x^0, x^a)$,

$$\langle \phi^a \rangle = x^a, \quad \langle \phi^0 \rangle = x^0. \tag{9}$$

We can parametrize the fields ϕ^a and ϕ^0 as

$$\phi^a = x^a + u^a, \quad \phi^0 = x^0 + u^0. \tag{10}$$

Here, the field u^a represents the deviation of the material coordinate from its equilibrium position, and is called the displacement field in the elasticity theory. The field u^0 can be identified as the fluctuating part of the $U(1)$ phase of the condensate (divided by the mass m of a particle).

Let us summarize the unbroken and broken generators for supersolids:

$$\begin{aligned} \text{unbroken}: &\bar{P}_\mu = P_\mu + Q_\mu, && \text{spacetime and internal translation,} \\ &\bar{J}_{ab} = J_{ab} + L_{ab}, && \text{spatial and internal rotation;} \\ \text{broken}: &Q_\mu = (Q_0, Q_a), && \text{internal } U(1) \times \mathbb{R}^d, \\ &L_{ab}, && \text{internal rotation,} \\ &B_a, && \text{Galilean boost.} \end{aligned} \tag{11}$$

Note that the ground state is invariant under the combinations of spacetime and internal translations/rotations [21]. The combined generators are written, for translation, as $\bar{P}_\mu = (-H - mQ, P_a + Q_a)$.

B. Covariant derivatives

The low-energy degrees of freedom are the coordinate of the coset space associated with the symmetry breaking (5).

We call those fields as Nambu-Goldstone (NG) fields. We can parametrize the coset space as

$$\Omega = e^{ix^\mu \bar{P}_\mu} e^{iv^a(x)B_a} e^{iu^\mu(x)Q_\mu} e^{i\theta^{ab}(x)L_{ab}}, \tag{12}$$

where $v^a(x)$, $u^\mu(x)$, and $\theta^{ab}(x)$ are the NG fields of Galilean boosts, $U(1)$ and internal translations, and internal rotations, respectively. The building blocks of the low-energy theory can be obtained from the Maurer-Cartan (MC) form,

$$\omega \equiv -i\Omega^{-1} d\Omega. \tag{13}$$

The MC form can be expanded by generators as

$$\omega = \omega^A_{\bar{P}} \bar{P}_A + \omega^{ab}_J \bar{J}_{ab} + \omega^\mu_Q Q_\mu + \omega^{ab}_L L_{ab} + \omega^a_B B_a. \tag{14}$$

The coefficients of the unbroken combined translation and rotational generator, $\bar{P}_A$, $\bar{J}_{ab}$, give the vielbein and spin connection,

$$\omega^A_{\bar{P}} = e^A_\mu dx^\mu, \quad \omega^{ab}_{\bar{J}} = \omega^{ab}_\mu dx^\mu. \tag{15}$$

The 1-forms proportional to the broken generators give the covariant derivatives D_A of NG fields,

$$\begin{aligned} \omega^\nu_Q &= e^A_\mu D_A u^\nu dx^\mu, \\ \omega^{ab}_L &= e^A_\mu D_A \theta^{ab} dx^\mu, \\ \omega^a_B &= e^A_\mu D_A v^a dx^\mu. \end{aligned} \tag{16}$$

By using the commutation relations, we have

$$\begin{aligned} e^{-iv^c B_c} P_a e^{iv^c B_c} &= P_a - v_a Q_0, \\ e^{-iv^c B_c} P_0 e^{iv^c B_c} &= P_0 - v^a P_a + \frac{1}{2} v^2 Q_0, \\ e^{-i\theta^a L_a} \bar{P}_\mu e^{i\theta^a L_a} &= \left(\bar{P}_0, P_a + \xi_a{}^b Q_b\right), \end{aligned} \tag{17}$$

where $\xi_a{}^b \in SO(d)$ is the rotational matrix associated with θ, $\xi_a{}^b \equiv (e^{i\theta\cdot L})_a^{\;b}$. Using those relations, we can identify the components of MC 1-forms as

$$\begin{aligned} \omega^0_{\bar{P}} &= dx^0, \\ \omega^a_{\bar{P}} &= dx^a - v^a dx^0, \\ \omega^0_Q &= du^0 - v_a dx^a + \frac{1}{2} v^2 dx^0 - \frac{\kappa}{2} v^a dv^b \epsilon_{ab}, \\ \omega^a_Q &= d\phi^b \xi_b^{\;a} - dx^a + v^a dx^0, \\ \omega^a_B &= dv^a. \end{aligned} \tag{18}$$

The term proportional to κ in ω^0_Q can exist only in $(2+1)$-dimensional spacetime. The part proportional to the rotational generator L_{ab} is written as

$$\omega^{ab}_L L_{ab} = -ie^{-i\theta^{ab} L_{ab}} d e^{i\theta^{ab} L_{ab}}. \tag{19}$$

From $\omega^A_{\bar{P}}$, we can read off the vielbein as

$$e^A = \begin{pmatrix} dx^0 \\ dx^a - v^a dx^0 \end{pmatrix}. \tag{20}$$

In terms of components, the vielbein and its inverse are given by

$$e^A_\mu = \delta^A_\mu - \delta^0_\mu v^A, \quad e^\mu_A \equiv \left(e^A_\mu\right)^{-1} = \delta^\mu_A + \delta^0_A v^\mu, \tag{21}$$

with $v^\mu = (0, v^a)$. They satisfy $e^\mu_A e^B_\mu = \delta^B_A$. We can read off the components of ω^0_Q, ω^a_Q, and ω^a_B as

$$\begin{aligned} \left(\omega^0_Q\right)_\mu &= \partial_\mu u^0 - v_a \delta^a_\mu + \frac{1}{2} v^2 \delta^0_\mu - \frac{\kappa}{2} \epsilon_{ab} v^a \partial_\mu v^b, \\ \left(\omega^a_Q\right)_\mu &= \partial_\mu \phi^b \xi^a_b - \delta^a_\mu + \delta^0_\mu v^a, \\ \left(\omega^a_B\right)_\mu &= \partial_\mu v^a. \end{aligned} \tag{22}$$

Accordingly, the covariant derivatives of the NG fields are given by

$$\begin{aligned} D_A u^0 &= e^\mu_A \left(\omega^0_Q\right)_\mu = \partial_A u^0 - \delta^a_A v_a + \delta^0_A \frac{v^2}{2} - \frac{\kappa}{2} \epsilon_{ab} v^a \partial_A v^b \\ &\quad + \delta^0_A v^c \left(\partial_c u^0 - v_c - \frac{\kappa}{2} \epsilon_{ab} v^a \partial_c v^b\right), \\ D_A u^a &= e^\mu_A \left(\omega^a_Q\right)_\mu = \partial_A \phi^b \xi^a_b - \delta^a_A + \delta^0_A v^a \\ &\quad + \delta^0_A v^c \left(\partial_c \phi^b \xi^a_b - \delta^a_c\right), \\ D_A v^a &= e^\mu_A \left(\omega^a_B\right)_\mu = \partial_A v^a + \delta^0_A v^b \partial_b v^a. \end{aligned} \tag{23}$$

Let us list the temporal and spatial components of the covariant derivatives separately:

$$\begin{aligned} D_0 u^0 &= \partial_0 u^0 + v^a \partial_a u^0 - \frac{1}{2} v^2 - \frac{\kappa}{2} \epsilon_{ab} v^a \partial_0 v^b - \frac{\kappa}{2} \epsilon_{ab} v^a v^c \partial_c v^b \\ &\equiv \mathcal{D}_0 u^0 - \frac{1}{2} v^2 - \frac{\kappa}{2} \epsilon_{ab} v^a \mathcal{D}_0 v^b, \\ D_a u^0 &= \partial_a u^0 - v_a - \frac{\kappa}{2} \epsilon_{bc} v^b \partial_a v^c, \\ D_0 u^a &= (\partial_0 \phi^b + v^c \partial_c \phi^b) \xi_b^{\;a} \equiv \mathcal{D}_0 \phi^b \xi_b^{\;a}, \\ D_b u^a &= \partial_b \phi^c \xi_c^{\;a} - \delta^a_b, \\ D_0 v^a &= \partial_0 v^a + v^b \partial_b v^a \equiv \mathcal{D}_0 v^a, \\ D_b v^a &= \partial_b v^a, \end{aligned} \tag{24}$$

where we defined the convective time derivative, $\mathcal{D}_0$, by[2] $\mathcal{D}_0 \equiv \partial_0 + v^a \partial_a$. The covariant derivatives (24) are the building blocks of the effective Lagrangian.

