Interband magnon drag in ferrimagnetic insulators

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We propose a new drag phenomenon, an interband magnon drag, and report on interaction effects and multiband effects in magnon transport of ferrimagnetic insulators. We study a spin-Seebeck coefficient S_{m}, a magnon conductivity σ_{m}, and a magnon thermal conductivity κ_{m} of interacting magnons for a minimal model of ferrimagnetic insulators using a $1/S$ expansion of the Holstein-Primakoff method, the linear-response theory, and a method of Green's functions. We show that the interband magnon drag enhances σ_{m} and reduces κ_{m}, whereas its total effects on S_{m} are small. This drag results from the interband momentum transfer induced by the magnon-magnon interactions. We also show that the higher-energy band magnons contribute to S_{m}, σ_{m}, and κ_{m} even for temperatures smaller than the energy difference between the two bands.

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I. INTRODUCTION

Magnon transport is the key to understanding spintronics and spin-caloritronics phenomena of magnetic insulators [1–3]. For example, a magnon spin current is vital for the spin Seebeck effect [2,4–7]. Magnon transport is important also for other relevant phenomena [8–13].

There are two key issues about magnon transport in ferrimagnetic insulators. One is about multiband effects. Yttrium iron garnet (YIG) is a ferrimagnetic insulator used in various spintronics or spin-caloritronics phenomena [1–3,8–12]. Its magnons have been often approximated as those of a ferromagnet. However, a study using its realistic model [14] showed that not only the lowest-energy band magnons, which could be approximated as those of a ferromagnet, but also the second-lowest-energy band magnons should be considered except for sufficiently low temperatures. Since the experiments using YIG are performed typically at room temperature [1–3,8,9,11,12], it is necessary to clarify the effects of the higher-energy band magnons on the magnon transport. The other is about interaction effects. The magnon-magnon interactions are usually neglected. However, their effects may be drastic in a ferrimagnet because they can induce the interband momentum transfer, which is expected to cause an interband magnon drag by analogy with various drag phenomena [15–40]. Nevertheless, it remains unclear how the magnon-magnon interactions affect the magnon transport.

In this paper, we provide the first step towards resolving the above issues and propose a new drag phenomenon, the interband magnon drag. We derive three transport coefficients of interacting magnons for a two-sublattice ferrimagnet and numerically evaluate their temperature dependences. We show that the interband magnon drag enhances a magnon conductivity and reduces a magnon thermal conductivity, whereas its total effects on a spin-Seebeck coefficient are small. We also show that the higher-energy band magnons contribute to these transport coefficients even for temperatures lower than the energy splitting of the two bands.

II. MODEL

Our ferrimagnetic insulator is described by

$$H = 2J\sum_{\langle i,j\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_j - h\sum_{i=1}^{N/2}S_i^z - h\sum_{j=1}^{N/2}S_j^z, \tag{1}$$

where the first term is the Heisenberg exchange interaction between nearest-neighbor spins, and the others are the Zeeman energy of a weak magnetic field ($|h| \ll J$). (The ground-state magnetization is aligned parallel to the magnetic field.) We have disregarded the dipolar interaction and the magnetic anisotropy, which are usually much smaller than J [14,41]. For concreteness, we consider a two-sublattice ferrimagnet on the body-centered cubic lattice (Fig. 1); i's and j's in Eq. (1) are site indices of the A and B sublattice, respectively. There are $N/2$ sites per sublattice. Our model can be regarded as a minimal model of ferrimagnetic insulators because a ferrimagnetic state, the spin alignments of which are given by $\boldsymbol{S}_i = {}^t(0\ 0\ S_A)$ for all i's and $\boldsymbol{S}_j = {}^t(0\ 0\ -S_B)$ for all j's, is stabilized for $J > 0$ with the weak magnetic field. We set $\hbar = 1$, $k_{\mathrm{B}} = 1$, and $a = 1$, where a is the lattice constant.

To describe magnons of our ferrimagnetic insulator, we rewrite Eq. (1) by using the Holstein-Primakoff method [42]. By applying the Holstein-Primakoff transformation [43–45] to Eq. (1) and using a $1/S$ expansion [43,44,46] and the Fourier transformation of magnon operators, we can write Eq. (1) in the form

$$H = H_{\mathrm{KE}} + H_{\mathrm{int}}. \tag{2}$$

Here H_{KE} represents the kinetic energy of magnons,

$$H_{\mathrm{KE}} = \sum_{\boldsymbol{q}}\left(a_{\boldsymbol{q}}^{\dagger}\ b_{\boldsymbol{q}}\right)\begin{pmatrix}\epsilon_{AA} & \epsilon_{AB}(\boldsymbol{q})\\ \epsilon_{AB}(\boldsymbol{q}) & \epsilon_{BB}\end{pmatrix}\begin{pmatrix}a_{\boldsymbol{q}}\\ b_{\boldsymbol{q}}^{\dagger}\end{pmatrix}, \tag{3}$$

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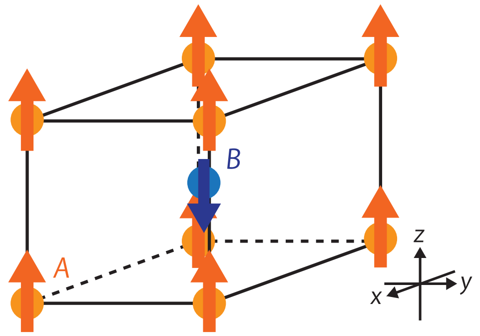

FIG. 1. Our ferrimagnetic insulator. The up or down arrows represent the spins on the A or B sublattice, respectively. The x, y, and z axes are also shown.

where $\epsilon_{AA} = 2J_{\mathbf{0}}S_B + h$, $\epsilon_{AB}(\boldsymbol{q}) = 2\sqrt{S_AS_B}J_{\boldsymbol{q}}$, $\epsilon_{BB} = 2J_{\mathbf{0}}S_A - h$, and $J_{\boldsymbol{q}} = 8J\cos\frac{q_x}{2}\cos\frac{q_y}{2}\cos\frac{q_z}{2}$; H_{int} represents the leading terms of magnon-magnon interactions,

$$H_{\text{int}} = -\frac{1}{N}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_2,\boldsymbol{q}_4}\delta_{\boldsymbol{q}_1+\boldsymbol{q}_2,\boldsymbol{q}_3+\boldsymbol{q}_4}\Bigg(2J_{\boldsymbol{q}_1-\boldsymbol{q}_3}a^{\dagger}_{\boldsymbol{q}_1}a_{\boldsymbol{q}_3}b^{\dagger}_{\boldsymbol{q}_4}b_{\boldsymbol{q}_2} + \sqrt{\frac{S_A}{S_B}}J_{\boldsymbol{q}_1}a_{\boldsymbol{q}_1}b^{\dagger}_{\boldsymbol{q}_2}b_{\boldsymbol{q}_3}b_{\boldsymbol{q}_4} + \sqrt{\frac{S_B}{S_A}}J_{\boldsymbol{q}_1}b_{\boldsymbol{q}_1}a^{\dagger}_{\boldsymbol{q}_2}a_{\boldsymbol{q}_3}a_{\boldsymbol{q}_4}\Bigg) + \text{H.c.} \tag{4}$$

We can also express H_{KE} as a two-band Hamiltonian by using the Bogoliubov transformation [43–45]:

$$H_{\text{KE}} = \sum_{\boldsymbol{q}}[\epsilon_{\alpha}(\boldsymbol{q})\alpha^{\dagger}_{\boldsymbol{q}}\alpha_{\boldsymbol{q}} + \epsilon_{\beta}(\boldsymbol{q})\beta_{\boldsymbol{q}}\beta^{\dagger}_{\boldsymbol{q}}], \tag{5}$$

where $\epsilon_{\alpha}(\boldsymbol{q}) = h + J_{\mathbf{0}}(S_B - S_A) + \Delta\epsilon_{\boldsymbol{q}}$, $\epsilon_{\beta}(\boldsymbol{q}) = -h + J_{\mathbf{0}}(S_A - S_B) + \Delta\epsilon_{\boldsymbol{q}}$, and $\Delta\epsilon_{\boldsymbol{q}} = \sqrt{J^2_{\mathbf{0}}(S_A+S_B)^2 - 4S_AS_BJ^2_{\boldsymbol{q}}}$. For $S_A > S_B$, we have $\epsilon_{\alpha}(\boldsymbol{q}) < \epsilon_{\beta}(\boldsymbol{q})$. Note that the Bogoliubov transformation is given by $a_{\boldsymbol{q}} = (U_{\boldsymbol{q}})_{A\alpha}\alpha_{\boldsymbol{q}} + (U_{\boldsymbol{q}})_{A\beta}\beta^{\dagger}_{\boldsymbol{q}}$ and $b^{\dagger}_{\boldsymbol{q}} = (U_{\boldsymbol{q}})_{B\alpha}\alpha_{\boldsymbol{q}} + (U_{\boldsymbol{q}})_{B\beta}\beta^{\dagger}_{\boldsymbol{q}}$, where $(U_{\boldsymbol{q}})_{A\alpha} = (U_{\boldsymbol{q}})_{B\beta} = \cosh\theta_{\boldsymbol{q}}$, $(U_{\boldsymbol{q}})_{A\beta} = (U_{\boldsymbol{q}})_{B\alpha} = -\sinh\theta_{\boldsymbol{q}}$, and these hyperbolic functions satisfy $\cosh 2\theta_{\boldsymbol{q}} = [J_{\mathbf{0}}(S_A+S_B)]/\Delta\epsilon_{\boldsymbol{q}}$ and $\sinh 2\theta_{\boldsymbol{q}} = (2\sqrt{S_AS_B}J_{\boldsymbol{q}})/\Delta\epsilon_{\boldsymbol{q}}$. Then, by using the Bogoliubov transformation, we can decompose H_{int} into the intraband and the interband components [47]. Because of these properties, our model is a minimal model to study the two key issues explained above.

III. DERIVATIONS OF TRANSPORT COEFFICIENTS

We consider three transport coefficients: a spin-Seebeck coefficient S_{m}, a magnon conductivity σ_{m}, and a magnon thermal conductivity κ_{m}. They are given by $S_{\text{m}} = L_{12}$, $\sigma_{\text{m}} = L_{11}$, and $\kappa_{\text{m}} = L_{22}$, where $L_{\mu\eta}$'s are defined as

$$\boldsymbol{j}_S = L_{11}\boldsymbol{E}_S + L_{12}\Big(-\frac{\nabla T}{T}\Big), \tag{6}$$

$$\boldsymbol{j}_Q = L_{21}\boldsymbol{E}_S + L_{22}\Big(-\frac{\nabla T}{T}\Big). \tag{7}$$

Here $\boldsymbol{j}_S$ and $\boldsymbol{j}_Q$ are magnon spin and heat, respectively, current densities, $\boldsymbol{E}_S$ is a nonthermal external field, and ∇T is a temperature gradient. (Note that one of the possible choices of $\boldsymbol{E}_S$ is a magnetic-field gradient [48].) $L_{21} = L_{12}$ holds owing to the Onsager reciprocal theorem. It should be noted that although κ_{m} is generally given by $\kappa_{\text{m}} = L_{22} - \frac{L_{21}L_{12}}{L_{11}}$, our definition $\kappa_{\text{m}} = L_{22}$ is sufficient to describe the thermal magnon transport at low temperatures at which the magnon picture is valid because the L_{22} gives the leading temperature dependence. Since a magnon chemical potential is zero in equilibrium, $\boldsymbol{j}_Q = \boldsymbol{j}_E$, where $\boldsymbol{j}_E$ is a magnon energy current density. Hereafter we focus on the magnon transport with $\boldsymbol{E}_S$ or $(-\nabla T/T)$ applied along the x axis.

