Slowly decaying real-time oscillations in instanton crystals

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Instanton crystal is a fascinating phase which is encountered when the minimum of the free energy corresponds to a configuration with an imaginary-time-dependent order parameter in the form of a chain of alternating instantons and anti-instantons. We present the results of the investigation of the real-time correlation functions of the order parameter in the instanton crystal phase. In order to obtain the correlation functions in real time, we formulate an original method of analytic continuation from imaginary times, which is easily adapted into an efficient numerical scheme for the computations. The resulting correlation functions exhibit nontrivial slowly decaying oscillations in real time, which is reminiscent of prethermal time crystals.

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I. INTRODUCTION

A. Instanton crystal

In standard theories of phase transitions, one uses the concept of the order parameter and spontaneous breaking of the symmetry in the phase when the order parameter is not equal to zero [1]. As one considers thermodynamic models starting with a time-independent Hamiltonian, physical real time t does not appear in thermodynamic quantities. In contrast, the imaginary time τ is a usual additional "coordinate" that arises in the formalism of quantum field theory which is nowadays a conventional tool for describing many body systems.

In many cases, like spin models, the spin integration variables $\mathbf{S}(\tau)$ contain the single imaginary time variable τ. In other cases, superconductivity, charge density waves, etc. the integration variables are functions of two variables of the type $\Delta(\tau, \tau')$. As concerns the order parameter usually found from the minimum of the free energy, it does not depend in all known cases for spin systems on τ at all, while in the two-time cases it may depend on the difference $\tau - \tau'$. At the same time, it is not clear whether a function of two variables τ and τ' playing the role of the order parameter may depend on $\tau + \tau'$, which would violate the imaginary time translation invariance. This possibility does not contradict to general principles but one should be sure that the free energy of such a state is minimal with respect to those of other states.

The question about the possibility of the imaginary time-dependent order parameter has been raised rather long ago [2,3]. In these publications, the order parameter was taken as a train of instantons and anti-instantons but the free energy turned out to be higher than the one for the static order parameter. More recently, the model was extended adding additional interaction terms in publications [4,5] but the free energies for the imaginary time-dependent states still exceeded the free energy of the corresponding state with the static order parameter. An attempt to construct soliton trains in imaginary time has been undertaken in Ref. [6]. In all these cases, however, the mathematical construction of solitons hinges on the reflectionless potentials of the underlying scattering problem. Generation of solitons using a supersymmetry approach has been discussed in a recent paper [7]. Apparently, in the models that can be solved exactly, solutions with the imaginary time-dependent order parameter are energetically unfavorable and one had to try more sophisticated models.

Fortunately, a proper model has been suggested in our recent publication [8]. This model contains the fermion part analogous to the one considered in Refs. [2–5] but the fermions interact additionally with a boson mode. The crucial difference with conventional electron-phonon models is that we include into consideration the interaction of currents created by fermions and bosons rather than interaction of densities. We have investigated the model in the mean-field approximation using both analytical and numerical methods and demonstrated existence of the phase transition from the static phase into a new state with the order parameter in a form of an instanton/anti-instanton train in the imaginary time. The transition can be either of the first or of the second order.

In Ref. [8], we restricted ourselves to the calculation of thermodynamic quantities. The discussion of the correlations of physical quantities in real time was left for the future study. The reason for that was that we deemed this topic highly nontrivial to deserve a separate work.

As such, this paper is devoted precisely to the study of real-time correlations in the instanton crystal. We find an interesting behavior which is specific for this thermodynamically stable state.

The real time t is introduced via the standard quantum mechanical replacement of the operators

$$\hat{A}(t) = e^{it(\hat{H}-\mu\hat{N})}\hat{A}e^{-it(\hat{H}-\mu\hat{N})}, \tag{1}$$

where $\hat{H}$ is the full (time-independent) Hamiltonian of the model and μ is chemical potential. Within this approach,

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we calculate certain correlation functions assuming that the system itself remains in the thermodynamic equilibrium.

The thermodynamic average $\langle \dots \rangle$ describing these functions equals

$$\langle \dots \rangle = Z^{-1} Tr[(\dots) \exp[-\beta(\hat{H} - \mu\hat{N})]], \tag{2}$$

while

$$Z = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{N})}],$$

is the partition function, and $\beta = 1/T$ is the inverse temperature. The brackets in Eq. (2) can contain products of real-time dependent operators like those, given in Eq. (1).

So, we consider purely thermodynamic model but are going to obtain nontrivial real-time dependent correlations. Of course, $\langle \hat{A}(t) \rangle = \langle \hat{A} \rangle$ cannot depend on t, and the perpetual motion is impossible.

B. Instanton crystal versus quantum time crystal

Recently, a lot of attention has been paid to the study of time crystals in several communities including condensed matter, atomic physics, quantum optics. This activity was triggered by Wilczek [9] who proposed a concept of quantum time crystals using a rather simple model that possessed a state with a current oscillating in time. However, a more careful consideration of the model [10] has led to the conclusion that this was not the equilibrium state. These publications were followed by discussions of the possibility of realization of a thermodynamically stable quantum time crystal [11–16].

Meanwhile, it has been realized that long living oscillations can exist in systems out of equilibrium [17–22]. In particular, in the systems under periodic pumping from external source. Several experimental works [23–25] have nicely confirmed that studying long-living or infinitely living time oscillations in such a context is a very interesting subject of research as well. Actually, nowadays, the term "quantum time crystal" relates to this type of systems. A review on this subject has appeared recently [26].

This shift of the interest from thermodynamic to nonequilibrium systems has been greatly affected by the "no-go" theorem proposed in Ref. [27]. According to the statements of this theorem, the thermodynamic time crystals are not possible. The proof looks mathematically convincing and, therefore, the study of the equilibrium systems was discontinued naturally in favour of the nonequilibrium systems, while the name for this interesting subject of research stayed the same.

In this paper, we demonstrate that the real-time correlation functions of the Instanton Crystal can oscillate without significant decay for the prolonged periods of time. Unfortunately, the analysis of the current paper does not allow us to say conclusively if these oscillations survive in the asymptotic limit $t \to \infty$. Overall, the instanton crystal is a genuine thermodynamic state obtained within a thermodynamic model using traditional methods of theoretical physics. The oscillating character of the correlation functions in real time is to some extent a "byproduct" of the equilibrium properties of this state. This is the reason why we avoid using the notion "time crystal" for the new phase found in our case.

Whether the observed oscillations are decaying or nondecaying, the sole fact of their appearance contradicts the statement of the "no-go" theorem. It turns out that, although their arguments are rather generally correct, the proof is not devoid of the problems. In particular, it was discussed in Ref. [26] that the proof in the case of nonzero temperature contains certain holes. We discuss the validity of the "no-go" theorem in regards to the instanton crystal state in Sec. VI.

Finally, we should point out that we perform all the calculations in the paper using the mean-field approximation, which is a standard first stage in the analysis of a new phase transition. We reserve the treatment of the effects of the fluctuations around the mean-field configurations for the future detailed study.

C. The structure of the paper

In Sec. II, we formulate the model and recapitulate the mean-field equations derived previously [8]. In Sec. III, we discuss the analytical continuation of correlation functions from the imaginary-time axis to the real-time axis. We show that the standard method of continuation via frequency domain is ambiguous in the case of the instanton crystal. Instead, we formulate a method of continuation directly in the time domain, which can be easily adapted into an efficient scheme for numerical computations. In Sec. IV, we consider the limit of negligible interaction between the fermions and the boson modes analytically. In Sec. V, we present the results of the numerical investigation of the correlation function in real time using an original scheme of computations. In Sec. VI we discuss the validity of the "no-go" theorem of Ref. [27] in the case of the instanton crystal state.

Section VII contains discussion and concluding remarks. In Appendix A, we discuss the connection between the correlation function studied in this paper and magnetic neutron scattering. In Appendix B, we outline the details of the numerical scheme used to produce the results of Sec. V. The scheme is based on the method discussed in Sec. III.

II. INSTANTON CRYSTAL

A. Model

The model exhibiting the instanton crystal state was proposed in Ref. [8] on the basis of the spin-fermion model with overlapping hot spots (SFMOHS) studied previously in Refs. [5,28–30]. In the Lagrangian formulation, the grand canonical partition function is given by the functional integral

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}\} \\ &= \int \exp\left(-S[\chi, \chi^+, a]\right) D\chi D\chi^+ Da, \end{aligned} \tag{3}$$

where the action of the model has the structure

$$S[\chi, \chi^+, a] = S_0 + S_{\mathrm{int}} + S_{\mathrm{B}} + S_{\mathrm{FB}}. \tag{4}$$

In Eq. (4), the term S_0 stands for the action of a system of noninteracting fermions:

$$\begin{aligned} S_0[\chi, \chi^+] = \sum_{\mathbf{p}} \int_0^\beta \chi_{\mathbf{p}}^+(\tau)[(\partial_\tau + \varepsilon_{\mathbf{p}}^+ - \mu)\check{1} \\ + \varepsilon_{\mathbf{p}}^- \check{\Sigma}_3]\chi_{\mathbf{p}}(\tau) d\tau. \end{aligned} \tag{5}$$

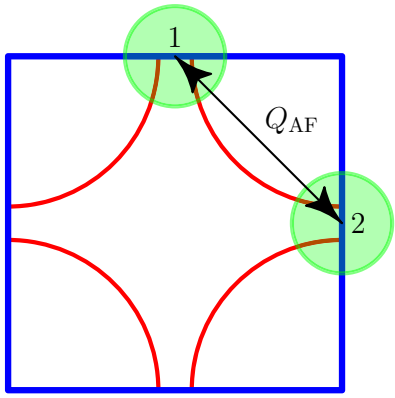

FIG. 1. Fermi surface (red) and overlapping hot spots (green).

These fermions live in two bands 1 and 2, which correspond to the two antinodal "hot regions" of SFMOHS (see Fig. 1 for the schematic representation of Fermi surface with hot regions). Four-component vectors

$$\chi_{\mathbf{p}}^{(+)}(\tau)=\left(\chi_{1\mathbf{p}}^{1(+)}(\tau),\chi_{1\mathbf{p}}^{2(+)}(\tau),\chi_{2\mathbf{p}}^{1(+)}(\tau),\chi_{2\mathbf{p}}^{2(+)}(\tau)\right) \tag{6}$$

contain as components Grasmann fields $\chi_{\alpha\mathbf{p}}^{s(+)}(\tau)$ that correspond to destruction (creation) operators for the fermions from the bands $s=1$ and 2 with spin projection labeled by $\alpha=1$ and 2. The energies $\varepsilon_{\mathbf{p}}^{\pm}$ are expressed in terms of the spectra $\varepsilon_{1,2}(\mathbf{p})$ in the bands 1 and 2 as

$$\varepsilon_{\mathbf{p}}^{\pm}=\tfrac{1}{2}(\varepsilon_1(\mathbf{p})\pm\varepsilon_2(\mathbf{p})). \tag{7}$$

The operators $\check{\Sigma}_i$, $i=1,\ 2,$ and 3 are Pauli matrices acting in the subspace of the bands 1 and 2, while $\check{1}$ is the identity operator acting in the same subspace.

