Curved-space Dirac description of elastically deformed monolayer graphene is generally incorrect

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Undistorted monolayer graphene has energy bands which cross at protected Dirac points. It elastically deforms, and much research has assumed the Dirac description persists, now in a curved space and coupled to a gauge field related to lattice strain. We show this is incorrect by using a real space gradient expansion to study how the Dirac equation derives from the tight-binding model. Generic spatially varying hopping functions give rise to large magnetic fields which spoil the truncation in derivatives. In the perturbative regime, the only consistent truncation to Dirac is one with nontrivial gauge field but in flat space. One can instead fine-tune the magnetic field to be small, and we derive the resulting differential condition that the hopping functions must satisfy to yield a consistent truncation to Dirac in curved space. We consider whether mechanical effects might impose this fine tuning but find this is not the case for a simple elastic membrane model.

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I. INTRODUCTION

Monolayer graphene has a band structure which contains two protected massless Dirac cones at the K and K' points. When undoped the bands are at half filling, putting the chemical potential at the Dirac points [1,2]. It can be bent beyond the linear regime [3], and when freely suspended, it naturally ripples [4,5]. When considering transport in such systems it is imperative to derive a continuum description of such distorted lattices.

Here we focus on the tight-binding model for graphene. For perturbative distortions, Fourier space calculations appear to show an effective description of Dirac fields in curved space coupled to a gauge field proportional to the strain of the lattice, originally for carbon nanotubes [6] and later for graphene [7–15]. It was noted early on that the magnetic field of this "strain gauge field" scales inversely with the lattice spacing, $\vec{A} \sim \mathcal{O}(1/a)$. This is naively concerning, as it appears the magnetic fields induced by strain and curvature would be very large; however, they are suppressed by the perturbative expansion. Based on these analyses, much work has assumed a curved-space Dirac description exists [16–25]. Our main result is to show that actually this inverse scaling of the magnetic field with the lattice spacing in fact does obstruct such a continuum Dirac description in curved space.

Here we derive an effective low-energy description from the tight-binding model via a real space gradient expansion, where we assume the hopping strengths vary on scales much larger than the lattice spacing. We consider both the leading Dirac term and the next two derivative corrections. We argue that the previous perturbative Dirac descriptions are inconsistent in that, while a correction to the frame-adding curvature exists, it is of the same order as the higher derivative term due to the magnetic field scaling as $1/a$. Hence in the perturbative case the correct gauged Dirac description is one in flat space. Worse still, in the nonlinear regime we argue that all higher derivative terms are of the same order and so there is no truncation whatsoever. In order to consistently keep curvature, the hopping strengths must be fine-tuned to remove large magnetic fields. When this unnatural fine tuning is made, we derive a gauged Dirac description nonlinearly in the hopping functions, which lives in a curved space with torsion-free spin connection.

II. THE LATTICE MODEL AND THE CONTINUUM

The tight-binding Hamiltonian for graphene is

$$H = \sum_{n,\vec{x}_A} \left(t_{n,\vec{x}_A+\frac{a\vec{\ell}_n}{2}} a^\dagger_{\vec{x}_A} b_{\vec{x}_A+a\vec{\ell}_n} + \text{H.c.} \right), \tag{1}$$

where $t_{n,\vec{x}}$ is the (real valued) hopping strength in one of the lattice translation directions n, $a^\dagger_{\vec{x}_A}$, $b^\dagger_{\vec{x}_B}$ are creation operators on the respective sublattices A and B, and $a\vec{\ell}_n$ are the translations from vertices in A to its neighbors, $\vec{\ell}_1 = (\sqrt{3}/2, 1/2)$, $\vec{\ell}_2 = (-\sqrt{3}/2, 1/2)$, $\vec{\ell}_3 = -\vec{\ell}_1 - \vec{\ell}_2$. Note we have put in an explicit lattice spacing a [it is implicitly in the lattice coordinates as, e.g., $\vec{x}_A = a(m_A\vec{a}_1 + n_A\vec{a}_2)$, $\vec{a}_{1,2} = \vec{\ell}_{1,2} - \vec{\ell}_3$]. Note that the t_n take values on the links, not the vertices. A general one-particle state is