[2]Under the action of a Galilean boost,

$$\begin{aligned} x^a &\mapsto x'^a = x^a + \beta^a t, \\ t &\mapsto t' = t, \\ v^a &\mapsto (v^a)' = v^a + \beta^a, \end{aligned}$$

the derivatives are transformed as $(\partial_0)' = \partial_0 - \beta^a \partial_a$ and $(\partial_a)' = \partial_a$. The vielbeins (20) are transformed covariantly under the Galilean transformation, $(dx^0)' = dx^0$, $(dx^a - v^a dx^0)' = dx^a - v^a dx^0$. The convective derivative $\mathcal{D}_0$ is Galilean-invariant, $(\mathcal{D}_0)' = (\partial_0)' + (v^a)'(\partial_a)' = \partial_0 - \beta^a \partial_a + (v^a + \beta^a)\partial_a = \mathcal{D}_0$. Note that the field u^0 is shifted under a Galilean boost (in the absence of κ) as $(u^0)' = u^0 + \boldsymbol{\beta} \cdot \boldsymbol{x} + \frac{1}{2}\beta^2 t$. We can explicitly check that the covariant derivative $D_0 u^0$ is indeed invariant under a Galilean boost:

$$\begin{aligned} \left[\partial_0 u^0 + \frac{1}{2}(\partial_a u^0)^2\right]' &= (\partial_0 - \boldsymbol{\beta} \cdot \nabla)\left(u^0 + \boldsymbol{\beta} \cdot \boldsymbol{x} + \frac{1}{2}\beta^2 t\right) \\ &\quad + \frac{1}{2}(\partial_a u^0 + \beta_a)^2 = \partial_0 u^0 + \frac{1}{2}(\partial_a u^0)^2. \end{aligned}$$

The field u^a is shifted under a Galilean boost as $(u^a)' = u^a - \beta^a t$ and $(\phi^a)' = \phi^a$.

C. Effective Lagrangian for supersolids and solids

For the broken spacetime symmetry, not every NG field leads to a physical gapless mode. Namely, there can be a redundancy in the parametrization of the modes and some of them can be expressed in other fields. Suppose that the commutator of the unbroken generator $\bar{P}_\mu$ with a broken generator X contains another broken generator X',

$$[\bar{P}_\mu, X] \sim X' + \cdots . \tag{25}$$

If we denote the NG fields for X and X' as π and π', we can impose the inverse Higgs constraint (IHC) of the form $D_\mu \pi' = 0$, to express π' via the derivative of π. However, whether one *should* impose the constraint depends on the microscopic details of the system [30]. This choice leads to the differences in the number of gapped physical modes.

Because of the algebra $[\bar{P}_a, B_b] = i\delta_{ab} Q_0$, we can express the boost NG field v^a in terms of u^0. By imposing the inverse Higgs constraint $D_a u^0 = 0$, the velocity field v^a is expressed as (in the absence[3] of κ)

$$v_a = \partial_a u^0. \tag{26}$$

Namely, we can identify v_a as the superfluid velocity. Since $[\bar{P}_a, L_{bc}] = -2i\eta_{a[b}Q_{c]}$, it is possible to impose the IHC $D_{[a}u^{b]} = 0$. This can be written as $\partial_{[b}u^{a]} - \theta^a_b = 0$ to the leading order in fields, and by this we can eliminate the antisymmetric part of $\partial_b u^a$, which results in the conventional elasticity theory [31–37] written in terms of the symmetric part, $\partial_{(b}u^{a)} = (\partial_b u^a + \partial_a u^b)/2$. However, whether one should impose the constraints to eliminate certain modes depends on the microscopic details of the system [30]. Although the nature of gapless modes does not depend on such a choice and hence is universal, it can lead to differences in the number of gapped modes.

Here, we *choose not to* impose $D_{[a}u^{b]} = 0$, and keep the rotational NG field in the Lagrangian. We will see that this leads to the Cosserat theory of elasticity. Substituting Eq. (26) to Eq. (24), the covariant derivatives are

$$\begin{aligned} D_0 u^0 &= \partial_0 u^0 + \frac{1}{2}(\partial_a u^0)^2, \\ D_0 u^a &= (\partial_0 \phi^b + \partial^c u^0 \partial_c \phi^b)\xi^a_b, \\ D_b u^a &= \partial_b \phi^c \xi^a_c - \delta^a_b, \\ D\theta^{ab} &= d\theta^{ab} + d\theta^a_c \theta^{cb} + O(\theta^3). \end{aligned} \tag{27}$$

In the absence of rotational NG fields, i.e., $\xi^a_b = \delta^a_b$, the first two coincide with the Galilean-invariant building blocks of the supersolid effective Lagrangian discussed in Ref. [38].

To the leading order in the numbers of fields and derivatives, the covariant derivatives are written as

$$D_0 u^0 \simeq \partial_0 u^0, \tag{28}$$

$$D_0 u^a \simeq \partial_0 u^a, \tag{29}$$

$$D_b u^a \simeq \partial_b u^a + \theta_b{}^a, \tag{30}$$

$$D\theta_{ab} \simeq d\theta_{ab}. \tag{31}$$

To the quadratic order in the number of fields and to the lowest order in derivatives, the effective Lagrangian for supersolids can be written in the form

$$\begin{aligned} L = {} & \frac{1}{2}K(\partial_0 u^0)^2 - \frac{1}{2}K_{ij}\partial^i u^0 \partial^j u^0 + \frac{1}{2}C_{ab}\partial_0 u^a \partial_0 u^b \\ & - \frac{1}{2}C_{iajb}\big(\partial_i u^a + \theta^a_i\big)\big(\partial_j u^b + \theta^b_j\big) \\ & + \frac{1}{2}C^{(\theta)}_{iajb}\partial_0\theta^a_i \partial_0 \theta^b_j - \frac{1}{2}C^{(\theta)}_{iabjcd}\partial^i\theta^{ab}\partial^j\theta^{cd}, \end{aligned} \tag{32}$$

where C_{iajb} are elastic constants. There can be an additional term,

$$L^{\mathrm{WZ}} = \rho_0 du^0 \wedge du^a \wedge \widetilde{e}_a, \tag{33}$$

where $\widetilde{e}_a \equiv \epsilon_{aa_2\ldots a_d} dx^{a_2} \wedge \cdots \wedge dx^{a_d}$ is a constant $(D-2)$-form. This term (33) appears as a Wess-Zumino term, and its derivation is discussed in Sec. IV.

The effective Lagrangian for solids can be obtained by setting $u^0 = 0$,

$$\begin{aligned} L = {} & \frac{1}{2}C_{ab}\partial_0 u^a \partial_0 u^b - \frac{1}{2}C_{iajb}\big(\partial_i u^a + \theta^a_i\big)\big(\partial_j u^b + \theta^b_j\big) \\ & + \frac{1}{2}C^{(\theta)}_{iajb}\partial_0\theta^a_i\partial_0\theta^b_j - \frac{1}{2}C^{(\theta)}_{iabjcd}\partial^i\theta^{ab}\partial^j\theta^{cd}. \end{aligned} \tag{34}$$

By choosing the elastic constants as

$$C_{iajb} = C_{ab}\delta_{ij}, \quad C^{(\theta)}_{iabjcd} = C^{(\theta)}_{abcd}\delta_{ij}, \tag{35}$$

the low-energy effective Lagrangian of a solid with translational and rotational NG fields, to the quadratic order, can be written in differential forms as

$$L = -C_{ab} d_\theta u^a \wedge \star d_\theta u^b - C^{(\theta)}_{abcd} d\theta^{ab} \wedge \star d\theta^{cd}, \tag{36}$$

where $\star$ denotes the Hodge dual operation and we have defined

$$d_\theta u^a \equiv du^a - \theta^a_b dx^b. \tag{37}$$

The appearance of the combination (37) is anticipated on symmetry grounds. The NG fields u^a and θ^a_b are transformed under infinitesimal translation and rotation as

$$u^a \mapsto u^a - \alpha^a - \beta^a_b x^b, \quad \theta^a_b \mapsto \theta^a_b - \beta^a_b, \tag{38}$$

where α^a and β^a_b are transformation parameters for the translation and rotation, respectively. Equation (37) is indeed invariant under these transformations. Intuitively speaking, the reason why the covariant derivative of u^a includes $\theta^a{}_b$ is that the translational NG field u^a is nontrivially transformed under rotations.

The Lagrangian (36) can be identified with that of the Cosserat elasticity theory [39,40]. The Cosserat theory is a

[3]When $\kappa \neq 0$ in $2+1$ dimensions, the condition $D_a u^0 = 0$ is written as $\partial_a u^0 - v_a - \frac{\kappa}{2}\epsilon_{bc}v^b\partial_a v^c = 0$. If we solve this for v_a perturbatively in the number of derivatives,

$$v_a = \partial_a u^0 - \frac{\kappa}{2}\epsilon_{bc}\partial^b u^0 \partial_a \partial^c u^0 + O(\partial^5).$$

Thus, the central extension with κ gives rise to a correction of the superfluid velocity with higher-order derivatives.

generalization of the conventional elasticity theory, and it contains rotational degrees of freedom, in addition to the displacement field. Such a generalization is necessary to explain the properties of porous materials [41,42] or mechanical metamaterials [43]. Because of the nontrivial covariant derivative (37), the rotational NG field ${\theta^a}_b$ has a mass term. This introduces characteristic scales in the theory, which are absent in the conventional elasticity. If we consider the long-wavelength limit, the mass term is dominant compared to the kinetic term for θ^a_b, and its equation of motion (EOM) is $\theta^a_b = \partial_{[b}u^{a]}$. At sufficiently low energies, those gapped modes can be ignored, and the remaining part for $\partial_b u^a$ is the symmetric part. In this way, the Lagrangian (36) reduces to the classical elasticity theory. Thus, we have shown that the Cosserat theory arises naturally from the coset construction, and it can be understood as a dynamical realization of the inverse Higgs phenomenon [20,44].