As usual, the imaginary time τ is defined for $0\leqslant\tau\leqslant\beta\equiv 1/T$, where T is the temperature. The fermionic fields $\chi_{\alpha\mathbf{p}}^{s}(\tau)$, $\chi_{\alpha\mathbf{p}}^{s+}(\tau)$ obey standard antiperiodic boundary conditions

$$\chi_{\alpha\mathbf{p}}^{s}(\tau+\beta)=-\chi_{\alpha\mathbf{p}}^{s}(\tau),\ \ \chi_{\alpha\mathbf{p}}^{s+}(\tau+\beta)=-\chi_{\alpha\mathbf{p}}^{s+}(\tau), \tag{8}$$

The second term S_{int} in Eq. (4) stands for the interaction between the fermions from different bands

$$S_{\mathrm{int}}[\chi,\chi^{+}]=-\frac{U_0}{4V}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}}\int_0^{\beta}d\tau(\chi_{\mathbf{p}_1}^{+}(\tau)\check{\Sigma}_2\chi_{\mathbf{p}_1+\mathbf{q}}(\tau))\times(\chi_{\mathbf{p}_2}^{+}(\tau)\check{\Sigma}_2\chi_{\mathbf{p}_2-\mathbf{q}}(\tau)), \tag{9}$$

where V is the volume of the system. Physically, this term corresponds to the local in time attraction of the fermionic loop currents $A_{\mathbf{q}}(\tau)$

$$A_{\mathbf{q}}(\tau)=\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\chi_{\mathbf{p}}^{+}\check{\Sigma}_2\chi_{\mathbf{p}+\mathbf{q}}(\tau). \tag{10}$$

The first two terms in Eq. (4) originate from SFMOHS. The last two ones, on other hand, were specifically introduced in Ref. [8] to stabilize the instanton crystal state. The third one, S_{B}, describes a system of currentlike bosonic modes labeled by different momenta $\mathbf{q}$:

$$S_{\mathrm{B}}[a]=\frac{1}{U_2}\sum_{\mathbf{q}}\int_0^{\beta}\left[\left|\frac{da_{\mathbf{q}}(\tau)}{d\tau}\right|^2+\omega_{\mathbf{q}}^2|a_{\mathbf{q}}(\tau)|^2\right]d\tau, \tag{11}$$

where $a_{\mathbf{q}}(\tau)$ are periodic complex fields satisfying

$$(a_{\mathbf{q}}(\tau))^{*}=a_{-\mathbf{q}}(\tau). \tag{12}$$

The fields $a_{\mathbf{q}}(\tau)$ correspond to the coordinates in the language of oscillator modes, and $da_{\mathbf{q}}(\tau)/d\tau$ correspond to their velocities.

Finally, the last term, S_{FB}, describes the coupling between the currentlike bosonic modes and the fermions:

$$S_{\mathrm{FB}}[\chi,\chi^{+},a]=-\sum_{\mathbf{p},\mathbf{q}}\int_0^{\beta}A_{\mathbf{q}}(\tau)\frac{da_{\mathbf{q}}(\tau)}{d\tau}d\tau. \tag{13}$$

The distinguishing feature of the currentlike modes is that, instead of the charge of the fermions, they couple to the fermionic loop currents $A_{\mathbf{q}}(\tau)$.

Starting with Eq. (3), the currentlike modes can be integrated out leading to the appearance of the effective nonlocal repulsion between the fermionic loop currents in addition to the local attraction due to the term (9). Then, the overall loop current interaction can be decoupled by the means of the Hubbard-Stratonovich transformation. Finally, one can integrate out the fermions to obtain an effective description of the partition function $\mathcal{Z}$ in terms of the functional integral over the real order parameter field $b(\tau)$. If we neglect the spatial fluctuations, we can write it as

$$\mathcal{Z}=\int\mathcal{D}[b(\tau)]\exp\left(-\beta\mathcal{F}[b]\right), \tag{14}$$

where the free energy functional is

$$\frac{\beta\mathcal{F}[b]}{V}=\iint_0^{\beta}d\tau d\tau' K^{-1}(\tau-\tau'|\omega_0)b(\tau)b(\tau')-2\int\frac{d\mathbf{p}}{(2\pi)^2}\mathrm{Tr}_{s,\tau}[\ln((\partial_\tau+\varepsilon_{\mathbf{p}}^{+}-\mu)\check{1}+\varepsilon_{\mathbf{p}}^{-}\check{\Sigma}_3-b(\tau)\check{\Sigma}_2)]. \tag{15}$$

Nonlocal kernel $K^{-1}(\tau-\tau'|\omega_0)$ is given by

$$K^{-1}(\tau-\tau'|\omega_0)=\frac{1}{U_0+U_2}\left[\delta(\tau-\tau')+\frac{U_2}{U_0}K_0(\tau-\tau'|\tilde{\omega}_0)\right], \tag{16}$$

where

$$K_0(\tau-\tau'|\tilde{\omega}_0)=\frac{\tilde{\omega}_0\cosh\left[\tilde{\omega}_0\left(\frac{\beta}{2}-|\tau-\tau'|\right)\right]}{2\sinh\frac{\beta\tilde{\omega}_0}{2}} \tag{17}$$

and

$$\tilde{\omega}_0=\sqrt{\frac{U_0}{U_0+U_2}}\omega_0 \tag{18}$$

is the renormalized frequency of the currentlike mode. The order parameter field $b(\tau)$ satisfies the periodic boundary conditions:

$$b(\tau)=b(\tau+\beta). \tag{19}$$

B. Mean-field analysis

Mean-field solutions correspond to the stationary configurations of the free energy functional $\mathcal{F}[b(\tau)]$:

$$b(\tau)+\frac{U_2}{U_0}\int_0^\beta d\tau' K_0(\tau-\tau'|\tilde{\omega}_0)b(\tau') = -(U_0+U_2)\int\frac{d\mathbf{p}}{(2\pi)^d}\mathrm{Tr}\{\check{\Sigma}_2\check{G}(\tau,\tau)\}, \tag{20}$$

where the fermion Green's function $\check{G}(\tau,\tau')$ satisfies

$$[(\partial_\tau+\varepsilon_{\mathbf{p}}^+)\check{\mathrm{I}}+\varepsilon_{\mathbf{p}}^-\check{\Sigma}_3-b(\tau)\check{\Sigma}_2]\check{G}(\tau,\tau')=-\check{\mathrm{I}}\delta(\tau-\tau'). \tag{21}$$

In the absence of nonlocal interaction, i.e., for $U_2=0$, the solutions of Eq. (20) are known analytically. The minimum of the free energy functional corresponds to the stationary configurations $b(\tau)\equiv\pm\gamma_T$, where the static gap γ_T is determined from the self-consistency equation

$$\frac{2}{U_0}=\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{\tanh\frac{\beta(\kappa_{\mathbf{p}}^{(0)}+\varepsilon_{\mathbf{p}}^+)}{2}+\tanh\frac{\beta(\kappa_{\mathbf{p}}^{(0)}-\varepsilon_{\mathbf{p}}^+)}{2}}{\kappa_{\mathbf{p}}^{(0)}} \tag{22}$$

with

$$\kappa_{\mathbf{p}}^{(0)}=\sqrt{(\varepsilon_{\mathbf{p}}^-)^2+\gamma_T^2}. \tag{23}$$

Besides that, there is also a series of saddle point solutions which correspond to the alternating chains of instantons and anti-instantons [2–5,8,31]. These instanton/anti-instanton solutions are expressed exactly in terms of Jacobi elliptic function $sn(x|k)$:

$$b(\tau)=k\gamma\,\mathrm{sn}(\gamma(\tau-\tau_0)|k). \tag{24}$$

The period of instanton/anti-instanton pair is

$$W=4K(k)/\gamma, \tag{25}$$

where $K(k)$ is the complete elliptic integral of the first kind. For a solution with m instanton/anti-instanton pairs, the requirement of periodicity leads to the condition

$$\beta=m\times\frac{4K(k)}{\gamma}. \tag{26}$$

In order to determine parameters k and γ unambiguously, Eq. (26) should be supplemented by generalized self-consistency equation

$$\frac{2}{U_0}=\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{|\varepsilon_{\mathbf{p}}^-|\left[\tanh\frac{\beta(\kappa_{\mathbf{p}}+\varepsilon^+)}{2}+\tanh\frac{\beta(\kappa_{\mathbf{p}}-\varepsilon^+)}{2}\right]}{\sqrt{\left((\varepsilon_{\mathbf{p}}^-)^2+\gamma^2\frac{(1-k)^2}{4}\right)\left((\varepsilon_{\mathbf{p}}^-)^2+\gamma^2\frac{(1+k)^2}{4}\right)}}, \tag{27}$$

where

$$\kappa_{\mathbf{p}}=|\varepsilon_{\mathbf{p}}^-|\sqrt{\frac{\left((\varepsilon_{\mathbf{p}}^-)^2+\gamma^2\frac{(1-k)^2}{4}\right)}{\left((\varepsilon_{\mathbf{p}}^-)^2+\gamma^2\frac{(1+k)^2}{4}\right)}}\frac{\Pi(n,\tilde{k})}{K(\tilde{k})}, \tag{28}$$

where

$$n=\frac{\gamma^2 k}{(\varepsilon_{\mathbf{p}}^-)^2+\gamma^2\frac{(1+k)^2}{4}},\qquad \tilde{k}=\frac{2\sqrt{k}}{1+k}. \tag{29}$$

Here, $\Pi(n,k)$ is the complete elliptic integral of the third kind. (One can find more details about elliptic integrals and functions, for example, in Refs. [32,33]). We should also point out that in the limit $k\to 1$, which corresponds to the dilute instanton/anti-instanton approximation, Eq. (27) reduces to Eq. (22).

As we have already mentioned, in the absence of nonlocal interaction, the minimum of the free energy functional corresponds to static order parameter field. However, when U_2 is turned on, the energy of the static configuration can get pushed higher than the energy of instanton configurations. In this case, the minimum corresponds to a solution with a particular number m of instanton/anti-instanton pairs. In Ref. [8], we demonstrated this phenomena both analytically (for $U_2/U_0\ll 1$) and numerically. In the latter case, we used an efficient procedure based on direct minimization of suitably discretized version of the free energy functional.

III. ANALYTIC CONTINUATION OF THE ORDER PARAMETER CORRELATION FUNCTION

A. Real- and imaginary-time correlation functions

The experimentally relevant quantity important for the characterization of the instanton crystal state is the dynamic correlation function of the loop currents:

$$C^{\mathcal{T}}(t)=\frac{1}{V\mathcal{Z}}\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}\mathcal{T}_t\hat{A}_0(t)\hat{A}_0\}, \tag{30}$$

where $\mathcal{T}_t$ is the time ordering operator along real time and

$$\hat{A}_0(t)=e^{it(\hat{H}-\mu\hat{N})}\hat{A}_0e^{-it(\hat{H}-\mu\hat{N})}. \tag{31}$$

Here, $\hat{A}_0$ is the operator counterpart of $A_{\mathbf{q}}(\tau)$ defined in Eq. (10) at $\mathbf{q}=0$:

$$\hat{A}_0=\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}c_{\mathbf{p}}^\dagger\check{\Sigma}_2c_{\mathbf{p}}, \tag{32}$$

where $c_{\mathbf{p}}^\dagger$ and $c_{\mathbf{p}}$ are fermion creation and destruction operators.