$$|\Psi(t)\rangle = \left(\sum_{\vec{y}_A} A_{\vec{y}_A}(t) a^\dagger_{\vec{y}_A} + \sum_{\vec{y}_B} B_{\vec{y}_B}(t) b^\dagger_{\vec{y}_B} \right) |0\rangle, \tag{2}$$

and the Schrödinger equation $i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle$ gives

$$i\hbar\partial_t A_{\vec{x}_A} = \sum_n t_{n,\vec{x}_A+\frac{a\vec{\ell}_n}{2}} B_{\vec{x}_A+a\vec{\ell}_n},$$

$$i\hbar\partial_t B_{\vec{x}_B} = \sum_n t_{n,\vec{x}_B-\frac{a\vec{\ell}_n}{2}} A_{\vec{x}_B-a\vec{\ell}_n}. \tag{3}$$

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First we take the hopping parameters to slowly vary, so we may write $t_{n,\vec{x}} = t_n(\vec{x})$, where $t_n(\vec{x})$ are smooth functions of the coordinates $x^i = (x, y)$. We may then think of a continuum limit as we refine the lattice taking $a \to 0$. We write the lattice wave functions A, B in terms of slowly varying wave functions F, G and a rapidly oscillating phase $\Phi(\vec{x})/a$,

$$A_{\vec{x}}(t) = F(t,\vec{x}) f(\vec{x}) e^{i\left(\frac{\Phi(\vec{x})}{a} + \frac{\phi(\vec{x})}{2}\right)},$$
$$B_{\vec{x}}(t) = G(t,\vec{x}) f(\vec{x}) e^{i\left(\frac{\Phi(\vec{x})}{a} - \frac{\phi(\vec{x})}{2}\right)}, \tag{4}$$

where we assume the phases Φ and ϕ, and the rescaling function f are smooth functions of x^i and a, so smooth in the continuum limit $a \to 0$. Finding a $\Phi(\vec{x})$ to eliminate oscillations in $\Psi(\vec{x})$ that diverge as $a \to 0$ means that it is unique up to $\mathcal{O}(a)$ corrections, which are naturally interpreted as gauge transformations and will come with a compensating transformation of the strain gauge field, as we will see below. The subleading correction $\phi(\vec{x})$, under which F, G have *opposite* charge, is similarly unfixed and can naturally be interpreted as an $SO(2)$ rotation of the frame, as we shall see shortly. As we are interested in low-energy states and the lattice theory is static, the only time dependence is seen in the slowly varying modulation functions F and G. However, at this stage we will not assume these are smooth in the $a \to 0$ limit, only that we may perform a gradient expansion in $a\partial_i$. Now expanding in a we may write the Schrödinger equation as the continuum equation,

$$\frac{i\hbar}{T}\gamma^0 \partial_t \Psi = \gamma^a (u_a + i a z_a + a^2 r_a)\Psi - i a \gamma^a \left(w_a^i - i a q_a^i\right)\partial_i \Psi + \frac{a^2}{2}\gamma^a v_a^{ij} \partial_i \partial_j \Psi + O(a^3), \tag{5}$$

where $\Psi = (F, G)$ is a complex spinor and we choose Dirac matrices $\gamma^A = (\gamma^0, \gamma^a) = (-i\sigma^3, \sigma^1, \sigma^2)$, where σ^i are Pauli matrices and $\partial_i = \partial/\partial x^i$ are lattice coordinate derivatives. The quantities $u_a, z_a, r_a, w_a^i, q_a^i, v_a^{ij}$ are real, and the ones we require here are

