Floquet integrability and long-range entanglement generation in the one-dimensional quantum Potts model

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We develop a Floquet protocol for long-range entanglement generation in the one-dimensional quantum Potts model, which generalizes the transverse-filed Ising model by allowing each spin to have $n > 2$ states. We focus on the case of $n = 3$, so that the model describes a chain of qutrits. The suggested protocol creates qutrit Bell-like pairs with nonlocal long-range entanglement that spans over the entire chain. We then conjecture that the proposed Floquet protocol is integrable and explicitly construct a few first nontrivial conserved quantities that commute with the stroboscopic evolution operator. Our analysis of the Floquet integrability relies on the deep connection between the quantum Potts model and a much broader class of models described by the Temperley-Lieb algebra. We work at the purely algebraic level and our results on Floquet integrability are valid for any representation of the Temperley-Lieb algebra. We expect that our findings can be probed with present experimental facilities using Rydberg programmable quantum simulators and can find various applications in quantum technologies.

I. INTRODUCTION

Over the past few years, a tremendous progress in the development of programmable quantum simulators of various nature has greatly pushed the research field in the direction of probing novel nonconventional states of quantum matter [1–7]. Recent achievements include investigations of exotic nonequilibrium many-body states and phase transitions in strongly correlated quantum systems, e.g., in quantum spin chains, such as the transverse-field Ising model (TFIM) [1,2] and its extensions [7]. Specifically, the use of a Rydberg programmable simulator enables one to study generalizations of TFIM, in which each spin has $n > 2$ states [7,8]. The n-state model with $\mathbb{Z}_n$ symmetry is known as the chiral clock model, whereas a model with a larger (S_n) symmetry comes under the name of the n-state Potts model. Both models exhibit incredibly rich physics and have been extensively. studied in the context of quantum phase transitions [1,7,8], critical phenomena [7], exotic quasiparticle excitations (e.g., mesonic and baryonic) [9], and integrable lattice models [10]. An additional interest to the n-state Potts model is motivated by the fact that it admits a description in terms of parafermions, particles obeying nontrivial quasilocal anyonic statistics, which is linked to topological quantum computing [11–13]. On the other hand, the n-state model corresponds to an array of qudits, which are promising for improving the performance of various quantum computational schemes and algorithms, e.g., by using them in the multiqubit gate decomposition [14–17]. In order to maximize the improvement, one has to use qudits

with a certain number of internal states that depends on the spatial topology and connectivity of a quantum system [16]. For example, in the case of a one-dimensional (1D) chain with all-to-all connectivity, the best performance is shown by qutrits (qudits with $n = 3$ internal states) [16].

One of the key challenges that arise in controllable spin chains and generic many-body systems is the generation of entanglement, which is a crucial resource for applications in quantum computing, simulation, and metrology. The case of long-range entanglement is traditionally of special interest, since it plays a key role in the understanding of various many-body phenomena, with the paradigmatic example being the quantum magnetism [18]. A powerful tool for generating states with a long-range entanglement and other nontrivial properties is provided by the periodic (Floquet) driving, which allows steering the dynamics to the desired state by a sequence of discrete time steps [19–28]. Recently, this method has been used in the realization of exotic nonequilibrium quantum many-body states, such as discrete time crystals [28] and quantum many-body scars [28,29]. A periodic driving protocol for on-demand generation of long-range entanglement has been suggested for a system of ultracold atoms in optical superlattices, a setup which simulates the 1D spin-1/2 Heisenberg model with time-dependent exchange interaction [19]. In this system, the consecutive switching of the interaction between the spins on even and odd links of the chain transforms the initial short-range entanglement between the the nearest-neighbor spins into the nonlocal one. A similar protocol was later studied for the case of the 1D TFIM, where

the nonlocal entanglement between the pairs of distant spins is generated by repeatedly switching the transverse field on and off [30–33]. A natural question is whether these long-range entanglement protocols can be extended beyond the spin-1/2 chains, such as the Heisenberg or Ising model, to the case of $\mathbb{Z}_n$ chains, e.g., n-state Potts model. This is of practical interest due to the aforementioned advantages for quantum computing that are offered by the qudit-based platforms.

From the fundamental point of view, it is important to emphasize that despite a significant amount of known Floquet protocols generating various nontrivial quantum states, all these cases are rather exceptional and require the system to be fine-tuned. In contrast, a generic interacting many-body quantum system subject to a periodic drive simply reaches an infinite temperature state. This happens because the energy is continuously pumped into the system and in the general case there are no conservation laws that can prevent the system from heating [34–36]. Thus, the situation is analogous to the phenomenon of termalization in statistical physics, which is characteristic of nonintegrable systems in the absence of disorder [37,38]. On the other hand, it is well known from statistical physics that integrable or localized many-body systems do not termalize due to a large number of conserved quantities (charges). Continuing the analogy between statistical models and systems with periodic driving, it is natural to expect that Floquet systems which do *not* heat up to an infinite temperature must be also in a certain sense integrable and possess an extensive number of conservations laws. This is indeed the case, and the field of Floquet integrability is a growing research area (for a review of recent results see, e.g., Ref. [20] and references therein), but a complete understanding is still missing. In particular, explicit construction of conservation laws for integrable Floquet protocols remains an open question.

In this paper, we propose a Floquet protocol for iterative generation of nonlocal entangled qutrit pairs in the 1D three-state Potts model, which describes a chain of qutrits. We show that by starting from a polarized state (i.e., the product state in which all qutrits are initially in one and the same internal state) and performing a state preparation scheme followed by a consequent switching of the transverse field on and off with a certain frequency, one arrives at a state consisting of qutrit pairs with increasingly long-ranged entanglement.

We then go one step further and argue that the existence of the suggested Floquet protocol is not merely a fortunate coincidence but a consequence of its integrability. Namely, we demonstrate the presence of emerging conservations laws in the parameter regime corresponding to the long-range entanglement generation. We explicitly construct the first few conserved charges and conjecture that one can similarly construct the higher ones. Using the fact that the Hamiltonian of the three-state Potts model can be thought of as a representation of a more general operator belonging to the so-called Temperley-Lieb algebra, we show that the long-range entanglement generating Floquet protocols for the TFIM and the Heisenberg models are also integrable as their Hamiltonians are nothing other as different representations of the same Temperley-Lieb-algebraic operator. Finally, we briefly discuss different driving regimes that do not result in the entanglement generation but nevertheless exhibit some interesting features,

although their detailed investigation is beyond the scope of the present work.

The rest of the paper is organized as follows. In Sec. II we introduce the 1D three-state Potts model and discuss an operator basis convenient for our purposes. Then, in Sec. III we construct the Floquet protocol, identify the parameter regime that leads to the entanglement generation, and present the resulting many-body state with the long-range entanglement between the qutrit pairs. In Sec. IV we show that the suggested Floquet protocol is integrable, present a few first nontrivial conserved charges, and extend our findings to other protocols related to the Temperley-Lieb algebraic models. We discuss our results and conclude in Sec. V.

II. THREE-STATE POTTS MODEL HAMILTONIAN

We consider the Potts model with $n = 3$ states per site, a generalization of the Ising model to spin variables taking three values. The Hamiltonian of the three-state Potts model on a chain of $2N$ sites can be written as

$$H = -JH_1 - fH_2, \qquad (1)$$

where J and f are real constants and for later convenience we separated the terms H_1 and H_2, which are given by

$$
\begin{aligned}
H_1 &= \sum_{j=1}^{2N-1} (X_j^\dagger X_{j+1} + X_j X_{j+1}^\dagger), \\
H_2 &= \sum_{j=1}^{2N} (Z_j + Z_j^\dagger).
\end{aligned}
\qquad (2)
$$

For concreteness, throughout this work we assume open boundary conditions although most of the results can be straightforwardly generalized to the periodic ones [39]. The operators X_j and Z_j in Eq. (2) satisfy the following relations:

$$
\begin{aligned}
X_j^3 &= 1, \quad Z_j^3 = 1, \\
X_j^2 &= X_j^\dagger = X_j^{-1}, \quad Z_j^2 = Z_j^\dagger = Z_j^{-1}, \\
X_j Z_j &= \omega Z_j X_j, \quad X_j Z_k = Z_k X_j \ (j \neq k),
\end{aligned}
\qquad (3)
$$

where $\omega = e^{2\pi i/3}$ is the principal cube root of unity. They act nontrivially on the jth site of the chain, i.e., $X_j = \mathbb{1} \otimes \dots \mathbb{1} \otimes X \otimes \mathbb{1} \dots \otimes \mathbb{1}$ and $Z_j = \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes Z \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$. For the operators X and Z we choose the following matrix representations:

$$
Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad
X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\qquad (4)
$$

Labelling the local basis states as $|l\rangle_j$, with $l \in \{0, 1, 2\}$, we have $Z_j^m |l\rangle_j = \omega^{lm} |l\rangle_j$ and $X_j^m |l\rangle_j = |l - m \mod 3\rangle_j$, where $m = 1, 2$. The matrices Z_j and X_j generalize the Pauli matrices σ_j^z and σ_j^x, correspondingly, and are commonly referred to as the shift (X_j) and clock (Z_j) matrices.

In Eq. (2), the term H_1 corresponds to the nearest-neighbor interaction between the spins, whereas the term H_2 plays the role of the transverse field. The operators X_j, Z_j, and their conjugates are related to each other by a unitary

transformation $W_j = \mathbb{1} \otimes \ldots \mathbb{1} \otimes W \otimes \mathbb{1} \ldots \otimes \mathbb{1}$, with

$$W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \tag{5}$$

which acts on the operators X_j and Z_j as

$$\begin{aligned} W_j X_j W_j^\dagger = Z_j, &\quad W_j Z_j W_j^\dagger = X_j^\dagger, \\ W_j X_j^\dagger W_j^\dagger = Z_j^\dagger, &\quad W_j Z_j^\dagger W_j^\dagger = X_j. \end{aligned} \tag{6}$$

Thus, the transformation $\prod_{j=1}^{2N} W_j$ applied to $H_{1,2}$ from Eq. (2) simply replaces $X_j \leftrightarrow Z_j$.

Unlike the Pauli matrices, Z_j and X_j alone do not form a group under multiplication [40], which makes them inconvenient for our purposes. Thus, we choose a different basis that satisfies the group properties. Namely, following Refs. [41,42], we introduce the operators [43]

$$\mathcal{J}_j^{\boldsymbol{m}} \equiv \mathcal{J}_j^{(m_1, m_2)} = \omega^{m_1 m_2} X_j^{m_1} Z_j^{m_2}, \tag{7}$$

with $0 \leqslant m_{1,2} \leqslant 2$. Using the representation (4), from Eq. (7) we obtain a unit matrix and eight unitary traceless matrices spanning the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Taking into account that $X_j^{m_1} Z_j^{m_2} = \omega^{m_1 m_2} Z_j^{m_2} X_j^{m_1}$ and $\omega^{2 m_2 n_1} = \omega^{-m_2 n_1}$, one can easily show that the operators (7) satisfy

$$\mathcal{J}_j^{\boldsymbol{m}} \mathcal{J}_j^{\boldsymbol{n}} = \omega^{-\boldsymbol{m} \times \boldsymbol{n}} \mathcal{J}_j^{\boldsymbol{m}+\boldsymbol{n}}, \tag{8}$$

where $\boldsymbol{m} \times \boldsymbol{n} \equiv m_1 n_2 - m_2 n_1$ and components of the vector $\boldsymbol{m} + \boldsymbol{n}$ are mod 3. This leads to the following commutation relations:

$$\left[\mathcal{J}_j^{\boldsymbol{m}}, \mathcal{J}_k^{\boldsymbol{n}} \right] = -2i \, \delta_{jk} \, \sin\left(\frac{2\pi}{3} \boldsymbol{m} \times \boldsymbol{n} \right) \mathcal{J}_j^{\boldsymbol{m}+\boldsymbol{n}}, \tag{9}$$

with δ_{jk} being the Kronecker symbol. In terms of the operators in Eq. (7) we have

$$\begin{aligned} H_1 &= \sum_{j=1}^{2N-1} \left[\mathcal{J}_j^{(2,0)} \mathcal{J}_{j+1}^{(1,0)} + \mathcal{J}_j^{(1,0)} \mathcal{J}_{j+1}^{(2,0)} \right], \\ H_2 &= \sum_{j=1}^{2N} \left[\mathcal{J}_j^{(0,1)} + \mathcal{J}_j^{(0,2)} \right]. \end{aligned} \tag{10}$$

Note that all terms in H_1 (H_2) commute with each other, whereas $[H_1, H_2] \neq 0$. Let us also mention that the Hamiltonian (1) is integrable at the critical point $J = f$ [10], and its superintegrable variations are known [44]. We now proceed with the discussion of a periodic driving protocol.

III. FLOQUET PROTOCOL

A. Preliminary remarks

We begin with a brief overview of the Floquet formalism for (isolated) time-dependent quantum systems. For a more detailed discussion see, e.g., Ref. [45]. The evolution operator for a time-dependent Hamiltonian $H(t)$ is given by the time-ordered exponential

$$U(t) = \mathbb{T} e^{-i \int_0^t d\tau H(\tau)}, \tag{11}$$

where $\mathbb{T}$ denotes the time ordering and we set $\hbar = 1$. According to the Floquet theorem, for periodic time dependence of

the Hamiltonian, $H(t + T) = H(t)$, one can rewrite Eq. (11) in the following way:

$$U_F(t) = P(t) e^{-it H_F}, \tag{12}$$

where H_F is time-independent effective (Floquet) Hamiltonian, whereas the operator $P(t)$ is periodic, $P(t + T) = P(t)$, and satisfies $P(mT) = \mathbb{1}$ for integer m. Thus, if one observes the system *stroboscopically*, i.e., only at times $t = nT$ with integer n, then the evolution operator becomes

$$U_F(T) = e^{-iT H_F}. \tag{13}$$

We note in passing that the Floquet Hamiltonian H_F can depend on the period duration T. Despite the simple form of Eq. (13), explicit construction of H_F remains in most cases extremely tedious, if not impossible. Remarkable exceptions are provided by the Lie-algebraic [20] and free-fermionic [46] Hamiltonians, for which one can obtain H_F quite easily.

An important and widely used class of periodic Hamiltonians corresponds to the so-called kicked models. A typical Hamiltonian is of the form

$$H(t) = g_1 H_1 + g_2 T \sum_{m \in \mathbb{Z}} \delta(t - mT) H_2, \tag{14}$$

which describes a sequence of instantaneous kicks by the term $g_2 H_2$ performed with a frequency $\omega = 2\pi/T$. In Eq. (14) we explicitly include the factor of T in the second term in order to fix the dimension of the Hamiltonian. Substituting $H(t)$ into Eq. (11) with $t = T$ and taking into account the δ-functional time dependence, we immediately obtain that the stroboscopic evolution operator factorizes and can be written as

$$U_F(T) = e^{-iT g_2 H_2} e^{-iT g_1 H_1}. \tag{15}$$

Thus, over the period T the evolution is governed solely by $g_1 H_1$, which is followed by the kick with $g_2 H_2$ in the end.

Alternatively, the stroboscopic Floquet protocol (15) can be obtained for the periodic steplike time dependence of the Hamiltonian:

$$H(t) = \begin{cases} H_1, & t \bmod T_1 + T_2 \in [0, T_1), \\ H_2, & t \bmod T_1 + T_2 \in [T_1, T_1 + T_2), \end{cases} \tag{16}$$

where H_1 and H_2 are time independent. We thus have $H(t + T_1 + T_2) = H(t)$ and the one-period stroboscopic Floquet operator is

$$U_F(T_1 + T_2) = e^{-iT_2 H_2} e^{-iT_1 H_1}. \tag{17}$$

Clearly, if H_1 and H_2 do not commute, then it is highly nontrivial to obtain the Floquet Hamiltonian H_F in a closed form. Indeed, in order to reduce the evolution operator from Eq. (15) or Eq. (17) to a single exponential as in Eq. (12), one has to sum the Baker-Campbell-Hausdorff series, which is only possible in a limited number of cases. Nevertheless, the form of the stroboscopic Floquet operator in Eq. (15) is already simple enough to work with and it has been investigated for various models and settings, see, e.g., Refs. [19,20,22–28,30–33] and references therein. In what follows we study the steplike stroboscopic Floquet protocol for the kicked three-state Potts model discussed in Sec. II.