D. Fractonic topological defects

Let us introduce topological defects. In a crystal, there are two kinds of defects, dislocations and disclinations. The former are associated with the translational-symmetry breaking $\mathbb{R}^d \to \mathbb{Z}^d$, and the latter are due to the broken rotational symmetry. Dislocations and disclinations can be identified with the multivalued [45] parts of the fields u^a and θ^a_b, respectively. We decompose the covariant derivatives of phonon fields into the continuous part and singular part as

$$d_A u^a = du^a - \theta^a_b \wedge dx^b + A^a, \quad d_{\mathcal{A}}\theta^a_b = d\theta^a_b + \mathcal{A}^a_b, \quad (39)$$

where we have introduced $A^a \equiv du^a_{(\mathrm{s})}$ and $\mathcal{A}^a_b \equiv d\theta^{ab}_{(\mathrm{s})}$, and the subscript (s) indicates the singular part.

The way the defect fields, A^a and $\mathcal{A}^a_b$, enter is the same as that of gauge fields. We can promote the translational and rotational transformations to local ones, $\alpha^a \mapsto \alpha^a(x)$ and $\beta^a_b \mapsto \beta^a_b(x)$, by simultaneously transforming A^a and $\mathcal{A}^a_b$ as

$$A^a \mapsto A^a + d\alpha^a + d\beta^a_b x^b, \quad \mathcal{A}^a_b \mapsto \mathcal{A}^a_b + d\beta^a_b. \quad (40)$$

The gauge-invariant field strengths of the defect gauge fields are given by

$$\star J^a \equiv dA^a + \mathcal{A}_{ab} \wedge dx^b, \quad \star \mathcal{J}^a_b \equiv d\mathcal{A}^a_b. \quad (41)$$

We can identify $\star J^a$ as the dislocation current, and $\star\mathcal{J}^a_b$ as the disclination current. The integration of $\star J^a$ over a surface S gives the Burgers vector, $b^a = \int_S \star J^a$. By taking the exterior derivative of Eq. (41), we obtain the following relations,

$$d\star J^a = \star\mathcal{J}^{ab} \wedge dx^b, \quad d\star\mathcal{J}^{ab} = 0. \quad (42)$$

The divergence of the dislocation current is equal to the disclination current. This relation corresponds to Eq. (1) and hence represents the fractonic feature of the defects. In this derivation, the geometric origin [46] of the fractonic behavior is manifest: the interrelation between the defect currents can be traced back to the semidirect product structure of the translational and rotational groups. The origin of those relations is the same as the Bianchi identities of torsion and curvature in a Riemann-Cartan spacetime [47,48].

Let us make several comments. We have shown that the inclusion of rotational NG field, which is a gapped mode, gives rise to the Cosserat elasticity theory, and the currents of dislocations/disclinations, which are defects associated with translations/rotations, satisfy the Bianchi identities (42). This construction depends only on the symmetry-breaking pattern and is generalizable to situations where translations and rotations are spontaneously broken, such as supersolids and vortex crystals. For each of those systems, we can construct a Cosserat-type theory via the inclusion of gapped rotational NG fields. The present derivation is for D-dimensional spacetime, where J^a and $\mathcal{J}^a_b$ are both $(D-2)$-form, and the dislocations and disclinations are $(D-3)$-dimensional objects. They are extended objects in general and the motions of disclinations are constrained by the relation (42). In $3+1$ dimensions, those defects are lines [49].

III. DUAL GAUGE THEORIES

In this section, we study the dual gauge theories of the effective theories of solids and supersolids constructed in the previous section. The dual transformations of Cosserat elasticity theory have been discussed in Refs. [50,51].

A. Dual gauge theory for solids

Let us discuss the properties of the dual gauge theory for solids. The derivation of the dual theories is straightforward and we give it in Appendix A1. An advantage of dualization is that, in the dual gauge theory, NG fields couple to the topological defects electrically.[4] For notational simplicity, let us here take the elastic constants to be of the form $C_{ab} = c\delta_{ab}$, $C_{abcd} = c'\delta_{ac}\delta_{bd}$. The Lagrangian of the dual gauge theory for solids reads

$$L_{\mathrm{dual}} = -\frac{c}{2} f^a \wedge \star f_a - \frac{c'}{2} f^{ab} \wedge \star f_{ab}, \quad (43)$$

where the gauge-invariant field strengths are defined as

$$f^a \equiv da^a, \quad f^{ab} \equiv da^{ab} + \bar{c}\, dx^b \wedge a^a, \quad (44)$$

with $\bar{c} \equiv c/c'$. The gauge fields are related to the original fields by $da^a = \star du^a$, $da^{ab} = \star d\theta^{ab}$. In the presence of topological defects, we also have source terms, $L_{\mathrm{s}} = a^a \wedge \star J_a + a^{ab} \wedge \star\mathcal{J}_{ab}$. A dislocation sources translational gauge field a^a, and a disclination sources rotational gauge field a^{ab}. The field strengths (44) are invariant under the following gauge transformations,

$$a^a \mapsto a^a + d\lambda^a, \quad a^{ab} \mapsto a^{ab} + d\rho^{ab} + \bar{c}\, dx^b \wedge \lambda^a, \quad (45)$$

where λ^a and ρ^{ab} are $(D-3)$-form transformation parameters. By varying the dual action with respect to the gauge fields, we obtain a type of Maxwell's equations,

$$d^\dagger f^a + \star^{-1}(\star f^{ab} \wedge dx^b) = J^a, \quad d^\dagger f^{ab} = \mathcal{J}^{ab}, \quad (46)$$

where $d^\dagger$ is the codifferential. By applying an exterior derivative on f^a and f^{ab} in Eq. (44), we obtain the Bianchi

[4]Note that we here treat topological defects as a background, meaning that they are not dynamical variables in the path integral. Incorporating dynamical topological defects in three (or higher) spatial dimensions in an effective field theory is a nontrivial problem, while the particle-vortex duality in 2+1 dimensions is well-established [52,53].

identities,

$$df^a = 0, \quad df^{ab} + \bar{c}\, dx^b \wedge f^a = 0. \tag{47}$$

Those equations correspond to the conservation of momentum and angular momentum, respectively. Equations (46) and (47) constitute the set of equations of motion in the presence of dislocations and disclinations.

To understand its dynamics, let us focus on $D = 2 + 1$ and rewrite the equations using electric and magnetic fields, which are introduced for each of f^a and f^{ab} by

$$f^a = \epsilon^a{}_b\left((E^b)_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij} B^b dx^i \wedge dx^j\right), \tag{48}$$

$$f^{ab} = \epsilon^{ab}\left(\mathsf{E}_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij} \mathsf{B}\, dx^i \wedge dx^j\right). \tag{49}$$

In the absence of topological defects, the equations of motion are written as follows:

(i) Maxwell's equations for translational phonons:

$$\partial^i (E^a)_i - \mathsf{E}^a = 0, \tag{50}$$

$$-\tilde{\partial}_i B^a + \partial_0 (E^a)_i - \epsilon^a{}_i \mathsf{B} = 0, \tag{51}$$

where $\tilde{\partial}_i \equiv \epsilon_{ij}\partial^j$.

(ii) Bianchi identity for translational phonons:

$$-\tilde{\partial}^i (E^a)_i + \partial_0 B^a = 0. \tag{52}$$

(iii) Maxwell's equations for rotational phonons:

$$\partial^i \mathsf{E}_i = 0, \tag{53}$$

$$-\tilde{\partial}_i \mathsf{B} + \partial_0 \mathsf{E}_i = 0. \tag{54}$$

(iv) Bianchi identity for rotational phonons:

$$-\tilde{\partial}^i \mathsf{E}_i + \partial_0 \mathsf{B} + \frac{\bar{c}}{2}\epsilon_c{}^i (E^c)_i = 0. \tag{55}$$

Using the Maxwell's equations and Bianchi identities,[5] we can derive closed equations for the translational electric fields $(E^a)_i$,

$$(\partial_0)^2 (E^a)_i = \tilde{\partial}_i \tilde{\partial}^j (E^a)_j + \epsilon^a_i \tilde{\partial}_j \partial^k (E^j)_k - \frac{\bar{c}}{2}\, \epsilon^a_i \epsilon_c{}^j (E^c)_j, \tag{56}$$

where $\tilde{\partial}_i \equiv \epsilon_{ij}\partial^j$ is the derivative in the transverse direction. If we see this in the momentum space, in the long-wavelength limit, $k \to 0$, the antisymmetric part of the translational electric fields $(E^a)_i$ satisfies $(\omega^2 - \bar{c})\epsilon_a{}^i (E^a)_i = 0$, where ω is the frequency. Thus, the antisymmetric part is gapped and its gap is given by $\sqrt{\bar{c}}$. At low energies, this part can be dropped, and the symmetric tensor gauge theory [10,11] is reproduced.