$$I^a u_a = \frac{2ie^{i\phi}}{3T}\sum_n e^{-i\partial_n \Phi} t_n, \;\; I^a w_a^i = \frac{2e^{i\phi}}{3T}\sum_n \ell_n^i e^{-i\partial_n \Phi} t_n,$$
$$I^a v_a^{ij} = \frac{2ie^{i\phi}}{3T}\sum_n \ell_n^i \ell_n^j e^{-i\partial_n \Phi} t_n, \;\; I^a z_a = -\frac{f^2}{2}\partial_i\left(\frac{w^i}{f^2}\right), \tag{6}$$

with $I^a = (1, i)$ and $\partial_n = \vec{\ell}_n \cdot \vec{\partial}$. We introduce the energy scale T and think of $t_n(\vec{x}) = T$ as the undeformed model.

III. ATTEMPTING TO TRUNCATE TO DIRAC

We will now truncate this continuum Schrödinger equation to first derivative order acting on Ψ. We should be suspicious about whether such a truncation is valid, but for now we will interpret (5) to order $O(a)$ as a curved-space Dirac equation coupled to a gauge field. We compose spacetime coordinates $x^\mu = (t, x^i)$ and define the metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -v^2 dt^2 + g_{ij}(\vec{x}) dx^i dx^j. \tag{7}$$

Here v will be the Fermi velocity for the undeformed model, $t_n(\vec{x}) = T$. This may be written in terms of a frame e_A^μ and its dual e_μ^A, where

$$e_0^t = \frac{1}{v}, \quad e_a^t = e_0^i = 0, \quad g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B, \tag{8}$$

with $\eta_{AB} = \mathrm{diag}(-1, +1, +1)$. We fix

$$f = \sqrt{|\det e_i^a|} = |g|^{1/4}, \tag{9}$$

which ensures that the $U(1)$ charge density of the Dirac theory is that of the original electrons,

$$J^0 = \sqrt{g}\bar{\Psi}\gamma^0 \Psi = (A, B)^\dagger \cdot (A, B). \tag{10}$$

This is equivalent to ensuring the lattice anticommutators $\{a_{\vec{x}}^\dagger, a_{\vec{y}}\} = \{b_{\vec{x}}^\dagger, b_{\vec{y}}\} = \delta_{\vec{x},\vec{y}}$ imply the correct curved-space anticommutator $\{\Psi^\dagger(\vec{x}), \Psi(\vec{y})\} = \frac{1}{\sqrt{g}}\delta^{(2)}(\vec{x} - \vec{y})$.

Now the Schrödinger equation (5), truncated to first derivatives on spinors, can be written as

$$a i e_a^\mu \gamma^a D_\mu \Psi = O(a^2), \tag{11}$$

where we have taken $v = 3aT/(2\hbar)$, and the covariant derivative is given in terms of a magnetic gauge field $A_\mu = (0, A_i)$ and spin connection, parameterized here by the spatial 1 form Ω_i,

$$D_t \Psi = \partial_t \Psi, \quad D_i \Psi = \left(\partial_i - i A_i + \frac{i}{2}\sigma_3 \Omega_i\right)\Psi, \tag{12}$$

and the frame and gauge field to this order $O(a)$ are

$$e_a^i = w_a^i, \quad A_i = -\frac{1}{a} e_i^a u_a. \tag{13}$$

Using the relations (6) and our choice of f we find

$$\Omega^i = \epsilon_{ab} e_a^j \partial_j e_b^i, \tag{14}$$

which is precisely the *torsion-free* spin connection that follows from the frame e_a^i. While we might expect that in the absence of lattice defects torsion vanishes in a continuum description, it is pleasing to see this explicitly emerge.