Since $\hat{A}_0$ conserves the number of particles, this definition of time dependency is in fact equivalent to the Heisenberg representation of operators. The correlation function $C^{\mathcal{T}}(t)$ can be measured, for example, by magnetic neutron scattering which is sensitive to the magnetic fields produced by the fluctuating loop currents (see Appendix A).

Analogously, one can define imaginary time counterpart of Eq. (30):

$$C^{\mathrm{M}}(\tau)=\frac{1}{V\mathcal{Z}}\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}\mathcal{T}_\tau\hat{A}_0(\tau)\hat{A}_0\}, \tag{33}$$

where $\mathcal{T}_\tau$ now denotes the ordering operator along the imaginary time and

$$\hat{A}_0(\tau)=e^{\tau(\hat{H}-\mu\hat{N})}\hat{A}_0e^{-\tau(\hat{H}-\mu\hat{N})}. \tag{34}$$

By directly applying the time ordering operators, one can also re-express $C^{\mathcal{T}}(t)$ and $C^{\mathrm{M}}(\tau)$ as

$$C^{\mathcal{T}}(t) = \frac{\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}e^{i|t|(\hat{H}-\mu\hat{N})}\hat{A}_0 e^{-i|t|(\hat{H}-\mu\hat{N})}\hat{A}_0\}}{V\mathcal{Z}}, \quad (35)$$

$$C^{\mathrm{M}}(\tau) = \frac{\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}e^{|\tau|(\hat{H}-\mu\hat{N})}\hat{A}_0 e^{-|\tau|(\hat{H}-\mu\hat{N})}\hat{A}_0\}}{V\mathcal{Z}}. \quad (36)$$

As one may notice from comparing Eqs. (35) and (36), correlation functions $C^{\mathcal{T}}(t)$ and $C^{\mathrm{M}}(\tau)$ are closely related: they can be obtained from each other by analytic continuation [34]. To describe it in more precise terms, let us introduce the following two functions of the complex argument $z = \tau + it$:

$$C^{>}(z) = \frac{\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}e^{z(\hat{H}-\mu\hat{N})}\hat{A}_0 e^{-z(\hat{H}-\mu\hat{N})}\hat{A}_0\}}{V\mathcal{Z}}, \quad (37)$$

$$C^{<}(z) = \frac{\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}e^{-z(\hat{H}-\mu\hat{N})}\hat{A}_0 e^{z(\hat{H}-\mu\hat{N})}\hat{A}_0\}}{V\mathcal{Z}}. \quad (38)$$

In the complex z plane, the function C^{M} is defined on the real axis, while the function $C^{\mathcal{T}}$ is defined on the imaginary axis. The greater function $C^{>}(z)$ is the analytic continuation of both $C^{\mathrm{M}}(\tau)$ and $C^{\mathcal{T}}(t)$ into the first quadrant $\mathrm{Re}z > 0$, $\mathrm{Im}z > 0$. Conversely, the lesser function $C^{<}(z)$ is the analytic continuation of both $C^{\mathrm{M}}(\tau)$ and $C^{\mathcal{T}}(t)$ into the third quadrant $\mathrm{Re}z < 0$, $\mathrm{Im}z < 0$:

$$C^{\mathrm{M}}(\tau) = C^{>}(\tau) \qquad \tau > 0, \quad (39)$$

$$C^{\mathrm{M}}(\tau) = C^{<}(\tau) \qquad \tau < 0, \quad (40)$$

$$C^{\mathcal{T}}(t) = C^{>}(it) \qquad t > 0, \quad (41)$$

$$C^{\mathcal{T}}(t) = C^{<}(it) \qquad t < 0. \quad (42)$$

In practice, the connection between real- and imaginary-time correlation functions is established usually first by converting everything into the frequency domain, and then performing the analytic continuation in the transformed domain. We shall demonstrate, however, that the procedure of the analytic continuation in the frequency domain is ambiguous for instanton crystal state. As a result, the function $C^{\mathcal{T}}(t)$ should be obtained from $C^{\mathrm{M}}(\tau)$ by analytic continuation directly in the time domain.

In conclusion of this section, we would also like to point out that the complex time plane is typically defined in such a way that the real time corresponds to the real axis while imaginary time corresponds to the imaginary axis of the complex plane. However, we found a complex time plane definition with interchanged axes more suitable for the presented work.

B. Analytic continuation in the frequency domain

To discuss continuation in the frequency domain, we need to introduce yet another two real-time correlation functions, retarded and advanced one:

$$C^{\mathrm{R}}(t) = -\frac{i\Theta(t)}{V\mathcal{Z}}\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}[\hat{A}_0(t), \hat{A}_0]\}, \quad (43)$$

$$C^{\mathrm{A}}(t) = \frac{i\Theta(-t)}{V\mathcal{Z}}\mathrm{Tr}\{e^{-\beta(\hat{H}-\mu\hat{N})}[\hat{A}_0(t), \hat{A}_0]\}, \quad (44)$$

where $\Theta(t)$ is the Heaviside theta-function.

For the real-time functions, we switch to the frequency domain by Fourier transform

$$C^{\mathcal{T},\mathrm{R,A}}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t - \eta|t|} C^{\mathcal{T},\mathrm{R,A}}(t). \quad (45)$$

The transformed real-time functions are related to each other by [35]

$$\mathrm{Re}C^{\mathcal{T}}(\omega) = \mp \mathrm{Im}C^{\mathrm{R/A}}(\omega)\coth\frac{\beta\omega}{2}, \quad (46)$$

$$\mathrm{Im}C^{\mathcal{T}}(\omega) = \mathrm{Re}C^{\mathrm{R/A}}(\omega). \quad (47)$$

On the other-hand, the imaginary-time function $C^{\mathrm{M}}(t)$ is expanded in Fourier series over Matsubara frequencies $\omega_n = 2\pi T n$, $n \in \mathbb{Z}$:

$$C^{\mathrm{M}}(i\omega_n) = \int_0^{\beta} d\tau e^{i\omega_n \tau} C^{\mathrm{M}}(\tau). \quad (48)$$

Finally, the retarded correlation function $C^{\mathrm{R}}(\omega)$ is obtained from $C^{\mathrm{M}}(i\omega_n > 0)$ by analytic continuation to the real axis in the upper complex frequency half plane: $C^{\mathrm{R}}(\omega) = -C^{\mathrm{M}}(\omega + i0)$. Analogously, the advanced correlation function $C^{\mathrm{A}}(\omega)$ is obtained from $C^{\mathrm{M}}(i\omega_n < 0)$ by analytic continuation to the real axis in the lower complex frequency half plane: $C^{\mathrm{A}}(\omega) = -C^{\mathrm{M}}(\omega - i0)$.

There are certain subtleties associated with continuation from a discrete set of Matsubara frequencies. For concreteness, let us consider the analytic continuation in the upper complex half plane. The accumulation point for the subset of Matsubara frequencies in the upper half plane is $+i\infty$. Correspondingly, the uniqueness of the continuation procedure is normally guaranteed by the regular behavior of $C^{\mathrm{M}}(i\omega_n)$ in the vicinity of the accumulation point, i.e., in the limit $i\omega_n \to +i\infty$.

This picture gets problematic if one considers the instanton crystal state. If the Instanton Crystal consists of m instanton/anti-instanton pairs, the period of the corresponding imaginary-time correlation function is β/m. As a consequence, coefficients $C^{\mathrm{M}}(i\omega_n)$ are nonzero only for $n = mk$, $k \in \mathbb{Z}$. Now, imagine we pick two subsequences of Matsubara frequencies, $i\omega_{n_k} = i\omega_{mk}$ and $i\omega'_{n_k} = i\omega_{mk-1}$, and then perform the analytic continuation either using only the values $C^{\mathrm{M}}(i\omega_{n_k})$ or using only the values $C^{\mathrm{M}}(i\omega'_{n_k})$. On one hand, one expects from the identity theorem of complex analysis that the results of these two continuations should coincide with result of the analytic continuation with the full set of Matsubara frequencies $C^{\mathrm{M}}(i\omega_n)$. On the other hand, the result of continuation using $C^{\mathrm{M}}(i\omega'_{n_k})$ is constant zero, while the result of continuation using $C^{\mathrm{M}}(i\omega_{n_k})$ is some nontrivial function. This allows us to conclude that analytic continuation in the frequency domain is ill defined in the case of instanton crystal state.

C. Direct analytic continuation in time domain

In the mean-field approximation, the imaginary-time correlation function $C^{\mathrm{M}}(\tau)$ has the form

$$C^{\mathrm{M}}(\tau) = \frac{4}{\beta}\int_0^\beta d\tau_0\left[\int \frac{d\mathbf{p}}{(2\pi)^2}\mathrm{Tr}\{\check{\Sigma}_2\check{G}_{\mathbf{p}}(\tau+\tau_0,\tau+\tau_0)\}\right] \times\left[\int \frac{d\mathbf{p}'}{(2\pi)^2}\mathrm{Tr}\{\check{\Sigma}_2\check{G}_{\mathbf{p}'}(\tau_0,\tau_0)\}\right]. \quad (49)$$

Equation (49) contains the averaging over the position τ_0 of the instanton lattice. This is the necessary procedure because the instanton crystal state configuration is invariant against the translations in the imaginary time τ (the periodic boundary conditions on the interval $(0, \beta)$ are implied), and one should properly average over the positions of the instanton lattice when calculating thermodynamic correlation functions.

In the following, it is convenient to introduce vector-function

$$\vec{S}_{\mathbf{p}}(\tau) = -\mathrm{Tr}\{\check{\vec{\Sigma}}\check{G}_{\mathbf{p}}(\tau,\tau)\} \quad (50)$$

and

$$\vec{S}(\tau) = \int \frac{d\mathbf{p}}{(2\pi)^2}\vec{S}_{\mathbf{p}}(\tau). \quad (51)$$

With the help of this definition we can rewrite Eq. (49) as

$$C^{\mathrm{M}}(\tau) = \frac{4}{\beta}\int_0^\beta d\tau_0 S_2(\tau+\tau_0)S_2(\tau_0). \quad (52)$$

The problem of analytic continuation of $C^{\mathrm{M}}(\tau)$ hence reduces to the analytic continuation of the function $S_2(\tau+\tau_0)$. Once it is done, the real-time function $C^{\mathcal{T}}(t)$ can be expressed as

$$C^{\mathcal{T}}(t) = \frac{4}{\beta}\int_0^\beta d\tau_0 S_2(\tau_0+it)S_2(\tau_0). \quad (53)$$

The starting point for the continuation of $S_2(\tau)$ is the gap equation (20). The nonlocal kernel $K^{-1}(\tau-\tau'|\omega_0)$ [see definition (16)] is the inverse to the direct kernel $K(\tau-\tau'|\omega_0)$, which is equal to (see Ref. [8])

$$K(\tau-\tau'|\omega_0) = (U_0+U_2)\delta(\tau-\tau') - U_2K_0(\tau-\tau'|\omega_0). \quad (54)$$

Applying the direct kernel to both sides of Eq. (20), we rewrite it equivalently as

$$b(\tau) = (U_0+U_2)\int \frac{d\mathbf{p}}{(2\pi)^2}S_{2\mathbf{p}}(\tau) - U_2\int_0^\beta d\tau' K_0(\tau-\tau'|\omega_0)\int \frac{d\mathbf{p}}{(2\pi)^2}S_{2\mathbf{p}}(\tau'), \quad (55)$$

where we also used the definition (50).