We express $L_{\mu\eta}$'s in terms of the correlation functions using the linear-response theory [23,49–54]. First, L_{12} is given by

$$L_{12} = \lim_{\omega\to 0}\frac{\Phi^{\text{R}}_{12}(\omega) - \Phi^{\text{R}}_{12}(0)}{i\omega}, \tag{8}$$

where $\Phi^{\text{R}}_{12}(\omega) = \Phi_{12}(i\Omega_n \to \omega + i\delta)$ $(\delta = 0+)$,

$$\Phi_{12}(i\Omega_n) = \int_0^{T^{-1}}d\tau e^{i\Omega_n\tau}\frac{1}{N}\langle T_{\tau}J^x_S(\tau)J^x_E\rangle, \tag{9}$$

and $\Omega_n = 2\pi Tn$ $(n > 0)$. Here T_{τ} is the time-ordering operator [51], and J^x_S and J^x_E are spin and energy, respectively, current operators. They are obtained from the continuity equations [55–57] (see Appendix A); the results are

$$J^x_S = -\sum_{\boldsymbol{q}}\sum_{l,l'=A,B}v^x_{ll'}(\boldsymbol{q})x^{\dagger}_{\boldsymbol{q}l}x_{\boldsymbol{q}l'}, \tag{10}$$

$$J^x_E = \sum_{\boldsymbol{q}}\sum_{l,l'=A,B}e^x_{ll'}(\boldsymbol{q})x^{\dagger}_{\boldsymbol{q}l}x_{\boldsymbol{q}l'}, \tag{11}$$

where $v^x_{ll'}(\boldsymbol{q}) = (1-\delta_{l,l'})\frac{\partial\epsilon_{AB}(\boldsymbol{q})}{\partial q_x}$, $x_{\boldsymbol{q}A} = a_{\boldsymbol{q}}$, $x_{\boldsymbol{q}B} = b^{\dagger}_{\boldsymbol{q}}$, $e^x_{BB}(\boldsymbol{q}) = -e^x_{AA}(\boldsymbol{q}) = \epsilon_{AB}(\boldsymbol{q})\frac{\partial\epsilon_{AB}(\boldsymbol{q})}{\partial q_x}$, and $e^x_{AB}(\boldsymbol{q}) = e^x_{BA}(\boldsymbol{q}) = \frac{1}{2}(\epsilon_{AA} - \epsilon_{BB})\frac{\partial\epsilon_{AB}(\boldsymbol{q})}{\partial q_x}$. In deriving Eqs. (10) and (11), we have omitted the corrections due to H_{int} because they may be negligible [23]. Then we can obtain L_{11} by replacing J^x_E in $\Phi_{12}(i\Omega_n)$ by J^x_S, and L_{22} by replacing $J^x_S(\tau)$ in $\Phi_{12}(i\Omega_n)$ by $J^x_E(\tau)$. Thus the derivation of L_{12} is enough in obtaining $L_{\mu\nu}$'s. In addition, since we can derive L_{12} in a similar way to the derivations of electron transport coefficients [23,33,50,54,58], we explain its main points below. (Note that the Bose-Einstein condensation of magnons is absent in our situation.)

By substituting Eqs. (10) and (11) into Eq. (9) and performing some calculations (for the details see Appendix B), we obtain

$$L_{12} = L^0_{12} + L'_{12}. \tag{12}$$

First, L^0_{12}, the noninteracting L_{12}, is given by (see Appendix B)

$$L^0_{12} = \frac{1}{\pi N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta}v^x_{\nu'\nu}(\boldsymbol{q})e^x_{\nu\nu'}(\boldsymbol{q})I^{(\text{I})}_{\nu\nu'}(\boldsymbol{q}), \tag{13}$$

where $v^x_{\nu'\nu}(\boldsymbol{q}) = \sum_{l,l'=A,B}v^x_{ll'}(\boldsymbol{q})(U_{\boldsymbol{q}})_{l\nu'}(U_{\boldsymbol{q}})_{l'\nu}$, $e^x_{\nu\nu'}(\boldsymbol{q}) = \sum_{l,l'=A,B}e^x_{ll'}(\boldsymbol{q})(U_{\boldsymbol{q}})_{l\nu}(U_{\boldsymbol{q}})_{l'\nu'}$, and

$$I^{(\text{I})}_{\nu\nu'}(\boldsymbol{q}) = \int_{-\infty}^{\infty}dz\frac{\partial n(z)}{\partial z}\text{Im}G^{\text{R}}_{\nu}(\boldsymbol{q},z)\text{Im}G^{\text{R}}_{\nu'}(\boldsymbol{q},z). \tag{14}$$

Here $n(z) = (e^{z/T}-1)^{-1}$, $G^{\text{R}}_{\alpha}(\boldsymbol{q},z) = [z - \epsilon_{\alpha}(\boldsymbol{q}) + i\gamma]^{-1}$, $G^{\text{R}}_{\beta}(\boldsymbol{q},z) = -[z + \epsilon_{\beta}(\boldsymbol{q}) + i\gamma]^{-1}$, and γ is the magnon damping. Next, L'_{12}, the leading correction due to the first-order perturbation of $H_{\rm int}$, is given by (see Appendix B)

$$L'_{12} = \frac{1}{\pi^2 N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu_1\nu_2,\nu_3,\nu_4} v^x_{\nu_1\nu_2}(\boldsymbol{q})e^x_{\nu_3\nu_4}(\boldsymbol{q}')V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}') \times\left[I^{(\rm I)}_{\nu_1\nu_2}(\boldsymbol{q})I^{(\rm II)}_{\nu_3\nu_4}(\boldsymbol{q}')+I^{(\rm II)}_{\nu_1\nu_2}(\boldsymbol{q})I^{(\rm I)}_{\nu_3\nu_4}(\boldsymbol{q}')\right], \tag{15}$$

where

$$I^{(\rm II)}_{\nu\nu'}(\boldsymbol{q}) = \int_{-\infty}^{\infty} dz n(z)\mathrm{Im}\left[G^{\rm R}_{\nu}(\boldsymbol{q},z)G^{\rm R}_{\nu'}(\boldsymbol{q},z)\right], \tag{16}$$

$V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}') = 4J_{\boldsymbol{q}-\boldsymbol{q}'}\sum_l (U_{\boldsymbol{q}})_{l\nu_1}(U_{\boldsymbol{q}})_{\bar{l}\nu_2}(U_{\boldsymbol{q}'})_{\bar{l}\nu_3}(U_{\boldsymbol{q}'})_{l\nu_4}$, and $\bar{l}$ is B or A for $l=A$ or B, respectively. Then we obtain

$$L_{11} = L^0_{11} + L'_{11},\ L_{22} = L^0_{22} + L'_{22}, \tag{17}$$

where L^0_{11}, L'_{11}, L^0_{22}, and L'_{22} are obtained by replacing $e^x_{\nu\nu'}(\boldsymbol{q})$ in Eq. (13) by $-v^x_{\nu\nu'}(\boldsymbol{q})$, $e^x_{\nu_3\nu_4}(\boldsymbol{q}')$ in Eq. (15) by $-v^x_{\nu_3\nu_4}(\boldsymbol{q}')$, $v^x_{\nu'\nu}(\boldsymbol{q})$ in Eq. (13) by $-e^x_{\nu'\nu}(\boldsymbol{q})$, and $v^x_{\nu_1\nu_2}(\boldsymbol{q})$ in Eq. (15) by $-e^x_{\nu_1\nu_2}(\boldsymbol{q})$, respectively.

Since we suppose that the magnon lifetime $\tau = (2\gamma)^{-1}$ is long enough to regard magnons as quasiparticles, we rewrite Eqs. (13) and (15) by taking the limit $\tau\to\infty$. First, Eq. (13) reduces to

$$L^0_{12} \sim L^0_{12\alpha} + L^0_{12\beta}, \tag{18}$$

where

$$L^0_{12\nu} \sim \frac{1}{N}\sum_{\boldsymbol{q}} v^x_{\nu\nu}(\boldsymbol{q})e^x_{\nu\nu}(\boldsymbol{q})\tau\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}. \tag{19}$$

(The detailed derivation is described in Appendix C.) This expression is consistent with that obtained in the Boltzmann theory with the relaxation-time approximation [59]. Equation (18) shows that $L^0_{12}\approx L^0_{12\alpha}$ at sufficiently low temperatures for $S_A > S_B$ owing to $\frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})} \gg \frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial\epsilon_\beta(\boldsymbol{q})}$. Similarly, we obtain

$$L^0_{11} \sim L^0_{11\alpha} + L^0_{11\beta},\ L^0_{22} \sim L^0_{22\alpha} + L^0_{22\beta}, \tag{20}$$

where $L^0_{11\nu}$ and $L^0_{22\nu}$ are obtained by replacing $e^x_{\nu\nu}(\boldsymbol{q})$ in Eq. (19) by $-v^x_{\nu\nu}(\boldsymbol{q})$ and by replacing $v^x_{\nu\nu}(\boldsymbol{q})$ by $-e^x_{\nu\nu}(\boldsymbol{q})$, respectively. Then, as we show in Appendix C, Eq. (15) reduces to

$$L'_{12} \sim L'_{12\text{-intra}} + L'_{12\text{-inter1}} + L'_{12\text{-inter2}}, \tag{21}$$

where $L'_{12\text{-intra}}$ is the correction due to the intraband interactions,

$$L'_{12\text{-intra}} = L'_{12\text{-intra-}\alpha} + L'_{12\text{-intra-}\beta}, \tag{22}$$

$$L'_{12\text{-intra-}\nu} = -\frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q})e^x_{\nu\nu}(\boldsymbol{q}')\tau V_{\nu\nu\nu\nu}(\boldsymbol{q},\boldsymbol{q}') \times\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial\epsilon_\nu(\boldsymbol{q}')}, \tag{23}$$

and $L'_{12\text{-inter1}}$ and $L'_{12\text{-inter2}}$ are the corrections due to the interband interactions,

$$L'_{12\text{-inter1}} = -\frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\alpha\alpha}(\boldsymbol{q})e^x_{\beta\beta}(\boldsymbol{q}')\tau V_{\alpha\alpha\beta\beta}(\boldsymbol{q},\boldsymbol{q}') \times\frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}\frac{\partial n[\epsilon_\beta(\boldsymbol{q}')]}{\partial\epsilon_\beta(\boldsymbol{q}')} - \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\beta\beta}(\boldsymbol{q})e^x_{\alpha\alpha}(\boldsymbol{q}')\tau V_{\beta\beta\alpha\alpha}(\boldsymbol{q},\boldsymbol{q}') \times\frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial\epsilon_\beta(\boldsymbol{q})}\frac{\partial n[\epsilon_\alpha(\boldsymbol{q}')]}{\partial\epsilon_\alpha(\boldsymbol{q}')}, \tag{24}$$

$$L'_{12\text{-inter2}} = L'_{12\text{-inter2-}\alpha} + L'_{12\text{-inter2-}\beta} = (L'_{\rm E\alpha} + L'_{\rm S\alpha}) + (L'_{\rm E\beta} + L'_{\rm S\beta}), \tag{25}$$

$$L'_{\rm E\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q})e^x_{\alpha\beta}(\boldsymbol{q}')\tau V_{\nu\nu\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') \times\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{n[\epsilon_\alpha(\boldsymbol{q}')]-n[-\epsilon_\beta(\boldsymbol{q}')]}{\epsilon_\alpha(\boldsymbol{q}')+\epsilon_\beta(\boldsymbol{q}')}, \tag{26}$$

$$L'_{\rm S\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\alpha\beta}(\boldsymbol{q})e^x_{\nu\nu}(\boldsymbol{q}')\tau V_{\alpha\beta\nu\nu}(\boldsymbol{q},\boldsymbol{q}') \times\frac{n[\epsilon_\alpha(\boldsymbol{q})]-n[-\epsilon_\beta(\boldsymbol{q})]}{\epsilon_\alpha(\boldsymbol{q})+\epsilon_\beta(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial\epsilon_\nu(\boldsymbol{q}')}. \tag{27}$$

Here the $V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}')$'s are given by

$$V_{\nu\nu\nu\nu}(\boldsymbol{q},\boldsymbol{q}') = V_{\alpha\alpha\beta\beta}(\boldsymbol{q},\boldsymbol{q}') = V_{\beta\beta\alpha\alpha}(\boldsymbol{q},\boldsymbol{q}') = 2J_{\boldsymbol{q}-\boldsymbol{q}'}\sinh 2\theta_{\boldsymbol{q}}\sinh 2\theta_{\boldsymbol{q}'}, \tag{28}$$

$$V_{\nu\nu\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') = V_{\alpha\beta\nu\nu}(\boldsymbol{q}',\boldsymbol{q}) = -2J_{\boldsymbol{q}-\boldsymbol{q}'}\sinh 2\theta_{\boldsymbol{q}}\cosh 2\theta_{\boldsymbol{q}'}. \tag{29}$$