We may understand the local freedom of shifting the phases of the two lattice fields A and B in Eq. (4) as a local frame rotation freedom,

$$\phi \to \phi + \delta\phi \;\Rightarrow\; \begin{cases} e_a^i & \to \quad e_a^i - \delta\phi\, \epsilon_{ab} e_b^i \\ A^i & \to \quad A^i \end{cases} \tag{15}$$

for an infinitesimal $\delta\phi$, together with a local gauge transform on the vector $A_a = e_a^i A_i$,

$$\Phi \to \Phi + \delta\Phi \;\Rightarrow\; \begin{cases} A_a & \to \quad A_a - \frac{1}{a}\partial_a \delta\Phi \\ e_a^i & \to \quad e_a^i - v_a^{ij}\partial_j \delta\Phi \end{cases} \tag{16}$$

for infinitesimal $\delta\Phi$.

Now we ask whether this truncation to first derivatives can be generally valid. We argue that for generic t_n it is not, as contributions from the standard Dirac kinetic terms $\gamma^a e_a^\mu(\nabla_\mu - iA_\mu)\Psi$ come at the same order as those from higher derivative terms in Eq. (5), such as $\gamma^a v_a^{ij}\partial_i\partial_j\Psi$. This may be seen in two ways:

(i) The gauge field in Eq. (13) goes as $A_i \sim 1/a$. The spinor Ψ responding to this will then generally have variation on scales that vanish as $a \to 0$, hence ruining the gradient expansion. We will discuss this explicitly for perturbative deformations.

(ii) The local phase symmetry in (16) is a gauge transformation for A_a, but e_a^i transforms too, and this is inconsistent with its interpretation as a frame which should be invariant.

On the latter issue there should be a continuum formulation of (5) written to manifest these local symmetries where the derivative expansion will be in covariant derivatives with respect to the gauge and frame symmetry. Consider a putative two-derivative term to match that in (5). It will have the form $a^2\gamma^a v_a^{ij}\tilde{D}_i\tilde{D}_j\Psi$ where $\tilde{D}_i$ are covariant. Note that their gauge and spin connections need not be the same as those of the leading Dirac theory. However, by taking the gauge field to scale as $\sim 1/a$, we then this term contains a contribution,

$$\frac{a^2}{2}\gamma^a v_a^{ij}\tilde{D}_i\tilde{D}_j\Psi \supset -ia^2\gamma^a v_a^{ij}B_i\partial_j\Psi, \tag{17}$$

for some gauge connection B_i. Then (13) gains a new term, $w_a^i = e_a^i + av_a^{ij}B_j$, and now if B_i transforms as $v_a^{ij}B_j \to v_a^{ij}B_j - \frac{1}{a}v_a^{ij}\partial_j\delta\Phi$; then we indeed see that e_a^i is invariant with B_i accounting for the transformation of w_a^i. However, then the gauge fields mix the contribution of covariant derivative terms between the partial derivative orders. This naturally extends to all higher derivative terms, with higher powers of the connection canceling the a suppression.

IV. TWO TRUNCATIONS TO DIRAC

In order to give a consistent truncation of our theory to the leading Dirac term we must tame the lattice scale gauge field by requiring $aA_i \to 0$ as $a \to 0$. There are two approaches.

A. Perturbative deformation

The lattice scale of the gauge field has been previously emphasized in [8]. In the derivation of curved-space Dirac of [7] and following work this was addressed using a perturbative expansion where

$$t_n = T(1 + \epsilon\delta t_n) \tag{18}$$

with $|\epsilon| \ll 1$. At $\epsilon = 0$ we return to the undeformed lattice Hamiltonian and can use standard momentum space tools. With our conventions a Dirac cone sits at the K point with wave vector $\vec{K} = \frac{1}{a}(\frac{4\pi}{3\sqrt{3}}, 0)$. This means that a slowly varying continuum field Ψ is related to lattice wave functions via $(A, B) = e^{i\vec{K}\cdot\vec{x}}\Psi$. Motivated by this, we will choose Φ to be this transformation at leading order,

$$\Phi = a\vec{K}\cdot x + \epsilon\chi(\vec{x}) = \frac{4\pi}{3\sqrt{3}}x + \epsilon\chi(\vec{x}). \tag{19}$$