Nonlocal kernel K_0 has an important property: it is the Green's function for a certain differential operator with periodic boundary conditions (see Ref. [8]):

$$\left[-\frac{d^2}{d\tau^2}+\omega_0^2\right]K_0(\tau-\tau'|\omega_0) = \omega_0^2\delta(\tau-\tau'). \quad (56)$$

Let us apply $[-\partial_\tau^2+\omega_0^2]$ to the both sides of Eq. (55):

$$\frac{d^2b(\tau)}{d\tau^2}+\omega_0^2 b(\tau) = (U_0+U_2)\int \frac{d\mathbf{p}}{(2\pi)^2}\frac{d^2S_{2\mathbf{p}}(\tau)}{d\tau^2} + U_0\omega_0^2\int \frac{d\mathbf{p}}{(2\pi)^2}S_{2\mathbf{p}}(\tau). \quad (57)$$

The obtained differential equation needs to be supplemented by another one which relates the derivatives of the vector-functions $\vec{S}_{\mathbf{p}}(\tau)$ to the order parameter field $b(\tau)$. To derive it, let us turn our attention to the fermion Green's function $\check{G}_{\mathbf{p}}(\tau,\tau')$. In addition to Eq. (21), it also satisfies

$$\check{G}_{\mathbf{p}}(\tau,\tau')[(-\overleftarrow{\partial}_{\tau'}+\varepsilon_{\mathbf{p}}^+)\check{1}+\varepsilon_{\mathbf{p}}^-\check{\Sigma}_3-b(\tau)\check{\Sigma}_2] = -\check{1}\delta(\tau-\tau'). \quad (58)$$

Subtracting Eq. (58) from Eq. (21) and putting $\tau'=\tau$, one obtains

$$(\partial_\tau+\partial_{\tau'})\check{G}_{\mathbf{p}}(\tau,\tau')|_{\tau'=\tau} \equiv \partial_\tau\check{G}_{\mathbf{p}}(\tau,\tau) = -[\varepsilon_{\mathbf{p}}^-\check{\Sigma}_3-b(\tau)\check{\Sigma}_2,\check{G}_{\mathbf{p}}(\tau,\tau)]. \quad (59)$$

Now, let us multiply both sides of equation (59) by $\check{\Sigma}_i$, $i=1,\ 2,$ and 3, and apply trace. Using the cyclic property of trace and commutation relations for Pauli matrices, one can write the resulting equations jointly as

$$\frac{d\vec{S}_{\mathbf{p}}(\tau)}{d\tau} = -2i\vec{B}_{\mathbf{p}}(\tau)\times\vec{S}_{\mathbf{p}}(\tau), \quad (60)$$

where

$$\vec{B}_{\mathbf{p}}(\tau) = (0,\quad -b(\tau),\quad \varepsilon_{\mathbf{p}}^-). \quad (61)$$

The necessary final touch is to get rid of the second derivative of $C_{2\mathbf{p}}(\tau)$ in Eq. (57). This can be achieved by applying Eq. (60) twice and leads to

$$\frac{d^2b(\tau)}{d\tau^2} = \omega_0^2\left[U_0\int \frac{d\mathbf{p}}{(2\pi)^2}S_{2\mathbf{p}}(\tau)-b(\tau)\right] + (U_0+U_2)\int \frac{d\mathbf{p}}{(2\pi)^2}4\varepsilon_{\mathbf{p}}^-[b(\tau)S_{3\mathbf{p}}(\tau)+\varepsilon_{\mathbf{p}}^-S_{2\mathbf{p}}(\tau)]. \quad (62)$$

Equations (60) and (62) constitute a closed system which the functions $b(\tau)$, $db(\tau)/d\tau$ and $\vec{S}_{\mathbf{p}}(\tau)$ satisfy: once the values of $b(\tau)$, $db(\tau)/d\tau$ and $\vec{S}_{\mathbf{p}}(\tau)$ at some τ are known, the functions at other points can be reconstructed by integrating Eqs. (60) and (62). Correspondingly, the analytic continuation of these functions can be facilitated by analytic continuation of the system of Eqs. (60) and (62).

After the analytic continuation of Eqs. (60) and (62) are done, it is convenient to restrict them to some line parallel to the imaginary axis (with our definition of complex time, real time runs along the imaginary axis). The equations along this line, which can be parametrized as $\tau+it$ with fixed τ, are obtained from Eqs. (60) and (62) by replacing

$$\frac{df(\tau)}{d\tau} \to -i\frac{df(\tau+it)}{dt}, \quad (63)$$

where $f(\tau)$ is one of the functions $b(\tau)$ and $\vec{S}_{\mathbf{p}}(\tau)$.

The resulting equations take the form

$$\frac{d\vec{S}_{\mathbf{p}}(\tau+it)}{dt} = 2\vec{B}_{\mathbf{p}}(\tau+it)\times\vec{S}_{\mathbf{p}}(\tau+it), \tag{64}$$

$$\begin{aligned}\frac{d^2b(\tau+it)}{dt^2} &= \omega_0^2\left[b(\tau+it) - U_0\int\frac{d\mathbf{p}}{(2\pi)^2}S_{2\mathbf{p}}(\tau+it)\right]\\ &- (U_0+U_2)\int\frac{d\mathbf{p}}{(2\pi)^2}4\varepsilon_{\mathbf{p}}^-[b(\tau+it)S_{3\mathbf{p}}(\tau+it)\\ &+ \varepsilon_{\mathbf{p}}^-S_{2\mathbf{p}}(\tau+it)].\end{aligned} \tag{65}$$

These equations allow one to determine $b(\tau+it)$, $db(\tau+it)/dt$ and $\vec{S}_{\mathbf{p}}(\tau+it)$ from their values at $t=0$. Moreover, the initial value of $db(\tau+it)/dt$ at $t=0$ can be related to the values of $b(\tau)$ and $\vec{S}_{\mathbf{p}}(\tau)$. To do that, let us differentiate Eq. (20) once:

$$\begin{aligned}&\frac{db(\tau)}{d\tau} + \frac{U_2}{U_0}\int_0^\beta d\tau'\frac{dK_0(\tau-\tau'|\tilde{\omega}_0)}{d\tau}b(\tau')\\ &= (U_0+U_2)\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{dS_{2\mathbf{p}}(\tau)}{d\tau}\\ &= -2i(U_0+U_2)\int\frac{d\mathbf{p}}{(2\pi)^2}\varepsilon_{\mathbf{p}}^-S_{1\mathbf{p}}(\tau).\end{aligned} \tag{66}$$

Here, we used Eq. (60) in the last line. Then, we can write

$$\begin{aligned}\left.\frac{db(\tau+it)}{dt}\right|_{t=0} &= i\frac{db(\tau)}{d\tau}\\ &= i\frac{U_2}{U_0}\int_0^\beta d\tau'\frac{dK_0(\tau-\tau'|\tilde{\omega}_0)}{d\tau}b(\tau')\\ &+ 2(U_0+U_2)\int\frac{d\mathbf{p}}{(2\pi)^2}\varepsilon_{\mathbf{p}}^-S_{1\mathbf{p}}(\tau).\end{aligned} \tag{67}$$

IV. ANALYTICAL RESULTS FOR THE REAL-TIME ORDER PARAMETER CORRELATION FUNCTION IN THE MEAN-FIELD APPROXIMATION AT $U_2 = 0$

In the previous section, we introduced the correlation functions of the order parameter both on the real-time axis, $C^{\mathcal{T}}(t)$, Eq. (30), and on the imaginary-time axis, $C^{\mathrm{M}}(\tau)$, Eq. (33). Now, we are going to calculate these functions at $U_2 = 0$. Of course, we have shown previously [8] that the instanton crystal state is not thermodynamically stable in the absence of nonlocal interaction. However, we have also seen [8] that even a small nonlocal interaction U_2 stabilizes the instanton crystal. As a result, calculations presented in this section allow us to demonstrate explicitly how the analytic continuation from τ to t is performed and how the instanton/anti-instanton train in imaginary time is transformed into oscillations in real time. In addition to that, for the physically relevant case of $U_2/U_0 \ll 1$, we may expect that the results are close to the results at $U_2 = 0$.

With the help of the mean-field equation (20) in the absence of nonlocal interaction, the correlation function $C^{\mathrm{M}}(\tau)$, Eq. (52), can be written in the form of a correlation function of the order parameter $b(\tau)$:

$$U_0^2C^{\mathrm{M}}(\tau) = \frac{1}{\beta}\int_0^\beta b(\tau+\tau_0)b(\tau_0)d\tau_0. \tag{68}$$

If we take into account the periodic properties of $b(\tau)$, this leads to the expression

$$U_0^2C^{\mathrm{M}}(\tau) = \frac{1}{W}\int_0^W b(\tau+\tau_0)b(\tau_0)d\tau_0, \tag{69}$$

where the period W of the instanton/anti-instanton chain is given by Eq. (25). The analytical continuation $\tau\to\tau+it$ in Eq. (69) is straightforward and we write

$$U_0^2C^{\mathrm{M}}(\tau+it) = \frac{1}{W}\int_0^W b(\tau+it+\tau_0)b(\tau_0)d\tau_0. \tag{70}$$

Now, we use a known Fourier series expansion for Jacobi elliptic function $\mathrm{sn}(x|k)$ [33] to write the function $b(\tau)$, Eq. (24), as

$$b(\tau) = \frac{4\pi}{W}\sum_{l=0}^{\infty}\frac{\sin\frac{2\pi(2l+1)\tau}{W}}{\sinh\frac{\pi(2l+1)W'}{W}}, \tag{71}$$

where

$$W' = 2K(k')/\gamma, \qquad k'^2 = 1-k^2. \tag{72}$$

The series expansion (71) can be continued to the strip $\tau+it$, $-W'/2 < t < W'/2$. This allows us to rewrite Eq. (70) in the form

$$\begin{aligned}&U_0^2C^{\mathrm{M}}(\tau+it)\\ &= \frac{16\pi^2}{W^3}\sum_{l,l'=0}^{\infty}\frac{1}{\sinh\frac{\pi(2l+1)W'}{W}\sinh\frac{\pi(2l'+1)W'}{W}}\\ &\times\int_{-W/2}^{W/2}\sin\frac{2\pi(2l+1)(\tau_0+\tau+it)}{W}\\ &\times\sin\frac{2\pi(2l'+1)\tau_0}{W}d\tau_0.\end{aligned} \tag{73}$$

Calculating the integral over τ_0 for $l, l' \geqslant 0$, we obtain

$$C^{\mathrm{M}}(\tau+it) = \frac{8\pi^2}{W^2}\sum_{l=0}^{\infty}\frac{\cos\frac{2\pi(2l+1)(\tau+it)}{W}}{\sinh^2\frac{\pi(2l+1)W'}{W}}. \tag{74}$$

It is clear that the function $C^{\mathrm{M}}(\tau+it)$ is analytic in the vicinity of zero, and the method of the analytical continuation from the imaginary time τ to the real one t works. At the same time, although the function $C^{\mathrm{M}}(\tau+it)$, Eq. (74), is explicitly periodic in τ, its periodicity is not evident in real time t.