Equation (24) shows that the interband components of the magnon-magnon interactions cause the energy-current-drag correction and the spin-current-drag correction, which are, in the case for $S_A > S_B$, the first and the second term, respectively, of Eq. (24). Furthermore, Eqs. (26) and (27) show that other interband components cause the energy-current-drag corrections $L'_{\rm E\nu}$'s and the spin-current-drag corrections $L'_{\rm S\nu}$'s. Since these interband components cause the interband momentum transfer, $L'_{12\text{-inter1}}$ and $L'_{12\text{-inter2}}$ are the corrections due to the interband magnon drag. The similar corrections are obtained for L'_{11} and L'_{22}:

$$L'_{11} \sim L'_{11\text{-intra}} + L'_{11\text{-inter1}} + L'_{11\text{-inter2}}, \tag{30}$$

$$L'_{22} \sim L'_{22\text{-intra}} + L'_{22\text{-inter1}} + L'_{22\text{-inter2}}, \tag{31}$$

where $L'_{11\text{-intra}}$ and $L'_{22\text{-intra}}$ are the corrections due to the intraband interactions,

$$L'_{11\text{-intra}} = L'_{11\text{-intra-}\alpha} + L'_{11\text{-intra-}\beta}, \tag{32}$$

$$L'_{11\text{-intra-}\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q})v^x_{\nu\nu}(\boldsymbol{q}')\tau V_{\nu\nu\nu\nu}(\boldsymbol{q},\boldsymbol{q}') \times\frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial\epsilon_\nu(\boldsymbol{q}')}, \tag{33}$$

$$L'_{22\text{-intra}} = L'_{22\text{-intra-}\alpha} + L'_{22\text{-intra-}\beta}, \tag{34}$$

$$L'_{22\text{-intra-}\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} e^x_{\nu\nu}(\boldsymbol{q})e^x_{\nu\nu}(\boldsymbol{q}')\tau V_{\nu\nu\nu\nu}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial\epsilon_\nu(\boldsymbol{q}')}, \tag{35}$$

and $L'_{11\text{-inter1}}$, $L'_{11\text{-inter2}}$, $L'_{22\text{-inter1}}$, and $L'_{22\text{-inter2}}$ are the corrections due to the interband interactions,

$$L'_{11\text{-inter1}} = \frac{4}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\alpha\alpha}(\boldsymbol{q})v^x_{\beta\beta}(\boldsymbol{q}')\tau V_{\alpha\alpha\beta\beta}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}\frac{\partial n[\epsilon_\beta(\boldsymbol{q}')]}{\partial\epsilon_\beta(\boldsymbol{q}')}, \tag{36}$$

$$L'_{11\text{-inter2}} = L'_{11\text{-inter2-}\alpha} + L'_{11\text{-inter2-}\beta}, \tag{37}$$

$$L'_{11\text{-inter2-}\nu} = -\frac{4}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q})v^x_{\alpha\beta}(\boldsymbol{q}')\tau V_{\nu\nu\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{n[\epsilon_\alpha(\boldsymbol{q}')] - n[-\epsilon_\beta(\boldsymbol{q}')]}{\epsilon_\alpha(\boldsymbol{q}') + \epsilon_\beta(\boldsymbol{q}')}, \tag{38}$$

$$L'_{22\text{-inter1}} = \frac{4}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} e^x_{\alpha\alpha}(\boldsymbol{q})e^x_{\beta\beta}(\boldsymbol{q}')\tau V_{\alpha\alpha\beta\beta}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}\frac{\partial n[\epsilon_\beta(\boldsymbol{q}')]}{\partial\epsilon_\beta(\boldsymbol{q}')}, \tag{39}$$

$$L'_{22\text{-inter2}} = L'_{22\text{-inter2-}\alpha} + L'_{22\text{-inter2-}\beta}, \tag{40}$$

$$L'_{22\text{-inter2-}\nu} = -\frac{4}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} e^x_{\nu\nu}(\boldsymbol{q})e^x_{\alpha\beta}(\boldsymbol{q}')\tau V_{\nu\nu\alpha\beta}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\frac{n[\epsilon_\alpha(\boldsymbol{q}')] - n[-\epsilon_\beta(\boldsymbol{q}')]}{\epsilon_\alpha(\boldsymbol{q}') + \epsilon_\beta(\boldsymbol{q}')}. \tag{41}$$

As well as $L'_{12\text{-inter1}}$ and $L'_{12\text{-inter2}}$, $L'_{11\text{-inter1}}$, $L'_{11\text{-inter2}}$, $L'_{22\text{-inter1}}$, and $L'_{22\text{-inter2}}$ are the interband magnon drag corrections.

IV. NUMERICAL RESULTS

We numerically evaluate S_m, σ_m, and κ_m. We set $J = 1$, $h = 0.02J$, and $(S_A, S_B) = (\frac{3}{2}, 1)$. $S_A : S_B = 3 : 2$ is consistent with a ratio of Fe^T to Fe^O sites in the unit cell of YIG [41]. The reason why $(S_A, S_B) = (\frac{3}{2}, 1)$ is considered is that the transition temperature derived in a mean-field approximation in this case with $J = 3$ meV at $h = 0$ [i.e., $T_\text{c} = (16/3)JS_A(S_B + 1) \sim 557$ K] is close to the Curie temperature of YIG, T_C. To perform the momentum summations numerically, we divide the first Brillouin zone into a N_q-point mesh and set $N_q = 24^3 (= N/2)$ (for more details, see Appendix D). The temperature range is chosen to be $0 < T \leqslant 10J (\sim 0.6T_\text{c})$ because a previous study [60] showed that the magnon theory in which the magnon-magnon interactions are considered in the first-order perturbation theory can reproduce the perpendicular spin susceptibility of MnF_2 up to about $0.6T_\text{N}$, where T_N is the Néel temperature. For simplicity, we determine τ by $\tau^{-1} = \gamma_0 + \gamma_1 T + \gamma_2 T^2$, where $\gamma_0 = 10^{-2}J$, $\gamma_1 = 10^{-4}$, and $\gamma_2 = 10^{-3}$. (The results shown below remain qualitatively unchanged at $h = 0.08J$ and $0.16J$, as shown in Appendix E.)

TABLE I. The effects of the drag terms on $L_{12}(= S_\text{m})$, $L_{11}(= \sigma_\text{m})$, and $L_{22}(= \kappa_\text{m})$. $|L_{12}|$ is enhanced by $L'_{12\text{-intra}}$ and reduced by $L'_{12\text{-inter1}}$ and $L'_{12\text{-inter2}}$. L_{11} is enhanced by $L'_{11\text{-intra}}$, $L'_{11\text{-inter1}}$, and $L'_{11\text{-inter2}}$. L_{22} is enhanced by $L'_{22\text{-intra}}$ and reduced by $L'_{22\text{-inter1}}$ and $L'_{22\text{-inter2}}$.

Transport coefficient	Intra term	Inter1 term	Inter2 term
$\lvert L_{12}\rvert$	Enhanced	Reduced	Reduced
L_{11}	Enhanced	Enhanced	Enhanced
L_{22}	Enhanced	Reduced	Reduced

We begin with the temperature dependence of S_m. Figure 2(a) shows that in the range of $0 < T \leqslant 2J$ $L_{12} \approx L^0_{12\alpha}$ holds, whereas for $T \geqslant 3J$ the contribution from $L^0_{12\beta}$ is non-negligible. For example, at $T = 6J$ we have $L^0_{12}/L^0_{12\alpha} \sim 0.7$. This result indicates that the higher-energy band magnons contribute to S_m even for $T < [\epsilon_\beta(\boldsymbol{q}) - \epsilon_\alpha(\boldsymbol{q})] = 7.96J$. This may be surprising because their contributions are believed to be negligible at such temperatures. Then, Fig. 2(a) shows that the magnitude of S_m is enhanced by the intraband correction $L'_{12\text{-intra}}[= L^{(a)}_{12} - L^0_{12}]$, whereas it is reduced by the interband corrections $L'_{12\text{-inter2}}[= L^{(b)}_{12} - L^{(a)}_{12}]$ and $L'_{12\text{-inter1}}[= L^0_{12} + L'_{12} - L^{(b)}_{12}]$ (Table I). Among these corrections, $L'_{12\text{-intra}}$ gives the largest contribution. (As we will see below, this contrasts with the result of L_{11} or L_{22}, for which the largest contribution comes from $L'_{11\text{-inter2}}$ or $L'_{22\text{-inter2}}$, respectively.) The reason why the interband magnon drag corrections $L'_{12\text{-inter2}}$ and $L'_{12\text{-inter1}}$ are small is that the energy-current-drag contributions and spin-current-drag contributions [e.g., $L'_{\text{E}\alpha}$ and $L'_{\text{S}\alpha}$ in Eq. (25)] are opposite in sign and are nearly canceled out. Figure 2(a) also shows $L^0_{12} + L'_{12} \approx L^0_{12}$. These results suggest that the total effects of the interband magnon drag on S_m are small.

We turn to σ_m and κ_m. Their temperature dependences are shown in Figs. 2(b) and 2(c). First, we see the β-band magnons contribute to L_{11} for $T \geqslant 4J$ and to L_{22} for $T \geqslant 3J$. This result is similar to that of L_{12} and indicates that the multiband effects are significant also for σ_m and κ_m. The largest effects on L_{22} are due to the property that $e^x_{\nu\nu}(\boldsymbol{q})$ includes $\epsilon_\nu(\boldsymbol{q})$ [more precisely, $e^x_{\alpha\alpha}(\boldsymbol{q}) = v^x_{\alpha\alpha}(\boldsymbol{q})\epsilon_\alpha(\boldsymbol{q})$ and $e^x_{\beta\beta}(\boldsymbol{q}) = -v^x_{\beta\beta}(\boldsymbol{q})\epsilon_\beta(\boldsymbol{q})$]. Then, Figs. 2(b) and 2(c) show that σ_m is enhanced by $L'_{11\text{-intra}}$, $L'_{11\text{-inter2}}$, and $L'_{11\text{-inter1}}$, and that κ_m is enhanced by $L'_{22\text{-intra}}$ and reduced by $L'_{22\text{-inter2}}$ and $L'_{22\text{-inter1}}$ (Table I). [Note that $L'_{\mu\eta\text{-intra}} = L^{(a)}_{\mu\eta} - L^0_{\mu\eta}$, $L'_{\mu\eta\text{-inter2}} = L^{(b)}_{\mu\eta} - L^{(a)}_{\mu\eta}$, and $L'_{\mu\eta\text{-inter1}} = L^0_{\mu\eta} + L'_{\mu\eta} - L^{(b)}_{\mu\eta}$.] In contrast to L'_{12}, the largest contributions to L'_{11} and L'_{22} come from $L'_{11\text{-inter2}}$ and $L'_{22\text{-inter2}}$, respectively. Since $L'_{11\text{-inter2}}$, $L'_{11\text{-inter1}}$, $L'_{22\text{-inter2}}$, and $L'_{22\text{-inter1}}$ are the interband magnon drag corrections, the above results suggest that the interband magnon drag enhances σ_m and reduces κ_m. This implies that the interband magnon drag could be used to enhance the spin current and to reduce the energy current. Since this drag results from the interband momentum transfer induced by the magnon-magnon interactions, its effects could be controlled by changing the band splitting energy considerably via external fields. (Such control is meaningful if and only if the magnon picture remains valid.) Note that for ferrimagnetic insulators the effects of the weak

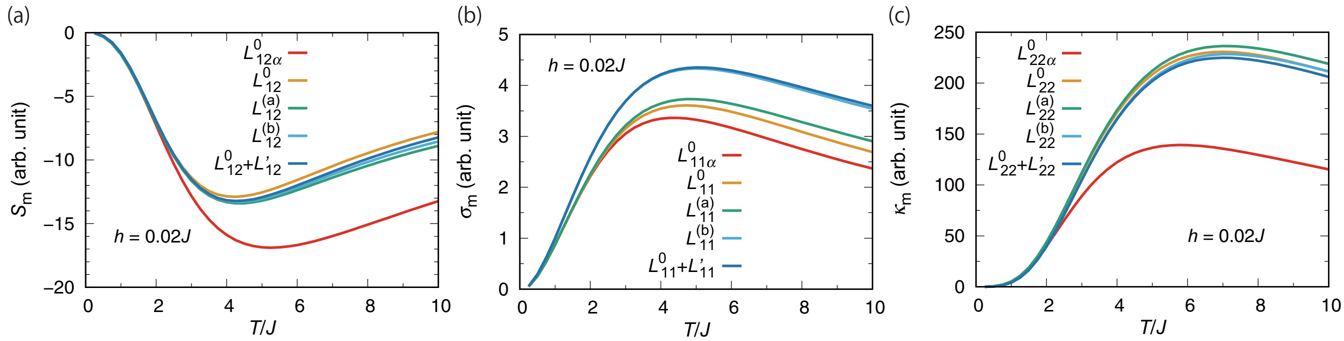

FIG. 2. The temperature dependences of (a) $S_m(=L_{12})$, (b) $\sigma_m(=L_{11})$, and (c) $\kappa_m(=L_{22})$ for $(S_A, S_B) = (\frac{3}{2}, 1)$ at $h = 0.02J$. $L^{(a)}_{\mu\eta}$ and $L^{(b)}_{\mu\eta}$ are defined as $L^{(a)}_{\mu\eta} = L^0_{\mu\eta} + L'_{\mu\eta\text{-intra}}$ and $L^{(b)}_{\mu\eta} = L^0_{\mu\eta} + L'_{\mu\eta\text{-intra}} + L'_{\mu\eta\text{-inter2}}$, respectively. Note that $L^0_{\mu\eta} = L^0_{\mu\eta\alpha} + L^0_{\mu\eta\beta}$ and $L'_{\mu\eta} = L'_{\mu\eta\text{-intra}} + L'_{\mu\eta\text{-inter1}} + L'_{\mu\eta\text{-inter2}}$. For S_m, the $L^0_{12\beta}$ is non-negligible for $T \geqslant 3J$ and the largest term of the drag terms is $L'_{12\text{-intra}}$, which enhances $|S_m|$. For σ_m, the $L^0_{11\beta}$ is non-negligible for $T \geqslant 4J$ and the largest term of the drag terms is $L'_{11\text{-inter2}}$, which enhances σ_m. For κ_m, the $L^0_{22\beta}$ is non-negligible for $T \geqslant 3J$ and the largest term of the drag terms is $L'_{22\text{-inter2}}$, which reduces κ_m. The effects of the other drag terms are summarized in Table I.

magnetic field on the band splitting energy are negligible because this energy for $h = 0$ is of the order of J. (The actual analysis about the possibility of controlling the interband magnon drag is a future problem.)