B. Kicked three-state Potts model

We now consider the time-dependent Hamiltonian of the three-state Potts model subject to periodically kicked transverse field H_1:

$$H(t) = -JH_1 - fT \sum_{m \in \mathbb{Z}} \delta(t - mT)H_2, \qquad (18)$$

where $H_{1,2}$ are given by Eq. (10). Thus, the time-dependent Hamiltonian $H(t)$ in Eq. (18) is of the form (14), with $g_1 = -J$ and $g_2 = -f$. Then, the one-period stroboscopic Floquet operator is given by Eq. (15) and reads as

$$U_F(T) = e^{ifTH_2} e^{iJTH_1}. \qquad (19)$$

It corresponds to the evolution for time T with the interaction Hamiltonian $-JH_1$, followed by an instantaneous kick by the uniform transverse field $-fH_2$. We are interested in the state of the system

$$|\psi(kT)\rangle = U_F^k(T)|\psi(0)\rangle \qquad (20)$$

after k periods of the protocol.

Let us denote by $\tilde{H}_1$ the interaction part of the Hamiltonian with the central link (i.e., that between the sites N and $N+1$) being switched off [47]:

$$\tilde{H}_1 = H_1 - (X_N^\dagger X_{N+1} + X_N X_{N+1}^\dagger). \qquad (21)$$

Then, we rewrite the one-period evolution operator by separating the part V_0 that acts only in the middle of the chain:

$$U_F(T) = e^{ifTH_2} e^{iJT\tilde{H}_1} V_0(T) \equiv \tilde{U}(T)V_0(T), \qquad (22)$$

where we introduced unitary operators V_0 and $\tilde{U}$. The former is given by

$$V_0 = \exp\{iJT\,(X_N^\dagger X_{N+1} + X_N X_{N+1}^\dagger)\} \qquad (23)$$

and acts nontrivially only at the Nth and $(N+1)$-th sites, i.e., over the central link of the chain. On the contrary, the operator

$$\tilde{U}(T) = e^{ifTH_2} e^{iJT\tilde{H}_1} \qquad (24)$$

acts nontrivially on the left and right halves of the chain and *not* across the central link. Therefore, $\tilde{U}$ does not entangle the left and right halves with each other and can be written in the factorized form

$$\tilde{U}(T) = \tilde{U}_L(T)\tilde{U}_R(T), \qquad (25)$$

where $\tilde{U}_{L(R)}$ acts on the left (right) half of the lattice and $[\tilde{U}_L, \tilde{U}_R] = 0$. Then, we rewrite the one-period Floquet operator in Eq. (22) as

$$U_F = \tilde{U}V_0 = V_1\tilde{U}, \quad V_1 = \tilde{U}V_0\tilde{U}^\dagger, \qquad (26)$$

were for the sake of readability we omitted the T dependence. Similarly, for two periods we have $U_F^2 = \tilde{U}V_0V_1\tilde{U}$, which can be written as

$$U_F^2 = V_1V_2\tilde{U}^2, \quad V_2 = \tilde{U}V_1\tilde{U}^\dagger. \qquad (27)$$

One can easily check that the evolution operator for k periods becomes

$$U_F^k = V_1 \ldots V_k \tilde{U}^k, \quad V_l = \tilde{U}V_{l-1}\tilde{U}^\dagger. \qquad (28)$$

The unitary operators V_k act nontrivially on both halves of the chain and entangle them. However, for some specific values of fT and JT the resulting entanglement has a very simple structure, as we show below.

C. Explicit form of V_k

Let us now find the explicit form of the operators V_k. We first rewrite the operator V_1 from Eq. (26) as

$$V_1 = \tilde{U}V_0\tilde{U}^\dagger = e^{ifT\,\mathrm{ad}_{H_2}}V_0$$

$$= \prod_{j=N}^{N+1} \prod_{n=1}^{2} \sum_{k=0}^{\infty} \frac{(ifT)^k}{k!} \mathrm{ad}_{\mathcal{J}_j^{(0,n)}}^k V_0, \qquad (29)$$

where $\mathrm{ad}_X^k Y \equiv [X, [X, \ldots [X, Y]]]$ is a k-fold nested commutator. In writing Eq. (29) we took into account that $[\tilde{H}_1, V_0] = 0$ and all terms in H_2 commute with each other, since the operators $\mathcal{J}_j^m$ commute on different sites. We also used a well-known identity $e^X Y e^{-X} = e^{\mathrm{ad}_X}Y$, valid for any X and Y in a Lie algebra [$\mathfrak{sl}(3, \mathbb{C})$ in our case]. To simplify Eq. (29) it is convenient to expand the exponential in V_0. From Eq. (23) we have (for details, see Appendix A)

$$V_0 = \mu\mathbb{1} + \nu\big[\mathcal{J}_N^{(1,0)}\mathcal{J}_{N+1}^{(2,0)} + \mathcal{J}_N^{(2,0)}\mathcal{J}_{N+1}^{(1,0)}\big], \qquad (30)$$

were we took into account Eq. (7) for the definition of the operators $\mathcal{J}_j^m$. The coefficients in Eq. (30) are given by

$$\mu = \nu + e^{-iTJ}, \quad \nu = e^{-iTJ}(e^{3iTJ} - 1)/3. \qquad (31)$$

It is now straightforward to calculate V_1. For generic values of fT the calculation of the adjoint action in Eq. (29) can be found in Appendix D. Importantly, the expression for V_1 becomes especially simple for $fT = \alpha_m$, where we denoted

$$\alpha_m = \frac{2\pi}{9}(3l - m), \qquad (32)$$

with $l \in \mathbb{Z}$ and $m \in \{0, 1, 2\}$. In this case, V_1 reads as

$$V_1 = \mu\mathbb{1} + \nu\big[\mathcal{J}_N^{(1,m)}\mathcal{J}_{N+1}^{(2,2m)} + \mathcal{J}_N^{(2,2m)}\mathcal{J}_{N+1}^{(1,m)}\big]. \qquad (33)$$

Obviously, the case $m = 0$ is trivial, since it results in $V_1 = V_0$.

Moreover, one can show that the operator V_2 from Eq. (27) also acquires a compact form for

$$fT = JT = \alpha_m, \qquad (34)$$

where α_m is given by Eq. (32). Using the results of Appendices C and D, we find

$$V_2 = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-1}^{(1,m)}\mathcal{J}_N^{(0,m)}\mathcal{J}_{N+1}^{(0,2m)}\mathcal{J}_{N+2}^{(2,2m)}$$
$$+ \mathcal{J}_{N-1}^{(2,2m)}\mathcal{J}_N^{(0,2m)}\mathcal{J}_{N+1}^{(0,m)}\mathcal{J}_{N+2}^{(1,m)}\big]. \qquad (35)$$

Similarly, under the conditions $JT = fT = \alpha_m$ and $2 \leqslant k \leqslant N$ one obtains the following expression for V_k:

$$V_k = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-k+1}^{(1,m)}\mathcal{J}_{N-k+2}^{(0,m)} \ldots \mathcal{J}_N^{(0,m)}$$
$$\times \mathcal{J}_{N+1}^{(0,2m)} \ldots \mathcal{J}_{N+k-1}^{(0,2m)}\mathcal{J}_{N+k}^{(2,2m)} + \mathrm{H.c.}\big], \qquad (36)$$

where in the conjugated term (denoted by "H.c."), one simply makes a replacement $m \leftrightarrow 2m$ in the upper indices of $\mathcal{J}_j^{(p,q)}$.

In terms of the operators X_j and Z_j one can write V_k as

$$
\begin{aligned}
V_k = \mu\mathbb{1} + \nu\omega^{2m}\big(&X_{N-k+1}Z_{N-k+1}^m \dots Z_N^m \\
&\times Z_{N+1}^{2m} \dots Z_{N+k-1}^{2m}X_{N+k}^2 Z_{N+k}^{2m} + \text{H.c.}\big),
\end{aligned} \quad (37)
$$

which follows from Eq. (7). We thus see that for $JT = fT = 2\pi(3l - m)/9$ the form of the operator V_k from Eq. (37) is quite simple, as well as its action on the chain. It only changes the internal state of qutrits on the sites $N - k + 1$ and $N + k$, whereas on the rest of the chain V_k either produces an extra phase factor or acts trivially.

Interestingly, for $k > N$ the form of V_k exhibits a peculiar structure. Before we proceed, let us rewrite V_k for later convenience as

$$
V_k = \mu\mathbb{1} + \nu(\mathcal{V}_k + \mathcal{V}_k^\dagger), \quad (38)
$$

which can be always done since V_0 has this form. Moreover, the operator $\mathcal{V}_k$ in Eq. (38) can be always written as

$$
\begin{aligned}
\mathcal{V}_k \equiv \mathcal{W}_k\tilde{\mathcal{W}}_k = &\prod_{j=1}^{N} \mathcal{J}_j^{(p_j(k),q_j(k))} \\
&\times \prod_{j=N+1}^{2N} \mathcal{J}_j^{(2p_{2N+1-j}(k),2q_{2N+1-j}(k))},
\end{aligned} \quad (39)
$$

because the upper indices of $\mathcal{J}_j^{(p_j(k),q_j(k))}$ in $\mathcal{V}_k$ are symmetric with respect to reflection across the Nth link of the chain. For brevity, let us focus on the structure of $\mathcal{W}_k = \prod_{j=1}^{N} \mathcal{J}_j^{(p_j(k),q_j(k))}$. Using the results of Appendices C and D, we obtain the following expressions for $\mathcal{W}_k$ with $k > N$:

$$
\begin{aligned}
\mathcal{W}_{N+l} &= \prod_{j=1}^{l-1} \mathcal{J}_j^{(0,2m)}\, \mathcal{J}_l^{(1,2m)} \prod_{j=l+1}^{N} \mathcal{J}_j^{(0,m)}, \\
\mathcal{W}_{2N+l} &= \prod_{j=1}^{N-l} \mathcal{J}_j^{(0,2m)}\, \mathcal{J}_{N-l+1}^{(1,0)} \prod_{j=N-l+2}^{N} \mathbb{1}_j, \\
\mathcal{W}_{3N+l} &= \prod_{j=1}^{l-1} \mathcal{J}_j^{(0,m)}\, \mathcal{J}_l^{(1,m)} \prod_{j=l+1}^{N} \mathbb{1}_j, \\
\mathcal{W}_{4N+l} &= \prod_{j=1}^{N-l} \mathcal{J}_j^{(0,m)}\, \mathcal{J}_{N-l+1}^{(1,2m)} \prod_{j=N-l+2}^{N} \mathcal{J}^{(0,2m)}, \\
\mathcal{W}_{5N+l} &= \prod_{j=1}^{l-1} \mathbb{1}_j\, \mathcal{J}_l^{(1,0)} \prod_{j=l+1}^{N} \mathcal{J}^{(0,2m)}, \\
\mathcal{W}_{6N+l} &= \prod_{j=1}^{N-l} \mathbb{1}_j\, \mathcal{J}_{N-l+1}^{(1,m)} \prod_{j=N-l+2}^{N} \mathcal{J}^{(0,m)},
\end{aligned} \quad (40)
$$

where $1 \leqslant l \leqslant N$. The structure of $\mathcal{W}_k$ in Eq. (40) is fairly complicated and one can see that the upper indices of $\mathcal{J}_j^{(p_j(k),q_j(k))}$ change completely across $\mathcal{W}_k$ several times as k increases from N to $6N$ [recall that N is *half* the length of the chain]. Quite remarkably, the expression for $\mathcal{W}_{6N+l}$ from Eq. (40) coincides with that for W_l with $1 \leqslant l \leqslant N$, as can be seen from Eq. (36). This means that the operators V_k are

periodic with respect to k and one has

$$
V_{6N+k} = V_k. \quad (41)
$$

Then, taking into account Eq. (28) we have $V_{6N+k} = \tilde{U}^{6N}V_k\tilde{U}^{\dagger\,6N}$, and one concludes that $[\tilde{U}^{6N}, V_k] = 0$ for any k. Our detailed analysis shows that in fact one has

$$
\tilde{U}^{6N} = \mathbb{1}, \quad (42)
$$

meaning that under the condition (34) the unitary operator $\tilde{U}$ from Eq. (24) is a permutation. This has far reaching consequences as one should be able to find explicitly the spectrum of the total Floquet operator U_F from Eq. (19). However, this lies beyond the scope of the present work and we leave it to future studies.

In the following we show explicitly that in the regime $fT = JT = \alpha_m$, with α_m given by Eq. (32), the Floquet protocol described in this section leads to the generation of long-range entanglement between the pairs of qutrits, and the resulting entanglement has a very simple form.

D. Long-range entangled state generation

The protocol consists of the preliminary state preparation and the generation of long-range entanglement itself. In the first stage, we start from an initial polarized state $|\psi_0\rangle$, i.e., the product state in which all qutrits are in one and the same internal state [48]. For instance, let us assume that every qutrit is initialized in the state $|0\rangle$. Thus, the initial state for the protocol reads

$$
|\psi(0)\rangle = \otimes_{j=1}^{2N} |0\rangle_j. \quad (43)
$$

Then, the state preparation procedure consists of evolving the state $|\psi(0)\rangle$ for k periods by the two-step Floquet protocol with the one period Floquet unitary $\tilde{U}^\dagger(T)$:

$$
|\tilde{\psi}(kT)\rangle = \tilde{U}^{\dagger k}(T)|\psi(0)\rangle = |\tilde{\Psi}(kT)\rangle_L \otimes |\tilde{\Psi}(kT)\rangle_R, \quad (44)
$$

where $\tilde{U}(T)$ is given by Eq. (24) and we took into account that it factorizes according to Eq. (25). From Eq. (24) we have

$$
\tilde{U}^\dagger(T) = e^{-iJT\tilde{H}_1}e^{-ifTH_2}, \quad (45)
$$

which corresponds to the stroboscopic evolution with the Hamiltonian

$$
H'(t) = fH_2 + J\sum_{n\in\mathbb{Z}} T\delta(t - nT)\tilde{H}_1, \quad (46)
$$

where $\tilde{H}_2$ is given by Eq. (21) and H_2 by Eq. (2). In other words, each period of the state preparation stage consists of evolving the state for time T with the Hamiltonian fH_2 (corresponding to the transverse field), followed by an instantaneous kick with the Hamiltonian $J\tilde{H}_1$, which corresponds to the interaction between the nearest neighbor spins on all links except for the central one [47].

At the next step, we perform the Floquet protocol with the full evolution operator $U_F(T)$ from Eqs. (19) and (22). After the first period of the protocol we obtain

$$
\begin{aligned}
|\psi(T)\rangle &= U_F(T)|\tilde{\psi}(kT)\rangle = V_1[\tilde{U}^\dagger(T)]^{k-1}|\psi(0)\rangle \\
&= V_1|\tilde{\Psi}[(k-1)T]\rangle_L \otimes |\tilde{\Psi}[(k-1)T]\rangle_R.
\end{aligned} \quad (47)
$$

Then, after k periods we arrive at the state

$$|\psi(kT)\rangle = U_F^k(T)|\tilde{\psi}(kT)\rangle = V_1 \ldots V_k|\psi(0)\rangle, \qquad (48)$$

where for U_F^k we used Eq. (28). We thus see that the evolution generated by the operator $\tilde{U}^{\dagger k}$ is eliminated, and the resulting state $|\psi(kT)\rangle$ in Eq. (48) is fully determined by the action of the string $V_1 \ldots V_k$ on the initial state $|\psi(0)\rangle$. Note that Eq. (48) is completely general and is valid for arbitrary values of fT and JT.