The gauge field is then

$$A_i = -\frac{\epsilon}{a}\left(\frac{2}{3}\sum_n \epsilon_{ij}\delta_n^j\delta t_n + \partial_i\chi\right), \tag{20}$$

and we see $\chi(\vec{x})$ parameterizes the gauge freedom. While this goes as $\sim 1/a$, the perturbative expansion in ϵ controls this. The geometry depends on this gauge, the frame leading to the metric,

$$g_{ij} = \delta_{ij} - \frac{4}{3}\epsilon\sum_n \ell_n^i\ell_n^j\delta t_n + \epsilon K^{ijk}\varepsilon^{k\ell}\partial_\ell\chi, \tag{21}$$

which has Ricci scalar curvature

$$R = \frac{4}{3}\epsilon\sum_n\left(\delta^{ij} - \ell_n^i\ell_n^j\right)\partial_i\partial_j\delta t_n + \epsilon K^{ijk}\varepsilon^{k\ell}\partial_i\partial_j\partial_\ell\chi, \tag{22}$$

where we have defined $K^{ijk} = -\frac{4}{3}\sum_n \ell_n^i\ell_n^j\ell_n^k$. Taking the K' Dirac point corresponds to taking $\Phi \to -\Phi$, $e_1^i \to -e_1^i$, and $A_i \to -A_i$, and the metric is invariant. For $\chi = 0$ these reproduce the results of [7] (when we consider the physical metric rather than the Weyl rescaled one [22]). Previously these results have been taken to show the perturbatively deformed tight-binding model is described by a curved-space Dirac equation. However, we explicitly see here the gauge freedom χ gives a physical contribution to the curvature. Noting that two-dimensional geometry is locally characterized by the Ricci scalar, in fact we can then choose any geometry, including a flat one, with an appropriate gauge choice χ. The second concerning feature highlighted in [8] is that while we have controlled $A_i \sim \epsilon/a$, we have done this at the cost of the spin connection being parametrically smaller, $\Omega_i \sim \epsilon$, as $a \to 0$. In [8] it is argued that the spin connection should be ignored, leaving only a frame and gauge field. We will demonstrate that for general perturbations it is inconsistent to ignore the spin connection but not the variation of the metric, as they are of the same order, and further that the corrections to the frame and spin connection have the same order as the two-derivative term. Consider in some region of size L the perturbative deformation to be $\phi = \chi = \delta t_3 = 0$ and $\delta t_1 = -\delta t_2 = \frac{\sqrt{3}x}{2L}$. This yields $A_i = (0, \frac{\epsilon}{a}\frac{x}{L})$, a Landau gauge magnetic field $B = \frac{1}{\ell_B^2}$ with $\ell_B = \sqrt{aL/\epsilon}$ the magnetic length which diverges as $a \to 0$ but may be parametrically larger than the lattice scale. At leading order $O(\epsilon)$ and $a \ll L$ this may be solved by Landau levels. Here we focus on the lowest level wave function, taking $\Psi = (0, e^{-\frac{x^2}{2\ell_B^2}})$. Now we can evaluate the Dirac term on this leading solution and compare this to the two-derivative term. Then at $O(\epsilon)$ we have $v_a^{ij} = \frac{1}{2}e_a^k K^{kmi}\epsilon_{mj}$, and find

$$iae_a^i\gamma^a\nabla_i\Psi = -\frac{ia\epsilon}{4L}\sigma^1\Psi,\ \ a^2\gamma^a v_a^{ij}\partial_i\partial_j\Psi = -\frac{ia\epsilon}{2L}\sigma^1\Psi, \tag{23}$$

so both go as $\sim a\epsilon/L$. We emphasize that the Dirac term is nonzero here due to the varying frame at $O(\epsilon)$. Thus this simple example demonstrates that the perturbative contribution to the frame in the Dirac term is the same order as higher derivative terms, and it is therefore inconsistent to consider it in isolation.