The periodicity and divergencies of the Jacobi elliptic functions are well-known. As such, $b(\tau+it)$ has the poles at $iW'/2$ and $W/2+iW'/2$, and the period along t equals W'. Hence, we can represent the function $C^{\mathcal{T}}(t)$ in the form of the Fourier series

$$C^{\mathcal{T}}(t) = \sum_{l=0}^{\infty}\sum_{L=0}^{\infty}C_{L,l}\cos\left(\frac{2\pi Lt}{W'}\right). \tag{75}$$

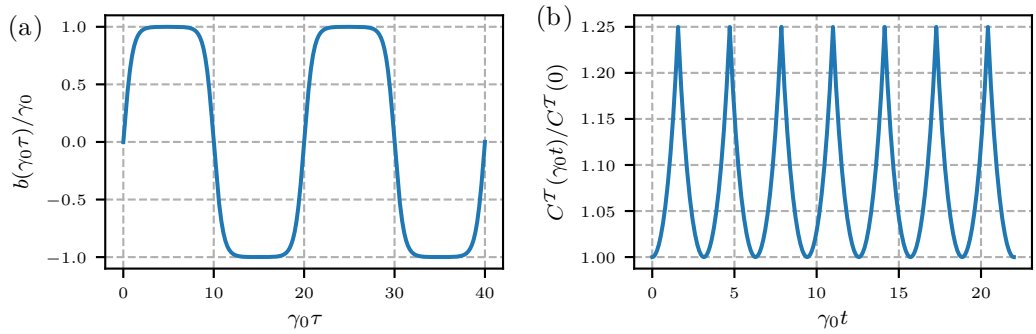

FIG. 2. Instanton lattice configuration and the corresponding real-time correlation function at $U_2 = 0$ and zero temperature. (a) Normalized order parameter of the instanton lattice $b(\gamma_0\tau)/\gamma_0$ as a function of dimensionless imaginary time $\gamma_0\tau$. The dimensionless period of instanton lattice is $\gamma_0 W = 20.0$. (b) Normalized correlation function of the order parameter $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ as a function of dimensionless real time $\gamma_0 t$ at $U_2 = 0$.

Setting $\tau = 0$ in Eq. (74), we calculate the coefficients $C_{L,l}$ to be

$$C_{L,l} = \frac{8\pi W'}{W^3} \frac{(-1)^L}{\sinh \frac{\pi(2l+1)W'}{W}} \frac{2l+1}{(2l+1)^2 + L^2} \tag{76}$$

and obtain the final expression for the correlation function

$$C^{\mathcal{T}}(t) = \frac{8\pi W'}{W^3} \sum_{l=0}^{\infty} \sum_{L=0}^{\infty} (-1)^L \frac{1}{\sinh \frac{\pi(2l+1)W'}{W}} \times \frac{2l+1}{(2l+1)^2 + L^2} \cos\left(\frac{2\pi L t}{W'}\right). \tag{77}$$

In the limit $k \to 1$ of the large period of the instanton/anti-instanton lattice, $K(k') \to \pi/2$, and Eq. (72) transforms into

$$W' \to \pi/\gamma. \tag{78}$$

We see from Eqs. (77) and (78) that, in this limit, the correlation function $C^{\mathcal{T}}(t)$ is periodic in real time with the frequency 2γ. This frequency equals to the excitation gap in SFMOHS studied previously [28–30].

To illustrate this discussion, we provide Fig. 2, where we present an example of the instanton lattice configuration at $U_2 = 0$ and the corresponding real-time correlation function obtained after analytic continuation. Panel (a) displays the normalized order parameter $b(\gamma_0\tau)/\gamma_0$ as the function of dimensionless imaginary time $\gamma_0\tau$. Here, γ_0 stands for the static gap at zero temperature [see Eq. (22)]. The order parameter configuration shown corresponds to the instanton crystal with dimensionless period $\gamma_0 W = 20.0$. Panel (b) displays the normalized order parameter correlation function $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ as the function of the dimensionless real time $\gamma_0 t$.

V. NUMERICAL RESULTS

In order to proceed with the actual numerical computations, we had to specify particular fermion dispersions $\varepsilon_{1,2}(\mathbf{p})$ [see Eq. (7)]. In agreement with our previous work [8], we chose it in the same form as it appears in SFMOHS:

$$\varepsilon_1(\mathbf{p}) = \alpha p_x^2 - \beta p_y^2 - \mu, \quad \varepsilon_2(\mathbf{p}) = \alpha p_y^2 - \beta p_x^2 - \mu, \tag{79}$$

where μ is the chemical potential. We also introduced an energy cutoff Λ limiting the width of the dispersion:

$$\frac{\alpha+\beta}{2}\left(p_x^2 + p_y^2\right) < \Lambda. \tag{80}$$

The specific values used throughout the computations were $\alpha = \beta = 1.0$, $\Lambda/\gamma_0 = 1.0$, and $\mu/\gamma_0 = 0.0$.

The configuration of the instanton crystal can be controlled by the following set of dimensionless parameters: $(U_0 + U_2)/\gamma_0$, the ratio U_2/U_0, the period of the lattice $\gamma_0 W$, the modified frequency of the boson mode $\gamma_0\tilde{\omega}_0$ and the number of the periods m. Here, γ_0 stands for the solution of the static gap equation (22) at zero temperature and in the absence of nonlocal term ($U_2 = 0$). We also used the static self-consistency equation (22) at zero temperature to fix the value of the parameter $(U_0 + U_2)/\gamma_0$.

These parameters are not completely independent. The period of the lattice $\gamma_0 W$ and their number m need to be determined from the dimensionless inverse temperature $\gamma_0\beta$ and other parameters by the condition of the minimum of the free-energy functional (15). Correspondingly, in the limit of zero temperature, $m \to +\infty$ and we determine only $\gamma_0 W$ from the same condition.

Despite these facts, it is convenient to treat these parameters as if they are independent for the purposes of the numeric investigation. Generally, we expect the properties of the instanton crystal configuration to vary smoothly with the control parameters, while the phase transitions correspond to the jumps in the parameter space.

Finally, we should mention that throughout the numerical calculations we assumed the limit of zero temperature.

In Fig. 3, we display the computed normalized real-time correlation functions $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ for varying values of the strength of the nonlocal interaction U_2/U_0. The values of the other parameters were kept fixed: $\gamma_0 W = 20.0$ and $\omega_0/\gamma_0 = 0.10$. The values of U_2/U_0 used for the calculations are listed in the figure. As we may observe, the oscillations with period W' [see Eq. (72)] which were present in the limit $U_2/U_0 = 0$ survive for small but finite values of U_2/U_0. At the same time, the amplitude of these surviving oscillations gets modulated, and the magnitude of the modulation grows together with the strength of the nonlocal interaction U_2/U_0.

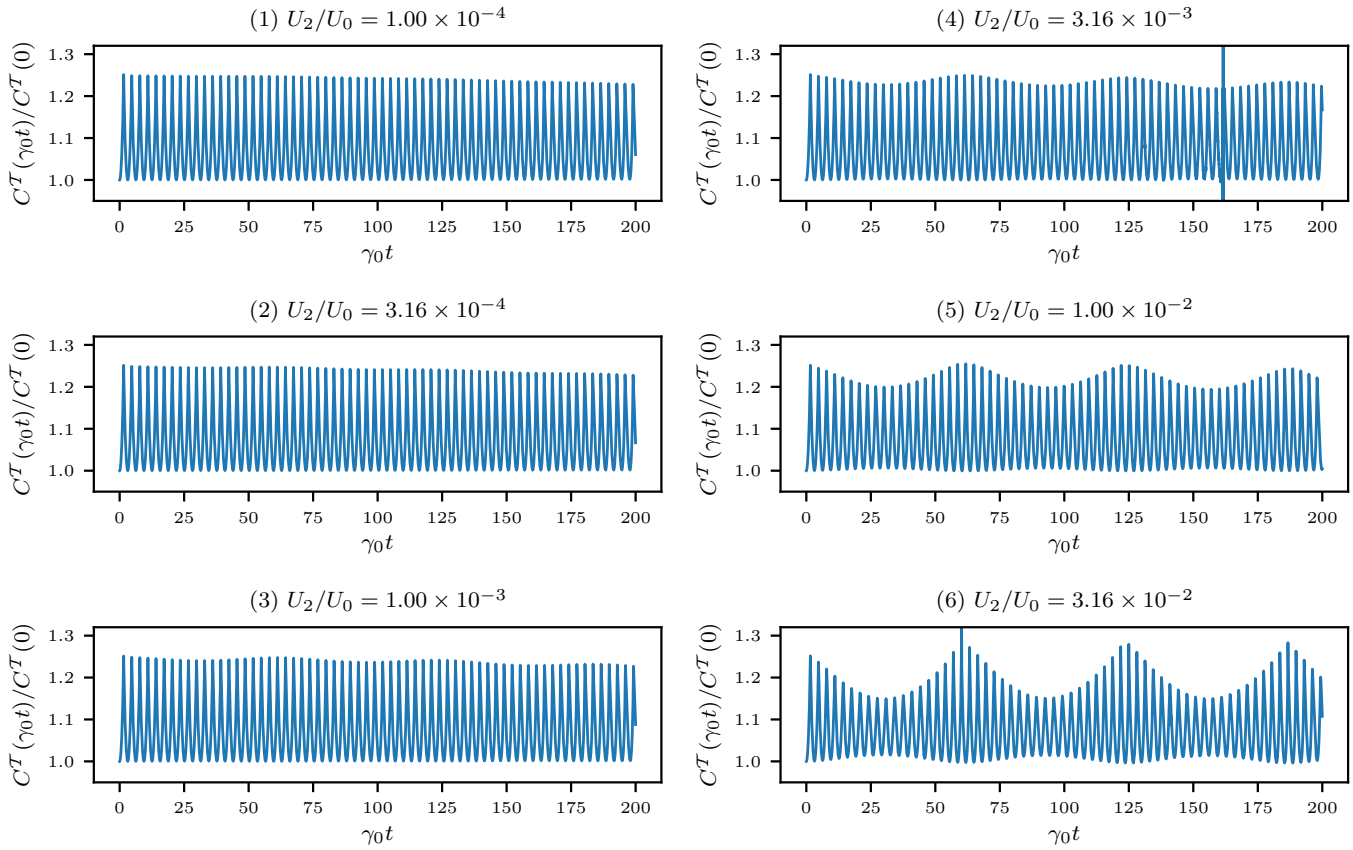

FIG. 3. Normalized real-time correlation function of the order parameter $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ for fixed $\omega_0/\gamma_0 = 0.10$ and $\gamma_0 W = 20.0$ but for varying strength of nonlocal interaction U_2/U_0.

It is natural to assume that the frequency of the modulation corresponds to the bosonic mode frequency. In Fig. 3, the period of the modulation does not change with variation of U_2/U_0, which correlates with the fact that we kept ω_0/γ_0 fixed. To test this hypothesis, we performed another series of calculations where the strength of the nonlocal interaction U_2/U_0 was fixed while we changed the frequency of the bosonic mode ω_0/γ_0. The results are displayed in Fig. 4. The period of the instanton lattice was again fixed at $\gamma_0 W = 20.0$, while we kept $U_2/U_0 = 0.1$. The values of ω_0/γ_0 are listed in the figure. As we may observe, the frequency of the modulation does indeed grow together with the bosonic frequency ω_0/γ_0.