In order to proceed with constructing the explicit form of the state $|\psi(kT)\rangle$ from Eq. (48), we choose $JT = fT = 2\pi(3l - m)/9$, so that the operators V_k have the simple form given in Eq. (37). In this case, one can clearly see that by applying V_k to the initial state $|\psi(0)\rangle$ we entangle only the sites $N - k + 1$ and $N + k$:

$$V_k|\psi(0)\rangle = |\Phi(\mu, \nu)\rangle_k \bigotimes_{j=1}^{2N}{}' |0\rangle_j, \qquad (49)$$

where the prime means that the tensor product does not include sites $j = N - k + 1$ and $j = N + k$. In Eq. (49) we denoted by $|\Phi(\mu, \nu)\rangle_j$ the maximally entangled two-qutrit Bell-like state

$$|\Phi(\mu, \nu)\rangle_j = \mu|0\rangle_{N-j+1}|0\rangle_{N+j} + \omega^{2m}\nu(|1\rangle_{N-j+1}|2\rangle_{N+j}$$
$$+ |2\rangle_{N-j+1}|1\rangle_{N+j}), \qquad (50)$$

with $m = 1, 2$ and μ and ν given by Eq. (31). One can easily see that the state $|\Phi(\mu, \nu)\rangle_j$ has the Schmidt rank $r = 3$.

Before we complete the discussion of the long-range entanglement generation, we also would like to mention a somewhat unrelated but peculiar observation. Imagine that instead of considering the regime in Eq. (34) one takes, say, $fT = 2\pi/9$ and $JT = 4\pi/9$. In this case, the operator V_k still contains only three terms, just like in Eqs. (37) and (38). For $fT = JT = \alpha_m$ one can immediately read off the form of $\mathcal{V}_k$ from Eq. (37), and it is extremely simple. On the contrary, for $fT = \alpha_{2m}$ and $JT = \alpha_m$ (or vice versa), the structure of $\mathcal{V}_k$ in Eq. (38) turns out to be quite complex and it drastically changes with k, as we discuss in more detail in Appendix D [see Eq. (D31)]. Neglecting the phase factor, one can write $\mathcal{V}_k$ as

$$\mathcal{V}_k \sim \prod_{j=k-1}^{0} X_{N-j}^{p_{j+1}} Z_{N-j}^{q_{j+1}} \prod_{j=1}^{k} X_{N+j}^{2p_j} Z_{N+j}^{2q_j}, \qquad (52)$$

so that the operator content of V_k in Eq. (38) is characterized by two $2N$-dimensional vectors $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$, containing the powers of X_j and Z_j, correspondingly, on all lattice sites:

$$\boldsymbol{P}_k = \{p_1(k), \ldots, p_N(k), 2p_N(k), \ldots, 2p_1(k)\}, \qquad (53)$$

and similarly for $\boldsymbol{Q}_k$. In the course of the Floquet protocol, the components of $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ are updated via Eq. (D34), which follows from the relation $V_k = \tilde{U}V_{k-1}\tilde{U}^{\dagger}$ in Eq. (28). Then, in order to gain insight on the structure of V_k for $fT = 2\pi/9$ and $JT = 4\pi/9$, let us plot the components of $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ for different values of k and see how they change across the chain as the number of periods k increases. Our findings are

Taking into account that for $JT = 2\pi(3l - m)/9$ we have $|\mu| = |\nu| = 1/\sqrt{3}$, it is also easy to check that the partial trace of $|\Phi(\mu, \nu)\rangle_j\langle\Phi(\mu, \nu)|$ with respect to either of the two subspaces gives $\mathbb{1}/3$. Therefore, the state $|\Phi(\mu, \nu)\rangle_j$ in Eq. (50) is indeed a maximally entangled one [49]. Then, from Eqs. (48) and (49) we immediately obtain the final state of the chain after k periods of the Floquet protocol:

$$|\psi(kT)\rangle = \bigotimes_{j=1}^{N-k} |0\rangle_j \bigotimes_{j=1}^{k} |\Phi(\mu, \nu)\rangle_j \bigotimes_{j=N+k+1}^{2N} |0\rangle_j, \qquad (51)$$

which is build of nonlocal qutrit pairs with a long-range entanglement. The result in Eq. (51) is valid for the number of periods $k \leqslant N$; otherwise, there are obviously not enough qutrit pairs to entangle. In particular, for $k = N$ the final state consists of N maximally entangled qutrit pairs, which are symmetrically distributed around the middle of the chain. We schematically illustrate the Floquet protocol in Fig. 1. Let us emphasize that the suggested Floquet protocol works properly only under a specific choice of the system parameters. Namely, the state (51) is obtained under the condition $fT = JT = \alpha_m$, with α_m given in Eq. (32), which guarantees the remarkably simple structure of the operators V_k in Eq. (37) and, consequently, that of the final state $|\psi(kT)\rangle$ in Eqs. (48) and (51).

illustrated in Fig. 2. One clearly sees that the components of $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ form a fractal pattern, which exhibits a large-scale structure resembling the Sierpiński carpet. It would be interesting to investigate the origin of this fractal behavior, as well as its possible physical consequences on the Floquet dynamics. However, this lies beyond the scope of the present paper and we leave it to future studies.

Returning to the regime of equal fT and JT, one may ask what is the physical reason behind the fine-tuning requirement in Eq. (34), which has to be satisfied for the Floquet protocol to generate the state with long-range entangled pairwise entanglement between qutrits. In Sec. IV we argue that Eq. (34) is nothing else than the *integrability condition* for the stroboscopic Floquet protocol (19). In this view it is quite natural that the protocol creates a state of a simple form, instead of simply heating the system up to infinite temperature.

IV. INTEGRABILITY OF THE FLOQUET PROTOCOL

A. General remarks and relation to the Temperley-Lieb algebra

We now show that the protocol consider in Sec. III can be viewed as a special case of a more general Floquet protocol. The reason is that the three-state Potts model is a *representation* of a Hamiltonian that belongs to the so-called Temperley-Lieb algebra. In this section we briefly overview the Temperley-Lieb algebra, construct a stroboscopic two-step Floquet protocol using the generators of the algebra and discuss the Floquet integrability of the protocol.
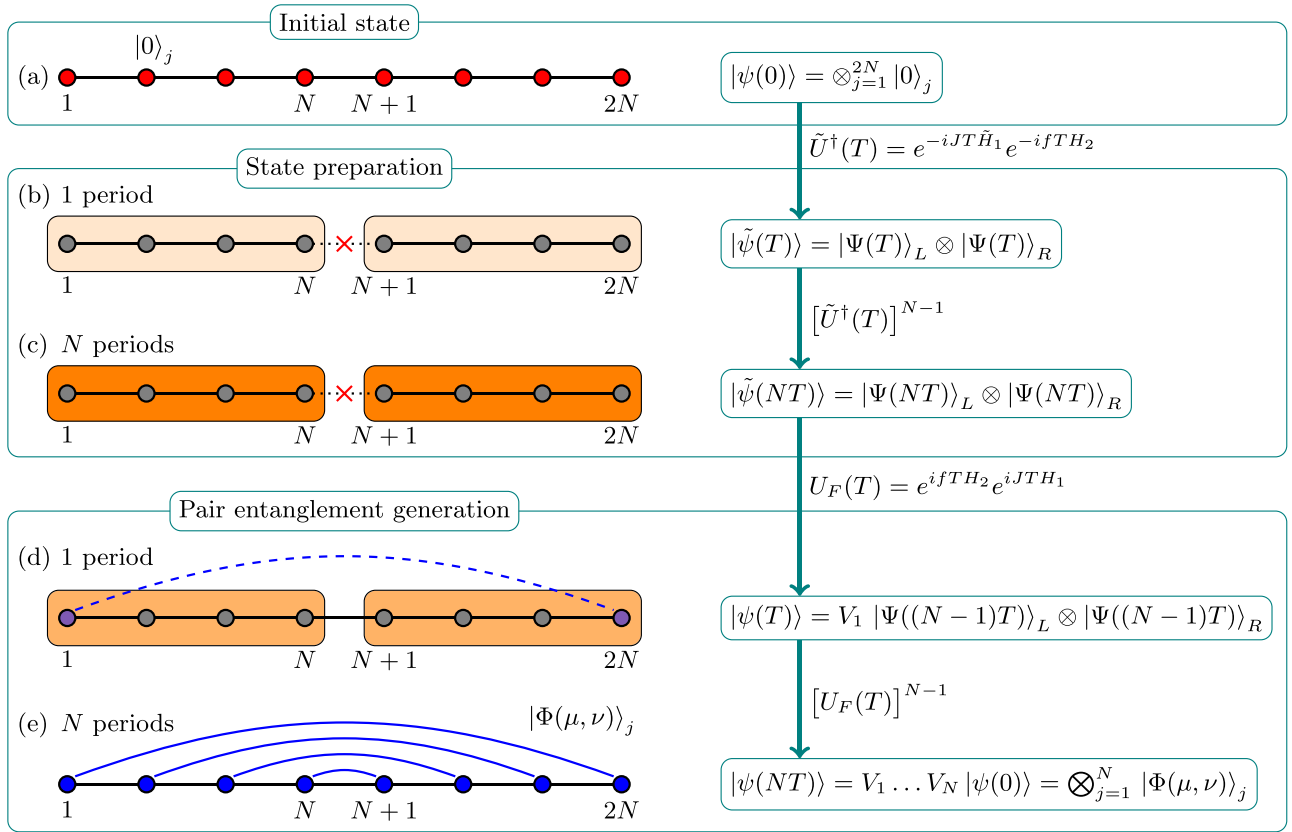
FIG. 1. Schematic illustration of the protocol for generating long-range entanglement in the three-state Potts model. The protocol starts with the initial state (a) and consists of the preliminary state preparation scheme [shown in (b) and (c)] followed by the procedure for generating nonlocal entanglement [displayed in (d) and (e)] between distant qutrit pairs. (a) At $t = 0$ every qutrit is in one and the same state $|0\rangle_j$, so that the chain initially is in the polarized product state $|\psi(0)\rangle$. At the first iteration of the state preparation scheme (b), the central link of the chain is switched off (shown by the red cross on the link) and the two halves of the chain are evolved independently with the unitary operator $\tilde{U}^\dagger(T)$, which is given by Eq. (45). This produces a bipartite state $|\tilde{\psi}(T)\rangle$ shown by two light colored blocks in (b). Then, after $N - 1$ more iterations of the state preparation scheme (c), the chain is in the bipartite state $|\tilde{\psi}(NT)\rangle$. The two halves of the chain are still factorized, but each of them is now highly entangled, as illustrated by the darker colored blocks in (c). This completes the state preparation scheme and one proceeds with the protocol generating entanglement between qutrit pairs. At the first period the protocol (d), one first reduces the amount of entanglement *inside* the left and right halves of the chain by the operator $U_F(T)$ and produces the state $|\tilde{\psi}((N - 1)T)\rangle$ [note that it is represented by the blocks of lighter color as compared to (c)]. In addition, by the operator V_1 one creates a two-particle entanglement *between* the left and right halves of the chain [blue dashed arc in (d)]. The result of every next period is also twofold: One further disentangles separately the states of the left and right halves of the chain and creates another two-site entanglement between them. As a result, after N periods of the protocol (e) the final state $|\psi(NT)\rangle$ is simply the product of maximally entangled two-qutrit states $|\Phi(\mu, \nu)\rangle_j$ [blue solid arcs in (e)].

Let u_j with $j = 1, \ldots, L - 1$ be the generators of the Temperley-Lieb algebra $\mathrm{TL}_L(\beta)$, where β is a complex parameter. The generators satisfy the defining relations

$$u_j^2 = \beta u_j, \tag{54a}$$

$$u_j u_{j \pm 1} u_j = u_j, \tag{54b}$$

$$[u_i, u_j] = 0, \quad |i - j| > 1. \tag{54c}$$

The elements (also called *words*) of $\mathrm{TL}_L(\beta)$ are obtained by multiplying the generators u_j in all possible ways. A word w is called *reduced* if it cannot be shortened with the help of the relations (54). Every reduced word $w \in \mathrm{TL}_L(\beta)$ can be written in the *Jones normal form* [50], namely as a sequence

of decreasing sequences of the generators:

$$w = (u_{j_1} u_{j_1 - 1} \ldots u_{k_1}) \ldots (u_{j_r} u_{j_r - 1} \ldots u_{k_r}), \tag{55}$$

where $0 < j_1 < \ldots < j_r < L$ and $0 < k_1 < \ldots < k_r < L$. It can be shown that the generator with the largest index appears in w only once. All reduced words formed of the generators $\{u_j\}_{j=1}^{L-1}$ span the basis in $\mathrm{TL}_L(\beta)$. Thus, the Temperley-Lieb algebra is finite dimensional and one can show that its dimensionality is

$$\dim \mathrm{TL}_L(\beta) = \frac{1}{L+1} \binom{2L}{L} = C_L, \tag{56}$$

which is the Lth Catalan number. Let us note that the defining relations (54) can be equivalently formulated using the so-called Gröbner-Shirshov basis [51]. To do so, one introduces
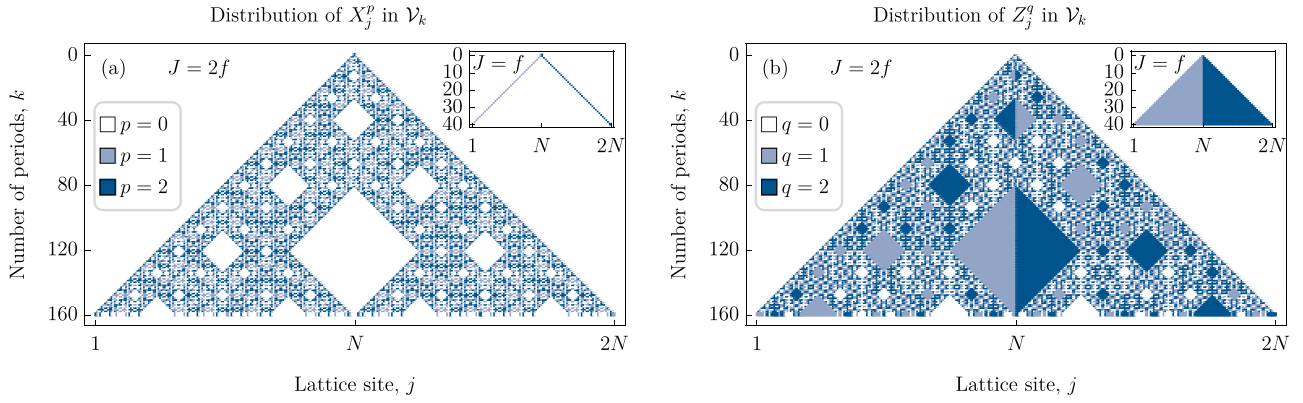
FIG. 2. Structure of $\mathcal{V}_k$ in Eqs. (38) and (52). (a) Distribution of X_j^p with $1 \leqslant j \leqslant 2N$ in $\mathcal{V}_k$ for $fT = 2\pi/9$ and $JT = 4\pi/9$ (main panel), chain length of $2N = 320$ sites, and the number of periods $0 \leqslant k \leqslant N = 160$. The color of a pixel with coordinates (j, k) encodes the power p of the operator X_j^p in the expression for $\mathcal{V}_k$ from Eq. (52). The inset shows the same distribution for $fT = JT = 2\pi/9$ and $2N = 80$. (b) Distribution of Z_j^q with $1 \leqslant j \leqslant 2N$ in $\mathcal{V}_k$. The color of a (j, k)-th pixel indicates the power q of Z_j^q in the expression for $\mathcal{V}_k$. The parameters on the main panel and the inset are the same as the corresponding ones on panel (a). Note that if one instead chooses $fT = 2\pi/9$ and $JT = 4\pi/9$ for both main panels, the distribution of X_j^p remains the same, whereas in the distribution of Z_j^q one simply swaps $q = 1$ and $q = 2$. Likewise, taking for the insets $fT = JT = 4\pi/9$, the inset in panel (a) does not change, whereas the inset in panel (b) gets reflected.

the operators $u_{i,j}$ defined as

$$u_{i,i} = u_i, \quad u_{i,i+1} = 1,$$
$$u_{i,j} = u_i u_{i-1} \ldots u_j, \quad i \geqslant j. \tag{57}$$

Then, one replaces the relation (54b) with the following two:

$$u_{i,j} u_i = u_{i-2,j} u_i,$$
$$u_j u_{i,j} = u_j u_{i,j+2}, \tag{58}$$

where $i > j$. One can easily check that Eqs. (54a), (54c), (57), and (58) are equivalent to the standard form of the defining relations in Eq. (54). However, the former are often much more convenient in practice.