V. DISCUSSIONS

We discuss the validity of our theory. Since H_{int} could be treated as perturbation except near T_C, we believe our theory is appropriate for describing the magnon transport for $T < T_C$. It may be suitable to treat the magnon-magnon interactions in the Holstein-Primakoff method because the unphysical processes that can appear in a $S = 1/2$ ferromagnet [61] are absent in our case. Then the effects of the magnon-phonon interactions may not change the results qualitatively. First, since the interaction-induced magnon polaron occurs only at several values of h [62], its effect can be avoided. Another effect is to cause the temperature dependence of τ [63,64], and it could be approximately considered as the temperature-dependent τ. Although the phonon-drag contributions might change S_m [21], experimental results [59] suggest that such contributions are small or negligible.

We make a short comment about the relation between our theory and the Boltzmann theory. Our theory is based on a method of Green's functions, which can describe the effects of the damping and the vertex corrections appropriately. In principle, these effects can be described also in the Boltzmann theory if the collision integral is treated appropriately [65]. However, in many analyses using the Boltzmann theory, the collision integral is evaluated in the relaxation-time approximation, in which the vertex corrections are completely omitted. Since our interband magnon drag comes from the vertex corrections due to the first-order perturbation of the quartic terms, the similar result might be obtained also in the Boltzmann theory if the interband components of the collision integral are treated appropriately.

We remark on the implications of our results. First, our interband magnon drag is distinct from a magnon drag in metals. For the latter, magnons drag an electron charge current via the second-order perturbation of a sd-type exchange interaction [25]. Second, the interband magnon drag is possible in various ferrimagnetic insulators and other magnetic systems, such as antiferromagnets [47,56,66] and spiral magnets [57,67]. Note that the possible ferrimagnetic insulators include not only YIG, but also some spinel ferrites, such as $CoFe_2O_4$ and $NiFe_2O_4$ [68,69]. Third, our theory can be extended to phonons and photons. Thus it may be useful for studying transport phenomena of various interacting bosons. Fourth, our results will stimulate further studies of YIG. For example, the reduction in $|S_m|$ due to the multiband effect could improve the differences between the voltages observed in the spin-Seebeck effect and obtained in the Boltzmann theory of the ferromagnet [59] at high temperatures because the voltage is proportional to S_m.

VI. CONCLUSION

We have studied S_m, σ_m, and κ_m of interacting magnons in the minimal model of ferrimagnetic insulators. We derived them by using the linear-response theory and treating the magnon-magnon interactions as perturbation. We showed that some interband components of the magnon-magnon interactions give the corrections to these transport coefficients. These corrections are due to the interband magnon drag, which is distinct from the magnon drag in metals. Then we numerically calculated the temperature dependences of S_m, σ_m, and κ_m for $(S_A, S_B) = (\frac{3}{2}, 1)$ and $h = 0.02J$. We showed that the total effects of the interband magnon drag on S_m become small, whereas it enhances σ_m and reduces κ_m. The latter result may suggest that the interband magnon drag could be used to enhance the spin current and reduce the energy current. For S_m, the interband corrections become small because they lead to the energy-current-drag contributions and spin-current-drag contributions, which are opposite in sign and are nearly canceled out. We also showed that the contributions from the higher-energy band magnons to S_m, σ_m, and κ_m are non-negligible even for temperatures lower than the band splitting. This result indicates the importance of the multiband effects.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATIONS OF EQS. (10) AND (11)

We explain the details of the derivations of J_S^x and J_E^x, Eqs. (10) and (11). As described in the main text, they are obtained from the continuity equations. Such a derivation is explained, for example, in Ref. [55].

We begin with the derivation of J_S^x. (Note that the following derivation, which is applicable to collinear magnets, can be extended to noncollinear magnets.) We suppose that the z component of a spin angular momentum, S_m^z, satisfies

$$\frac{dS_m^z}{dt} + \nabla \cdot \boldsymbol{j}_m^{(S)} = 0, \tag{A1}$$

where $\boldsymbol{j}_m^{(S)}$ is a spin current operator at site m. Using this equation, we have

$$\frac{d}{dt}\Big(\sum_m \boldsymbol{R}_m S_m^z\Big) = -\sum_m \boldsymbol{R}_m \nabla \cdot \boldsymbol{j}_m^{(S)} = \sum_m \boldsymbol{j}_m^{(S)} = \boldsymbol{J}_l^{(S)}. \tag{A2}$$

Here l is A or B when the sum $\sum_m$ takes over sites on the A or the B sublattice, respectively. In deriving the second equal in Eq. (A2), we have omitted the surface contributions. J_S^x is given by the x component of $\boldsymbol{J}_S$, where

$$\boldsymbol{J}_S = \boldsymbol{J}_A^{(S)} + \boldsymbol{J}_B^{(S)}. \tag{A3}$$

Combining Eq. (A2) with the Heisenberg equation of motion, we obtain

$$\boldsymbol{J}_l^{(S)} = i\sum_m \boldsymbol{R}_m [H, S_m^z], \tag{A4}$$

where H is the Hamiltonian of the system considered. Then, since we focus on the magnon system described by $H = H_{\mathrm{KE}} + H_{\mathrm{int}}$, where H_{KE} and H_{int} are given in the main text, and treat H_{int} as perturbation, we replace H in Eq. (A4) by H_{KE} and S_m^z in Eq. (A4) either by $S_A - a_m^\dagger a_m$ for $l = A$ or by $-S_B + b_m^\dagger b_m$ for $l = B$; as a result, we obtain

$$\boldsymbol{J}_A^{(S)} = i\sum_{\langle i,j\rangle}\sum_m \boldsymbol{R}_m \big[h_{ij}^0, S_A - a_m^\dagger a_m\big], \tag{A5}$$

$$\boldsymbol{J}_B^{(S)} = i\sum_{\langle i,j\rangle}\sum_m \boldsymbol{R}_m \big[h_{ij}^0, -S_B + b_m^\dagger b_m\big], \tag{A6}$$

where $H_{\mathrm{KE}} = \sum_{\langle i,j\rangle} h_{ij}^0$ and $h_{ij}^0 = (2JS_B + \delta_{i,j}h)a_i^\dagger a_i + (2JS_A - \delta_{i,j}h)b_j^\dagger b_j + 2J\sqrt{S_AS_B}(a_i^\dagger b_j^\dagger + a_i b_j)$. Note that the replacement of H by H_{KE} may be suitable because the corrections due to H_{int} are next-leading terms; and that the replacement of S_m^z by $S_A - a_m^\dagger a_m$ or by $-S_B + b_m^\dagger b_m$ corresponds to the Holstein-Primakoff transformation of the ferrimagnet. After some algebra, we can write Eqs. (A5) and (A6) as follows:

$$\boldsymbol{J}_A^{(S)} = -i2J\sqrt{S_AS_B}\sum_{\langle i,j\rangle} \boldsymbol{R}_i (a_i b_j - a_i^\dagger b_j^\dagger), \tag{A7}$$

$$\boldsymbol{J}_B^{(S)} = i2J\sqrt{S_AS_B}\sum_{\langle i,j\rangle} \boldsymbol{R}_j (a_i b_j - a_i^\dagger b_j^\dagger). \tag{A8}$$

Combining these equations with Eq. (A3), we have

$$\boldsymbol{J}_S = -i2J\sqrt{S_AS_B}\sum_{\langle i,j\rangle} (\boldsymbol{R}_i - \boldsymbol{R}_j)(a_i b_j - a_i^\dagger b_j^\dagger). \tag{A9}$$

Then, by using the Fourier coefficients of the magnon operators,

$$a_i = \sqrt{\frac{2}{N}}\sum_{\boldsymbol{q}} a_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}, \quad b_j^\dagger = \sqrt{\frac{2}{N}}\sum_{\boldsymbol{q}} b_{\boldsymbol{q}}^\dagger e^{i\boldsymbol{q}\cdot\boldsymbol{R}_j}, \tag{A10}$$

we can rewrite Eq. (A9) as follows:

$$\boldsymbol{J}_S = -2J\sqrt{S_AS_B}\sum_{\boldsymbol{q}} \frac{\partial J_{\boldsymbol{q}}}{\partial \boldsymbol{q}}(a_{\boldsymbol{q}} b_{\boldsymbol{q}} + a_{\boldsymbol{q}}^\dagger b_{\boldsymbol{q}}^\dagger) = -\sum_{\boldsymbol{q}} \frac{\partial \epsilon_{AB}(\boldsymbol{q})}{\partial \boldsymbol{q}}(x_{\boldsymbol{q}B}^\dagger x_{\boldsymbol{q}A} + x_{\boldsymbol{q}A}^\dagger x_{\boldsymbol{q}B}), \tag{A11}$$

where $J_{\boldsymbol{q}} = J\sum_{j=1}^{z} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)} = 8J\cos\frac{q_x}{2}\cos\frac{q_y}{2}\cos\frac{q_z}{2}$, $\epsilon_{AB}(\boldsymbol{q}) = 2J\sqrt{S_AS_B}J_{\boldsymbol{q}}$, $x_{\boldsymbol{q}A} = a_{\boldsymbol{q}}$, and $x_{\boldsymbol{q}B} = b_{\boldsymbol{q}}^\dagger$. Note that z is the number of nearest-neighbor sites ($z = 8$). The x component of Eq. (A11) gives Eq. (10).