Finally, we performed a series of more physically realistic computations where the instanton lattice period was not chosen arbitrarily but was determined from the condition of the minimum of the free energy. The results are presented in Fig. 5. We kept the strength of the nonlocal interaction fixed at $U_2/U_0 = 0.2$, while the bosonic frequency was varied. The values of the modified bosonic frequency $\tilde{\omega}_0/\gamma_0$ and of the instanton lattice period $\gamma_0 W$ are listed in the figure as usual. In principle, we may again observe the structure of modulated oscillations with period W' which we noted in Figs. 3 and 4. At the same time, it is important to point out that the modulation dominates the oscillations in this case.

VI. VALIDITY OF THE "NO-GO" THEOREM IN THE CASE OF INSTANTON CRYSTAL

In the publication [27] by Watanabe and Oshikawa (WO), a theorem has been proposed that the time crystal is not possible in the thermodynamic equilibrium and therefore one can speak of this phenomenon only including nonequilibrium processes like, e.g., pumping, many body localization, etc. (for a recent review, see Ref. [26]). There are many definitions of the time crystals but we follow just the one used in Ref. [27]. The arguments of Ref. [27] are based on scaling and counting powers of the volume V entering physical quantities. Moreover, the proof of Watanabe and Oshikawa (WO) excludes even the possibility of the time dependence of the order parameter correlation function. However, there are certain problems in their proof which render the theorem not applicable to the case of the instanton crystal.

WO consider a correlation function of two quite general operators $\hat{A}$ and $\hat{B}$ composed of local operators $a(\mathbf{r})$ and $b(\mathbf{r})$ in a form of integrals over the space

$$\hat{A} = \frac{1}{V}\int_V a(\mathbf{r}) d^d\mathbf{r}, \quad \hat{B} = \frac{1}{V}\int_V b(\mathbf{r}) d^d\mathbf{r} \tag{81}$$

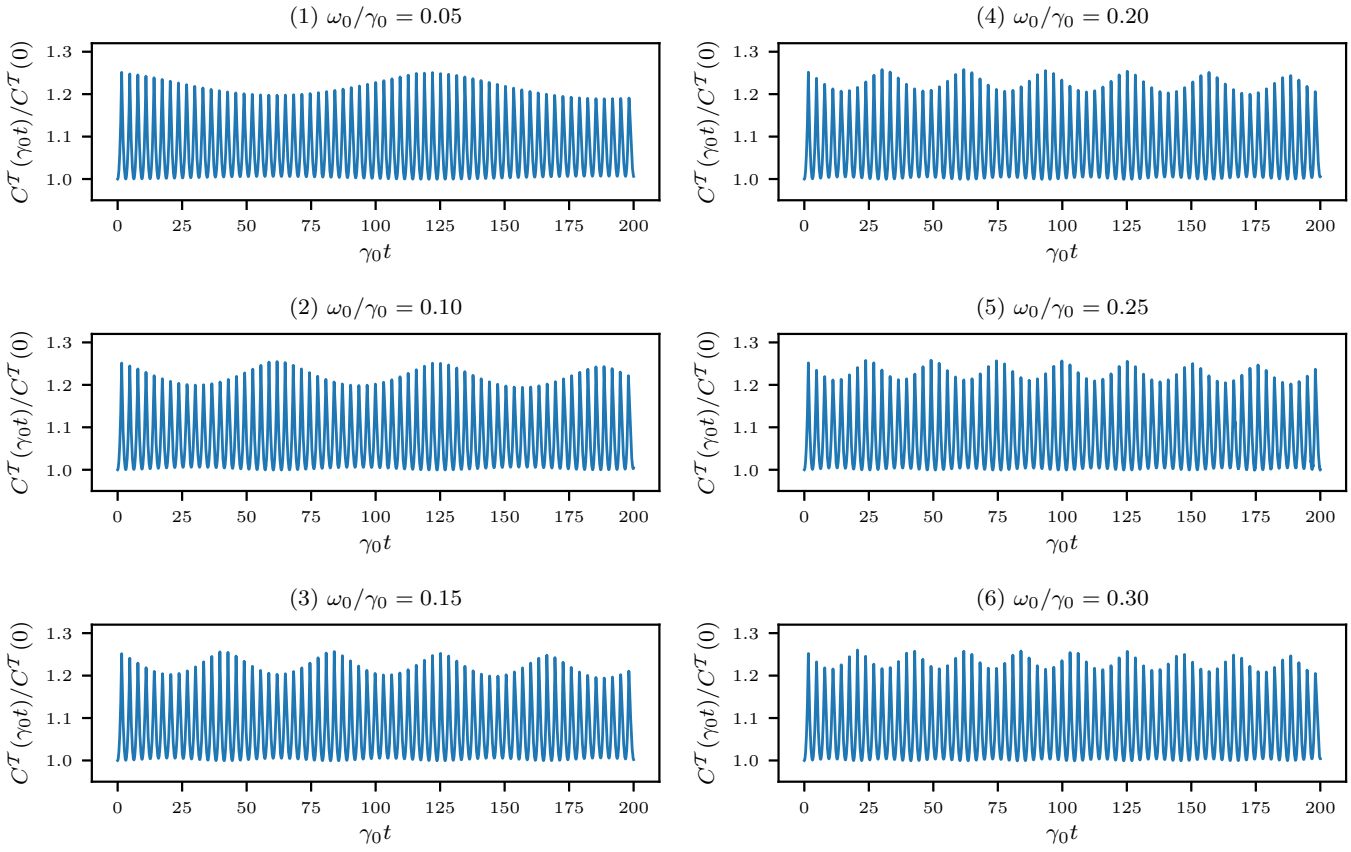

FIG. 4. Normalized real-time correlation function of the order parameter $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ for fixed $U_2/U_0 = 0.10$ and $\gamma_0 W = 20.0$ but for varying frequency of bosonic mode ω_0/γ_0.

(we have normalized here the operators $\hat{A}$ and $\hat{B}$ multiplying the integrals over $\mathbf{r}$ by V^{-1}). With this choice, the operators $\hat{A}$ and $\hat{B}$ scale with the volume as V^0 and, using the locality of the operators $a(\mathbf{r})$ and $b(\mathbf{r})$, one can assume that the commutators containing the Hamiltonian $\hat{H}$ scale as

$$[\hat{A}, \hat{H}] \propto V^0, \quad [\hat{B}, \hat{H}] \propto V^0, \ [[\hat{A}, \hat{H}], \hat{A}] \propto V^{-1}, \tag{82}$$

where [,] are commutators. According to the assumption by WO, the Hamiltonian $\hat{H}$ describes a completely general model with finite range interactions, and one can make estimates simply starting with Eqs. (81) and (82) without assuming anything else.

A. Zero temperature

The main part of the proof of the theorem by WO is based on the use of the Cauchy-Schwarz inequality applied at zero temperature $T = 0$ to a general model with a general Hamiltonian $\hat{H}$. The main claim of the theorem is that the nondecaying real-time dependence is impossible in any macroscopic thermodynamically stable system. According to the authors, the restriction $T = 0$ should not be so important, and they suggest in the second part of the paper a discussion at finite temperature based on the rigorous bound for spin systems [36].

However, making scaling with V and putting $T = 0$ is not an unambiguous procedure and the result depends on the problem considered. Generally, one can take either the limit 1) $T \to 0$ and then $V \to \infty$ or 2) first $V \to \infty$ and then $T \to 0$. What they have in mind is apparently the case 1) when temperature T can be smaller that the energy of the first higher level.

If so, one should assume that in the limit $V \to \infty$ the first excited state is separated from the ground state by a finite gap Δ_0 that does not vanish in the limit $V \to \infty$. Certainly, geometrically finite systems having a finite distance between the levels exist and the limit $T \ll \Delta$ is possible but one cannot scale time with volume in the way as it was done by WO. In any case, the model of fermions and bosonic modes considered by us cannot be described by first putting $T = 0$ and only then $V \to \infty$. We start with a model of a metal with a level separation $\Delta = (\nu V)^{-1}$, where ν is the density of states, and fix a finite T, which means that we consider explicitly the case 2). Studying properties of the system in the thermodynamic limit $V \to \infty$, we cannot keep the restriction $T \ll \Delta$ because $\Delta \to 0$. Therefore the proof at $T = 0$ presented in Ref. [27] cannot be applied in our model, and the Cauchy-Schwarz inequality does not help.

We already considered the real-time correlation functions $C^{>}(z)$ and $C^{<}(z)$ in Sec. III A. For comparison with experiments, it is convenient to use their combination $C^{\mathrm{K}}(t)$:

$$C^{\mathrm{K}}(t) = C^{>}(t) + C^{<}(t) = \frac{1}{\mathcal{Z}} \mathrm{Tr}\{e^{-\beta \hat{H}_g}\{\hat{A}(t), \hat{A}(0)\}\}, \tag{83}$$

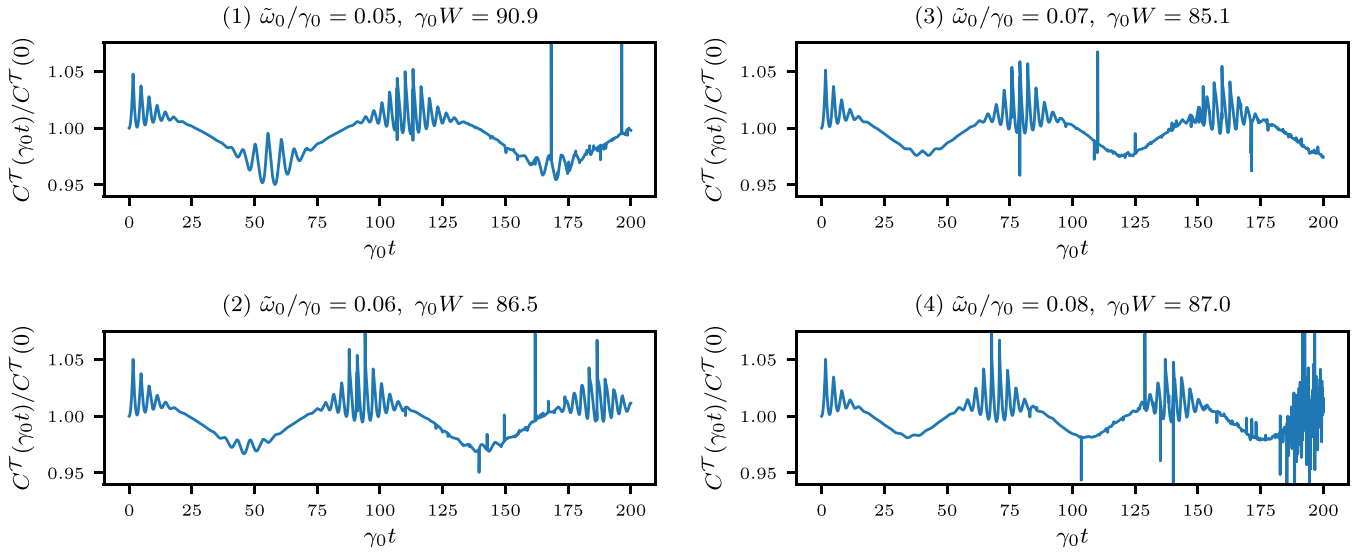

FIG. 5. Normalized real-time correlation function of the order parameter $C^{\mathcal{T}}(\gamma_0 t)/C^{\mathcal{T}}(0)$ for fixed $U_2/U_0 = 0.20$ and varying modified frequency of the bosonic mode $\tilde{\omega}_0/\gamma_0$. The period of the instanton lattice was determined from the minimum of the free energy.

where $\hat{H}_g = \hat{H} - \mu\hat{N}$ and $\hat{A}(t)$ is given by Eq. (1). As we calculate thermodynamic averages, we also need operators depending on the imaginary time τ:

$$\hat{A}(it) = e^{i\hat{H}_g t}\hat{A}e^{-i\hat{H}_g t}. \tag{84}$$

Estimating bounds is not what is usually done in physics. In this section we simply want to understand in simple terms whether our results on the dependence of the function $C^{\mathrm{K}}(t)$ on time t really contradict to the "no-go" theorem [27] that forbids any time dependence of the function $C^{\mathrm{K}}(t)$ in the limit $V \to \infty$ or it is a special property of the instanton crystal.