Quite remarkably, the Temperley-Lieb algebra possesses numerous representations that correspond to various paradigmatic physical models, such as the TFIM, spin-1/2 XXZ spin chain, and the n-state Potts model [52]. We are interested in the representation corresponding to the three-state Potts model on a chain with M sites and open boundary conditions:

$$u_{2j} = \frac{1}{\sqrt{3}}(1 + X_j X_{j+1}^\dagger + X_j^\dagger X_{j+1}), \quad 1 \leqslant j < M,$$
$$u_{2j-1} = \frac{1}{\sqrt{3}}(1 + Z_j + Z_j^\dagger), \quad 1 \leqslant j \leqslant M. \tag{59}$$

One can easily check that the operators u_j in Eq. (59) satisfy the defining relations in Eq. (54) with $\beta = \sqrt{3}$ and thus form a representation of $\mathrm{TL}_{2M}(\sqrt{3})$.

We now consider the following linear combinations belonging to $\mathrm{TL}_{2M}(\beta)$:

$$H_1 = \sum_{j=1}^{M-1} u_{2j}, \quad H_2 = \sum_{j=1}^{M} u_{2j-1}. \tag{60}$$

Assuming that u_j are Hermitian, we treat $H_{1,2}$ as abstract Hamiltonians for which one can use any Hermitian representation, in particular the one in Eq. (59). We then construct a

stroboscopic two-step Floquet protocol,

$$U_F = e^{-iT_2 H_2} e^{-iT_1 H_1}, \tag{61}$$

where T_k is either the time period over which the dynamics is governed by H_k, as in Eq. (17), or $T_k \equiv g_k T$ as in Eq. (15) corresponding to the kicked protocol. In the latter case T_k are allowed to be negative. Thus, taking $M = 2N$, $T_1 = -\sqrt{3}JT$, $T_2 = -\sqrt{3}fT$, and using the representation (59), we reduce Eq. (61) to the Floquet protocol in Eq. (19) up to a constant phase. However, in what follows we mostly work with the general case in Eq. (61), hereinafter referred to as the Temperley-Lieb algebraic Floquet protocol.

B. Integrability of the Temperley-Lieb algebraic Floquet protocol and its conservation laws

Note that the Floquet operator in Eq. (61) can be written in the form

$$U_F = \prod_{j=1}^{M} (\mathbb{1} + x_2 u_{2j-1}) \prod_{j=1}^{M-1} (\mathbb{1} + x_1 u_{2j}), \tag{62}$$

where we used Eq. (54) and denoted $x_k = (e^{-i\beta T_k} - 1)/\beta$. In this form, the Floquet evolution operator U_F resembles the transfer matrix of a two-dimensional classical *integrable* lattice model. This similarity suggests that the Floquet protocol (62) is also integrable in some sense [20]. This is indeed the case, and the notion of integrability in the context of Floquet dynamics should be understood in the following way. Let us rewrite Eq. (61) in the form of a single exponential [as in Eq. (13)]:

$$U_F = e^{-iTH_F}, \tag{63}$$

where H_F is an effective Floquet Hamiltonian. Then, for an integrable stroboscopic Floquet protocol with the one-period evolution operator U_F there is a macroscopically large number of conserved quantities (charges) Q_n that commute with the

effective Hamiltonian H_F and with each other:

$$[Q_n, H_F] = 0, \quad [Q_n, Q_m] = 0. \quad (64)$$

Obviously, due to Eq. (63), the charges Q_n also commute with the Floquet evolution operator U_F, which eliminates the need to calculate H_F explicitly. Importantly, the conserved charges are required to be local, i.e., expressible as a linear combination of terms with a finite support:

$$Q_n = \sum_{l \lesssim n} \sum_j q^{(n)}_{j,j+1,\dots j+l}. \quad (65)$$

The support of $q^{(n)}_{j,j+1,\dots j+l}$ increases with n, but for a local charge Q_n remains finite. For the Temperley-Lieb algebraic Floquet protocol the operators $q^{(n)}_{j,j+1,\dots j+l}$ are multilinear in the generators u_j and correspond to the reduced words (55) in the Temperley-Lieb algebra.

We now proceed with looking for local conserved charges Q_n that commute with the Temperley-Lieb algebraic Floquet evolution operator U_F in Eq. (61). It is convenient to rewrite the integrability condition $[Q_n, U_F] = 0$ as

$$e^{iT_2 H_2} Q_n e^{-iT_2 H_2} = e^{-iT_1 H_1} Q_n e^{iT_1 H_1}. \quad (66)$$

In order to find the first conserved charge one can simply make the most general ansatz for Q_n, which is homogeneous and consists of terms that are at most bilinear in the generators u_j. Thus, keeping in mind that the terms on even and odd sites play distinct roles [see Eq. (62)], for the first conserved charge we make the following ansatz:

$$Q_1 = H_1 + a_0 H_2 + \sum_{j=1}^{M-1}(b_0 u_{2j} u_{2j+1} + b_1 u_{2j-1} u_{2j})$$
$$+ \sum_{j=1}^{M-1}(c_0 u_{2j+1} u_{2j} + c_1 u_{2j} u_{2j-1}), \quad (67)$$

where $H_{1,2}$ are given by Eq. (60) and $a_0, b_{0,1}, c_{0,1}$ are yet unknown coefficients to be determined from Eq. (66). Anticipating the result, it is convenient to rewrite the ansatz (67) as

$$Q_1 = H_1 + a_0 H_2 + a_1 [H_1, H_2] + a_2 \mathcal{A}, \quad (68)$$

where from Eqs. (54) and (60) one has [53]

$$[H_1, H_2] = \sum_{j=1}^{2M-2}(-1)^j [u_j, u_{j+1}] \quad (69)$$

and we introduced the operator

$$\mathcal{A} = \sum_{j=1}^{2M-2}\{u_j, u_{j+1}\}, \quad (70)$$

with $\{\cdot, \cdot\}$ being the anticommutator. The coefficients $a_{1,2}$ in Eq. (68) are related to those in Eq. (67) via $b_0 = c_1 = a_2 + a_1$ and $b_1 = c_0 = a_2 - a_1$. Even though the number of free parameters in the ansatz (68) is reduced, detailed analysis shows that Eq. (67) does not lead to any new solutions to Eq. (66), apart from the one in Eq. (68).

We then substitute the ansatz (68) for Q_1 into the integrability condition (66) and check whether it can be satisfied for

some values of a_k. Remarkably, one can perform the unitary transformations in Eq. (66) analytically and obtain a closed form expression for $e^{iT_k H_k} Q_1 e^{-iT_k H_k}$, with $k = 1, 2$. We discuss the details of this calculation in Appendix F. Using Eqs. (F7)–(F11), we find that condition (66) is satisfied if one has

$$T_1 = T_2 = T, \quad (71)$$

and the coefficients a_k in Eq. (68) are given by

$$a_0 = 1, \quad a_1 = \frac{i}{2\beta}\sin\beta T, \quad a_2 = -\frac{1}{\beta}\sin^2\frac{\beta T}{2}. \quad (72)$$

Thus, the local charge Q_1 in Eq. (68) with the coefficients a_k from Eq. (72) provides an *exact* conservation law of the Temperley-Lieb algebraic Floquet protocol (61), since it commutes with the one-period evolution operator $U_F = e^{-iT_2 H_2} e^{-iT_1 H_1}$. We emphasize that since Q_1 exists only when $T_1 = T_2$, the same condition is required for all higher-order conserved charges as well.

Note that we were able to derive Q_1 analytically because the adjoint action $\exp\{s \, \mathrm{ad}_{H_{1,2}}\}$ on the terms linear and bilinear in the Temperley-Lieb generators u_j can be calculated in a closed form. Unfortunately, for higher-order terms this procedure quickly becomes cumbersome. One can still try to find a few higher-order charges with brute force by simply using an ansatz (65) and requiring that its commutator with the evolution operator in the form (62) is zero. Proceeding in this way, setting $T_1 = T_2 = T$ and using the relations (57) and (58), for the second conserved charge we obtain

$$Q_2 = Q_2^{(2)} + Q_2^{(3)} + Q_2^{(4)} + Q_2^{(\mathrm{edge})}, \quad (73)$$

where $Q_2^{(n)}$ is a term that contains multilinear products of n generators u_j and acts in the bulk of the chain, whereas $Q_2^{(\mathrm{edge})}$ is the boundary term, which appears due to the fact that the Temperley-Lieb generators are defined on a chain with open boundary conditions. Explicitly, for $Q_2^{(2)}$ we have

$$Q_2^{(2)} = b_2 [H_1, H_2] + c_2 \mathcal{A}, \quad (74)$$

where $\mathcal{A}$ is given by Eq. (70) and the coefficients are

$$b_2 = \frac{i}{2\beta}(\beta^2 + 2\cos\beta T)\tan\frac{\beta T}{2},$$
$$c_2 = \frac{1}{2\beta}\left(\beta^2 - 4\sin^2\frac{\beta T}{2}\right). \quad (75)$$

The terms $Q_2^{(n)}$, with $3 \leqslant n \leqslant 4$, are more complicated and for the sake of readability we introduce the short-hand notations

$$C_j^- = [u_j, u_{j+1}], \quad C_j^+ = \{u_j, u_{j+1}\}. \quad (76)$$

Then, in terms of the generator we have

$$Q_2^{(3)} = b_3 \sum_{j=1}^{2M-3}(-1)^j([u_j, C_{j+1}^+] - \{u_j, C_{j+1}^-\})$$
$$+ c_3 \sum_{j=1}^{2M-3}[u_j, C_{j+1}^-], \quad (77)$$

where the coefficients are given by

$$b_3 = -\frac{i}{2}\tan\frac{\beta T}{2}, \quad c_3 = 1. \quad (78)$$

The next term is given by

$$Q_2^{(4)} = \sum_{j=1}^{2M-4} (-1)^j (b_4 [C_j^-, C_{j+2}^-] + c_4 [C_j^+, C_{j+2}^+])$$
$$+ d_4 \sum_{j=1}^{2M-4} ([C_j^-, C_{j+2}^+] + [C_j^+, C_{j+2}^-]), \quad (79)$$

where the coefficients read as

$$b_4 = \frac{i}{2\beta} \sin \beta T, \quad d_4 = -\frac{1}{\beta} \sin^2 \frac{\beta T}{2},$$
$$c_4 = -\frac{i}{\beta} \sin^2 \frac{\beta T}{2} \tan \frac{\beta T}{2}. \quad (80)$$

Finally, for the boundary term one has

$$Q_2^{(\text{edge})} = -(u_1 + u_{2M-1}) + b_{\text{edge}}(C_1^+ + C_{2M-2}^+)$$
$$+ c_{\text{edge}}(C_1^- - C_{2M-2}^-), \quad (81)$$

with the constants

$$b_{\text{edge}} = \frac{2}{\beta} \sin^2 \frac{\beta T}{2}, \quad c_{\text{edge}} = \frac{i}{\beta} \cos \beta T \tan \frac{\beta T}{2}. \quad (82)$$

We have checked that Q_2 commutes with both Q_1 in Eq. (68) and U_F in Eq. (62) for $T_1 = T_2 = T$. Note that Q_2 in Eq. (73) already includes the terms with up to four generators u_j, even though the previous charge Q_1 in Eq. (68) contains at most bilinear terms. The reason is that we are dealing with open boundary conditions, and it is well known that in the absence of translational invariance there only exist conserved charges whose maximal support is even. For integrable spin chains with open boundary conditions this is shown in Ref. [54].

We have also obtained the next conserved charge Q_3 and verified that it commutes with Q_1, Q_2, and U_F under the same condition. The explicit form of Q_3 is extremely bulky and not illuminating. For this reason, we do not present it here. Expressions for higher-order charges Q_n are even more complicated, in particular because of the boundary terms which

proliferate for larger n [39]. Despite the fact that the general form of Q_n is missing, we strongly believe that it should be possible to obtain it in the closed form. We thus conjecture that for $T_1 = T_2$ the Temperley-Lieb algebraic Floquet protocol in Eq. (61) is integrable for arbitrary β and there exists a macroscopically large number of local conserved charges Q_n that commute with the Floquet evolution operator U_F and represent the conservation laws of the Floquet Hamiltonian H_F. We leave the proof of our conjecture for future work. Let us emphasize that integrability of the Temperley-Lieb algebraic Floquet protocol automatically extends to *every* representation of the Temperley-Lieb algebra, even to non-Hermitian ones. In particular, the results of this section cover the protocol considered in Sec. IV for the three-state Potts model, as well as the ones for the TFIM [31] and the Heisenberg model [19], since all three models correspond to different representation of the Temperley-Lieb algebra.

C. Three-step stroboscopic Floquet protocol and its integrability

One can easily obtain a slightly more general result. Namely, consider the following evolution operator

$$U_F'(\lambda) = e^{-i\lambda T H_1} e^{-i T H_2} e^{-i(1-\lambda) T H_1}, \quad (83)$$

with $\lambda \in \mathbb{R}$, which corresponds to a three-step stroboscopic Floquet protocol. On the other hand, $U_F'(\lambda)$ in Eq. (83) is nothing other than the adjoint action of H_1 on the two-step Floquet evolution operator U_F from Eq. (61) at the integrable point $T_1 = T_2 = T$, i.e., $U_F'(\lambda) = e^{-i\lambda T H_1} U_F e^{i\lambda T H_1}$. Therefore, U_F' obviously commutes with an operator

$$Q_n'(\lambda) = e^{-i\lambda T H_1} Q_n e^{i\lambda T H_1}, \quad (84)$$

where Q_n is the nth conserved charge of the two-step protocol U_F with $T_1 = T_2 = T$. This means that the three-step Floquet protocol (83) is integrable by construction. Using the results of Appendix F, namely Eqs. (F7), (F10), and (F11), we can immediately find the explicit form of the first nontrivial charge Q_1', which reads

$$Q_1'(\lambda) = H_1 + H_2 + \frac{i}{\beta} \cos \frac{\beta T}{2} \sin \frac{(1-2\lambda)\beta T}{2} [H_1, H_2] - \frac{1}{\beta} \left[\sin^2 \frac{\lambda \beta T}{2} + \sin^2 \frac{(1-\lambda)\beta T}{2} \right] \mathcal{A}$$
$$\frac{4}{\beta^2} \sin \frac{\lambda \beta T}{2} \sin \frac{(1-\lambda)\beta T}{2} \left\{ i \sin \frac{(1-2\lambda)\beta T}{2} \mathcal{K}_0 + \cos \frac{(1-2\lambda)\beta T}{2} \mathcal{P}_0 - \cos \frac{\beta T}{2} (2H_1 + \mathcal{R}_0) \right\}$$
$$- \frac{16}{\beta^3} \sin^2 \frac{\lambda \beta T}{2} \sin^2 \frac{(1-\lambda)\beta T}{2} \mathcal{S}_0, \quad (85)$$

where $\mathcal{A}$ is given by Eq. (70), and the operators $\mathcal{K}_0$, $\mathcal{P}_0$, $\mathcal{R}_0$, and $\mathcal{S}_0$ are defined in Eq. (F4). Taking $\lambda = 0$ in Eq. (85) we recover Eq. (68) with the coefficients given by Eq. (72), whereas for $\lambda = 1$ we obtain Eq. (68) with H_1 and H_2 being swapped. Likewise, one immediately sees that the Floquet protocol

$$U_F''(\mu) = e^{-i(1-\mu)T H_2} e^{-i T H_1} e^{i\mu T H_2}, \quad (86)$$

with $\mu \in \mathbb{R}$ is also integrable by construction, with the charges given by $Q_n''(\mu) = e^{i\mu T H_2} Q_n e^{-i\mu T H_2}$. Ex-

plicit form of $Q_1''(\mu)$ follows from Eqs. (F7), (F10), and (F11).