We note it *is* consistent to truncate to Dirac if we ignore corrections to the frame, as then for $a \ll L$ these terms are small corrections to the flat gauged Dirac term.

B. Fine tuning

In order to preserve curvature we are forced to fine-tune the gauge field so that $aA_i \to 0$ as $a \to 0$, which implies the condition $u_a \to 0$ in this limit. The condition $u_a = 0$ is solved by

$$t_1 = \frac{\sin[(\vec{\ell}_2 - \vec{\ell}_3)\cdot\vec{\partial}\Phi]}{\sin[(\vec{\ell}_1 - \vec{\ell}_2)\cdot\vec{\partial}\Phi]}t_3, t_2 = \frac{\sin[(\vec{\ell}_1 - \vec{\ell}_3)\cdot\vec{\partial}\Phi]}{\sin[(\vec{\ell}_2 - \vec{\ell}_1)\cdot\vec{\partial}\Phi]}t_3, \quad (24)$$

so that Φ determines the ratios $t_{1,2}/t_3$. However, we wish to specify couplings t_n. We can solve (24) for $Q_x = \partial_x\Phi$ and $Q_y = \partial_y\Phi$ independently as functions of t_n [26]. Explicitly, if we define

$$X = e^{\frac{\sqrt{3}i}{2}Q_x}, \quad Y = e^{\frac{3i}{2}Q_y}, \quad (25)$$

then we find

$$Y = -\frac{t_3 X}{t_2 + t_1 X^2}, \quad X^4 + \frac{t_1^2 + t_2^2 - t_3^3}{t_1 t_2}X^2 + 1 = 0, \quad (26)$$

where the two roots of the quadratic in X^2 yield the inequivalent Dirac points. In the case of homogeneous t_n where we can Fourier transform and work in momentum space, $\vec{Q}/a$ given by the two roots are the wave vectors of the two Dirac points.

However, given $\vec{Q}$ one can only integrate to find a phase Φ if the integrability condition $\partial_x Q_y = \partial_y Q_x$ holds. Using the explicit solution for $\vec{Q}$ above, we find this condition can be written neatly as the constraint that the t_n must obey,

$$\sum_n V_n^i \frac{\partial_i t_n}{t_n} = 0, V_m^i = \sum_n (K^{ijk}\ell_m^j \ell_n^k - \frac{1}{2}\ell_m^i)t_n^2. \quad (27)$$

We will physically interpret this constraint shortly.

An important solution to this is constant but unequal t_n [27], which corresponds to constant nonzero strain. As we show below, in this case the metric has changed but is still constant, and so there is no geometric curvature. This corresponds to a misalignment of the lattice and ambient space coordinates. Using the preserved translational symmetry one can still work in momentum space, finding that the location of the Dirac cone has moved.

Another special solution where the t_n are spatially varying is when they are equal, so $t_n = t(\vec{x})$. This corresponds to a pure expansion lattice distortion, with vanishing strain.

Suppose the above condition (27) holds so we may find a solution to (24) written as $\Phi = f(t_{1,2}/t_3)$. Then a second solution is given by $\Phi = -f(t_{1,2}/t_3)$, corresponding to the other Dirac point. For the two solutions we find the frames,

$$\begin{pmatrix} e_1^x & e_1^y \\ e_2^x & e_2^y \end{pmatrix} = -\frac{1}{\sqrt{3}Tt_3}\mathbf{R}'\cdot\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \Delta & 0 \\ t_1^2 - t_2^2 & \sqrt{3}t_3^2 \end{pmatrix}, \quad (28)$$

where $\mathbf{R}' = (\begin{smallmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{smallmatrix})$ is a frame rotation with $\theta = \phi + \partial_y\Phi$ and Δ is given by the two solutions of