In a similar way, we can obtain the expression of J_E^x. (The following derivation is similar to that for an antiferromagnet [56].) First, we suppose that the Hamiltonian at site m, h_m, satisfies

$$\frac{dh_m}{dt} + \nabla \cdot \boldsymbol{j}_m^{(E)} = 0, \tag{A12}$$

where $\boldsymbol{j}_m^{(E)}$ is an energy current operator at site m. Because of this relation, the energy current operator $\boldsymbol{J}_E$ can be determined from

$$\boldsymbol{J}_E = \boldsymbol{J}_A^{(E)} + \boldsymbol{J}_B^{(E)}, \tag{A13}$$

where $\boldsymbol{J}_l^{(E)}$ is given by

$$\boldsymbol{J}_l^{(E)} = i\sum_{m,n} \boldsymbol{R}_n [h_m, h_n], \tag{A14}$$

the sum $\sum_m$ take over sites on the A or the B sublattice, and the sum $\sum_n$ take over sites on sublattice l. Then, to calculate the commutator in Eq. (A14), we consider the contributions only from H_{KE} and neglect the corrections due to H_{int}, as in the derivation of $\boldsymbol{J}_l^{(S)}$. As a result, h_m for $m \in A$ is given by

$$h_{mA}^0 = (2S_BzJ + h)a_m^\dagger a_m + \sqrt{S_AS_B}\sum_j J_{mj}(a_m b_j + a_m^\dagger b_j^\dagger), \tag{A15}$$

and that for $m \in B$ is given by

$$h_{mB}^0 = (2S_AzJ - h)b_m^\dagger b_m + \sqrt{S_AS_B}\sum_i J_{im}(a_i b_m + a_i^\dagger b_m^\dagger). \tag{A16}$$

Here $m \in A$ or B means that m is on the A or B sublattice, respectively, and $J_{ij} = J_{ji} = J$ for nearest-neighbor sites i and j. Note that $\sum_{i=1}^{N/2} h_{iA}^0 + \sum_{j=1}^{N/2} h_{jB}^0 = H_{\mathrm{KE}}$. In our definition, the energy current operator includes the conribution from the Zeeman energy [see Eqs. (A14)–(A16)]. Combining Eqs. (A15) and (A16) with Eqs. (A13) and (A14), we have

$$\boldsymbol{J}_E = i\sum_{m,n} \boldsymbol{R}_n\big[h_{mA}^0, h_{nA}^0\big] + i\sum_{m,n} \boldsymbol{R}_n\big[h_{mB}^0, h_{nB}^0\big] + i\sum_{m,n} \boldsymbol{R}_n\big[h_{mA}^0, h_{nB}^0\big] + i\sum_{m,n} \boldsymbol{R}_n\big[h_{mB}^0, h_{nA}^0\big]. \tag{A17}$$

Then we can calculate the commutators in Eq. (A17) by using the commutation relations of the magnon operators and the identities $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = [A, B]C + B[A, C]$; the results are

$$\left[h^0_{mA}, h^0_{nA}\right] = S_A S_B \sum_j J_{mj} J_{nj} (a^\dagger_n a_m - a^\dagger_m a_n), \tag{A18}$$

$$\left[h^0_{mB}, h^0_{nB}\right] = S_A S_B \sum_i J_{im} J_{in} (b_m b^\dagger_n - b_n b^\dagger_m), \tag{A19}$$

$$\begin{aligned}\left[h^0_{mA}, h^0_{nB}\right] = {} & S_A S_B \sum_j J_{mn} J_{mj} (b_j b^\dagger_n - b_n b^\dagger_j) \\ & + [2Jz(S_A - S_B) - 2h]\sqrt{S_A S_B} J_{mn} \\ & \times (a_m b_n - a^\dagger_m b^\dagger_n) \\ & + S_A S_B \sum_i J_{mn} J_{ni} (a^\dagger_i a_m - a^\dagger_m a_i),\end{aligned} \tag{A20}$$

$$\begin{aligned}\left[h^0_{mB}, h^0_{nA}\right] = {} & S_A S_B \sum_j J_{mn} J_{nj} (b_m b^\dagger_j - b_j b^\dagger_m) \\ & + [-2Jz(S_A - S_B) + 2h]\sqrt{S_A S_B} J_{nm} \\ & \times (a_n b_m - a^\dagger_n b^\dagger_m) \\ & + S_A S_B \sum_i J_{nm} J_{mi} (a^\dagger_n a_i - a^\dagger_i a_n).\end{aligned} \tag{A21}$$

By substituting these equations into Eq. (A17) and performing some calculations, we obtain

$$\begin{aligned}\boldsymbol{J}_E = {} & 2i \sum_{m,n,j} (\boldsymbol{R}_n - \boldsymbol{R}_m) S_A S_B J_{nj} J_{jm} a^\dagger_n a_m \\ & - 2i \sum_{m,n,i} (\boldsymbol{R}_n - \boldsymbol{R}_m) S_A S_B J_{ni} J_{im} b_n b^\dagger_m \\ & + i \sum_{m,n} (\boldsymbol{R}_n - \boldsymbol{R}_m)[2Jz(S_A - S_B) - 2h]\sqrt{S_A S_B} \\ & \times J_{mn}(a_m b_n - a^\dagger_m b^\dagger_n).\end{aligned} \tag{A22}$$

As in the derivation of $\boldsymbol{J}_S$, we can rewrite Eq. (A22) by using the Fourier coefficients of the magnon operators [Eq. (A10)]; as a result, we have

$$\begin{aligned}\boldsymbol{J}_E = {} & -\sum_{\boldsymbol{q}} 2\sqrt{S_A S_B} J_{\boldsymbol{q}} 2\sqrt{S_A S_B} \frac{\partial J_{\boldsymbol{q}}}{\partial \boldsymbol{q}} (a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}} - b_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}}) \\ & - [J_{\boldsymbol{0}}(S_A - S_B) - h] 2\sqrt{S_A S_B} \frac{\partial J_{\boldsymbol{q}}}{\partial \boldsymbol{q}} (a_{\boldsymbol{q}} b_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}}).\end{aligned} \tag{A23}$$

Since $\epsilon_{AA} = 2J_{\boldsymbol{0}} S_B + h$, $\epsilon_{BB} = 2J_{\boldsymbol{0}} S_A - h$, and $\epsilon_{AB}(\boldsymbol{q}) = 2\sqrt{S_A S_B} J_{\boldsymbol{q}}$, we can write Eq. (A23) as follows:

$$\begin{aligned}\boldsymbol{J}_E = {} & -\sum_{\boldsymbol{q}} \epsilon_{AB}(\boldsymbol{q}) \frac{\partial \epsilon_{AB}(\boldsymbol{q})}{\partial \boldsymbol{q}} (a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}} - b_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}}) \\ & + \sum_{\boldsymbol{q}} \frac{1}{2} (\epsilon_{AA} - \epsilon_{BB}) \frac{\partial \epsilon_{AB}(\boldsymbol{q})}{\partial \boldsymbol{q}} (a_{\boldsymbol{q}} b_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}}) \\ = {} & \sum_{\boldsymbol{q}} \sum_{l,l'=A,B} \boldsymbol{e}_{ll'}(\boldsymbol{q}) x^\dagger_{\boldsymbol{q}l} x_{\boldsymbol{q}l'},\end{aligned} \tag{A24}$$

where $\boldsymbol{e}_{AA}(\boldsymbol{q}) = -\boldsymbol{e}_{BB}(\boldsymbol{q}) = -\epsilon_{AB}(\boldsymbol{q}) \frac{\partial \epsilon_{AB}(\boldsymbol{q})}{\partial \boldsymbol{q}}$ and $\boldsymbol{e}_{AB}(\boldsymbol{q}) = \boldsymbol{e}_{BA}(\boldsymbol{q}) = \frac{1}{2}(\epsilon_{AA} - \epsilon_{BB}) \frac{\partial \epsilon_{AB}(\boldsymbol{q})}{\partial \boldsymbol{q}}$. Equation (A24) for the x component is Eq. (11).

APPENDIX B: DERIVATIONS OF EQS. (13) AND (15)

We derive Eqs. (13) and (15). As described in the main text, their derivations can be done in a similar way to the derivations of electron transport coefficients [23,50,54,58]: the transport coefficients can be derived by using a method of Green's functions [51]. We first derive L^0_{12}, the noninteracting L_{12}, and then derive L'_{12}, the leading correction to L_{12} due to the first-order perturbation of H_{int}.

First, we derive L^0_{12}, Eq. (13). Substituting Eqs. (10) and (11) into Eq. (9), we have

$$\begin{aligned}\Phi_{12}(i\Omega_n) = {} & -\frac{1}{N} \sum_{\boldsymbol{q},\boldsymbol{q}'} \sum_{l_1,l_2,l_3,l_4=A,B} v^x_{l_1 l_2}(\boldsymbol{q}) e^x_{l_3 l_4}(\boldsymbol{q}') \\ & \times \int_0^{T^{-1}} d\tau e^{i\Omega_n \tau} \langle T_\tau x^\dagger_{\boldsymbol{q}l_1}(\tau) x_{\boldsymbol{q}l_2}(\tau) x^\dagger_{\boldsymbol{q}'l_3} x_{\boldsymbol{q}'l_4} \rangle \\ = {} & -\frac{1}{N} \sum_{\boldsymbol{q},\boldsymbol{q}'} \sum_{l_1,l_2,l_3,l_4=A,B} v^x_{l_1 l_2}(\boldsymbol{q}) e^x_{l_3 l_4}(\boldsymbol{q}') \\ & \times G^{(\text{II})}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n),\end{aligned} \tag{B1}$$

where $\Omega_n = 2\pi T n$ with $n > 0$. (Note that the n and m used in this section are different from those used in Appendix A.) Equation (B1) provides a starting point to derive L^0_{12} and L'_{12}. To derive L^0_{12}, we calculate $G^{(\text{II})}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n)$ in the absence of H_{int} by using Wick's theorem [51]; the result is

$$\begin{aligned}G^{(\text{II})}_{l_1 l_2 l_3 l_4}(\boldsymbol{q}, \boldsymbol{q}'; i\Omega_n) = {} & \delta_{\boldsymbol{q},\boldsymbol{q}'} T \sum_m G_{l_2 l_3}(\boldsymbol{q}, i\Omega_n + i\Omega_m) \\ & \times G_{l_4 l_1}(\boldsymbol{q}, i\Omega_m),\end{aligned} \tag{B2}$$

where $G_{ll'}(\boldsymbol{q}, i\Omega_m)$ is the magnon Green's function in the sublattice basis with $\Omega_m = 2\pi T m$ and an integer m,

$$G_{ll'}(\boldsymbol{q}, i\Omega_m) = -\int_0^{T^{-1}} d\tau e^{i\Omega_m \tau} \langle T_\tau x_{\boldsymbol{q}l}(\tau) x^\dagger_{\boldsymbol{q}l'} \rangle. \tag{B3}$$

Then the magnon operators in the sublattice basis, $x_{\boldsymbol{q}l}$ and $x^\dagger_{\boldsymbol{q}l}$, are connected with those in the band basis, $x_{\boldsymbol{q}\nu}$ and $x^\dagger_{\boldsymbol{q}\nu}$, through the Bogoliubov transformation,

$$x_{\boldsymbol{q}l} = \sum_{\nu=\alpha,\beta} (U_{\boldsymbol{q}})_{l\nu} x_{\boldsymbol{q}\nu}, \tag{B4}$$

where $x_{\boldsymbol{q}\alpha} = \alpha_{\boldsymbol{q}}$, $x_{\boldsymbol{q}\beta} = \beta^\dagger_{\boldsymbol{q}}$, $(U_{\boldsymbol{q}})_{A\alpha} = (U_{\boldsymbol{q}})_{B\beta} = \cosh\theta_{\boldsymbol{q}}$, and $(U_{\boldsymbol{q}})_{A\beta} = (U_{\boldsymbol{q}})_{B\alpha} = -\sinh\theta_{\boldsymbol{q}}$; as described in the main text, these hyperbolic functions satisfy $\cosh 2\theta_{\boldsymbol{q}} = \frac{J_{\boldsymbol{0}}(S_A+S_B)}{\Delta\epsilon_{\boldsymbol{q}}}$ and $\sinh 2\theta_{\boldsymbol{q}} = \frac{2\sqrt{S_A S_B} J_{\boldsymbol{q}}}{\Delta\epsilon_{\boldsymbol{q}}}$. Thus $G_{ll'}(\boldsymbol{q}, i\Omega_m)$ is related to the magnon Green's function in the band basis, $G_\nu(\boldsymbol{q}, i\Omega_m)$:

$$G_{ll'}(\boldsymbol{q}, i\Omega_m) = \sum_{\nu=\alpha,\beta} (U_{\boldsymbol{q}})_{l\nu} (U_{\boldsymbol{q}})_{l'\nu} G_\nu(\boldsymbol{q}, i\Omega_m), \tag{B5}$$