V. DISCUSSION AND CONCLUSIONS

Here we summarize the main results of the present work and formulate some open questions for future research. In the first part of the paper (Sec. III) we have proposed a stroboscopic Floquet protocol for generating very simple albeit nonlocal pairwise entanglement between distant qutrits in the

1D three-state Potts model with periodically kicked transverse field. We consider a realistic and experimentally relevant case of a finite chain with $2N$ sites and open boundary conditions. The protocol consists of two main stages. At the first stage we perform a state preparation procedure and transform an initial polarized state of the chain (i.e., every qutrit is in one and the same internal state) into a bipartite state in which the two halves of the chain are completely isolated but each of them is highly entangled. At the second stage of the protocol we iteratively eliminate the entanglement *inside* the left and right halves of the chain and at the same time create a simple but nonlocal pairwise entanglement *between* them. At the end of the second stage the system is in product state of maximally entangled nonlocal Bell-like qutrit pairs. The protocol is illustrated in Fig. 1. Note that the protocol requires tuning the transverse field switching frequency to a specific value. We argue that the reason for this condition is deeply rooted into the Floquet integrability of the protocol.

The second part of the paper (Sec. IV) is dedicated to the idea of Floquet integrability, which is understood as the presence of local conserved charges that commute with the Floquet evolution operator U_F and, consequently, with an effective Floquet Hamiltonian H_F. Motivated by the fact that the three-state Potts model can be thought of as a representation of the Temperley-Lieb algebra, which has remarkably many different representations corresponding to other paramount physical models, we rewrite the stroboscopic two step Floquet protocol in terms of the Temperley-Lieb algebra generators. We then find the first two nontrivial conservation laws of the Temperley-Lieb algebraic Floquet protocol, and explicitly construct the corresponding conserved charges. We then conjecture that the general closed form expression for the conserved charges can be found, although the proof of our conjecture is beyond the scope of this work.

APPENDIX A: DERIVATION OF V_0 IN EQ. (30)

The easiest way to derive Eq. (30) is by using the representation (59) of the Temperley-Lieb algebra:

$$u_{2N} = \frac{1}{\sqrt{3}}(\mathbb{1} + X_N^\dagger X_{N+1} + X_N X_{N+1}^\dagger). \tag{A1}$$

We then write V_0 from Eq. (23) as

$$V_0 = \exp\{iJT \, (X_N^\dagger X_{N+1} + X_N X_{N+1}^\dagger)\} = e^{-iJT} e^{\sqrt{3}iJTu_{2N}}. \tag{A2}$$

Taking into account that $u_{2N}^2 = \sqrt{3}u_{2N}$ according to Eq. (54), we immediately obtain

$$V_0 = e^{-iTJ}\left(\mathbb{1} + \sum_{k=1}^{+\infty} \frac{(\sqrt{3}iJT)^k}{k!} 3^{(k-1)/2} u_{2N}\right) = e^{-iTJ}\left[\mathbb{1} + \frac{1}{\sqrt{3}}(e^{3iJT} - 1)u_{2N}\right]. \tag{A3}$$

Using Eq. (A1) for u_{2N} we arrive at

$$V_0 = \mu\mathbb{1} + \nu(X_N^\dagger X_{N+1} + X_N X_{N+1}^\dagger), \quad \mu = \nu + e^{-iJT}, \quad \nu = \frac{1}{3}e^{-iJT}(e^{3iJT} - 1). \tag{A4}$$

We then write $X_j = \mathcal{J}_j^{(1,0)}$ and $X_j^\dagger = X_j^2 = \mathcal{J}_j^{(2,0)}$, as follows from Eq. (7), and obtain Eq. (30) of the main text.

APPENDIX B: GENERALIZED HYPERBOLIC FUNCTIONS

In this Appendix we review basic properties of the generalized hyperbolic functions and present some relations that are useful for our purposes. We closely follow the discussion in Ref. [55]. Generalized hyperbolic functions $H_{n,k}(x)$ of order n and kth kind are solutions to an ordinary differential equation,

$$\frac{d^n}{dz^n}H_{n,k}(z) = H_{n,k}(z), \quad 0 \leqslant k \leqslant n - 1. \tag{B1}$$

They have the following series representation:

$$H_{n,k}(z) = \sum_{r=0}^{+\infty} \frac{z^{nr+k}}{(nr+k)!}, \quad z \in \mathbb{C}, \tag{B2}$$

from which one immediately obtains

$$\frac{d}{dz}H_{n,k}(z) = H_{n,k-1}(z), \quad H_{n,-1} = H_{n,n-1}(z). \tag{B3}$$

Clearly, Eq. (B2) for $n = 2$ reduces to the series for usual hyperbolic functions, i.e., $H_{2,0} = \cosh z$ and $H_{2,1} = \sinh z$. We are interested in the case $n = 3$, for which Eq. (B2) gives

$$H_{3,0}(z) \equiv h_0(z) = \frac{1}{3}\left[e^z + 2e^{-z/2}\cos\frac{\sqrt{3}z}{2}\right] = \frac{1}{3}(e^z + e^{\omega z} + e^{\omega^2 z}),$$

$$H_{3,1}(z) \equiv h_1(z) = \frac{1}{3}\left[e^z - 2e^{-z/2}\cos\left(\frac{\sqrt{3}z}{2} + \frac{\pi}{3}\right)\right] = \frac{1}{3}(e^z + \omega e^{\omega z} + \omega^2 e^{\omega^2 z}), \tag{B4}$$

$$H_{3,2}(z) \equiv h_2(z) = \frac{1}{3}\left[e^x - 2e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2} - \frac{\pi}{3}\right)\right] = \frac{1}{3}(e^z + \omega^2 e^{\omega z} + \omega e^{\omega^2 z}),$$

where we introduced the functions $h_k(z)$ for brevity and $\omega = e^{2\pi i/3}$, as in the rest of the paper. One immediately sees that $h_k(z)$ satisfy

$$\begin{aligned}
h_0(\omega z) &= h_0(z), & h_0(\omega^2 z) &= h_0(z),\\
h_1(\omega z) &= \omega^2 h_1(z), & h_1(\omega^2 z) &= \omega h_1(z)\\
h_2(\omega z) &= \omega h_2(z) & h_2(\omega^2 z) &= \omega^2 h_2(z),
\end{aligned} \tag{B5}$$

and one has $e^z = \sum_{l=0}^{2} h_l(z)$. Let us combine $h_k(z)$ into a circulant matrix

$$\mathbb{H}(z) = \begin{bmatrix} h_0(z) & h_2(z) & h_1(z) \\ h_1(z) & h_0(z) & h_2(z) \\ h_2(z) & h_1(z) & h_0(z) \end{bmatrix}. \tag{B6}$$

One can show that the matrix $\mathbb{H}(z)$ satisfies $\det \mathbb{H}(z) = 1$ and the following group property:

$$\mathbb{H}(z_1)\mathbb{H}(z_2) = \mathbb{H}(z_2)\mathbb{H}(z_1) = \mathbb{H}(z_1 + z_2). \tag{B7}$$

Thus, for the generalized hyperbolic functions of order 3 we have

$$\begin{aligned}
h_0(z_1 + z_2) &= h_0(z_1)h_0(z_2) + h_1(z_1)h_2(z_2) + h_2(z_1)h_1(z_2),\\
h_1(z_1 + z_2) &= h_0(z_1)h_1(z_2) + h_1(z_1)h_0(z_2) + h_2(z_1)h_2(z_2),\\
h_2(z_1 + z_2) &= h_0(z_1)h_2(z_2) + h_1(z_1)h_1(z_2) + h_2(z_1)h_0(z_2).
\end{aligned} \tag{B8}$$

The symmetry relations (B5) can be written in a compact form

$$\mathbb{H}(\omega^m z) = Z^{-m}\mathbb{H}(z)Z^m, \tag{B9}$$

where $0 \leqslant m \leqslant 2$ and the matrix $Z = \text{diag}\{1, \omega, \omega^2\}$ coincides with the one given in Eq. (4). One can also easily check the following interesting relations:

$$\sum_{l=0}^{2} h_l(x)h_l(y) = \frac{1}{3}\left[e^{x+y} + 2e^{-(x+y)/2}\cos\frac{\sqrt{3}(x-y)}{2}\right],$$

$$\sum_{l=0}^{2} h_{l+k \bmod 3}(x)h_l(y) = \frac{1}{3}\left[e^{x+y} - 2e^{-(x+y)/2}\cos\left(\frac{\sqrt{3}(x-y)}{2} + (-1)^{k-1}\frac{\pi}{3}\right)\right], \quad k = 1, 2. \tag{B10}$$

Finally, we note that Eqs. (B7) [with $n \times n$ circulant matrix] and (B9) [with $0 \leqslant m \leqslant n - 1$] also hold for the generalized hyperbolic functions of order n [55–57], and all other relations can be easily extended to the case of arbitrary n.

APPENDIX C: ADJOINT ACTIONS

In this Appendix we present a detailed derivation of the adjoint actions $e^{i\alpha\, \text{ad}\, \mathcal{J}_j^m} \mathcal{J}_j^p$ and $e^{i\alpha\, \text{ad}\, \mathcal{J}_{j+\ell}^m \mathcal{J}_{j+\ell+1}^n} \mathcal{J}_j^p \mathcal{J}_{j+1}^q$, which we then use in Appendix D to obtain the explicit forms for V_1, V_2, and V_k.

1. Adjoint action of $\mathcal{J}_j^{(m_1,m_2)}$ on $\mathcal{J}_j^{(p_1,p_2)}$

Using Eq. (9) for the commutator $[\mathcal{J}_j^m, \mathcal{J}_j^p] \equiv \mathrm{ad}_{\mathcal{J}_j^m} \mathcal{J}_j^p$ multiple times, we obtain:

$$\mathrm{ad}_{\mathcal{J}_j^m}^k \mathcal{J}_j^p = (-2i)^k \sin^k\left(\frac{2\pi}{3} \, \boldsymbol{m} \times \boldsymbol{p}\right) \mathcal{J}_j^{p+km}, \tag{C1}$$

where the components of the two-dimensional vector $\boldsymbol{p} + k\,\boldsymbol{m}$ are mod 3. Therefore, we have the following adjoint action:

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^m}} \mathcal{J}_j^p = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \mathrm{ad}_{\mathcal{J}_j^m}^n \mathcal{J}_j^p = \sum_{k=0}^{2}\sum_{n=0}^{\infty} \frac{(i\alpha)^{3n+k}}{(3n+k)!} \mathrm{ad}_{\mathcal{J}_j^m}^{3n+k} \mathcal{J}_j^p = \sum_{k=0}^{2} h_k(\alpha\,\xi_{m,p}) \mathcal{J}_j^{p+km}, \tag{C2}$$

where we denoted $\xi_{m,p} = 2\sin[2\pi\,(\boldsymbol{m} \times \boldsymbol{p})/3]$, and $h_k(z)$ are the generalized hyperbolic functions of order 3 and kth kind, given by Eq. (B4).

2. Adjoint action of $\mathcal{J}_{j+\ell}^{(m_1,m_2)} \mathcal{J}_{j+\ell+1}^{(n_1,n_2)}$ on $\mathcal{J}_j^{(p_1,p_2)} \mathcal{J}_{j+1}^{(q_1,q_2)}$

We proceed with calculating the adjoint actions containing more than two operators, i.e., the commutators of the form

$$\mathrm{ad}_{\mathcal{J}_{j+\ell}^m \mathcal{J}_{j+\ell+1}^n} \mathcal{J}_j^p \mathcal{J}_{j+1}^q = \left[\mathcal{J}_{j+\ell}^m \mathcal{J}_{j+\ell+1}^n, \mathcal{J}_j^p \mathcal{J}_{j+1}^q\right]. \tag{C3}$$

Clearly, the resulting expression differs from zero only if $\ell = 0, \pm 1$. Taking $\ell = 0$, Eq. (C3) yields

$$\mathrm{ad}_{\mathcal{J}_j^m \mathcal{J}_{j+1}^n} \mathcal{J}_j^p \mathcal{J}_{j+1}^q = -2i \sin\left[\frac{2\pi}{3}(\boldsymbol{m} \times \boldsymbol{p} + \boldsymbol{n} \times \boldsymbol{q})\right] \mathcal{J}_j^{p+m} \mathcal{J}_{j+1}^{q+n}, \tag{C4}$$

where we took into account that the operators $\mathcal{J}_j^m$ commute on different sites and used Eq. (8). Thus, for the adjoint action we obtain

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^m \mathcal{J}_{j+1}^n}} \mathcal{J}_j^p \mathcal{J}_{j+1}^q = \sum_{k=0}^{2} h_k\left(\alpha\,\zeta_{m,p}^{n,q}\right) \mathcal{J}_j^{p+km} \mathcal{J}_{j+1}^{q+kn}, \tag{C5}$$

where $\zeta_{m,p}^{n,q} = 2\sin[2\pi(\boldsymbol{m} \times \boldsymbol{p} + \boldsymbol{n} \times \boldsymbol{q})/3]$. Note that by taking $\boldsymbol{n} = \boldsymbol{q} = (0,0)$ or $\boldsymbol{m} = \boldsymbol{p} = (0,0)$, we simply reduce Eq. (C5) to Eq. (C2).

The result for $\ell = \pm 1$ can be easily found in a similar way. In this case one has

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{j+1}^m \mathcal{J}_{j+2}^n}} \mathcal{J}_j^p \mathcal{J}_{j+1}^q = \sum_{k=0}^{2} h_k(\alpha\xi_{m,q}) \mathcal{J}_j^p \mathcal{J}_{j+1}^{q+km} \mathcal{J}_{j+2}^{kn},$$

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{j-1}^m \mathcal{J}_j^n}} \mathcal{J}_j^p \mathcal{J}_{j+1}^q = \sum_{k=0}^{2} h_k(\alpha\xi_{n,p}) \mathcal{J}_{j-1}^{km} \mathcal{J}_j^{p+kn} \mathcal{J}_{j+1}^q, \tag{C6}$$

with $\xi_{r,s}$ given after Eq. (C2).

APPENDIX D: DERIVATION OF V_1 IN EQ. (33), V_2 in Eq. (35), and V_k in Eq. (36)

In this Appendix we present a detailed derivations of the operators V_1, V_2, and V_k. The most important relations obtained this Appendix are summarized in D 4.