[5]See Appendix A3 for more detailed analysis of excitation spectra based on the dual gauge theory.

B. Dual gauge theory for supersolids

Let us discuss the dual gauge theory for supersolids. We here start with the following quadratic Lagrangian,

$$L = -\frac{1}{2} C_{ab} d_\theta u^a \wedge \star d_\theta u^b - \frac{1}{2} C_{abcd} d\theta^{ab} \wedge \star d\theta^{ab} - \frac{c_0}{2} du^0 \wedge \star du^0 - g\, du^0 \wedge du^a \wedge \widetilde{e}_a, \tag{57}$$

where $\widetilde{e}_a \equiv \epsilon_{aa_2 \ldots a_d} dx^{a_2} \wedge \cdots \wedge dx^{a_d}$ is a constant $(D-2)$-form. The last term arises as a Wess-Zumino term, as we discuss later. This effective theory contains a gapped rotational NG mode θ^a_b, and is a Cosserat-type theory for supersolids. In the low-energy limit, it reduces to the conventional effective theory for supersolids [12]. The dual gauge theory for supersolids is given by

$$L_{\rm dual} = -\frac{c_0}{2} f^0 \wedge \star f^0 - \frac{c}{2} f^a \wedge \star f_a - \frac{c'}{2} f^{ab} \wedge \star f_{ab} - g \star f_0 \wedge \star f^a \wedge \widetilde{e}_a. \tag{58}$$

Similarly to the case of solids, the field strengths satisfy the following Bianchi identities,

$$\begin{aligned} df^A &= 0, \\ df^{ab} + \bar{c}\, dx^b \wedge f^a &= 0, \end{aligned} \tag{59}$$

where $A = 0, i$. By varying the dual action (58) with respect to the gauge fields, one obtains the corresponding Maxwell's equations,

$$\begin{aligned} d^\dagger (f^0 + g_0 \star f^a \wedge \widetilde{e}_a) &= J^0, \\ d^\dagger (f^a - \bar{g} \star f^0 \wedge \widetilde{e}^a) + \star^{-1}(\star f^{ab} \wedge dx^b) &= J^a, \\ d^\dagger f^{ab} &= \mathcal{J}^{ab}, \end{aligned} \tag{60}$$

where we set $g_0 \equiv g/c_0$ and $\bar{g} \equiv g/c$.

Similarly to the case of solids, let us focus on 2+1 dimensions and introduce the electric and magnetic fields for superfluid phonons by

$$f^0 = e_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij} b\, dx^i \wedge dx^j, \tag{61}$$

in addition to the corresponding expressions for translational and rotational gauge fields. Then, the last term of Eq. (58) is written as

$$-g[b\,(E^a)_a + e_a B^a] dx^0 \wedge dx^1 \wedge dx^2. \tag{62}$$

These are mixed $\boldsymbol{E} \cdot \boldsymbol{B}$-type terms and are responsible for a generalized Witten effect [15]. Indeed, Maxwell's equations for superfluid phonons are written in terms of electric and magnetic fields as

$$\partial^i e_i - g_0 \partial^i B_i = -(J_0)_0, \tag{63}$$

$$\partial_0 e_i - \tilde{\partial}_i b + g_0[\tilde{\partial}_i (E^a)_a - \partial_0 B_i] = -(J_0)_i. \tag{64}$$

As pointed out in Ref. [15], Eq. (63) indicates that a vortex acquires a magnetic charge of the translational gauge field because of the topological term, which means that a vortex carries a crystalline angular momentum. For the full set of equations of motion written by electric and magnetic fields, see Appendix A2.

IV. WESS-ZUMINO TERMS FOR SUPERSOLIDS

The Lagrangian built from the covariant derivatives (24) is exactly invariant under the symmetry transformations. The effective Lagrangian can also contain Wess-Zumino terms [54,55], which are invariant only up to total derivatives. Such terms arise when the Lie algebra of the symmetry group allows for nontrivial central extensions. They can be identified by finding invariant and closed $(D+1)$-forms made out of MC forms. In this section, we discuss the Wess-Zumino terms for supersolids and their phenomenological implications.

A. Derivation of Wess-Zumino terms

In this section, we derive the Wess-Zumino terms for supersolids. Below, we consider the low-energy limit and do not consider the rotational NG fields θ^a_b. We also consider the case where the superfluid density is small, which is typically the case [38]. We first note that the Maurer-Cartan forms satisfy the following equations,

$$\begin{aligned} &d\omega_{\bar P}^0 = 0, \quad d\omega_{\bar P}^a = \omega_{\bar P}^0 \wedge \omega_B^a, \quad d\omega_B^a = 0,\\ &d\omega_Q^0 = \omega_{\bar P}^a \wedge \omega_B^a - \frac{\kappa}{2}\epsilon_{ab}\omega_B^a \wedge \omega_B^b,\\ &d\omega_Q^a = -\omega_{\bar P}^0 \wedge \omega_B^a = -d\omega_{\bar P}^a, \end{aligned} \tag{65}$$

where the term proportional to κ exists only in $2+1$ dimensions.

To look for Wess-Zumino terms, we have to find invariant and closed $(D+1)$-forms. Let us define the mass current $j_{\rm m}$ in d-spatial dimensions by

$$\star j_{\rm m} \equiv \frac{\rho_0}{d!}\epsilon_{a_1\cdots a_d} d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_d}, \tag{66}$$

where ρ_0 is the mass density in the ground state. The current is trivially conserved, $d\star j_{\rm m} = 0$. Note that $d\phi^a = \omega_Q^a + \omega_{\bar P}^a$. We have the following closed $(D+1)$-form,

$$\Omega_{D+1} = \omega_B^a \wedge (\omega_Q)_a \wedge \star j_{\rm m} = -\omega_B^a \wedge (\omega_{\bar P})_a \wedge \star j_{\rm m}, \tag{67}$$

where the latter equality follows from $d\phi^a \wedge d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_d}\epsilon_{a_1\cdots a_d} = 0$. The closedness of Ω_{D+1} can be checked using Eq. (65). The $(D+1)$-form Ω_{D+1} can be written as

$$dv^a \wedge (dx_a - v_a dx^0)\wedge \star j_{\rm m} = d\left[\left(v^a dx_a - \frac{v^2}{2}dx^0\right)\wedge \star j_{\rm m}\right]. \tag{68}$$

Therefore, we can write down the corresponding Wess-Zumino term in D dimensions as

$$L_D^{\rm WZ} = \left(v^a dx_a - \frac{v^2}{2}dx^0\right)\wedge \star j_{\rm m}. \tag{69}$$

Note that it can be written as

$$L_D^{\rm WZ} = \left(du^0 - \omega_Q^0\right)\wedge \star j_{\rm m}. \tag{70}$$

The one-form ω_Q^0 is exactly invariant under Galilean boosts. Thus, the term proportional to ω_Q^0 should be already taken into account by the coset construction. Thus, we can adopt the following form instead of Eq. (70),[6]

$$L_D^{\rm WZ'} = du^0 \wedge \star j_{\rm m}. \tag{71}$$

This expression is consistent with the interpretation of $\star j_{\rm m}$ as the mass current. When the superfluid density is small, ρ_0 indeed equals the mass density at the equilibrium. Hereafter, we consider this situation. This term was introduced in Ref. [38] for $(3+1)$-dimensional supersolids, and it appears as a Wess-Zumino term in the current construction. It is a total derivative, and does not affect the equations of motion. Still, this term changes the identification of Noether currents and is needed to reproduce the centrally extended algebra, $[B_a, P_b] = -i\delta_{ab}Q_0$. In the presence of vortices, the term induces the interactions of vortices with lattice phonons. Under a Galilean boost, $L_D^{\rm WZ'}$ is transformed as

$$\begin{aligned} \delta L_D^{\rm WZ'} = \delta L_D^{\rm WZ} &= \left(\beta^a dx_a + \frac{\beta^2}{2}dx^0\right)\wedge \star j_{\rm m}\\ &= d\left[\left(\beta^a x_a + \frac{\beta^2}{2}x^0\right)\wedge \star j_{\rm m}\right], \end{aligned} \tag{72}$$

which is a total derivative.

In 2+1 dimensions, the Galilean algebra allows for another central extension parametrized by κ. Correspondingly, we have another invariant 4-form,

$$\Omega'_4 \equiv \frac{\kappa}{2}\epsilon_{ab}\,\omega_B^a \wedge \omega_B^b \wedge \star j_{\rm m}. \tag{73}$$

It produces the following Wess-Zumino term,

$$L_{D=3}^{\rm WZ2} = \frac{\kappa}{2}\epsilon_{ab} v^a dv^b \wedge \star j_{\rm m}. \tag{74}$$

Under a Galilean boost, this term is shifted by a total derivative,

$$\delta L_{D=3}^{\rm WZ2} = \frac{\kappa}{2}\epsilon_{ab}\beta^a dv^b \wedge \star j_{\rm m} = d\left[\frac{\kappa}{2}\epsilon_{ab}\beta^a v^b \wedge \star j_{\rm m}\right]. \tag{75}$$

If we use the IHC $v_a = \partial_a u^0$, the term (74) also is a total derivative, and the EOMs are not affected in the absence of topological defects.