In order to answer this question, we simply consider the limit of short times t and expand the function $C^{\mathrm{K}}(t)$ in t. The first terms of the expansion equal

$$C^{\mathrm{K}}(t) - C^{\mathrm{K}}(0) = \frac{1}{\mathcal{Z}}\sum_n e^{-\beta E_n}\Big[-\frac{it}{2}\langle n|[\hat{H}_g,\hat{A}]\hat{A}|n\rangle - \frac{it}{2}\langle n|\hat{A}[\hat{H}_g,\hat{A}]|n\rangle\Big] - \frac{t^2}{2}\langle n|[\hat{H}_g,\hat{A}]^2|n\rangle. \tag{85}$$

The linear in t terms cancel each other as they should. The quadratic in t term is more interesting. Using the standard Heisenberg equation in the imaginary time

$$\frac{d\hat{A}(\tau)}{d\tau} = [\hat{H}_g, \hat{A}(\tau)], \tag{86}$$

we bring Eq. (85) to the form

$$\begin{aligned} C^{\mathrm{K}}(t) - C^{\mathrm{K}}(0) &\approx -\frac{t^2}{2\mathcal{Z}}\sum_n e^{-\beta E_n}\langle n|[\hat{H}_g,\hat{A}]^2|n\rangle \\ &= -\frac{t^2}{2\mathcal{Z}}\sum_n e^{-\beta E_n}\langle n|[\hat{H}_g,\hat{A}(\tau)]^2|n\rangle \\ &= -\frac{t^2}{2\mathcal{Z}}\sum_n e^{-\beta E_n}\langle n|\frac{d\hat{A}(\tau)}{d\tau}\frac{d\hat{A}(\tau)}{d\tau}|n\rangle. \end{aligned} \tag{87}$$

Now, using Eq. (81) and introducing a correlation function $L(\mathbf{r}-\mathbf{r}')$

$$L(\mathbf{r}-\mathbf{r}') = \frac{1}{\mathcal{Z}}\mathrm{Tr}\Big[e^{-\beta\hat{H}_g}\frac{da(\tau,\mathbf{r})}{d\tau}\frac{da(\tau,\mathbf{r}')}{d\tau}\Big], \tag{88}$$

we bring Eq. (87) to the form

$$C^{\mathrm{K}}(t) - C^{\mathrm{K}}(0) = -\frac{t^2}{2}\frac{1}{V}\int L(\mathbf{r}-\mathbf{r}')d\mathbf{r}'. \tag{89}$$

The function $a(\tau,\mathbf{r})$ in our model corresponds to the local loop currents which are local quantities. Local quantities are also assumed in Ref. [27]. At first glance, this should lead to a finite value of the integral over $\mathbf{r}'$ in Eq. (89) in any thermodynamic system, which would give in the limit $V \to \infty$ the zero value of $C^{\mathrm{K}}(t) - C^{\mathrm{K}}(0)$.

The only possibility to avoid this scenario is formation of a long range order in space such that the correlation function $L(\mathbf{r}) \to const$ in the limit $|\mathbf{r}| \to \infty$. Then, the integration over $\mathbf{r}'$ in Eq. (89) would give an additional volume V and the $C^{\mathrm{K}}(t) - C^{\mathrm{K}}(0)$ would become time-dependent. At the same time, the function $L(\mathbf{r}-\mathbf{r}')$, Eq. (88), contains finite derivative of the order parameter $a(\tau)$ with respect to the imaginary time τ. To the best of our knowledge, only the instanton crystal proposed in Ref. [8] possesses this property.

Of course, the consideration suggested in this section is not sufficient for proving the existence of the slowly decaying real time oscillations but it does indicate what has been missed in the arguments presented in Ref. [27]. For the explicit proof, one should perform calculations like those carried out in the previous sections.

We emphasize that the very possibility of the existence of the slowly decaying real time oscillations is the consequence of the existence of the instanton crystal and one cannot expect them in other systems.

B. Finite temperature

The proof of second part of "no-go" theorem of Ref. [27], which deals with the case of finite temperature, is based on the exact Lieb-Robinson bound for spin systems [36]. Crucially, Watanabe and Oshikawa assume that Lieb-Robinson bound necessarily implies the existence of a similar bound in the Fourier representation. However, as it is pointed out in Ref. [26], this assumption is incorrect because Lieb-Robinson bound is satisfied only at finite times. This fact renders the proof for the second part of the "no-go" theorem invalid.

VII. DISCUSSION AND OUTLOOK

A. Summary

In this paper, we considered the model of instanton crystal, i.e., spontaneously broken state characterized by a periodic structure of the order parameter in imaginary time. Staying in the context of the mean-field approximation, we have focused on the dynamical autocorrelation function $C^{\mathcal{T}}(t)$ of the order parameter. In the model we studied, the said function describes the dynamical correlations of a macroscopic arrangement of loop currents. As such, the relevant correlation function can be, in principle, accessed by means of magnetic neutron scattering which should be sensitive to the magnetic fields induced by the loop currents.

In order to compute the dynamical correlation function, we have performed the analytical continuation from imaginary to real times. As we have shown, the typical procedure based on the fourier representation of correlation functions is ambiguous in this case. Instead, we proposed the method of continuation directly in time domain, which is based on analytical continuation of a system of differential equations satisfied by the mean-field configuration. The important feature of the method is that it is straightforward to implement numerically.

The results of our calculations indicate that $C^{\mathcal{T}}(t)$ exhibits nontrivial slowly decaying oscillations. Peculiar feature of these oscillations is that their amplitude is periodically modulated, moreover, the period of the modulation corresponds to the bosonic mode frequency. Unfortunately, these results are not enough to make a definitive conclusion about the behavior of $C^{\mathcal{T}}(t)$ in the limit $t \to +0$. One possibility is that the decay saturates at some point and the oscillations survive at long times. Another possibility, which is rather plausible, is that the oscillations die out, however the correlation function saturates at some nonzero constant. In this scenario, an instanton crystal in imaginary time would behave like a prethermal time crystal in real time.

B. Discussion

All the considerations in the paper are rather theoretical and highly involved. To somewhat remedy that, we suggest the following qualitative picture of what is going on.

Our picture follows from the initial loop currents statical picture. The system in the ordered phase can be either in the $+1$ and -1 states (depending on the direction of the loop currents) forming classical macrosopic bits. The phase of the instanton crystal is characterized by an imaginary-time-dependent order parameter $b(\tau)$. We interpret its oscillating behavior as a possibility for the system to be in both $+1$ and -1 states simultaneously forming a state

$$|\psi\rangle = \alpha|1\rangle + \beta|-1\rangle \tag{90}$$

with real α and β satisfying the normalization condition $\alpha^2 + \beta^2 = 1$. In the language of the order parameter $b(\tau)$ the degeneracy with respect to α and β corresponds to an arbitrary position τ_0 of the instanton/anti-instanton lattice. In the limit of a large period of the lattice, the probability of the finding system either in the state $|1\rangle$ or in state $|-1\rangle$ is close to unity but, generally, it is a superposition of the both.

The period of the oscillations can rather easily be extracted from the picture presented here. In the case $k \to 1$ corresponding to the large period $4K(k)/\gamma$ of the instanton lattice, the gap in the spectrum 2γ in the limit $k \to 1$ should produce oscillations in real time with the period $\pi\gamma^{-1}$, which is standard for a 2-level system. Our result obtained in this limit is $2K(k')/\gamma$. Using the limiting value $K(k' \to 0) = \pi/2$ we obtain the same period $\pi\gamma^{-1}$.

So, the properties of the real-time behavior of the instanton crystal correlate with those of a macroscopic qubit. The slowness of oscillation decay corresponds to the suppression of the decoherence. It is a collective effect originating from the special long-range order present in the instanton crystal.

C. Outlook

Since all the results have been obtained using the mean-field approximation, the important open question is: what is the influence of the fluctuations around the mean-field configuration?

As we have argued in Ref. [8], the fluctuations should not destroy the instanton crystal state: We have considered a system with at least two spatial dimensions, and the imaginary time acts as an additional dimension as well. At the same time, the order parameter in the absence of instantons corresponds to discrete $\mathbb{Z}_2$-symmetry breaking.

However, the form of the dynamical correlation function may be influenced by the inclusion of fluctuations. We believe that the important role in this regards may play specifically fluctuations in imaginary time. As we have mentioned, in the absence of nonlocal interaction, the instanton configuration is a saddle point: there are fluctuation modes with negative energy. In the limit of large period of instanton lattice, these modes correspond to the displacement of instantons and anti-instantons relative to each other. The introduction of nonlocal interaction stabilizes the instanton configuration and brings the energies of these "displacement" modes above zero. Nevertheless, we expect these modes to play the role of the important low-energy excitations. In conclusion, we should say that we are hoping to touch the subject of the fluctuations in the future work.

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APPENDIX A: MAGNETIC NEUTRON SCATTERING AND CORRELATION FUNCTION $C^{\mathcal{T}}(t)$

In this section, we show that the correlation function $C^{\mathcal{T}}(t)$ computed previously can in principle be directly measured. The model for the instanton crystal considered here has been suggested for interacting loop currents and a boson mode that both oscillating in space with the vector $\mathbf{Q}_{\rm AF}$. Studying experimentally inelastic magnetic neutron scattering might be a proper tool to measure such magnetic structures and we show now that the correlation function $C^{\mathcal{T}}(t)$ is just what is needed to obtain the scattering cross-section.

Considering neutrons scattered by the instanton crystal, we start with a Hamiltonian $\mathcal{H}$ containing both the neutrons and the instanton crystal. We write it in the form

$$\mathcal{H} = \hat{H}_n + \hat{H} + \hat{V}_{\rm int}. \tag{A1}$$

In Eq. (A1), $\hat{H}_n$ is the Hamiltonian of free neutrons

$$\hat{H}_n = \sum_p E_{\mathbf{p}} d_p^+ d_p, \tag{A2}$$

where $E_{\mathbf{p}} = \mathbf{p}^2/2M$ is the kinetic energy, and d_p^+, d_p, are creation and destruction operators of neutrons. The Hamiltonian $\hat{H}$ describes the system of interacting electrons and bosonic mode studied here, and $\hat{V}_{\rm int}$ stands for the interaction of neutron magnetic moments with the magnetic field created by electron currents. At the moment, we write this term without going into details of the interaction in the form

$$\hat{V}_{\rm int} = \Gamma \sum_{\mathbf{q}} d_{\mathbf{q}}^+ d_{\mathbf{q}+\mathbf{Q}_{\rm AF}} \hat{A}_0, \tag{A3}$$

where operator $\hat{A}_0$ has been introduced in Eq. (30).