$$\Delta^2 = \left(\sum_n t_n^2\right)^2 - 2\sum_n t_n^4. \quad (29)$$

The plus sign above corresponds to the solution which, for undeformed t_n, gives the K point. The minus sign is for the solution giving the K' point for undeformed t_n. For both of these, the corresponding spatial metric that the Dirac field sees, which we will call the "electrometric," is

$$g_{ij} = e_i^a e_j^b \eta_{ab} = \frac{3T^2}{\Delta^2}\sum_n \left(\delta_{ij} - \frac{4}{3}\ell_n^i \ell_n^j\right)t_n^2. \quad (30)$$

In the special case of equal $t_n = t(\vec{x})$, then the electrometric is Weyl flat, $g_{ij} = [T/t(\vec{x})]^2\delta_{ij}$. Recalling the strain gauge field vanishes, this case can then be interpreted as massless Dirac fields coupled only to a curved geometry and no strain gauge field.

Strictly imposing the constraint (27) gives an exactly vanishing strain gauge field. However, if we define the map from the lattice to the hopping functions with a subleading behavior,

$$t_{n,\vec{x}} = t_n(\vec{x}, a) = \bar{t}_n(\vec{x}) + a\tau_n(\vec{x}) + O(a^2), \quad (31)$$

and for the phase we write

$$\Phi(\vec{x}, a) = \bar{\Phi}(\vec{x}) + a\chi(\vec{x}) + O(a^2) \quad (32)$$

and impose the constraint only on the $\bar{t}_n$ so it is not exactly satisfied, but is as $a \to 0$, then we may retain an $O(a^0)$ strain gauge field,

$$A_i = \pm\frac{\Delta}{2\sqrt{3}\bar{t}_1\bar{t}_2}\begin{pmatrix} -\frac{\bar{t}_1+\bar{t}_2}{(\bar{t}_1+\bar{t}_2)^2-\bar{t}_3^2} & \frac{\bar{t}_1-\bar{t}_2}{(\bar{t}_1-\bar{t}_2)^2-\bar{t}_3^2} & \frac{4\bar{t}_1\bar{t}_2\bar{t}_3}{\Delta^2} \\ \frac{1}{\sqrt{3}}\frac{\bar{t}_1-\bar{t}_2}{(\bar{t}_1+\bar{t}_2)^2-\bar{t}_3^2} & -\frac{1}{\sqrt{3}}\frac{\bar{t}_1+\bar{t}_2}{(\bar{t}_1-\bar{t}_2)^2-\bar{t}_3^2} & -\frac{4}{\sqrt{3}}\frac{(t_1^2-t_2^2)t_1t_2}{\Delta^2 t_3} \end{pmatrix}\begin{pmatrix} \tau_+ \\ \tau_- \\ \tau_3 \end{pmatrix} - \partial_i\chi, \quad (33)$$

where $\tau_\pm = \tau_1 \pm \tau_2$. The signs correspond to the solutions as for the frame in Eq. (28). Taking $\Phi \to \Phi + a\,\delta\Phi$, then A_i transforms by a gauge transformation, $A_i \to A_i - \partial_i\delta\Phi$, and the frame is invariant to leading order in a as we require.

For general spatially varying hopping functions satisfying the constraint (27) we can define a wave vector $\vec{Q}[t_n(\vec{x})]$ by using exactly the same expressions as in Eqs. (25) and (26) for the homogeneous case, but now with varying $t_n(\vec{x})$. Explicit calculation of the magnetic field of $\vec{A}$ from its expression above shows it is simply related to the curl of this wave vector $\vec{Q}$ as

$$\frac{1}{a}(\partial_x Q_y - \partial_y Q_x) = F_{xy} = \partial_x A_y - \partial_y A_x. \quad (34)$$

Hence up to a gauge transformation we see that $\vec{Q}/a$ is equal to the gauge field $\vec{A}$. Finally, we see that the constraint condition (27), which is that $\nabla \times \vec{Q}$ vanishes, is in fact just the requirement that the strain magnetic field is indeed finite as $a \to 0$.