1. Adjoint action of H_2 on $\mathcal{J}_j^{(p_1,p_2)}$ and expression for V_1 in Eq. (33)

In order to explicitly calculate V_1 in Eq. (29), let us consider the adjoint action of the transverse field H_2, given by Eq. (10), on $\mathcal{J}_j^p$. Using Eq. (C2), we obtain

$$e^{i\alpha \, \mathrm{ad}_{H_2}} \mathcal{J}_j^p = \prod_{n=1}^{2} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(0,n)}}} \mathcal{J}_j^p = \prod_{n=1}^{2} \sum_{k=0}^{\infty} \frac{(ifT)^k}{k!} \mathrm{ad}_{\mathcal{J}_j^{(0,n)}}^k \mathcal{J}_j^p = \sum_{k,q=0}^{2} h_k(\alpha\,\xi_{(0,1),p}) h_q(\alpha\,\xi_{(0,2),p}) \mathcal{J}_j^{(p_1,p_2+k+2q)}, \tag{D1}$$

where $\xi_{(0,n),p} = -2\sin(2np_1\pi/3)$ and $1 \leqslant j \leqslant 2N$. Keeping in mind that $p_2 + k + 2q$ should be taken mod 3, we can rewrite Eq. (D1) as

$$\prod_{n=1}^{2} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(0,n)}}} \mathcal{J}_j^p = \sum_{k=0}^{2} G_k^{(1,2)}(-\alpha, p_1) \mathcal{J}_j^{(p_1,p_2+k)}. \tag{D2}$$

In deriving Eq. (D2) we used the explicit expressions $\xi_{(0,m),\boldsymbol{p}} = -2\sin[2mp_1\pi/3]$ and introduced the function

$$G_k^{(m,n)}(\alpha, p) \equiv \sum_{l=0}^{2} h_{l+k \bmod 3}(2\alpha \sin[2mp\pi/3]) \, h_l(2\alpha \sin[2np\pi/3]). \tag{D3}$$

Taking into account that $p \in \{0, 1, 2\}$ and using the relations (B10), we obtain

$$G_k^{(1,2)}(\alpha, p) = \left\{ \delta_{p,0} + \frac{1}{3}(\delta_{p,1} + \delta_{p,2})(1 + 2\cos 3\alpha) \right\} \delta_{k,0} + \frac{1}{3}\left\{ \delta_{p,1}\left[1 + 2\cos\left(3\alpha - \frac{2\pi}{3}\right)\right] + \delta_{p,2}\left[1 + 2\cos\left(3\alpha + \frac{2\pi}{3}\right)\right] \right\}\delta_{k,1}$$

$$+ \frac{1}{3}\left\{ \delta_{p,1}\left[1 + 2\cos\left(3\alpha + \frac{2\pi}{3}\right)\right] + \delta_{p,2}\left[1 + 2\cos\left(3\alpha - \frac{2\pi}{3}\right)\right] \right\}\delta_{k,2}$$

$$= \delta_{p,0}\delta_{k,0} + \frac{1}{3}(1 - \delta_{p,0})\left[1 + 2\cos\left(3\alpha - \frac{2pk\pi}{3}\right)\right]. \tag{D4}$$

Then, we further simplify the adjoint action in Eq. (D2) and it becomes

$$\prod_{n=1}^{2} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(0,n)}}} \mathcal{J}_j^{(p_1,p_2)} = \delta_{p_1,0}\mathcal{J}_j^{(p_1,p_2)} + \frac{1}{3}(1 - \delta_{p_1,0})\sum_{k=0}^{2}\left[1 + 2\cos\left(3\alpha + \frac{2p_1 k\pi}{3}\right)\right]\mathcal{J}_j^{(p_1,p_2+k)}. \tag{D5}$$

Changing the summation index to $p_1 q$ and taking into account that for $p_1 \in \{1, 2\}$ one has $p_1^2 \bmod 3 = 1$, we write

$$e^{i\alpha \, \mathrm{ad}_{H_2}} \mathcal{J}_j^{(p_1,p_2)} = \delta_{p_1,0}\mathcal{J}_j^{(0,p_2)} + \frac{1}{3}(1 - \delta_{p_1,0})\sum_{k=0}^{2}\left[1 + 2\cos\left(3\alpha + \frac{2k\pi}{3}\right)\right]\mathcal{J}_j^{(p_1,p_2+kp_1)}. \tag{D6}$$

It is now clear that by choosing

$$\alpha = \alpha_m = \frac{2\pi}{9}(3l - m), \quad \text{with} \quad l \in \mathbb{Z}, \ m \in \{0, 1, 2\}, \tag{D7}$$

the expression (D6) simplifies, since in the sum over k only the term with $k = m$ differs from zero. We then have

$$e^{i\alpha_m \, \mathrm{ad}_{H_2}} \mathcal{J}_j^{(p_1,p_2)} = \delta_{p_1,0}\mathcal{J}_j^{(0,p_2)} + (1 - \delta_{p_1,0})\mathcal{J}_j^{(p_1,p_2+mp_1)}, \quad 1 \leqslant j \leqslant 2N. \tag{D8}$$

Thus, using Eq. (23) for V_0 and the definition of V_1 in Eq. (29), for $fT = \alpha_m$ we obtain

$$V_1 = e^{i\alpha_m \, \mathrm{ad}_{H_2}} V_0 = \mu\mathbb{1} + \nu\left[\mathcal{J}_N^{(1,m)}\mathcal{J}_{N+1}^{(2,2m)} + \mathcal{J}_N^{(2,2m)}\mathcal{J}_{N+1}^{(1,m)}\right], \tag{D9}$$

which is the expression for V_1 in Eq. (33). Clearly, V_1 produces a maximally entangled two-qutrit state on the sites N and $N + 1$.

2. Adjoint action of $\tilde{H}_1$ on $\mathcal{J}_j^{(p_1,p_2)}$ with $j = 1, N, N + 1, 2N$, and expression for V_2 in Eq. (35)

Let us now proceed with showing that the explicit form of V_2 is given by Eq. (35). To do so, we consider the adjoint action

$$e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_N^{\boldsymbol{p}} = \exp\left\{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N-1}^{(2,0)}\mathcal{J}_N^{(1,0)}}\right\} \exp\left\{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N-1}^{(1,0)}\mathcal{J}_N^{(2,0)}}\right\} \mathcal{J}_N^{\boldsymbol{p}},$$

$$e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N+1}^{\boldsymbol{p}} = \exp\left\{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N+1}^{(2,0)}\mathcal{J}_{N+2}^{(1,0)}}\right\} \exp\left\{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N+1}^{(1,0)}\mathcal{J}_{N+2}^{(2,0)}}\right\} \mathcal{J}_{N+1}^{\boldsymbol{p}}, \tag{D10}$$

where we took into account that $\tilde{H}_1$, given by Eqs. (10) and (21), does not contain the terms that act on the central link between the sites N and $N + 1$. Using Eq. (C6) with $\boldsymbol{q} = (0, 0)$ and Eq. (C5), from the first line of Eq. (D10) we have

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N-1}^{(2,0)}\mathcal{J}_N^{(1,0)}}} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N-1}^{(1,0)}\mathcal{J}_N^{(2,0)}}} \mathcal{J}_N^{\boldsymbol{p}} = e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{N-1}^{(2,0)}\mathcal{J}_N^{(1,0)}}} \sum_{k=0}^{2} h_k(\alpha\xi_{(2,0),\boldsymbol{p}}) \mathcal{J}_{N-1}^{(k,0)} \mathcal{J}_N^{(p_1+2k,p_2)}$$

$$= \sum_{k,q=0}^{2} h_k(\alpha\xi_{(2,0),\boldsymbol{p}}) h_q\left(\alpha\zeta_{(2,0),(k,0)}^{(1,0),(p_1+2k,p_2)}\right) \mathcal{J}_{N-1}^{(k+2q,0)} \mathcal{J}_N^{(p_1+2k+q,p_2)}$$

$$= \sum_{k=0}^{2} G_k^{(1,2)}(\alpha, p_2) \mathcal{J}_{N-1}^{(2k,0)} \mathcal{J}_N^{(p_1+k,p_2)}, \tag{D11}$$

where we used the fact that $k + 2q$ and $p_1 + 2k + q$ are $\bmod 3$, along with the expressions $\xi_{(2,0),\boldsymbol{p}} = 2\sin(4p_2\pi/3)$ and $\zeta_{(2,0),(k,0)}^{(1,0),(p_1+2k,p_2)} = 2\sin[2p_2\pi/3]$, and took into account Eq. (D3) for the function $G_k^{(m,n)}(\alpha, p)$. Then, using Eq. (D4) and repeating

the steps leading to Eq. (D6), we immediately obtain

$$e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_N^{\boldsymbol{p}} = \delta_{p_2,0} \mathcal{J}_N^{(p_1,0)} + \frac{1}{3}(1 - \delta_{p_2,0}) \sum_{k=0}^{2} \left[1 + 2\cos\left(3\alpha - \frac{2k\pi}{3} \right) \right] \mathcal{J}_{N-1}^{(2kp_2,0)} \mathcal{J}_N^{(p_1+kp_2,p_2)}. \tag{D12}$$

It is easy to see that one similarly has

$$e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N+1}^{\boldsymbol{p}} = \delta_{p_2,0} \mathcal{J}_{N+1}^{(p_1,0)} + \frac{1}{3}(1 - \delta_{p_2,0}) \sum_{k=0}^{2} \left[1 + 2\cos\left(3\alpha - \frac{2k\pi}{3} \right) \right] \mathcal{J}_{N+1}^{(p_1+kp_2,p_2)} \mathcal{J}_{N+2}^{(2kp_2,0)}. \tag{D13}$$

Just like with Eq. (D6), we can simplify Eqs. (D12) and (D13) by using $\alpha = \alpha_m$ from Eq. (D7). In this case only the term with $k = 2m$ does not vanish in Eqs. (D12), (D12), and one obtains

$$e^{i\alpha_m \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_N^{\boldsymbol{p}} = \delta_{p_2,0} \mathcal{J}_N^{(p_1,0)} + (1 - \delta_{p_2,0}) \mathcal{J}_{N-1}^{(mp_2,0)} \mathcal{J}_N^{(p_1+2mp_2,p_2)},$$

$$e^{i\alpha_m \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N+1}^{\boldsymbol{p}} = \delta_{p_2,0} \mathcal{J}_N^{(p_1,0)} + (1 - \delta_{p_2,0}) \mathcal{J}_{N+1}^{(p_1+2mp_2,p_2)} \mathcal{J}_{N+2}^{(mp_2,0)}. \tag{D14}$$

Then, in order to find the explicit form of V_2 from Eq. (27), we need to calculate

$$V_2 = \tilde{U} V_1 \tilde{U}^\dagger = e^{ifT \, \mathrm{ad}_{H_2}} e^{iJT \, \mathrm{ad}_{\tilde{H}_1}} V_1, \tag{D15}$$

where V_1 is given by Eq. (D9). Taking $JT = \alpha_n$ and $fT = \alpha_s$, with α_m given by Eq. (D7), we obtain

$$e^{i\alpha_s \, \mathrm{ad}_{H_2}} e^{i\alpha_n \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_N^{(1,m)} \mathcal{J}_{N+1}^{(2,2m)} = e^{i\alpha_s \, \mathrm{ad}_{H_2}} \mathcal{J}_{N-1}^{(nm,0)} \mathcal{J}_N^{(1+2nm,m)} \mathcal{J}_{N+1}^{(2+4nm,2m)} \mathcal{J}_{N+2}^{(2nm,0)}$$

$$= \mathcal{J}_{N-1}^{(nm,snm)} \mathcal{J}_N^{(1+2nm,m+s(1+2nm))} \mathcal{J}_{N+1}^{(2+nm,2m+s(2+nm))} \mathcal{J}_{N+2}^{(2nm,2snm)}. \tag{D16}$$

The resulting expression for V_2 is especially simple for $m = n = s$, which corresponds to $fT = JT = \alpha_m = 2\pi(3l - m)/9$. In this case V_2 is given by

$$V_2 = \mu \mathbb{1} + \nu \left[\mathcal{J}_{N-1}^{(1,m)} \mathcal{J}_N^{(0,m)} \mathcal{J}_{N+1}^{(0,2m)} \mathcal{J}_{N+2}^{(2,2m)} + \mathrm{H.c.} \right], \tag{D17}$$

where we took into account that for $m \in \{1, 2\}$ one has $m^2 \bmod 3 = 1$ and $m^3 \bmod 3 = m$. Setting $l = 1$ and $m = 2$ we obtain $fT = JT = 2\pi/9$ and V_2 reduces to Eq. (35). Using Eq. (7), one can rewrite Eq. (D17) as

$$V_2 = \mu \mathbb{1} + \nu \left(X_{N-1} Z_{N-1}^m Z_N^m Z_{N+1}^{2m} X_{N+2}^2 Z_{N+2}^{2m} + \mathrm{H.c.} \right), \tag{D18}$$

and we see that V_2 entangles the spins on sites $N - 1$ and $N + 2$, while only changing the *phase* on the sites N and $N + 1$. On the other hand, for $n = 2m$ and $s = m$ [which corresponds to $fT = \alpha_m$ and $JT = \alpha_{2m}$, with α_n given in Eq. (D7)], we have

$$V_2 = \mu \mathbb{1} + \nu \left[\mathcal{J}_{N-1}^{(2,2m)} \mathcal{J}_N^{(2,0)} \mathcal{J}_{N+1}^{(1,0)} \mathcal{J}_{N+2}^{(1,m)} + \mathrm{H.c.} \right] = \mu \mathbb{1} + \nu \left(X_{N-1}^2 Z_{N-1}^{2m} X_N^2 X_{N+1} X_{N+2} Z_{N+2}^m + \mathrm{H.c.} \right). \tag{D19}$$

The effect of V_2 in this case is more complicated as compared to Eq. (D17), since it now shifts the *states* on sites N and $N + 1$.

For the adjoint actions $e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_j^{\boldsymbol{p}}$ with $j = 1$ and $2N$, the result follows immediately from Eqs. (D12) and (D13). Replacing $N + 1$ with 1 in Eq. (D13) gives the expression for $e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_1^{\boldsymbol{p}}$, and changing N to $2N$ in Eq. (D12) we obtain the result for $e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{2N}^{\boldsymbol{p}}$.