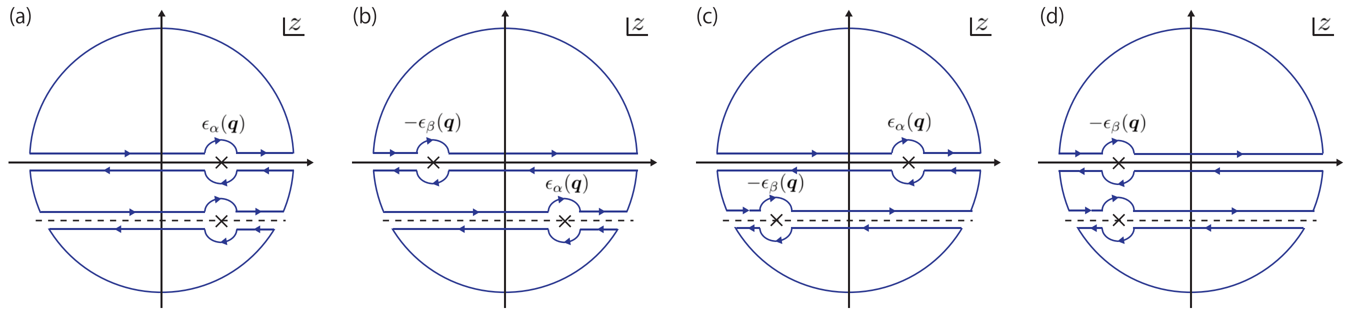

FIG. 3. The contours used for the integrations in (a) $G^{(\mathrm{II})}_{\alpha\alpha}(\boldsymbol{q}, i\Omega_n)$, (b) $G^{(\mathrm{II})}_{\alpha\beta}(\boldsymbol{q}, i\Omega_n)$, (c) $G^{(\mathrm{II})}_{\beta\alpha}(\boldsymbol{q}, i\Omega_n)$, and (d) $G^{(\mathrm{II})}_{\beta\beta}(\boldsymbol{q}, i\Omega_n)$. The horizontal dashed lines correspond to $\mathrm{Im}z = -\Omega_n$.

where

$$G_\alpha(\boldsymbol{q}, i\Omega_m) = \frac{1}{i\Omega_m - \epsilon_\alpha(\boldsymbol{q})},\ G_\beta(\boldsymbol{q}, i\Omega_m) = -\frac{1}{i\Omega_m + \epsilon_\beta(\boldsymbol{q})}. \tag{B6}$$

Combining Eq. (B5) with Eqs. (B2) and (B1), we have

$$\begin{aligned}\Phi_{12}(i\Omega_n) &= -\frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta} v^x_{\nu'\nu}(\boldsymbol{q}) e^x_{\nu\nu'}(\boldsymbol{q}) \\ &\quad\times T\sum_m G_\nu(\boldsymbol{q}, i\Omega_{n+m}) G_{\nu'}(\boldsymbol{q}, i\Omega_m) \\ &= -\frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta} v^x_{\nu'\nu}(\boldsymbol{q}) e^x_{\nu\nu'}(\boldsymbol{q}) G^{(\mathrm{II})}_{\nu\nu'}(\boldsymbol{q}, i\Omega_n),\end{aligned} \tag{B7}$$

where

$$v^x_{\nu'\nu}(\boldsymbol{q}) = \sum_{l_1,l_2=A,B} v^x_{l_1 l_2}(\boldsymbol{q}) (U_{\boldsymbol{q}})_{l_1\nu'} (U_{\boldsymbol{q}})_{l_2\nu}, \tag{B8}$$

$$e^x_{\nu\nu'}(\boldsymbol{q}) = \sum_{l_3,l_4=A,B} e^x_{l_3 l_4}(\boldsymbol{q}) (U_{\boldsymbol{q}})_{l_3\nu} (U_{\boldsymbol{q}})_{l_4\nu'}. \tag{B9}$$

Then we can rewrite $G^{(\mathrm{II})}_{\nu\nu'}(\boldsymbol{q}, i\Omega_n)$ in Eq. (B7) as follows:

$$\begin{aligned}G^{(\mathrm{II})}_{\nu\nu'}(\boldsymbol{q}, i\Omega_n) &= \int_{\mathrm{C}} \frac{dz}{2\pi i} n(z) G_\nu(\boldsymbol{q}, i\Omega_n + z) G_{\nu'}(\boldsymbol{q}, z) \\ &\quad + T[G_\nu(\boldsymbol{q}, i\Omega_n) G_{\nu'}(\boldsymbol{q}, 0) \\ &\quad + G_\nu(\boldsymbol{q}, 0) G_{\nu'}(\boldsymbol{q}, -i\Omega_n)],\end{aligned} \tag{B10}$$

where $n(z)$ is the Bose distribution function, $n(z) = (e^{z/T} - 1)^{-1}$, and C is one of the contours shown in Fig. 3. Using Eqs. (B10) and (B6), we obtain

$$G^{(\mathrm{II})}_{\nu\nu'}(\boldsymbol{q}, i\Omega_n) = \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) \{G^{\mathrm{R}}_\nu(\boldsymbol{q}, z + i\Omega_n)[G^{\mathrm{R}}_{\nu'}(\boldsymbol{q}, z) - G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z)] + [G^{\mathrm{R}}_\nu(\boldsymbol{q}, z) - G^{\mathrm{A}}_\nu(\boldsymbol{q}, z)] G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z - i\Omega_n)\}, \tag{B11}$$

where $G^{\mathrm{R}}_\nu(\boldsymbol{q}, z)$ is the retarded magnon Green's function,

$$G^{\mathrm{R}}_\alpha(\boldsymbol{q}, z) = \frac{1}{z - \epsilon_\alpha(\boldsymbol{q}) + i\gamma}, \quad G^{\mathrm{R}}_\beta(\boldsymbol{q}, z) = -\frac{1}{z + \epsilon_\beta(\boldsymbol{q}) + i\gamma}, \tag{B12}$$

$G^{\mathrm{A}}_\nu(\boldsymbol{q}, z)$ is the advanced one, and γ is the magnon damping. By combining Eq. (B11) with Eq. (B7) and performing the analytic continuation $i\Omega_n \to \omega + i\delta$ with $\delta = 0+$, we have

$$\begin{aligned}\Phi^{\mathrm{R}}_{12}(\omega) = \Phi_{12}(i\Omega_n \to \omega + i\delta) &= -\frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta} v^x_{\nu'\nu}(\boldsymbol{q}) e^x_{\nu\nu'}(\boldsymbol{q}) \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) \\ &\quad\times \{G^{\mathrm{R}}_\nu(\boldsymbol{q}, z + \omega)[G^{\mathrm{R}}_{\nu'}(\boldsymbol{q}, z) - G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z)] + [G^{\mathrm{R}}_\nu(\boldsymbol{q}, z) - G^{\mathrm{A}}_\nu(\boldsymbol{q}, z)] G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z - \omega)\}.\end{aligned} \tag{B13}$$

By using $G(z + \omega) = G(z) + \omega\frac{\partial G(z)}{\partial z} + O(\omega^2)$ and performing the partial integration, we obtain

$$\begin{aligned}L^0_{12} &= \lim_{\omega\to 0} \frac{\Phi^{\mathrm{R}}_{12}(\omega) - \Phi^{\mathrm{R}}_{12}(0)}{i\omega} \\ &= -\frac{1}{4N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta} v^x_{\nu'\nu}(\boldsymbol{q}) e^x_{\nu\nu'}(\boldsymbol{q}) \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} [G^{\mathrm{R}}_\nu(\boldsymbol{q}, z) G^{\mathrm{R}}_{\nu'}(\boldsymbol{q}, z) - 2G^{\mathrm{R}}_\nu(\boldsymbol{q}, z) G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z) + G^{\mathrm{A}}_\nu(\boldsymbol{q}, z) G^{\mathrm{A}}_{\nu'}(\boldsymbol{q}, z)] \\ &= \frac{1}{N}\sum_{\boldsymbol{q}}\sum_{\nu,\nu'=\alpha,\beta} v^x_{\nu'\nu}(\boldsymbol{q}) e^x_{\nu\nu'}(\boldsymbol{q}) \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} \mathrm{Im}G^{\mathrm{R}}_\nu(\boldsymbol{q}, z) \mathrm{Im}G^{\mathrm{R}}_{\nu'}(\boldsymbol{q}, z).\end{aligned} \tag{B14}$$

In deriving this equation, we have used the symmetry relations $v^x_{\nu'\nu}(\boldsymbol{q}) = v^x_{\nu\nu'}(\boldsymbol{q})$ and $e^x_{\nu\nu'}(\boldsymbol{q}) = e^x_{\nu'\nu}(\boldsymbol{q})$. Equation (B14) is Eq. (13).

Next, we derive L'_{12}, Eq. (15). By using Eq. (B1), we can write the correction due to the first-order perturbation of H_{int} as follows:

$$\Delta\Phi_{12}(i\Omega_n) = +\frac{1}{N}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{l_1,l_2,l_3,l_4=A,B} v^x_{l_1l_2}(\boldsymbol{q})e^x_{l_3l_4}(\boldsymbol{q}')\int_0^{T^{-1}} d\tau e^{i\Omega_n\tau}\int_0^{T^{-1}} d\tau_1\langle T_\tau x^\dagger_{\boldsymbol{q}l_1}(\tau)x_{\boldsymbol{q}l_2}(\tau)x^\dagger_{\boldsymbol{q}'l_3}x_{\boldsymbol{q}'l_4}H_{\mathrm{int}}(\tau_1)\rangle. \tag{B15}$$

[Note that H_{int} has been defined in Eq. (4).] By using Wick's theorem [51], we can calculate $\langle T_\tau x^\dagger_{\boldsymbol{q}l_1}(\tau)x_{\boldsymbol{q}l_2}(\tau)x^\dagger_{\boldsymbol{q}'l_3}x_{\boldsymbol{q}'l_4}H_{\mathrm{int}}(\tau_1)\rangle$; the result is

$$\langle T_\tau x^\dagger_{\boldsymbol{q}l_1}(\tau)x_{\boldsymbol{q}l_2}(\tau)x^\dagger_{\boldsymbol{q}'l_3}x_{\boldsymbol{q}'l_4}H_{\mathrm{int}}(\tau_1)\rangle = -\frac{1}{N}\sum_{l_5,l_6,l_7,l_8=A,B} V_{l_5l_6l_7l_8}(\boldsymbol{q},\boldsymbol{q}')G_{l_5l_1}(\boldsymbol{q},\tau_1-\tau)G_{l_2l_6}(\boldsymbol{q},\tau-\tau_1)G_{l_7l_3}(\boldsymbol{q}',\tau_1)G_{l_4l_8}(\boldsymbol{q}',-\tau_1), \tag{B16}$$

where $G_{ll'}(\boldsymbol{q},\tau) = T\sum_m e^{-i\Omega_m\tau}G_{ll'}(\boldsymbol{q},i\Omega_m)$,

$$V_{l_5l_6l_7l_8}(\boldsymbol{q},\boldsymbol{q}') = \begin{cases} 4J_{\boldsymbol{0}} & (l_5=l_6=l,\, l_7=l_8=\bar{l}),\\ 4J_{\boldsymbol{q}-\boldsymbol{q}'} & (l_5=l_8=l,\, l_6=l_7=\bar{l}),\\ 2J_{\boldsymbol{q}'}\sqrt{\frac{S_A}{S_B}} & (l_5=l_6=B,\, l_7=l,\, l_8=\bar{l}),\\ 2J_{\boldsymbol{q}}\sqrt{\frac{S_A}{S_B}} & (l_5=l,\, l_6=\bar{l},\, l_7=l_8=B),\\ 2J_{\boldsymbol{q}'}\sqrt{\frac{S_B}{S_A}} & (l_5=l_6=A,\, l_7=l,\, l_8=\bar{l}),\\ 2J_{\boldsymbol{q}}\sqrt{\frac{S_B}{S_A}} & (l_5=l,\, l_6=\bar{l},\, l_7=l_8=A), \end{cases} \tag{B17}$$

and $\bar{l}=B$ or A for $l=A$ or B, respectively. Then, by substituting Eq. (B16) into Eq. (B15) and carrying out the integrations, we obtain

$$\Delta\Phi_{12}(i\Omega_n) = -\frac{1}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{l_1,l_2,\ldots,l_8=A,B} v^x_{l_1l_2}(\boldsymbol{q})e^x_{l_3l_4}(\boldsymbol{q}')V_{l_5l_6l_7l_8}(\boldsymbol{q},\boldsymbol{q}')T^2\sum_{m,m'}G_{l_5l_1}(\boldsymbol{q},i\Omega_m)G_{l_2l_6}(\boldsymbol{q},i\Omega_{n+m}) \times G_{l_7l_3}(\boldsymbol{q}',i\Omega_{n+m'})G_{l_4l_8}(\boldsymbol{q}',i\Omega_{m'}). \tag{B18}$$