Wess-Zumino terms change the identification of the currents. The term (71) leads to the following additional contribution to the boost current,

$$(\star j_B^a)_{\rm WZ} = \frac{\partial L_D^{\rm WZ'}}{\partial du^0}\delta^a u^0 = x^a(\star j_{\rm m}), \tag{76}$$

where $\delta^a u^0$ indicates the variation of u^0 under a Galilean boost in the ath direction. The boost current is written as

$$\star j_B^a = -t(\star p^a) + x^a(\star j_{\rm m}), \tag{77}$$

where p^a denotes the translational current. With this contribution, we can reproduce the centrally extended algebra,

$$\begin{aligned} \langle [P_a, B^b]\rangle &= \left\langle \left[P_a, \int_V \star j_B^b\right]\right\rangle = \left\langle \int_V (-i\partial_a(x^b \star j_{\rm m}))\right\rangle\\ &= -i\delta_a^b\left\langle \int_V \star j_{\rm m}\right\rangle = -i\delta_a^b\langle Q_0\rangle. \end{aligned} \tag{78}$$

[6]The coefficient ρ_0 may slightly deviate as an effect of a nonzero superfluid fraction.

B. Scattering of quasiparticles off topological defects

The Wess-Zumino terms (71) and (74) are total derivatives and do not affect the EOM in the absence of topological defects. However, they change the identification of Noether currents, and when topological defects are present, these terms induce nontrivial scattering effects.[7] Here, we discuss such processes induced by Eqs. (71) and (74). Since their coefficients are determined by the symmetry algebra, the coupling constants of those processes are model-independent.

Let us first discuss the consequence of Eq. (71). We here consider a vortex in a 3 + 1-dimensional supersolid located along the z direction at $x = y = 0$. Such a vortex configuration can be expressed by $\epsilon_{abc}\partial^a\partial^b u^0 = (2\pi/m)\delta_c(\boldsymbol{x}_\mathrm{T})$, where $\delta_c(\boldsymbol{x}_\mathrm{T}) \equiv n_c\delta(x)\delta(y)$ is the transverse delta function and $\boldsymbol{n} = \hat{\boldsymbol{z}}$ is the unit tangent vector along the vortex. In the presence of a superfluid vortex, the Lagrangian density can be written as

$$L^{\mathrm{WZ}'}_{D=4} = -\frac{\rho_0}{3!m}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc}\partial_\mu\partial_\nu\varphi\phi^a\partial_\rho\phi^b\partial_\sigma\phi^c = -\pi\frac{\rho_0}{m}\epsilon_{abc}u^a\dot{u}^b\delta^c(\boldsymbol{x}_\mathrm{T}) + O(u^3). \tag{79}$$

The leading-order term describes elastic scatterings of lattice phonons off superfluid vortices in a supersolid. We denote the energy/momentum and polarization of the incoming/scattered particles as $\{(\omega, \boldsymbol{k}), \boldsymbol{\epsilon}\}$ and $\{(\omega', \boldsymbol{k}'), \boldsymbol{\epsilon}'\}$, respectively. For definiteness, we consider the incoming lattice phonon that is incident perpendicularly to the vortex, $\boldsymbol{k}\cdot\boldsymbol{n} = 0$, and assume that it is transversely polarized, $\boldsymbol{k}\cdot\boldsymbol{\epsilon} = 0$. We also assume that the scattered phonon is also transversely polarized, and the polarization vectors of the initial and scattered phonons are in the plane perpendicular to the vortex, $\boldsymbol{\epsilon}\cdot\boldsymbol{n} = \boldsymbol{\epsilon}'\cdot\boldsymbol{n} = 0$. The current collision geometry reduces the problem effectively to two spatial dimensions. The matrix element of this elastic scattering process is[8]

$$i\mathcal{M} = 2\frac{1}{\rho_0}\frac{\pi\rho_0}{m}i\omega\epsilon_{abc}\epsilon^a\epsilon^b n^c = \frac{2\pi}{m}\omega(\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}')\cdot\boldsymbol{n} = \frac{2\pi}{m}\omega\sin\theta, \tag{80}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. We here assume that the supersolid is isotropic. The 1-body final-state phase space is given by

$$d\Pi_b \equiv 2\pi\delta(\omega-\omega')\frac{1}{2\omega_b}\frac{d^2k_b}{(2\pi)^2}. \tag{81}$$

Since the superfluid vortex does not break time translations, the scattering conserves the energy, $\omega = \omega'$, which is reflected in the phase-space delta function, $2\pi\delta(\omega-\omega')$. The infinitesimal cross section of a vortex line element $d\ell$ is given by

$$d\sigma_{a\to b} = \frac{1}{2\omega}\frac{1}{v_a}|\mathcal{M}_{a\to b}|^2 d\Pi_b d\ell = \frac{1}{2\omega}\frac{1}{v_\mathrm{T}}\left(\frac{4\pi^2}{m^2}\omega^2\sin^2\theta\right)\left(\frac{d\theta d\ell}{4\pi}\frac{1}{v_\mathrm{T}^2}\right) = \frac{\pi}{2}\frac{\omega}{m^2 v_\mathrm{T}^3}\sin^2\theta d\theta d\ell, \tag{82}$$

where v_T is the velocity of transverse phonons. The differential cross section per unit vortex length is written as

$$\frac{d^2\sigma}{d\theta d\ell} = \frac{\pi}{2}\frac{k}{m^2 v_\mathrm{T}^2}\sin^2\theta, \tag{83}$$

which is obtained in Ref. [38]. The cross section is linearly proportional to the momentum k and it is largest when the scattering angle is $\pi/2$. This is the dominant elastic scattering process of lattice phonons off vortices when the superfluid density is small.

Let us now consider a dislocation in a (3 + 1)-dimensional supersolid. The term $du^0 \wedge \star j_\mathrm{m}$ leads to the conversion of lattice phonons and superfluid phonons on a dislocation. The existence of a dislocation leads to the following multivalued part of u^a,

$$\epsilon_{bcd}\partial^b\partial^c u^a = b^a\delta_d(\boldsymbol{x}_\mathrm{T}), \tag{84}$$

where b^a is the Burgers vector. The interaction Lagrangian is written as

$$L^{\mathrm{WZ}'}_{D=4} = \frac{\rho_0}{3!}du^0\wedge d\phi^a\wedge d\phi^b\wedge d\phi^c\epsilon_{abc} = -\frac{\rho_0}{2}u^0\wedge d^2\phi^a\wedge d\phi^b\wedge d\phi^c\epsilon_{abc} = -\rho_0 u^0\epsilon_{abc}b^a\dot{u}^b\delta^c(\boldsymbol{x}_\mathrm{T})dx^0\wedge dx^1\wedge dx^2\wedge dx^3 + O(u^3). \tag{85}$$

This results in the conservation of a superfluid phonon with energy/momentum $(\omega, \boldsymbol{k})$ to a lattice phonon $(\omega', \boldsymbol{k}')$ with polarization $\boldsymbol{\epsilon}'$ via the scattering off a dislocation. The matrix element of this process is

$$i\mathcal{M} = \frac{1}{\sqrt{\rho_s}\sqrt{\rho_0}}\rho_0 i\omega' b^a\epsilon'^b\epsilon_{abc}n^c = i\sqrt{\frac{\rho_0}{\rho_s}}\omega'(\boldsymbol{b}\times\boldsymbol{\epsilon}')\cdot\boldsymbol{n}, \tag{86}$$

where $\boldsymbol{n}$ is the tangent vector to the dislocation. The infinitesimal cross section is

$$d\sigma = \frac{1}{2\omega}\frac{1}{v_s}|\mathcal{M}|^2\left(\frac{d\theta d\ell}{4\pi}\frac{1}{v^2}\right) = \frac{\rho_0}{\rho_s}\frac{k}{8\pi v^2}|(\boldsymbol{b}\times\boldsymbol{\epsilon}')\cdot\boldsymbol{n}|^2 d\theta d\ell, \tag{87}$$

where v_s and v are the velocities of superfluid phonons and lattice phonons, respectively, and θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. The cross section is determined by the relative orientations of the Burgers vector $\boldsymbol{b}$, the polarization $\boldsymbol{\epsilon}'$ of the final-state lattice phonon, and the direction $\boldsymbol{n}$ of the dislocation. For example, the cross section vanishes for a screw dislocation, for which $\boldsymbol{b}$ is parallel to $\boldsymbol{n}$. We also note that the cross section is enhanced at small superfluid densities.

[7] The elastic scattering of lattice phonons off superfluid vortices through Eq. (71) is studied in Ref. [38].