We assume that the interaction constant Γ is small and use for calculation of transition rate the perturbation theory in Γ.

The initial state of neutron and system at $t \to -\infty$ is given by a function $\Psi_0 = |\mathbf{p}\rangle \otimes |\Phi_i\rangle$, where $\mathbf{p}$ is the initial momentum of neutron, and Φ_i is the initial state of the instanton crystal. We write the Schrödinger equation for the function $|\Psi(t)\rangle$ in the interaction representation as

$$i\frac{d|\Psi(t)\rangle}{dt} = \hat{V}_{\rm int}(t)|\Psi(t)\rangle, \tag{A4}$$

where

$$\hat{V}_{\rm int}(t) = \Gamma e^{\lambda t} \sum_{\mathbf{q}} e^{i\hat{H}_n t} d_{\mathbf{q}}^+ d_{\mathbf{q}+\mathbf{Q}_{\rm AF}} e^{-i\hat{H}_n t} \hat{A}_0(t). \tag{A5}$$

The parameter λ is infinitesimally small, $\lambda \to +0$ and is, as usual, introduced as adiabatic switching of the interaction at $t = -\infty$. Time-dependent operator of loop currents $\hat{A}_0(t)$ has been defined in Eq. (31).

Equation (A4) is solved representing $|\Psi(t)\rangle$ as

$$\begin{aligned}|\Psi(t)\rangle &= \mathcal{T} e^{-i\int_{-\infty}^{t} dt' \hat{V}_{\rm int}(t')} |\Psi_0\rangle \\ &\approx \left[1 - i\int_{-\infty}^{t} dt' \hat{V}_{\rm int}(t')\right] |\Psi_0\rangle.\end{aligned} \tag{A6}$$

We are interested in a matrix element a_{fi} of the transition between the initial state i and the final state f of the form $|\Psi_f\rangle = |\mathbf{p}'\rangle \otimes |\Phi_f\rangle$ and we write

$$a_{fi} = -i\Gamma \int_{-\infty}^{t} dt' e^{\lambda t'} e^{i(E_{\mathbf{p}'} - E_{\mathbf{p}})t'} \langle \Phi_f | \hat{A}_0(t') | \Phi_i \rangle. \tag{A7}$$

Then, we obtain

$$\begin{aligned}|a_{fi}|^2 = \Gamma^2 \int_{-\infty}^{t} dt' e^{\lambda t'} \int_{-\infty}^{t} dt'' e^{\lambda t''} e^{-i(E_{\mathbf{p}'} - E_{\mathbf{p}})(t' - t'')} \\ \times \langle \Phi_i | \hat{A}_0(t') | \Phi_f \rangle \langle \Phi_f | \hat{A}_0(t'') | \Phi_i \rangle.\end{aligned} \tag{A8}$$

Up until now, we considered particular initial and finite states of instanton crystal. In reality, we need to sum over unobserved final states $|\Phi_f\rangle$ and perform thermal averaging over initial states $|\Phi_i\rangle$, which leads us to the following expression:

$$\begin{aligned}\overline{|a_{fi}|^2} &= \lim_{\lambda \to +0} \frac{\Gamma^2}{\mathcal{Z}} \iint_{-\infty}^{t} dt' dt'' e^{\lambda(t' + t'')} \\ &\quad \times e^{-i(E_{\mathbf{p}'} - E_{\mathbf{p}})(t' - t'')} \mathrm{Tr}\{e^{-\beta(\hat{H} - \mu\hat{N})} \hat{A}_0(t') \hat{A}_0(t'')\} \\ &= \lim_{\lambda \to +0} \Gamma^2 V \iint_{-\infty}^{t} dt' dt'' e^{\lambda(t' + t'')} \\ &\quad \times e^{-i(E_{\mathbf{p}'} - E_{\mathbf{p}})(t' - t'')} C^{>}(t' - t'').\end{aligned} \tag{A9}$$

Now we want to take the limit $t \to +\infty$ and calculate the transition rate. It is convenient to start with the Fourier transform $C^{>}(\omega)$ of the function $C^{>}(t' - t'')$:

$$C^{>}(t' - t'') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} C^{\omega}(\omega) e^{-i\omega(t' - t'')}, \tag{A10}$$

where

$$C^{>}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} C^{>}(t). \tag{A11}$$

Then, we have

$$\begin{aligned}&\lim_{\lambda \to +0} \iint_{-\infty}^{t} dt_1 dt_2 e^{\lambda(t_1 + t_2)} e^{i(E_{\mathbf{p}} - E_{\mathbf{p}'})t} C^{\mathcal{T}}(t_1 - t_2) \\ &\quad = \frac{1}{2\pi} \lim_{\lambda \to +0} \int_{-\infty}^{\infty} d\omega \iint_{-\infty}^{t} dt_1 dt_2 e^{\lambda(t_1 + t_2)} e^{-i\omega(t_1 - t_2)} \\ &\qquad \times e^{i(E_{\mathbf{p}} - E_{\mathbf{p}'})(t_1 - t_2)} C^{\mathcal{T}}(\omega) \\ &\quad = \frac{1}{2\pi} \lim_{\lambda \to +0} \int_{-\infty}^{\infty} d\omega \frac{2\lambda e^{2\lambda t}}{\lambda^2 + (\omega - E_{\mathbf{p}} + E_{\mathbf{p}'})^2} C^{\mathcal{T}}(\omega),\end{aligned} \tag{A12}$$

and, finally,

$$\begin{aligned}&\frac{d}{dt}\overline{|a_{fi}|^2} \\ &\quad = \frac{\Gamma^2 V}{2\pi} \int_{-\infty}^{\infty} d\omega C^{>}(\omega) \lim_{\lambda \to +0} \frac{2\lambda}{\lambda^2 + (\omega - E_{\mathbf{p}} + E_{\mathbf{p}'})^2} \\ &\quad = \Gamma^2 V \int_{-\infty}^{\infty} d\omega C^{>}(\omega) \delta(\omega - E_{\mathbf{p}} + E_{\mathbf{p}'}) \\ &\quad = \Gamma^2 V C^{>}(E_{\mathbf{p}} - E_{\mathbf{p}'}).\end{aligned}$$

This is actually the standard way of derivation of the scattering cross-section in quantum mechanics. One can conclude

that the magnetic neutron scattering cross-section is directly given by the function $C^{>}(E_{\mathbf{p}} - E_{\mathbf{p}'})$ and can be written as

$$\frac{d^2\sigma(\mathbf{q},E)}{dEd\Omega} \propto \Gamma^2 V C^{>}(E)\delta(\mathbf{q} - \mathbf{Q}_{\mathrm{AF}}), \tag{A13}$$

where E is transferred energy and $\mathbf{q}$ transferred momentum. In its turn, $C^{>}(t)$ is directly related to the function $C^{\mathcal{T}}(t)$, as we have discussed in Sec. III.

APPENDIX B: NUMERICAL SCHEME FOR ANALYTIC CONTINUATION

For an instanton crystal consisting of m instanton/anti-instanton pairs, we denote the period of the pair as $W = \beta/m$. The numerical scheme that we developed in our previous paper [8] can be used to compute $b(\tau_i)$ and $\vec{C}_{\mathbf{p}}(\tau_i)$ for a discrete set of N equidistant points τ_i belonging to a single period: $0 \leqslant \tau_i < W$. Consequently, solving the system of Eqs. (64) and (65) for each of τ_i, one obtains the values of $b(\tau)$ and $\vec{S}_{\mathbf{p}}(\tau)$ at points $\tau_i + it_j$. With this knowledge, the correlation function can be approximated as

$$C^{\mathcal{T}}(t_j) = \frac{4}{N}\sum_i S_2(\tau_i + it_j)S_2(\tau_i), \tag{B1}$$

which is the discretized version of Eq. (52). Note also that we used the periodicity of the instanton crystal to reduce the sum to a single period.

The important technical detail which deserves to be mentioned is the way the initial conditions for $db(\tau + i0)/dt$ are computed. Using the periodicity in the imaginary time, we rewrite Eq. (67) as

$$\left.\frac{db(\tau + it)}{dt}\right|_{t=0} = i\frac{U_2}{U_0}\int_0^W d\tau' \frac{d\tilde{K}_0(\tau - \tau'|\tilde{\omega}_0)}{d\tau} b(\tau') + 2(U_0 + U_2)\int \frac{d\mathbf{p}}{(2\pi)^2}\varepsilon_{\mathbf{p}}^{-} S_{1\mathbf{p}}(\tau), \tag{B2}$$

where

$$\tilde{K}_0(\tau - \tau'|\tilde{\omega}_0) = \sum_{k=0}^{m-1} K_0(\tau - \tau' - kW|\tilde{\omega}_0) = \frac{\tilde{\omega}_0 \cosh\left[\tilde{\omega}_0\left(\frac{W}{2} - |\tau - \tau'|\right)\right]}{2\sinh\frac{W\tilde{\omega}_0}{2}}. \tag{B3}$$

Note that the functional form of $\tilde{K}_0(\tau - \tau'|\tilde{\omega}_0)$ is identical to the functional form of $K_0(\tau - \tau'|\tilde{\omega}_0)$. The only difference is that the inverse temperature β is replaced with the instanton crystal period W.

Correspondingly,

$$\frac{d\tilde{K}_0(\tau - \tau'|\tilde{\omega}_0)}{d\tau} = -\mathrm{sign}(\tau - \tau') \times \frac{\tilde{\omega}_0 \sinh\left[\tilde{\omega}_0\left(\frac{W}{2} - |\tau - \tau'|\right)\right]}{2\sinh\frac{W\tilde{\omega}_0}{2}}. \tag{B4}$$

Finally, we can write the discretized version of Eq. (B2) as

$$\left.\frac{db(\tau_i + it)}{dt}\right|_{t=0} = i\frac{U_2}{U_0}\frac{W}{N}\sum_j \frac{d\tilde{K}_0(\tau_i - \tau_j|\tilde{\omega}_0)}{d\tau} b(\tau_j) + 2(U_0 + U_2)\int \frac{d\mathbf{p}}{(2\pi)^2}\varepsilon_{\mathbf{p}}^{-} S_{1\mathbf{p}}(\tau_i), \tag{B5}$$

where it is assumed that $\mathrm{sign}(0) = 0$.

In the absence of nonlocal interaction, $U_2 = 0$, functions $b(\tau + it)$ has the poles along the lines $\tau = 0$ and $\tau = W/2$. This poles survive even for finite U_2. In order to avoid them, we chose the set of time points τ_i to be $\tau = W/N(i - 1/2)$ for $1 \leqslant i \leqslant N$. As in our previous work [8], we used programming language JULIA [37] to perform the numerical calculations presented in the paper.

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