Furthermore, we can rewrite this equation by using the Bogoliubov transformation [i.e., Eq. (B4)]; the result is

$$\Delta\Phi_{12}(i\Omega_n) = -\frac{1}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu_1,\nu_2,\nu_3,\nu_4=\alpha,\beta} v^x_{\nu_1\nu_2}(\boldsymbol{q})e^x_{\nu_3\nu_4}(\boldsymbol{q}')V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}')\Delta G^{(\mathrm{II})}_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n), \tag{B19}$$

where

$$V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}') = \sum_{l_5,l_6,l_7,l_8=A,B} V_{l_5l_6l_7l_8}(\boldsymbol{q},\boldsymbol{q}')(U_{\boldsymbol{q}})_{l_5\nu_1}(U_{\boldsymbol{q}})_{l_6\nu_2}(U_{\boldsymbol{q}'})_{l_7\nu_3}(U_{\boldsymbol{q}'})_{l_8\nu_4}, \tag{B20}$$

$$\Delta G^{(\mathrm{II})}_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n) = T^2\sum_{m,m'}G_{\nu_1}(\boldsymbol{q},i\Omega_m)G_{\nu_2}(\boldsymbol{q},i\Omega_{n+m})G_{\nu_3}(\boldsymbol{q}',i\Omega_{n+m'})G_{\nu_4}(\boldsymbol{q}',i\Omega_{m'}). \tag{B21}$$

Since $v^x_{\nu_1\nu_2}(\boldsymbol{q})$ and $e^x_{\nu_3\nu_4}(\boldsymbol{q}')$ are odd functions in term of q_x and q'_x, respectively, and $G_\nu(\boldsymbol{q},i\Omega_m)$'s are even functions, the finite terms of $V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}')$ in Eq. (B19), i.e., the terms which are finite even after carrying out $\sum_{\boldsymbol{q},\boldsymbol{q}'}$, come only from $V_{ABBA}(\boldsymbol{q},\boldsymbol{q}') = V_{BAAB}(\boldsymbol{q},\boldsymbol{q}') = 4J_{\boldsymbol{q}-\boldsymbol{q}'}$ [Eq. (B17)]; because of this property, we can replace Eq. (B20) by

$$V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}') = \sum_{l=A,B}4J_{\boldsymbol{q}-\boldsymbol{q}'}(U_{\boldsymbol{q}})_{l\nu_1}(U_{\boldsymbol{q}})_{\bar{l}\nu_2}(U_{\boldsymbol{q}'})_{\bar{l}\nu_3}(U_{\boldsymbol{q}'})_{l\nu_4}. \tag{B22}$$

Then, as in $G^{(\mathrm{II})}_{\nu\nu'}(\boldsymbol{q},i\Omega_n)$ [Eq. (B10)], we can replace the sums in Eq. (B21) by the corresponding integrals:

$$\begin{aligned}\Delta G^{(\mathrm{II})}_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}';i\Omega_n) &= \left[\int_C\frac{dz}{2\pi i}n(z)G_{\nu_1}(\boldsymbol{q},z)G_{\nu_2}(\boldsymbol{q},z+i\Omega_n)+A\right]\left[\int_{C'}\frac{dz'}{2\pi i}n(z')G_{\nu_3}(\boldsymbol{q}',z'+i\Omega_n)G_{\nu_4}(\boldsymbol{q}',z')+A'\right]\\ &= G^{(\mathrm{II})}_{\nu_2\nu_1}(\boldsymbol{q},i\Omega_n)G^{(\mathrm{II})}_{\nu_3\nu_4}(\boldsymbol{q}',i\Omega_n),\end{aligned} \tag{B23}$$

where $A = T[G_{\nu_1}(\boldsymbol{q},0)G_{\nu_2}(\boldsymbol{q},i\Omega_n)+G_{\nu_1}(\boldsymbol{q},-i\Omega_n)G_{\nu_2}(\boldsymbol{q},0)]$, $A' = T[G_{\nu_3}(\boldsymbol{q}',i\Omega_n)G_{\nu_4}(\boldsymbol{q}',0)+G_{\nu_3}(\boldsymbol{q}',0)G_{\nu_4}(\boldsymbol{q}',-i\Omega_n)]$, and C or C' is one of the contours shown in Fig. 3. By substituting Eq. (B11) into Eq. (B23) and performing the analytic continuation $i\Omega_n \to \omega+i\delta$ ($\delta=0+$), we have

$$\begin{aligned}\Delta\Phi^{\mathrm{R}}_{12}(\omega) &= \Delta\Phi_{12}(i\Omega_n\to\omega+i\delta)\\ &= -\frac{1}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'}\sum_{\nu_1,\nu_2,\nu_3,\nu_4=\alpha,\beta} v^x_{\nu_1\nu_2}(\boldsymbol{q})e^x_{\nu_3\nu_4}(\boldsymbol{q}')V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}')\end{aligned}$$

$$\times \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z)\{[G^{\rm R}_{\nu_1}(\boldsymbol{q},z) - G^{\rm A}_{\nu_1}(\boldsymbol{q},z)]G^{\rm R}_{\nu_2}(\boldsymbol{q},z+\omega) + G^{\rm A}_{\nu_1}(\boldsymbol{q},z-\omega)[G^{\rm R}_{\nu_2}(\boldsymbol{q},z) - G^{\rm A}_{\nu_2}(\boldsymbol{q},z)]\}$$

$$\times \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z')\{G^{\rm R}_{\nu_3}(\boldsymbol{q}',z'+\omega)[G^{\rm R}_{\nu_4}(\boldsymbol{q}',z') - G^{\rm A}_{\nu_4}(\boldsymbol{q}',z')] + [G^{\rm R}_{\nu_3}(\boldsymbol{q}',z') - G^{\rm A}_{\nu_3}(\boldsymbol{q}',z')]G^{\rm A}_{\nu_4}(\boldsymbol{q}',z'-\omega)\}. \quad \text{(B24)}$$

Then, by performing the calculations similar to the derivation of Eq. (B14), we obtain

$$L'_{12} = \lim_{\omega\to 0} \frac{\Delta\Phi^{\rm R}_{12}(\omega) - \Delta\Phi^{\rm R}_{12}(0)}{i\omega}$$
$$= \frac{1}{4\pi^2 i N^2} \sum_{\boldsymbol{q},\boldsymbol{q}'} \sum_{\nu_1,\nu_2,\nu_3,\nu_4=\alpha,\beta} v^x_{\nu_1\nu_2}(\boldsymbol{q}) e^x_{\nu_3\nu_4}(\boldsymbol{q}') V_{\nu_1\nu_2\nu_3\nu_4}(\boldsymbol{q},\boldsymbol{q}')[F^{\rm (I)}_{\nu_1\nu_2}(\boldsymbol{q})F^{\rm (II)}_{\nu_3\nu_4}(\boldsymbol{q}') + F^{\rm (II)}_{\nu_1\nu_2}(\boldsymbol{q})F^{\rm (I)}_{\nu_3\nu_4}(\boldsymbol{q}')], \quad \text{(B25)}$$

where

$$F^{\rm (I)}_{\nu\nu'}(\boldsymbol{q}) = -\frac{1}{2}\int_{-\infty}^{\infty} dz \frac{\partial n(z)}{\partial z}[G^{\rm R}_{\nu}(\boldsymbol{q},z)G^{\rm R}_{\nu'}(\boldsymbol{q},z) + G^{\rm A}_{\nu}(\boldsymbol{q},z)G^{\rm A}_{\nu'}(\boldsymbol{q},z) - 2G^{\rm A}_{\nu}(\boldsymbol{q},z)G^{\rm R}_{\nu'}(\boldsymbol{q},z)]$$
$$= 2\int_{-\infty}^{\infty} dz \frac{\partial n(z)}{\partial z} {\rm Im}G^{\rm R}_{\nu}(\boldsymbol{q},z){\rm Im}G^{\rm R}_{\nu'}(\boldsymbol{q},z), \quad \text{(B26)}$$

$$F^{\rm (II)}_{\nu\nu'}(\boldsymbol{q}') = \int_{-\infty}^{\infty} dz' n(z')[G^{\rm R}_{\nu}(\boldsymbol{q}',z')G^{\rm R}_{\nu'}(\boldsymbol{q}',z') - G^{\rm A}_{\nu}(\boldsymbol{q}',z')G^{\rm A}_{\nu'}(\boldsymbol{q}',z')]$$
$$= 2i\int_{-\infty}^{\infty} dz' n(z')[{\rm Re}G^{\rm R}_{\nu}(\boldsymbol{q}',z'){\rm Im}G^{\rm R}_{\nu'}(\boldsymbol{q}',z') + {\rm Im}G^{\rm R}_{\nu}(\boldsymbol{q}',z'){\rm Re}G^{\rm R}_{\nu'}(\boldsymbol{q}',z')]. \quad \text{(B27)}$$

A combination of Eqs. (B26), (B27), and (B25) gives Eq. (15).

APPENDIX C: DERIVATIONS OF EQS. (18), (19), AND (21)–(27)

We explain the details of the derivations of Eqs. (18), (19), and (21)–(27). These equations are obtained by deriving the expressions of L^0_{12} and L'_{12} in the limit $\tau \to \infty$, where $\tau = (2\gamma)^{-1}$ is the magnon lifetime.

First, we derive Eqs. (18) and (19). Using Eq. (B12), we have

$$ {\rm Im}G^{\rm R}_{\alpha}(\boldsymbol{q},z) = -\frac{\gamma}{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2}, \quad \text{(C1)}$$

$$ {\rm Im}G^{\rm R}_{\beta}(\boldsymbol{q},z) = \frac{\gamma}{[z+\epsilon_\beta(\boldsymbol{q})]^2+\gamma^2}. \quad \text{(C2)}$$

Since $\tau \to \infty$ corresponds to $\gamma \to 0$, we can express $I^{\rm (I)}_{\nu\nu'}(\boldsymbol{q})$ [i.e., Eq. (14)] in this limit as follows:

$$I^{\rm (I)}_{\alpha\alpha}(\boldsymbol{q}) \sim \frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}\int_{-\infty}^{\infty} dz \frac{\gamma^2}{\{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2\}^2} = \frac{\pi}{2\gamma}\frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}, \quad \text{(C3)}$$

$$I^{\rm (I)}_{\beta\beta}(\boldsymbol{q}) \sim \frac{\pi}{2\gamma}\frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial\epsilon_\beta(\boldsymbol{q})}, \quad \text{(C4)}$$

$$I^{\rm (I)}_{\alpha\beta}(\boldsymbol{q}) = I^{\rm (I)}_{\beta\alpha}(\boldsymbol{q}) \sim 0. \quad \text{(C5)}$$

Combining these equations with Eq. (13), we have

$$L^0_{12} \sim L^0_{12\alpha} + L^0_{12\beta}, \quad \text{(C6)}$$

$$L^0_{12\nu} = \frac{1}{N}\sum_{\boldsymbol{q}} v^x_{\nu\nu}(\boldsymbol{q}) e^x_{\nu\nu}(\boldsymbol{q}) \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial\epsilon_\nu(\boldsymbol{q})}\tau. \quad \text{(C7)}$$

These are Eqs. (18) and (19).

Next, we derive Eqs. (21)–(27). Since L'_{12} is given by Eq. (15), the remaining task is to derive the expression of $I^{\rm (II)}_{\nu\nu'}(\boldsymbol{q})$ in the limit $\tau \to \infty$. By performing the similar calculations to the derivations of Eqs. (C3)–(C5), we obtain

$$\int_{-\infty}^{\infty} dz n(z){\rm Re}G^{\rm R}_{\alpha}(\boldsymbol{q},z){\rm Im}G^{\rm R}_{\alpha}(\boldsymbol{q},z) = -\gamma\int_{-\infty}^{\infty} dz n(z)\frac{z-\epsilon_\alpha(\boldsymbol{q})}{\{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2\}^2}$$
$$= -\gamma\int_{-\infty}^{\infty} dz n(z)\frac{\partial}{\partial z}\left\{-\frac{1}{2}\frac{1}{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2}\right\} \sim -\frac{\pi}{2}\frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}, \quad \text{(C8)}$$