[8] The scattering processes involving vortices have been studied in the EFT approach, for example, in Refs. [56–58]. In the computation of scattering amplitudes, we need to take into account the noncanonical normalization of the kinetic term [56],

$$\mathcal{L} = \frac{\rho_0}{2}[(\dot{u}^a)^2 - v^2(\partial_i u^a)^2] + \frac{\rho_s}{2}(\dot{u}^0)^2 + \dots.$$

Next, let us consider the term (74), which can exist in 2+1 dimensions. In the presence of vortices, this term induces nontrivial interactions among NG fields. In 2+1 dimensions, a vortex is a pointlike object. If we consider a vortex placed at the origin, we have $\epsilon_{ab}\partial^a\partial^b u^0 = \frac{2\pi}{m}\delta(\boldsymbol{x})$, where $\delta(\boldsymbol{x}) \equiv \delta(x)\delta(y)$, and we have the following terms,

$$\frac{2\pi\kappa\rho_0}{m}\delta(\boldsymbol{x})\big(\dot{u}^0\,\nabla\cdot\boldsymbol{u} - \dot{u}^a\partial_a u^0 + O(u^3)\big). \tag{88}$$

This indicates that superfluid phonons and lattice phonons can be converted via the scatterings off a superfluid vortex. Let us consider the process of a transverse lattice phonon with $(\omega, \boldsymbol{k})$ and polarization $\boldsymbol{\epsilon}$ converted on a vortex into a superfluid phonon with $(\omega', \boldsymbol{k}')$. The matrix element is

$$i\mathcal{M} = \frac{1}{\sqrt{\rho_0\rho_s}}\frac{2\pi\kappa\rho_0}{m}[-(-i\omega)(-i\boldsymbol{k}')\cdot\boldsymbol{\epsilon}] = \frac{2\pi\kappa}{m}\sqrt{\frac{\rho_0}{\rho_s}}\omega\boldsymbol{k}'\cdot\boldsymbol{\epsilon}. \tag{89}$$

The infinitesimal cross section is computed as

$$\begin{aligned} d\sigma &= \frac{1}{2\omega}\frac{1}{v}|\mathcal{M}|^2 d\Pi_b \\ &= \frac{1}{2\omega}\frac{1}{v}\left(\frac{4\pi^2\kappa^2}{m^2}\frac{\rho_0}{\rho_s}\omega^2(\boldsymbol{k}'\cdot\boldsymbol{\epsilon})^2\right)\left(\frac{d\theta}{4\pi}\frac{1}{v_s^2}\right) \\ &= \frac{\pi}{2}\frac{\rho_0}{\rho_s}\frac{\kappa^2}{m^2 v_s^2}kk'^2\sin^2\theta d\theta, \end{aligned} \tag{90}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$.

Finally, let us emphasize that the coupling constants associated with the Wess-Zumino terms are fixed by the symmetry algebra, and the cross sections associated with those terms are model-independent predictions of the effective theory.

V. SUMMARY

We derived effective field theories of gapless phases with fractons, such as solids and supersolids, using a coset construction. We found that a dynamical realization of the inverse Higgs phenomenon naturally leads to the Cosserat theory of elasticity. The topological defects appear as the singular parts of the NG fields, and the corresponding currents obey the relation (42), which plays a key role so that the disclinations behave as fractons. The derivation clarifies the geometric origin of the fractonic nature: it comes from the semidirect product structure of the translational and rotational groups. The current construction can be applied to systems where the translational symmetry is broken, and we can understand why fractons appear in solids, supersolids, vortex crystals, and so on. We identified Wess-Zumino terms in supersolids, which differ by total derivatives under symmetry transformation. Those terms induce nontrivial scattering processes involving topological defects in a supersolid phase. We gave examples of the computations of scattering cross sections of such processes. When the superfluid density is small, the coupling constants of these processes are fixed by the algebra, and hence are model-independent.

ACKNOWLEDGMENTS

We thank G.-Y. Cho and K.-S. Kim for valuable discussions. Y.H. and Y.-H.Q. are supported by the Korean Ministry of Education, Science, and Technology, Gyeongsangbuk-do Provincial Government, and Pohang City Government for Independent Junior Research Groups at the Asia Pacific Center for Theoretical Physics (APCTP). Y.H. is supported by the National Research Foundation (NRF) funded by the Ministry of Science of Korea (Grant No. 2020R1F1A1076267).

APPENDIX A: DETAILS ON DUAL GAUGE THEORIES

In this Appendix, we provide the derivation of dual gauge theories of the effective Lagrangians of solids and supersolids. We also derive the corresponding Maxwell's equations written in terms of electric and magnetic fields, and discuss their dynamical properties.

1. Derivation of dual gauge theories

In this section, we give the derivation of the dual gauge theories for solids and supersolids. Since the gauge theory of solids can be obtained from that of supersolids, we here discuss supersolids. We start with the partition function of supersolids $Z = \int[\mathcal{D}u_a][\mathcal{D}\theta_{ab}]e^{iS[u^a,\theta^{ab},u^0]}$, where the action is given by

$$S[u^0, u^a, \theta^{ab}] = -\int_{\mathcal{M}_D}\left[\frac{1}{2}C_{ab}d_\theta u^a\wedge\star d_\theta u^b + \frac{1}{2}C_{abcd}d\theta^{ab}\wedge\star d\theta^{ab} + \frac{1}{2}C_{00}du^0\wedge\star du^0 + g\,du^0\wedge du^a\wedge\widetilde{e}_a\right], \tag{A1}$$

where $\mathcal{M}_D$ is a D-dimensional spacetime manifold. For notational simplicity, we organize u^0 and u^a as a four-vector as $u^A = (u^0, u^a)$, and the constants are also organized accordingly. We write the action as

$$S[u^A, \theta^{ab}] = -\int_{\mathcal{M}_D}\left[\frac{1}{2}C_{AB}d_\theta u^A\wedge\star d_\theta u^B + \frac{1}{2}C_{abcd}d\theta^{ab}\wedge\star d\theta^{ab} + g\,du^0\wedge du^a\wedge\widetilde{e}_a\right]. \tag{A2}$$

By introducing auxiliary fields, (τ^A, σ^{ab}), the partition function can be written as $Z = \int[\mathcal{D}u^A][\mathcal{D}\theta^{ab}][\mathcal{D}\tau^A][\mathcal{D}\sigma^{ab}]e^{iS[u^A,\theta^{ab},\tau^A,\sigma^{ab}]}$, where the action is given by

$$S = \int_{\mathcal{M}_D}\left[\frac{1}{2}C^{-1}_{AB}\tau^A\wedge\star\tau^B + \tau^A\wedge\star d_\theta u_A + \frac{1}{2}C^{-1}_{abcd}\sigma^{ab}\wedge\star\sigma^{cd} + \sigma^{ab}\wedge\star d\theta_{ab} - g\,du^0\wedge du^a\wedge\widetilde{e}_a\right]. \tag{A3}$$

By doing a variation with respect to τ^a and σ^{ab}, we obtain

$$-C^{-1}_{AB}\tau_B = d_\theta u^A, \quad -C^{-1}_{abcd}\sigma_{cd} = d\theta^{ab}. \tag{A4}$$

The path-integration of (the smooth part of) u^A and θ^{ab} leads to the following equations of motion,

$$d \star \tau^A = 0, \quad d \star \sigma^{ab} + dx^b \wedge \star\tau^a = 0, \tag{A5}$$

which are conservation laws of $U(1)$ charge $(\tau^0)_0$, momentum density $(\tau^a)_0$, and angular momentum density$(\sigma^{ab})_0$, respectively. The equations above can be solved explicitly by introducing $(D-2)$-form gauge fields a^A, a^{ab} as

$$\star\tau_A = -C_{AB}da^B, \quad \star\sigma_{ab} = -C_{abcd}da^{cd} - C_{ac}dx_b \wedge a^c. \tag{A6}$$

By comparing with Eq. (A4), we obtain the relation between the dual gauge fields and the original fields as

$$da^A = \star du^A, \quad da^{ab} = \star d\theta^{ab}. \tag{A7}$$

Substituting Eq. (A6) to Eq. (A3) gives the following dual effective Lagrangian,

$$L_{\rm dual} = -\frac{1}{2}C_{AB}f^A \wedge \star f^B - \frac{1}{2}C_{abcd}f^{ab} \wedge \star f^{cd} - g \star f_0 \wedge \star f^a \wedge \widetilde{e}_a. \tag{A8}$$

To simplify the expression, let us choose isotropic elastic constants as $C_{ab} = c\delta_{ab}$ and $C_{abcd} = c'\delta_{ac}\delta_{bd}$, and $C_{00} = c_0$. Then, we have $\star\tau_a = -da_a$, and $\star\sigma_{ab} = -da_{ab} - \bar{c}\, dx_b \wedge a_a$ with $\bar{c} \equiv c/c'$. In this case, the dual Lagrangian reads

$$L_{\rm dual} = -\frac{c_0}{2}f^0 \wedge \star f^0 - \frac{c}{2}f^a \wedge \star f_a - \frac{c'}{2}f^{ab} \wedge \star f_{ab} - g \star f_0 \wedge \star f^a \wedge \widetilde{e}_a, \tag{A9}$$

where the field strengths are defined as[9]

$$f^A \equiv da^A, \quad f^{ab} \equiv da^{ab} + \bar{c}\, dx^b \wedge a^a. \tag{A10}$$