3. Adjoint action of $\tilde{H}_1$ on $\mathcal{J}_j^{(p_1,p_2)}$ with $j \neq 1, N, N + 1, 2N$, and expression for V_k in Eq. (36)

Finally, we derive the explicit expression for V_k in Eq. (36). Let us first calculate the following adjoint action, with $j \neq 1, N, N + 1, 2N$:

$$e^{i\alpha \, \mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_j^{\boldsymbol{p}} = \exp\left\{ i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(2,0)} \mathcal{J}_{j+1}^{(1,0)} + \mathcal{J}_j^{(1,0)} \mathcal{J}_{j+1}^{(2,0)}} \right\} \exp\left\{ i\alpha \, \mathrm{ad}_{\mathcal{J}_{j-1}^{(2,0)} \mathcal{J}_j^{(1,0)} + \mathcal{J}_{j-1}^{(1,0)} \mathcal{J}_j^{(2,0)}} \right\} \mathcal{J}_j^{\boldsymbol{p}}, \tag{D20}$$

where we used Eqs. (10) and (21) for $\tilde{H}_1$ and took into account that the operators $\mathcal{J}_j^{(k,0)}$ with $k = 1, 2$ commute. For the first adjoint action in Eq. (D20) we can simply use Eq. (D11) with N replaced by j, which yields

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{j-1}^{(2,0)} \mathcal{J}_j^{(1,0)}}} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_{j-1}^{(1,0)} \mathcal{J}_j^{(2,0)}}} \mathcal{J}_j^{\boldsymbol{p}} = \sum_{k=0}^{2} G_k^{(1,2)}(\alpha, p_2) \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_j^{(p_1+k,p_2)}. \tag{D21}$$

Then, using Eqs. (C5) and (C6), for the remaining adjoint action in Eq. (D20) we obtain

$$e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(2,0)} \mathcal{J}_{j+1}^{(1,0)}}} e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(1,0)} \mathcal{J}_{j+1}^{(2,0)}}} \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_j^{(p_1+k,p_2)}$$

$$= e^{i\alpha \, \mathrm{ad}_{\mathcal{J}_j^{(2,0)} \mathcal{J}_{j+1}^{(1,0)}}} \sum_{r=0}^{2} h_r(\alpha \xi_{(1,0),(p_1+k,p_2)}) \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_j^{(p_1+k+r,p_2)} \mathcal{J}_{j+1}^{(2r,0)}$$

$$= \sum_{r,s=0}^{2} h_r(\alpha \xi_{(1,0),(p_1+k,p_2)}) h_s(\alpha \zeta_{(2,0),(p_1+k+r,p_2)}^{(1,0),(2r,0)}) \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_{j}^{(p_1+k+r+2s,p_2)} \mathcal{J}_{j+1}^{(2r+s,0)}$$

$$= \sum_{r=0}^{2} G_r^{(1,2)}(\alpha, p_2) \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_{j}^{(p_1+k+r,p_2)} \mathcal{J}_{j+1}^{(2r,0)}, \tag{D22}$$

where we took into account that $\xi_{(1,0),(p_1+k,p_2)} = 2\sin[2p_2\pi/3]$ and $\zeta_{(2,0),(p_1+k+r,p_2)}^{(1,0),(2r,0)} = 2\sin[4p_2\pi/3]$. Thus, combining Eqs. (D20), (D21), and (D22), we have

$$e^{i\alpha\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_j^{\boldsymbol{p}} = \sum_{k,r=0}^{2} G_k^{(1,2)}(\alpha, p_2)\, G_r^{(1,2)}(\alpha, p_2)\, \mathcal{J}_{j-1}^{(2k,0)} \mathcal{J}_{j}^{(p_1+k+r,p_2)} \mathcal{J}_{j+1}^{(2r,0)} = \delta_{p_2,0} \mathcal{J}_j^{(p_1,p_2)}$$

$$+ \frac{1}{9}(1-\delta_{p_2,0}) \sum_{k,r=0}^{2} \left[1 + 2\cos\left(3\alpha - \frac{2k\pi}{3}\right)\right]\left[1 + 2\cos\left(3\alpha - \frac{2r\pi}{3}\right)\right] \mathcal{J}_{j-1}^{(2kp_2,0)} \mathcal{J}_{j}^{(p_1+(r+k)p_2,p_2)} \mathcal{J}_{j+1}^{(2rp_2,0)}, \tag{D23}$$

where we used Eq. (D4) and the fact that $\delta_{p,0}(1 - \delta_{p,0}) \equiv 0$. Clearly, Eq. (D23) greatly simplifies if one takes α_s from Eq. (D7). In this case the sum over k and r in Eq. (D23) contains only one nonzero term corresponding to $k = r = 2s$, and we obtain

$$e^{i\alpha_s\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_j^{(p_1,p_2)} = \delta_{p_2,0} \mathcal{J}_j^{(p_1,0)} + (1-\delta_{p_2,0}) \mathcal{J}_{j-1}^{(s\,p_2,0)} \mathcal{J}_{j}^{(p_1+s\,p_2,p_2)} \mathcal{J}_{j+1}^{(s\,p_2,0)}, \quad j \neq 1, N, N+1, 2N. \tag{D24}$$

Let us now calculate V_3 using the relation

$$V_3 = e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} V_2 = e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}}\ e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}}\ e^{ifT\,\mathrm{ad}_{H_2}} V_0. \tag{D25}$$

In what follows we consider the cases $fT = JT = \alpha_m$ and $fT = \alpha_m, JT = \alpha_{2m}$ separately.

a. $fT = JT = \alpha_m$

We first assume that $fT = JT = \alpha_m$, so that V_2 is given by Eq. (D17). Then, we have

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N-1}^{(1,m)} \stackrel{(D24)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N-2}^{(1,0)} \mathcal{J}_{N-1}^{(2,m)} \mathcal{J}_{N}^{(1,0)} \stackrel{(D8)}{=} \mathcal{J}_{N-2}^{(1,m)} \mathcal{J}_{N-1}^{(2,0)} \mathcal{J}_{N}^{(1,m)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N}^{(0,m)} \stackrel{(D14)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N-1}^{(1,0)} \mathcal{J}_{N}^{(2,m)} \stackrel{(D8)}{=} \mathcal{J}_{N-1}^{(1,m)} \mathcal{J}_{N}^{(2,0)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N+1}^{(0,2m)} \stackrel{(D14)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N+1}^{(1,2m)} \mathcal{J}_{N+2}^{(2,0)} \stackrel{(D8)}{=} \mathcal{J}_{N+1}^{(1,0)} \mathcal{J}_{N+2}^{(2,2m)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N+2}^{(0,2m)} \stackrel{(D24)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N+1}^{(2,0)} \mathcal{J}_{N+2}^{(1,2m)} \mathcal{J}_{N+3}^{(2,0)} \stackrel{(D8)}{=} \mathcal{J}_{N+1}^{(2,2m)} \mathcal{J}_{N+2}^{(1,0)} \mathcal{J}_{N+3}^{(2,2m)}. \tag{D26}$$

Using Eqs. (D17) and (D25) for V_2 and V_3, correspondingly, and multiplying the results in Eq. (D26), we obtain

$$V_3 = \mu\mathbb{1} + \nu\big(\mathcal{J}_{N-2}^{(1,m)} \mathcal{J}_{N-1}^{(0,m)} \mathcal{J}_{N}^{(0,m)} \mathcal{J}_{N+1}^{(0,2m)} \mathcal{J}_{N+2}^{(0,2m)} \mathcal{J}_{N+3}^{(2,2m)} + \mathrm{H.c.}\big), \tag{D27}$$

where all factors of ω, which appear [see Eq. (8)] from the products of two operators on the same site, cancel each other. We see that for $fT = JT = \alpha_m$ the action of V_3 is similar to that of V_2 from Eq. (D17). Namely, V_3 entangles the sites $N - 2$ and $N + 3$, whereas on all sites in between it only rotates the phase.

Then, using Eqs. (D35) and (D36), one can easily show by induction that for $fT = JT = \alpha_m$ and $1 \leqslant k < N$ the operator V_k has the following form:

$$V_k = \mu\mathbb{1} + \nu\left[\mathcal{J}_{N-k+1}^{(1,m)} \prod_{j=k-2}^{0} \mathcal{J}_{N-j}^{(0,m)} \prod_{j=1}^{k-1} \mathcal{J}_{N+j}^{(0,2m)} \mathcal{J}_{N+k}^{(2,2m)} + \mathrm{H.c.}\right]. \tag{D28}$$

Quite remarkably, it only entangles the sites $N - k + 1$ and $N + k$, while on the rest of the sites $N - k + 1 < j < N + k$ its effect is a simple phase rotation. Due to this fact, the operator $V_1 V_2 \ldots V_k$ in Eq. (48) produces an entangled state of a very simple product form, as discussed at the end of Sec. III.

b. $fT = \alpha_m, JT = \alpha_{2m}$

Let us now investigate what happens if one chooses different values of fT and JT, and consider $fT = \alpha_m, JT = \alpha_{2m}$. In this case V_2 is given by Eq. (D19), and instead of Eq. (D26) we need the relations

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N-1}^{(2,2m)} \stackrel{(D24)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N-2}^{(1,0)} \mathcal{J}_{N-1}^{(0,2m)} \mathcal{J}_{N}^{(1,0)} \stackrel{(D8)}{=} \mathcal{J}_{N-2}^{(1,m)} \mathcal{J}_{N-1}^{(0,2m)} \mathcal{J}_{N}^{(1,m)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}} e^{iJT\,\mathrm{ad}_{\tilde{H}_1}} \mathcal{J}_{N}^{(2,0)} \stackrel{(D14)}{=} e^{ifT\,\mathrm{ad}_{H_2}} \mathcal{J}_{N}^{(2,0)} \stackrel{(D8)}{=} \mathcal{J}_{N}^{(2,2m)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}}e^{iJT\,\mathrm{ad}_{\tilde{H}_1}}\mathcal{J}_{N+1}^{(1,0)} \overset{(D14)}{=} e^{ifT\,\mathrm{ad}_{H_2}}\mathcal{J}_{N+1}^{(1,0)} \overset{(D8)}{=} \mathcal{J}_{N+1}^{(1,m)},$$

$$e^{ifT\,\mathrm{ad}_{H_2}}e^{iJT\,\mathrm{ad}_{\tilde{H}_1}}\mathcal{J}_{N+2}^{(1,m)} \overset{(D24)}{=} e^{ifT\,\mathrm{ad}_{H_2}}\mathcal{J}_{N+1}^{(2,0)}\mathcal{J}_{N+2}^{(0,m)}\mathcal{J}_{N+3}^{(2,0)} \overset{(D8)}{=} \mathcal{J}_{N+1}^{(2,2m)}\mathcal{J}_{N+2}^{(0,m)}\mathcal{J}_{N+3}^{(2,2m)}. \tag{D29}$$

Using Eq. (D19) for V_2 and multiplying the terms in Eq. (D29), from Eq. (D25) for V_3 we obtain

$$V_3 = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-2}^{(1,m)}\mathcal{J}_{N-1}^{(0,2m)}\mathbb{1}_N\mathbb{1}_{N+1}\mathcal{J}_{N+2}^{(0,m)}\mathcal{J}_{N+3}^{(2,2m)} + \mathrm{H.c.}\big]. \tag{D30}$$

At the first sight, the resulting expression for V_3 in Eq. (D30) looks even simpler as compared to Eq. (D27). However, the situation turns out to be more complicated, because for $fT = \alpha_m$ and $JT = \alpha_{2m}$, the expression for V_k strongly depends on k, in contrast to the case of $fT = JT = \alpha_m$ [see Eq. (D28)]. Indeed, using Eqs. (D35) and (D36) it can be easily shown that the next few V_k are given by

$$V_4 = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-3}^{(2,2m)}\mathcal{J}_{N-2}^{(1,2m)}\mathcal{J}_{N-1}^{(0,2m)}\mathcal{J}_{N}^{(1,m)}\mathcal{J}_{N+1}^{(2,2m)}\mathcal{J}_{N+2}^{(0,m)}\mathcal{J}_{N+3}^{(2,m)}\mathcal{J}_{N+4}^{(1,m)} + \mathrm{H.c.}\big],$$

$$V_5 = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-4}^{(1,m)}\mathcal{J}_{N-3}^{(1,0)}\mathcal{J}_{N-2}^{(1,0)}\mathcal{J}_{N-1}^{(1,0)}\mathcal{J}_{N}^{(0,m)}\mathcal{J}_{N+1}^{(0,2m)}\mathcal{J}_{N+2}^{(2,0)}\mathcal{J}_{N+3}^{(2,0)}\mathcal{J}_{N+4}^{(2,0)}\mathcal{J}_{N+5}^{(2,2m)} + \mathrm{H.c.}\big], \tag{D31}$$

$$V_6 = \mu\mathbb{1} + \nu\big[\mathcal{J}_{N-5}^{(2,2m)}\mathcal{J}_{N-4}^{(0,m)}\mathbb{1}_{N-3}\mathcal{J}_{N-2}^{(1,m)}\mathbb{1}_{N-1}\mathcal{J}_{N}^{(1,2m)}\mathcal{J}_{N+1}^{(2,m)}\mathbb{1}_{N+2}\mathcal{J}_{N+3}^{(2,2m)}\mathbb{1}_{N+4}\mathcal{J}_{N+5}^{(0,2m)}\mathcal{J}_{N+6}^{(1,m)} + \mathrm{H.c.}\big],$$

where we assumed that $N > 6$ to avoid dealing with the boundaries. We see that the form of V_k drastically changes with increasing k. Thus, Eq. (D31) suggests that in the case $fT = \alpha_m$ and $JT = \alpha_{2m}$ the general expression for V_k cannot be written in a closed form.

Nevertheless, writing $V_k = \mu\mathbb{1} + \nu(\mathcal{V}_k + \mathrm{H.c.})$ as in Eq. (38), one can obtain a recursive relation between the indices of $\mathcal{J}_j^{(p,q)}$ in $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$. Indeed, for every $1 \leqslant k < N$ we can write $\mathcal{V}_k$ in the form

$$\mathcal{V}_k = \mathcal{J}_{N-k+1}^{(p_{N-k+1},q_{N-k+1})}\cdots\mathcal{J}_{N}^{(p_N,q_N)}\mathcal{J}_{N+1}^{(2p_N,2q_N)}\cdots\mathcal{J}_{N+k}^{(2p_{N-k+1},2q_{N-k+1})}. \tag{D32}$$

Then, for $\mathcal{V}_{k+1} = \tilde{U}\mathcal{V}_k\tilde{U} = e^{ifT\,\mathrm{ad}_{H_2}}e^{iJT\,\mathrm{ad}_{\tilde{H}_1}}\mathcal{V}_k$ we can write

$$e^{i\alpha_m\,\mathrm{ad}_{H_2}}e^{i\alpha_{2m}\,\mathrm{ad}_{\tilde{H}_1}}\mathcal{J}_{N-k+1}^{(p_{N-k+1},q_{N-k+1})}\cdots\mathcal{J}_{N}^{(p_N,q_N)}\mathcal{J}_{N+1}^{(2p_N,2q_N)}\cdots\mathcal{J}_{N+k}^{(2p_{N-k+1},2q_{N-k+1})}$$

$$= \mathcal{J}_{N-k}^{(p'_{N-k},q'_{N-k})}\cdots\mathcal{J}_{N}^{(p'_N,q'_N)}\mathcal{J}_{N+1}^{(2p'_N,2q'_N)}\cdots\mathcal{J}_{N+k+1}^{(2p'_{N-k},2q'_{N-k})}. \tag{D33}$$

The relation (D33) can be viewed simply as a linear transformation between the indices of $\mathcal{J}_j^{(p_j,q_j)}$, which can be shown to be

$$(p'_{N-k}, q'_{N-k}) = (2m\,q_{N-k+1}, 2\,q_{N-k+1}),$$

$$(q'_{N-k+1}, q'_{N-k+1}) = (2m\,q_{N-k+2} + p_{N-k+1} + 2m\,q_{N-k+1}, 2\,q_{N-k+2} + m\,p_{N-k+1}),$$

$$\vdots$$

$$(p'_{N-j}, q'_{N-j}) = (2m\,q_{N-j+1} + p_{N-j} + 2m\,q_{N-j} + 2m\,q_{N-j-1}, 2\,q_{N-j+1} + m\,p_{N-j} + 2\,q_{N-j-1}),$$

$$\vdots$$

$$(p'_{N-1}, q'_{N-1}) = (2m\,q_N + p_{N-1} + 2m\,q_{N-1} + 2m\,q_{N-2}, 2\,q_N + m\,p_{N-1} + 2\,q_{N-2}),$$

$$(p'_N, q'_N) = (p_N + m\,q_N + 2m\,q_{N-1}, m\,p_N + 2\,q_N + 2\,q_{N-1}). \tag{D34}$$

Using Eq. (D34) we generate Fig. 2, which shows that the distributions of X_j^p and Z_j^q in $\mathcal{V}_k$ exhibit a fractal structure similar to the Sierpiński carpet.