V. FINE-TUNING AND EMBEDDING

Can this fine-tuning of Eq. (27) arise from mechanical considerations? Consider an almost flat embedding into $\mathbb{R}^3$ with coordinates (X^i, Z) given by

$$X^i = x^i + \epsilon v^i(\vec{x}), \quad Z = \sqrt{\epsilon} h(\vec{x}), \tag{35}$$

with height function h and strain field v^i. Linearizing in ϵ the induced metric of this embedding is

$$g_{ij}^{(\mathrm{ind})} = \delta_{ij} + 2\epsilon\sigma_{ij}, \quad \sigma_{ij} = \tfrac{1}{2}(\partial_i v_j + \partial_j v_i + \partial_i h \partial_j h), \tag{36}$$

with indices lowered/raised using δ_{ij} and σ_{ij} as the strain tensor. Assuming the hopping parameters depend on bond length and not angle [28], then to $O(\epsilon)$,

$$t_n \simeq T(1 - \epsilon\beta\sigma_{ij}\ell_n^i\ell_n^j), \tag{37}$$

where β can be estimated for graphene as $\beta \simeq 3.3$ [29]. We may neatly express our fine-tuning condition (27) as $K^{ijk}\partial_k\sigma_{ij} = 0$. Consider the canonical elastic energy,

$$E_{\mathrm{mech}} = \int d^2\vec{x}\left(\frac{\kappa}{2}(\partial^2 h)^2 + \mu\sigma_{ij}^2 + \frac{\lambda}{2}\sigma_{ii}^2\right), \tag{38}$$

with bending rigidity κ and Lamé coefficients μ and λ. Varying the strain field v^i yields $\mu\partial_j\sigma_{ji} + \frac{\lambda}{2}\partial_i\sigma_{jj} = 0$. Assuming $\mu, \lambda > 0$, then this is generally incompatible with the previous fine-tuning condition. Thus this membrane energetics does not impose the necessary constraint on strain for a curved-space Dirac description.

In the event that the constraint is satisfied, the perturbation to the electrometric is $g_{ij} \simeq \delta_{ij} + 2\epsilon\beta\sigma_{ij}$. As emphasized in Refs. [10,22], this is not the same as the induced metric in Eq. (36). Nonperturbatively, the $t_n(\vec{x})$ will be a functional of the induced metric $g_{ij}^{(\mathrm{ind})}(\vec{x})$, and the map to the electrometric is given by Eq. (30).

VI. CONCLUSION

We have argued that contrary to graphene folklore, the tight-binding model with generic slow spatial variation of the hopping functions does not have a curved-space Dirac description coupled to a strain gauge field. We find a continuum spinor description, but this is generally obstructed from truncating to first spinor derivatives by large magnetic fields. Making the t_n vary perturbatively cannot solve this, although it does allow a consistent flat space Dirac description with strain gauge field. However, for generic slow nonperturbative variation of t_n there is no Dirac description at all. Related issues have been noted in the lattice literature when studying Euclidean theories on the honeycomb lattice as well [30]. Standard examples of constant uniaxial strain, which have been well studied in the literature [27], do not in any way disagree with this result, as a constant strain can be interpreted as a constant metric deformation, which induces no curvature.

One may obtain a curved-space Dirac description if one fine-tunes the variation of the hopping functions. However, this fine-tuning appears unnatural, and we have shown simple membrane energetics will not impose it. Thus we believe that elastically deformed graphene monolayers do not generally have a curved-space Dirac description. Likewise, optical lattice constructions of graphenelike lattices [31] will need to be highly fine-tuned to recover curved-space Dirac. This clearly has important implications for using graphene and other lattice systems as a laboratory to study curved-space quantum field theory and analog gravity, as well as requiring a new paradigm to understand transport.

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