The field strengths (A10) are invariant under the following gauge transformations,

$$a^A \mapsto a^A + d\lambda^A, \quad a^{ab} \mapsto a^{ab} + d\rho^{ab} + \bar{c}\, dx^b \wedge \lambda^a, \tag{A11}$$

where λ^A and ρ^{ab} are $(D-3)$-form gauge parameters. Note that the rotational gauge field is shifted by the transformation parameter of the translational gauge field. Accordingly, the field strengths satisfy the following Bianchi identities,

$$\begin{aligned} df^A &= 0, \\ df^{ab} + \bar{c}\, dx^b \wedge f^a &= 0. \end{aligned} \tag{A12}$$

Topological defects enter as sources to the dual gauge fields, $L_s = C_{AB}a^A \wedge \star J^B + C_{abcd}a^{ab} \wedge \star \mathcal{J}^{cd}$. By varying the dual action with respect to the gauge field a^A and a^{ab}, we obtain the equations of motion as

$$\begin{aligned} d^\dagger\left(f^0 + g_0 \star f^a \wedge \widetilde{e}_a\right) &= J^0, \\ d^\dagger\left(f^a - \bar{g} \star f^0 \wedge \widetilde{e}_a\right) + \star^{-1}(\star f^{ab} \wedge dx^b) &= J^a, \\ d^\dagger f^{ab} &= \mathcal{J}^{ab}, \end{aligned} \tag{A13}$$

where we set $g_0 \equiv g/c_0$ and $\bar{g} \equiv g/c$. The equations of motion (A13) together with Bianchi identities (A12) describe the dynamics of superfluid phonons, translational phonons, and rotational phonons in D spacetime dimensions in the presence of vortices, dislocations, and disclinations.

2. Equations of motion in terms of electric and magnetic fields

Let us rewrite the EOMs using electric and magnetic fields. We here consider 2+1 dimensions. Electric and magnetic fields for superfluid phonons, lattice phonons, and rotational phonons are introduced by

$$f^0 = e_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij} b\, dx^i \wedge dx^j, \tag{A14}$$

$$f^a = \epsilon^a{}_b\left((E^b)_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij}B^b dx^i \wedge dx^j\right), \tag{A15}$$

$$f^{ab} = \epsilon^{ab}\left(\mathsf{E}_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ij}\mathsf{B}\, dx^i \wedge dx^j\right). \tag{A16}$$

The topological term can be written as

$$-g \star f_0 \wedge \star f^a \wedge \widetilde{e}_a = -g[b\,(E^a)_a + e_a B^a]dx^0 \wedge dx^1 \wedge dx^2. \tag{A17}$$

Those are crossed $\boldsymbol{E}\cdot\boldsymbol{B}$-type terms and are responsible for the generalized Witten effect.

Maxwell's equations for superfluid phonons are now written as

$$\partial^i e_i - g_0 \partial^i B_i = -(J_0)_0, \tag{A18}$$

$$\partial_0 e_i - \tilde{\partial}_i b + g_0[\tilde{\partial}_i (E^a)_a - \partial_0 B_i] = -(J_0)_i, \tag{A19}$$

where $\tilde{\partial}_i \equiv \epsilon_{ij}\partial^j$.

Maxwell's equations for translational phonons are

$$\partial^i (E^a)_i + \bar{g}\partial^a b - \mathsf{E}^a = \epsilon^a_b (J^b)_0, \tag{A20}$$

$$-\tilde{\partial}_i B^a + \partial_0 (E^a)_i + \bar{g}\left(\delta^a_i\, \partial_0 b - \tilde{\partial}_i e^a\right) - \epsilon_{ai}\mathsf{B} = \epsilon^a_b (J^b)_i. \tag{A21}$$

Maxwell's equations for rotational phonons are

$$\partial^i \mathsf{E}_i = -\frac{1}{2}\epsilon_{ab}(\mathcal{J}^{ab})_0, \tag{A22}$$

$$-\tilde{\partial}_i \mathsf{B} + \partial_0 \mathsf{E}_i = -\frac{1}{2}\epsilon_{ab}(\mathcal{J}^{ab})_i. \tag{A23}$$

Bianchi identities are written as

$$-\tilde{\partial}^i e_i + \partial_0 b = 0, \tag{A24}$$

$$-\tilde{\partial}^i (E^a)_i + \partial_0 B^a = 0, \tag{A25}$$

$$-\tilde{\partial}^i \mathsf{E}_i + \partial_0 \mathsf{B} + \frac{\bar{c}}{2}\epsilon_c{}^i (E^c)_i = 0. \tag{A26}$$

3. Excitations in solids

Let us look at the properties of excitations in solids in terms of the dual gauge theory. The EOMs for solids can be obtained by setting $f^0 = 0$. In the absence of topological defects, the EOMs are written as follows:

[9] In the general case, $f^a \equiv da^a$ and $f^{ab} \equiv da^{ab} + (C^{-1})^{cdab}C_{ce}\, dx_d \wedge a^e$.

(i) Maxwell's equations for translational phonons:

$$\partial^i (E^a)_i - \mathsf{E}^a = 0, \tag{A27}$$

$$-\tilde{\partial}_i B^a + \partial_0 (E^a)_i - \epsilon^a{}_i \mathsf{B} = 0. \tag{A28}$$

(ii) Bianchi identity for translational phonons:

$$-\tilde{\partial}^i (E^a)_i + \partial_0 B^a = 0. \tag{A29}$$

(iii) Maxwell's equations for rotational phonons:

$$\partial^i \mathsf{E}_i = 0, \tag{A30}$$

$$-\tilde{\partial}_i \mathsf{B} + \partial_0 \mathsf{E}_i = 0. \tag{A31}$$

(iv) Bianchi identity for rotational phonons:

$$-\tilde{\partial}^i \mathsf{E}_i + \partial_0 \mathsf{B} + \frac{\bar{c}}{2} \epsilon_c{}^i (E^c)_i = 0. \tag{A32}$$

Let us write the EOM in term of electric fields. The translational electric fields $(E^a)_i$ have four components. Because of the Gauss law, $\partial_a \partial^i (E^a)_i = 0$, there are three physical degrees of freedom. The EOM for $(E^a)_i$ can be derived as

$$\begin{aligned}(\partial_0)^2 (E^a)_i &= \tilde{\partial}_i \partial_0 B^a + \epsilon^a_i \partial_0 \mathsf{B} \\ &= \tilde{\partial}_i \tilde{\partial}^j (E^a)_j + \epsilon^a_i \left[\tilde{\partial}^j \mathsf{E}_j - \frac{\bar{c}}{2} \epsilon_c{}^j (E^c)_j \right] \\ &= \tilde{\partial}_i \tilde{\partial}^j (E^a)_j + \epsilon^a_i \tilde{\partial}_j \partial^k (E^j)_k - \frac{\bar{c}}{2} \epsilon^a_i \epsilon_c{}^j (E^c)_j.\end{aligned} \tag{A33}$$

To discuss the nature of linear excitations, let us write the EOM in the momentum space,

$$\omega^2 (E^a)_i - k^2 \tilde{n}_i (E^a)_j \tilde{n}^j - k^2 \epsilon^a_i \tilde{n}_j (E^j)_l n^l - \frac{\bar{c}}{2} \epsilon^a_i \epsilon_c{}^j (E^c)_j = 0, \tag{A34}$$

where $k \equiv \sqrt{k_i k^i}$, $\tilde{k}_i \equiv \epsilon_{ij} k^j$ is the transverse vector (to k_i), and $n_i \equiv k_i/k$, $\tilde{n}_i \equiv \tilde{k}_i/k$ are unit vectors in the longitudinal and transverse directions, respectively. In the long-wavelength limit, $k \to 0$, the antisymmetric part of the translational electric fields $(E^a)_i$ satisfies

$$(\omega^2 - \bar{c}) \epsilon_a{}^i (E^a)_i = 0. \tag{A35}$$

Hence, as a result of the coupling to the rotational electric and magnetic fields, the antisymmetric part, $\epsilon_a{}^i (E^a)_i$, acquires a gap, and its gap is given by $\omega = \sqrt{\bar{c}}$. The other two modes are gapless. Because of the gap, the antisymmetric part $\epsilon_a{}^i (E^a)_i$ can be dropped at low energies [50], and the remaining electric field is symmetric. The symmetric tensor gauge theory [12] is reproduced in this way.