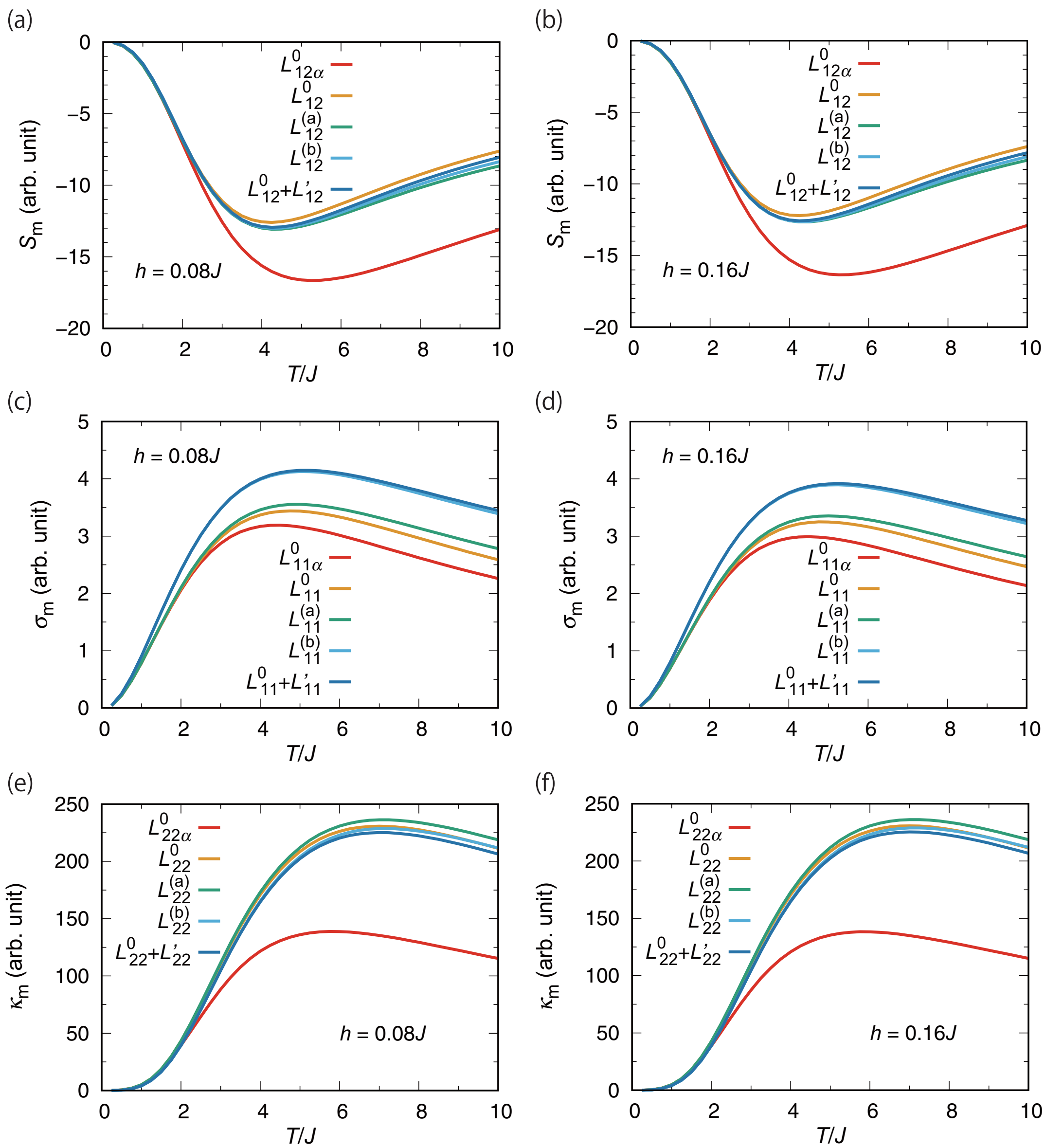

FIG. 4. The temperature dependences of $S_{\rm m}(=L_{12})$, $\sigma_{\rm m}(=L_{11})$, and $\kappa_{\rm m}(=L_{22})$ at $h=0.08J$ and $0.16J$. h is $0.08J$ in (a), (c), and (e) and $0.16J$ in (b), (d), and (f). $L^{(a)}_{\mu\eta}$ and $L^{(b)}_{\mu\eta}$ are defined as $L^{(a)}_{\mu\eta}=L^0_{\mu\eta}+L'_{\mu\eta\text{-intra}}$ and $L^{(b)}_{\mu\eta}=L^0_{\mu\eta}+L'_{\mu\eta\text{-intra}}+L'_{\mu\eta\text{-inter2}}$, respectively. Note that $L^0_{\mu\eta}=L^0_{\mu\eta\alpha}+L^0_{\mu\eta\beta}$ and $L'_{\mu\eta}=L'_{\mu\eta\text{-intra}}+L'_{\mu\eta\text{-inter1}}+L'_{\mu\eta\text{-inter2}}$.

$$\int_{-\infty}^{\infty} dzn(z)\mathrm{Re}G^{\rm R}_{\alpha}(\boldsymbol{q},z)\mathrm{Im}G^{\rm R}_{\beta}(\boldsymbol{q},z)=\gamma\int_{-\infty}^{\infty} dzn(z)\frac{z-\epsilon_\alpha(\boldsymbol{q})}{\{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2\}\{[z+\epsilon_\beta(\boldsymbol{q})]^2+\gamma^2\}}\sim-\pi\frac{n[-\epsilon_\beta(\boldsymbol{q})]}{\epsilon_\alpha(\boldsymbol{q})+\epsilon_\beta(\boldsymbol{q})}, \tag{C9}$$

$$\int_{-\infty}^{\infty} dzn(z)\mathrm{Re}G^{\rm R}_{\beta}(\boldsymbol{q},z)\mathrm{Im}G^{\rm R}_{\alpha}(\boldsymbol{q},z)=\gamma\int_{-\infty}^{\infty} dzn(z)\frac{z+\epsilon_\beta(\boldsymbol{q})}{\{[z+\epsilon_\beta(\boldsymbol{q})]^2+\gamma^2\}\{[z-\epsilon_\alpha(\boldsymbol{q})]^2+\gamma^2\}}\sim\pi\frac{n[\epsilon_\alpha(\boldsymbol{q})]}{\epsilon_\alpha(\boldsymbol{q})+\epsilon_\beta(\boldsymbol{q})}, \tag{C10}$$

$$\begin{aligned}\int_{-\infty}^{\infty} dzn(z)\mathrm{Re}G^{\rm R}_{\beta}(\boldsymbol{q},z)\mathrm{Im}G^{\rm R}_{\beta}(\boldsymbol{q},z)&=-\gamma\int_{-\infty}^{\infty} dzn(z)\frac{z+\epsilon_\beta(\boldsymbol{q})}{\{[z+\epsilon_\beta(\boldsymbol{q})]^2+\gamma^2\}^2}\\&=-\gamma\int_{-\infty}^{\infty} dzn(z)\frac{\partial}{\partial z}\left\{-\frac{1}{2}\frac{1}{[z+\epsilon_\beta(\boldsymbol{q})]^2+\gamma^2}\right\}\sim-\frac{\pi}{2}\frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial\epsilon_\beta(\boldsymbol{q})}.\end{aligned} \tag{C11}$$

By combining these equations with Eq. (16), we can express $I^{(\rm II)}_{\nu\nu'}(\boldsymbol{q})$ in the limit $\tau\to\infty$ as follows:

$$I^{(\rm II)}_{\alpha\alpha}(\boldsymbol{q})\sim-\pi\frac{\partial n[\epsilon_\alpha(\boldsymbol{q})]}{\partial\epsilon_\alpha(\boldsymbol{q})}, \tag{C12}$$

$$I^{(\rm II)}_{\beta\beta}(\boldsymbol{q})\sim-\pi\frac{\partial n[\epsilon_\beta(\boldsymbol{q})]}{\partial\epsilon_\beta(\boldsymbol{q})}, \tag{C13}$$

$$I^{(\rm II)}_{\alpha\beta}(\boldsymbol{q})=I^{(\rm II)}_{\beta\alpha}(\boldsymbol{q})\sim\pi\frac{n[\epsilon_\alpha(\boldsymbol{q})]-n[-\epsilon_\beta(\boldsymbol{q})]}{\epsilon_\alpha(\boldsymbol{q})+\epsilon_\beta(\boldsymbol{q})}. \tag{C14}$$

Substituting these equations and Eqs. (C3)–(C5) into Eq. (15), we obtain

$$L'_{12} \sim L'_{12\text{-intra}} + L'_{12\text{-inter1}} + L'_{12\text{-inter2}}, \tag{C15}$$

where

$$L'_{12\text{-intra}} = \sum_{\nu=\alpha,\beta} L'_{12\text{-intra-}\nu}, \tag{C16}$$

$$L'_{12\text{-intra-}\nu} = -\frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q}) e^x_{\nu\nu}(\boldsymbol{q}')\tau V_{\nu\nu\nu\nu}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial \epsilon_\nu(\boldsymbol{q}')}, \tag{C17}$$

$$L'_{12\text{-inter1}} = \sum_{\nu=\alpha,\beta}\left\{-\frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q}) e^x_{\bar{\nu}\bar{\nu}}(\boldsymbol{q}')\tau V_{\nu\nu\bar{\nu}\bar{\nu}}(\boldsymbol{q},\boldsymbol{q}') \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}\frac{\partial n[\epsilon_{\bar{\nu}}(\boldsymbol{q}')]}{\partial \epsilon_{\bar{\nu}}(\boldsymbol{q}')}\right\}, \tag{C18}$$

and

$$L'_{12\text{-inter2}} = \sum_{\nu=\alpha,\beta}(L'_{\text{E}\nu} + L'_{\text{S}\nu}), \tag{C19}$$

$$L'_{\text{E}\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\nu\nu}(\boldsymbol{q}) e^x_{\alpha\beta}(\boldsymbol{q}')V_{\nu\nu\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\tau \times \frac{\partial n[\epsilon_\nu(\boldsymbol{q})]}{\partial \epsilon_\nu(\boldsymbol{q})}\frac{n[\epsilon_\alpha(\boldsymbol{q}')] - n[-\epsilon_\beta(\boldsymbol{q}')]}{\epsilon_\alpha(\boldsymbol{q}') + \epsilon_\beta(\boldsymbol{q}')}, \tag{C20}$$

$$L'_{\text{S}\nu} = \frac{2}{N^2}\sum_{\boldsymbol{q},\boldsymbol{q}'} v^x_{\alpha\beta}(\boldsymbol{q}) e^x_{\nu\nu}(\boldsymbol{q}')V_{\alpha\beta\nu\nu}(\boldsymbol{q},\boldsymbol{q}')\tau \times \frac{n[\epsilon_\alpha(\boldsymbol{q})] - n[-\epsilon_\beta(\boldsymbol{q})]}{\epsilon_\alpha(\boldsymbol{q}) + \epsilon_\beta(\boldsymbol{q})}\frac{\partial n[\epsilon_\nu(\boldsymbol{q}')]}{\partial \epsilon_\nu(\boldsymbol{q}')}. \tag{C21}$$

In Eq. (C18), $\bar{\nu} = \beta$ or α for $\nu = \alpha$ or β, respectively. Equations (C15)–(C21) are Eqs. (21)–(27).

APPENDIX D: REMARK ON THE NUMERICAL CALCULATION

To calculate $L^0_{\mu\eta}$ and $L'_{\mu\eta}$ numerically, we perform the momentum summations using a N_q-point mesh of the first Brillouin zone. Since the sublattice of our ferrimagnetic insulator is described by a set of primitive vectors, $\boldsymbol{a}_1 = {}^t(1\ 0\ 0)$, $\boldsymbol{a}_2 = {}^t(0\ 1\ 0)$, and $\boldsymbol{a}_3 = {}^t(0\ 0\ 1)$, the primitive vectors for the reciprocal lattice are $\boldsymbol{b}_1 = {}^t(2\pi\ 0\ 0)$, $\boldsymbol{b}_2 = {}^t(0\ 2\pi\ 0)$, and $\boldsymbol{b}_3 = {}^t(0\ 0\ 2\pi)$. Thus, in the periodic boundary condition, momentum $\boldsymbol{q}$ is written in the form

$$\boldsymbol{q} = \frac{m_x}{N_x}\boldsymbol{b}_1 + \frac{m_y}{N_y}\boldsymbol{b}_2 + \frac{m_z}{N_z}\boldsymbol{b}_3, \tag{D1}$$

where $0 \leqslant m_x < N_x$, $0 \leqslant m_y < N_y$, and $0 \leqslant m_z < N_z$ with $N_xN_yN_z = N_q = N/2$. As a result, the first Brillouin zone is divided into the $(N_xN_yN_z)$-point mesh. In the numerical calculation, we set $N_x = N_y = N_z = 24$ (i.e., $N_q = 24^3$).

APPENDIX E: NUMERICAL RESULTS AT $h = 0.08J$ AND $0.16J$

We present the additional results of the numerical calculations, the temperature dependences of S_{m}, σ_{m}, and κ_{m} at $h = 0.08J$ and $0.16J$. They are shown in Figs. 4(a)–4(f). Comparing these figures with Fig. 2, we see the results obtained at $h = 0.08J$ and $0.16J$ are similar to those obtained at $h = 0.02J$. Namely, the properties obtained at $h = 0.02J$ remain qualitatively unchanged for other values of h.

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