4. Summary: Adjoint action $e^{i\alpha_s\,\mathrm{ad}_{H_2}}e^{i\alpha_m\,\mathrm{ad}_{\tilde{H}_1}}$ on $\mathcal{J}_j^{(p_1,p_2)}$

For convenience, here we summarize the most important relations derived in this Appendix. For the values of $fT = \alpha_s$ and $JT = \alpha_m$, where α_r is given in Eq. (D7), the adjoint action $e^{i\alpha_s\,\mathrm{ad}_{H_2}}e^{i\alpha_m\,\mathrm{ad}_{\tilde{H}_1}}$ provides the following mapping:

$$e^{i\alpha_m\,\mathrm{ad}_{H_2}}e^{i\alpha_s\,\mathrm{ad}_{\tilde{H}_1}}: \quad \mathcal{J}_j^{(p_1,p_2)} \to \mathcal{J}_{j-1}^{(sp_2,msp_2)}\mathcal{J}_j^{(p_1+sp_2,(1+ms)p_2+mp_1)}\mathcal{J}_{j+1}^{(sp_2,msp_2)}, \quad j \neq 1, N, N+1, 2N, \tag{D35}$$

and similarly for the boundary and central terms:

$$\mathcal{J}_j^{(p_1,p_2)} \to \mathcal{J}_j^{(p_1+2sp_2,(1+2ms)p_2+mp_1)}\mathcal{J}_{j+1}^{(sp_2,msp_2)}, \quad j = 1, N+1,$$

$$\mathcal{J}_j^{(p_1,p_2)} \to \mathcal{J}_{j-1}^{(sp_2,msp_2)}\mathcal{J}_j^{(p_1+2sp_2,(1+2ms)p_2+mp_1)}, \quad j = N, 2N. \tag{D36}$$

Eqs. (D35) and (D36) are easily obtained by combining Eqs. (D8), (D14), and (D24).

Choosing $s = m \in \{1, 2\}$ and taking into account that $m^2 \bmod 3 = 1$ and $m^3 \bmod 3 = m$, from Eq. (D35) we obtain

$$e^{i\alpha_m\,\mathrm{ad}_{H_2}}e^{i\alpha_m\,\mathrm{ad}_{\tilde{H}_1}}: \quad \mathcal{J}_j^{(p_1,p_2)} \to \mathcal{J}_{j-1}^{(mp_2,p_2)}\mathcal{J}_j^{(p_1+mp_2,2p_2+mp_1)}\mathcal{J}_{j+1}^{(mp_2,p_2)}, \quad j \neq 1, N, N+1, 2N, \tag{D37}$$

whereas Eq. (D36) reduces to

$$
\begin{aligned}
\mathcal{J}_j^{(p_1,p_2)} &\to \mathcal{J}_j^{(p_1+2mp_2,mp_1)} \mathcal{J}_{j+1}^{(mp_2,p_2)}, \quad j = 1, N+1, \\
\mathcal{J}_j^{(p_1,p_2)} &\to \mathcal{J}_{j-1}^{(mp_2,p_2)} \mathcal{J}_j^{(p_1+2mp_2,mp_1)}, \quad j = N, 2N.
\end{aligned}
\tag{D38}
$$

APPENDIX E: UNEQUAL SPLITTING OF THE CHAIN

In this Appendix we discuss a more general entanglement generating Floquet protocol based on switching off links other than the central one. Choosing the central link in Eq. (21) is not crucial and one can easily switch off any other Mth link [between sites M and $M+1$], with $1 \leqslant M \leqslant 2N-1$. Then, assuming $M \geqslant 2$ (we are not interesting the case of $M=1$ since the protocol consists of a single period) and taking *the same* value for $fT = JT = \alpha_m$ as in Eqs. (32), one simply has to replace $N \to M$ in Eq. (36) for V_k, which now becomes

$$
V_k = \mu \mathbb{1} + \nu \big[\mathcal{J}_{M-k+1}^{(1,m)} \mathcal{J}_{M-k+2}^{(0,m)} \cdots \mathcal{J}_M^{(0,m)} \mathcal{J}_{M+1}^{(0,2m)} \cdots \mathcal{J}_{M+k-1}^{(0,2m)} \mathcal{J}_{M+k}^{(2,2m)} + \text{H.c.} \big], \quad 2 \leqslant k \leqslant K.
\tag{E1}
$$

The upper bound on k is given by

$$
K = \min(M, 2N - M)
\tag{E2}
$$

and it depends on whether the eliminated Mth link is closer to the left or right boundary of the chain. Clearly, V_k in Eq. (E1) only changes the internal state of qutrits on the sites $M-k+1$ and $M+k$ whereas on the rest of the chain V_k either produces an extra phase factor or acts trivially. As a consequence, switching off the Mth link limits the number of generated entangled qutrit pairs to K, which is maximal for $M = N$. Indeed, the entanglement generating protocol also requires some minor modifications. After k periods of the state preparation part of the protocol *followed* by k periods of the entanglement generating part of the protocol, the state of the chain is $|\psi(kT)\rangle = V_1 \ldots V_k \otimes_{j=1}^{2N} |0\rangle_j$, as given by Eq. (48). Then, using Eq. (E1) it is straightforward to rewrite $|\psi(kT)\rangle$ as

$$
|\psi(kT)\rangle = \bigotimes_{j=1}^{M-k} |0\rangle_j \bigotimes_{j=1}^{k} |\Phi(\mu,\nu)\rangle_j \bigotimes_{j=M+k+1}^{2N} |0\rangle_j,
\tag{E3}
$$

where $|\Phi(\mu,\nu)\rangle_j$ is given by Eq. (50) with the replacement $N \to M$. Finally, taking into account that $k \leqslant K$, we immediately obtain

$$
\begin{aligned}
|\psi(KT)\rangle &= \bigotimes_{j=1}^{M} |\Phi(\mu,\nu)\rangle_j \bigotimes_{j=2M+1}^{2N} |0\rangle_j, \quad 1 \leqslant M < N, \\
|\psi(KT)\rangle &= \bigotimes_{j=1}^{2(M-N)} |0\rangle_j \bigotimes_{j=1}^{2N-M} |\Phi(\mu,\nu)\rangle_j, \quad N < M \leqslant 2N-1.
\end{aligned}
\tag{E4}
$$

Thus, by switching off the Mth link one obtains K entangled qutrit pairs, whereas the remaining $2|N-M|$ qutrits are not affected and stay in their initial states. For $M = N$ one has $K = N$ and Eq. (E4) agrees with Eq. (51).

APPENDIX F: ADJOINT ACTIONS IN THE TEMPERLEY-LIEB ALGEBRA

In this Appendix we present some useful algebraic relations satisfied by the elements of the Temperley-Lieb algebra $\mathrm{TL}_{2M}(\beta)$. From Eqs. (54) and (60) one obviously has

$$
\mathrm{ad}_{H_1} H_2 \equiv [H_1, H_2] = \sum_{j=1}^{2M-2} (-1)^j [u_j, u_{j+1}].
\tag{F1}
$$

A little less obvious is the observation that there is a closed form expression for any number of nested commutators:

$$
\mathrm{ad}_{H_m}^k H_n = \big[\underbrace{H_m, [H_m, \ldots [H_m}_{k}, H_n]] \big] = \begin{cases} \mathcal{F}_k^{(m)}(\beta), & k = 2l-1, \\ \mathcal{G}_k^{(m)}(\beta), & k = 2l, \end{cases}
\tag{F2}
$$

where $m, n \in \{1, 2\}$, $m \neq n$, and l is a positive integer. Explicitly, the functions $\mathcal{F}_k^{(m)}$ and $\mathcal{G}_k^{(m)}$ read as

$$
\begin{aligned}
\mathcal{F}_k^{(1)}(\beta) &= \beta^{k-1}[H_1, H_2] + (2^k - 2)\beta^{k-2}\mathcal{K}_0, \quad \mathcal{F}_k^{(2)}(\beta) = -\beta^{k-1}[H_1, H_2] + (2^k - 2)\beta^{k-2}\mathcal{K}_1, \\
\mathcal{G}_k^{(1)}(\beta) &= -4\beta^{k-2}H_1 + \beta^{k-2}(\beta\mathcal{A} - 2\mathcal{R}_0) + (2^k - 2)\beta^{k-2}\mathcal{P}_0 - (2^{k+1} - 8)\beta^{k-3}\mathcal{S}_0, \\
\mathcal{G}_k^{(2)}(\beta) &= -4\beta^{k-2}\big[H_2 - \tfrac{1}{2}(u_1 + u_{2M-1})\big] + \beta^{k-2}(\beta\mathcal{A} - 2\mathcal{R}_1) + (2^k - 2)\beta^{k-2}\mathcal{P}_1 - (2^{k+1} - 8)\beta^{k-3}\mathcal{S}_1,
\end{aligned}
\tag{F3}
$$

where $\mathcal{A}$ is given by Eq. (70) and we introduced the following k-independent operators ($n = 0, 1$):

$$\mathcal{K}_n = \sum_{j=1}^{M-2+n} [u_{2j-n}u_{2j+2-n}, u_{2j+1-n}], \quad \mathcal{R}_n = \sum_{j=1}^{M-2+n} (u_{2j-n}u_{2j+1-n}u_{2j+2-n} + u_{2j+2-n}u_{2j+1-n}u_{2j-n}),$$

$$\mathcal{P}_n = \sum_{j=1}^{M-2+n} \{u_{2j-n}u_{2j+2-n}, u_{2j+1-n}\}, \quad \mathcal{S}_n = \sum_{j=1}^{M-2+n} u_{2j-n}u_{2j+2-n}. \tag{F4}$$

The proof is by induction and is left as an exercise. One can also check that

$$\mathrm{ad}_{H_m}^k \mathcal{A} = \begin{cases} \beta\,\mathcal{F}_k^{(m)}(\beta) + 2^k\beta^{k-1}\mathcal{K}_{m-1}, & k = 2l-1, \\ \beta\,\mathcal{G}_k^{(m)}(\beta) + 2^k\beta^{k-2}(\beta\mathcal{P}_{m-1} - 2\mathcal{S}_{m-1}), & k = 2l. \end{cases} \tag{F5}$$

Equations (F2)–(F5) allow us to calculate various adjoint actions in a closed form. Let us consider

$$\tilde{H}_n(s) \equiv e^{sH_m}H_n\,e^{-sH_m} = e^{s\,\mathrm{ad}_{H_m}}H_n = \sum_{k=0}^{+\infty} \frac{s^k}{k!}\mathrm{ad}_{H_m}^k H_n = H_n + \sum_{l=1}^{+\infty} \frac{s^{2l-1}}{(2l-1)!}\mathcal{F}_{2l-1}^{(m)}(\beta) + \sum_{l=1}^{+\infty} \frac{s^{2l}}{(2l)!}\mathcal{G}_{2l}^{(m)}(\beta). \tag{F6}$$

Then, using Eq. (F3), we immediately obtain

$$\tilde{H}_2(s) = e^{sH_1}H_2\,e^{-sH_1} = H_2 - \frac{8}{\beta^2}\sinh^2\frac{s\beta}{2}H_1 + \frac{1}{\beta}\sinh s\beta\,[H_1, H_2] + \frac{4}{\beta^2}\sinh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{K}_0$$

$$+ \frac{2}{\beta^2}\sinh^2\frac{s\beta}{2}(\beta\mathcal{A} - 2\mathcal{R}_0) + \frac{4}{\beta^2}\cosh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{P}_0 - \frac{16}{\beta^3}\sinh^4\frac{s\beta}{2}\mathcal{S}_0. \tag{F7}$$

Similarly, for $\tilde{H}_1(s)$ one has the following expression:

$$\tilde{H}_1(s) = e^{sH_2}H_1\,e^{-sH_2} = H_1 - \frac{8}{\beta^2}\sinh^2\frac{s\beta}{2}\left[H_2 - \frac{1}{2}(u_1 + u_{2M-1})\right] - \frac{1}{\beta}\sinh s\beta\,[H_1, H_2]$$

$$+ \frac{4}{\beta^2}\sinh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{K}_1 + \frac{2}{\beta^2}\sinh^2\frac{s\beta}{2}(\beta\mathcal{A} - 2\mathcal{R}_1) + \frac{4}{\beta^2}\cosh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{P}_1 - \frac{16}{\beta^3}\sinh^4\frac{s\beta}{2}\mathcal{S}_1. \tag{F8}$$

In the same way one can calculate

$$e^{sH_m}[H_m, H_n]e^{-sH_m} = \sum_{k=0}^{+\infty} \frac{s^k}{k!}\mathrm{ad}_{H_m}^{k+1}H_n = [H_m, H_n] + \sum_{l=1}^{+\infty} \frac{s^{2l-1}}{(2l-1)!}\mathcal{G}_{2l}^{(m)}(\beta) + \sum_{l=1}^{+\infty} \frac{s^{2l}}{(2l)!}\mathcal{F}_{2l+1}^{(m)}(\beta), \tag{F9}$$

which yields

$$e^{sH_1}[H_1, H_2]e^{-sH_1} = \cosh s\beta\,[H_1, H_2] - \frac{4}{\beta}\sinh s\beta H_1 + \frac{4}{\beta}\sinh\frac{3s\beta}{2}\sinh\frac{s\beta}{2}\mathcal{K}_0$$

$$+ \frac{1}{\beta}\sinh s\beta(\beta\mathcal{A} - 2\mathcal{R}_0) + \frac{4}{\beta}\cosh\frac{3s\beta}{2}\sinh\frac{s\beta}{2}\mathcal{P}_0 - \frac{16}{\beta^2}\sinh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{S}_0,$$

$$e^{sH_2}[H_2, H_1]e^{-sH_2} = \cosh s\beta\,[H_2, H_1] - \frac{4}{\beta}\sinh s\beta\left[H_2 - \frac{1}{2}(u_1 + u_{2M-1})\right] + \frac{4}{\beta}\sinh\frac{3s\beta}{2}\sinh\frac{s\beta}{2}\mathcal{K}_1$$

$$+ \frac{1}{\beta}\sinh s\beta(\beta\mathcal{A} - 2\mathcal{R}_1) + \frac{4}{\beta}\cosh\frac{3s\beta}{2}\sinh\frac{s\beta}{2}\mathcal{P}_1 - \frac{16}{\beta^2}\sinh s\beta\,\sinh^2\frac{s\beta}{2}\mathcal{S}_1. \tag{F10}$$

Proceeding as in Eq. (F6), it is also easy to show that

$$e^{sH_1}\mathcal{A}e^{-sH_1} = \mathcal{A} + \beta(\tilde{H}_2(s) - H_2) + \frac{1}{\beta}\sinh 2s\beta\,\mathcal{K}_0 + \frac{2}{\beta^2}\sinh^2 s\beta\,(\beta\mathcal{P}_0 - 2\mathcal{S}_0),$$

$$e^{sH_2}\mathcal{A}e^{-sH_2} = \mathcal{A} + \beta(\tilde{H}_1(s) - H_1) + \frac{1}{\beta}\sinh 2s\beta\,\mathcal{K}_1 + \frac{2}{\beta^2}\sinh^2 s\beta\,(\beta\mathcal{P}_1 - 2\mathcal{S}_1). \tag{F11}$$

It is now straightforward to check whether the ansatz (68) for the first conserved charge satisfies the integrability condition (66). Using Eqs. (F7), (F8), (F10), and (F11), one finds that $[U_F, Q_1] = 0$ only if $T_1 = T_2$ and the coefficients a_k are given by Eq. (72).

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[40] Note that one can define an operator $Y_j = X_j Z_j$, which then satisfies the relations $X_j Y_j = \omega Y_j X_j$ and $Y_j Z_j = \omega Z_j Y_j$, similar to those for X_j and Z_j in Eq. (3). However, it is easy to see that neither $\{X_j, Y_j, Z_j\}$ nor $\{X_j^m, Y_j^p, Z_j^q\}$ with $m, p, q \in \{0, 1, 2\}$ form a group or a Lie algebra. For this reason it is more convenietn to work with the basis of $\mathfrak{sl}(3, \mathbb{C})$ in Eq. (7).

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