To obtain the modes at finite k, let us project $(E^a)_i$ to the longitudinal and transverse directions for each index. We have the following projected components,

$$TT \equiv \tilde{n}_a (E^a)_i \tilde{n}^i, \quad LT \equiv n_a (E^a)_i \tilde{n}^i, \quad TL \equiv \tilde{n}_a (E^a)_i n^i. \tag{A36}$$

As we stated earlier, $LL = 0$ because of the Gauss law. Using the relation $\epsilon_{ij} = \tilde{n}_i n_j - n_i \tilde{n}_j$, we can write the antisymmetric part as $\epsilon_a{}^i (E^a)_i = TL - LT$. The EOMs are written in the matrix form as

$$\begin{pmatrix} \omega^2 - k^2 - \frac{\bar{c}}{2} & k^2 + \frac{\bar{c}}{2} & 0 \\ \frac{\bar{c}}{2} & \omega^2 - k^2 - \frac{\bar{c}}{2} & 0 \\ 0 & 0 & \omega^2 - k^2 \end{pmatrix} \begin{pmatrix} LT \\ TL \\ TT \end{pmatrix} = \mathbf{0}. \tag{A37}$$

The transverse-transverse sector is decoupled and represents a gapless mode. There is a mixing between longitudinal-transverse and transverse-longitudinal sectors. As a result, there will be one gapless and one gapped mode from this sector.

Instead of electric fields, we can write the EOMs using the magnetic fields as follows:

$$\begin{aligned}(\partial_0)^2 \mathsf{B} &= \tilde{\partial}^i \partial_0 \mathsf{E}_i - \frac{\bar{c}}{2} \epsilon_a{}^i \partial_0 (E^a)_i \\ &= \tilde{\partial}^i \tilde{\partial}_i \mathsf{B} - \frac{\bar{c}}{2} \epsilon_{ai} (\tilde{\partial}^i B^a + \epsilon^{ai} \mathsf{B}) \\ &= \partial^2 \mathsf{B} - \bar{c}\, \mathsf{B} + \frac{\bar{c}}{2} \partial_a B^a,\end{aligned} \tag{A38}$$

$$\begin{aligned}(\partial_0)^2 (\partial_a B^a) &= \tilde{\partial}^i \partial_a \partial_0 (E^a)_i \\ &= \tilde{\partial}^i \partial_a \left[\tilde{\partial}_i B^a + \epsilon^a_i \mathsf{B} \right] \\ &= \partial^2 \partial_a B^a - \partial^2 \mathsf{B},\end{aligned} \tag{A39}$$

$$\begin{aligned}(\partial_0)^2 (\tilde{\partial}_a B^a) &= \tilde{\partial}^i \tilde{\partial}_a \partial_0 (E^a)_i \\ &= \tilde{\partial}^i \tilde{\partial}_a \left[\tilde{\partial}_i B^a + \epsilon^a_i \mathsf{B} \right] \\ &= \partial^2 (\tilde{\partial}_a B^a).\end{aligned} \tag{A40}$$

Those equations can be written in the momentum space in the following matrix form,

$$\begin{pmatrix} \omega^2 - k^2 - \bar{c} & \bar{c}/2 & 0 \\ k^2 & \omega^2 - k^2 & 0 \\ 0 & 0 & \omega^2 - k^2 \end{pmatrix} \begin{pmatrix} \mathsf{B} \\ n_a B^a \\ \tilde{n}_a B^a \end{pmatrix} = \mathbf{0}. \tag{A41}$$

The longitudinal part of B^a is mixed with the rotational magnetic field B. The transverse part of B^a is decoupled and stays gapless.

[1] R. M. Nandkishore and M. Hermele, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).
[2] M. Pretko, X. Chen, and Y. You, Int. J. Mod. Phys. A **35**, 2030003 (2020).
[3] C. Chamon, Phys. Rev. Lett. **94**, 040402 (2005).
[4] J. Haah, Phys. Rev. A **83**, 042330 (2011).
[5] B. Yoshida, Phys. Rev. B **88**, 125122 (2013).
[6] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **92**, 235136 (2015).
[7] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **94**, 235157 (2016).
[8] C. Xu, arXiv:cond-mat/0602443.
[9] C. Xu, Phys. Rev. B **74**, 224433 (2006).
[10] M. Pretko, Phys. Rev. B **95**, 115139 (2017).
[11] M. Pretko, Phys. Rev. B **96**, 035119 (2017).

[12] M. Pretko and L. Radzihovsky, Phys. Rev. Lett. **120**, 195301 (2018).
[13] A. Gromov, Phys. Rev. X **9**, 031035 (2019).
[14] L. Bidussi, J. Hartong, E. Have, J. Musaeus, and S. Prohazka, arXiv:2111.03668.
[15] M. Pretko and L. Radzihovsky, Phys. Rev. Lett. **121**, 235301 (2018).
[16] D. X. Nguyen, A. Gromov, and S. Moroz, SciPost Phys. **9**, 076 (2020).
[17] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).
[18] C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
[19] E. A. Ivanov and V. I. Ogievetsky, JETP Lett. **23**, 606 (1976).
[20] I. Low and A. V. Manohar, Phys. Rev. Lett. **88**, 101602 (2002).
[21] A. Nicolis, R. Penco, and R. A. Rosen, Phys. Rev. D **89**, 045002 (2014).
[22] G. Goon, A. Joyce, and M. Trodden, Phys. Rev. D **90**, 025022 (2014).
[23] Y. Hidaka, T. Noumi, and G. Shiu, Phys. Rev. D **92**, 045020 (2015).
[24] Y. Hidaka, Y. Hirono, and R. Yokokura, Phys. Rev. Lett. **126**, 071601 (2021).
[25] M. J. Landry, arXiv:2101.02210.
[26] F. Peña Benitez, arXiv:2107.13884.
[27] A. Andreev and I. Lifshits, Zh. Eksp. Teor. Fiz. **56**, 2057 (1969).
[28] S. Weinberg, *The Quantum Theory of Fields, Vol. 1: Foundations* (Cambridge University Press, 2005).
[29] R. Jackiw and V. P. Nair, Phys. Lett. B **480**, 237 (2000).
[30] A. Nicolis, R. Penco, F. Piazza, and R. A. Rosen, J. High Energy Phys. 11 (2013) 055.
[31] L. Landau and E. Lifshitz, *Theory of Elasticity*, Course of Theoretical Physics, Vol. 7 (Elsevier, New York, 1986).
[32] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989).
[33] J. Marsden, *Mathematical Foundations of Elasticity* (Dover, New York, 1994).
[34] P. Chaikin, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).
[35] L. Zubov, *Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies* (Springer, Berlin, 1997).
[36] A. J. Beekman, J. Nissinen, K. Wu, K. Liu, R.-J. Slager, Z. Nussinov, V. Cvetkovic, and J. Zaanen, Phys. Rep. **683**, 1 (2017).
[37] A. J. Beekman, J. Nissinen, K. Wu, and J. Zaanen, Phys. Rev. B **96**, 165115 (2017).
[38] D. Son, Phys. Rev. Lett. **94**, 175301 (2005).
[39] E. Cosserat and F. Cosserat, *Théorie de Corps Déformables* (Hermann, Paris, 1909).
[40] A. C. Eringen, in *Microcontinuum Field Theories* (Springer, 1999), pp. 101–248.
[41] R. Lakes, Int. J. Solids Struct. **22**, 55 (1986).
[42] J. Yang and R. S. Lakes, J. Biomech. **15**, 91 (1982).
[43] Z. Rueger and R. S. Lakes, Phys. Rev. Lett. **120**, 065501 (2018).
[44] E. Ivanov and V. Ogievetsky, Theor. Math. Phys. **25**, 1050 (1975).
[45] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific, Singapore, 2008).
[46] A. Gromov, Phys. Rev. Lett. **122**, 076403 (2019).
[47] E. Cartan, *Riemannian Geometry in an Orthogonal Frame: From Lectures Delivered by Élie Cartan at the Sorbonne in 1926-1927* (World Scientific, River Edge, NJ, 2001).
[48] A. Yavari and A. Goriely, Arch. Ration. Mech. Anal. **205**, 59 (2012).
[49] S. Pai and M. Pretko, Phys. Rev. B **97**, 235102 (2018).
[50] A. Gromov and P. Surówka, SciPost Phys. **8**, 065 (2020).
[51] L. Radzihovsky and M. Hermele, Phys. Rev. Lett. **124**, 050402 (2020).
[52] A. Karch and D. Tong, Phys. Rev. X **6**, 031043 (2016).
[53] N. Seiberg, T. Senthil, C. Wang, and E. Witten, Ann. Phys. **374**, 395 (2016).
[54] L. V. Delacrétaz, A. Nicolis, R. Penco, and R. A. Rosen, Phys. Rev. Lett. **114**, 091601 (2015).
[55] G. Goon, K. Hinterbichler, A. Joyce, and M. Trodden, J. High Energy Phys. 06 (2012) 004.
[56] S. Endlich, A. Nicolis, R. Rattazzi, and J. Wang, J. High Energy Phys. 04 (2011) 102.
[57] A. Nicolis, arXiv:1108.2513.
[58] B. Horn, A. Nicolis, and R. Penco, J. High Energy Phys. 10 (2015) 153.