Quantum effective action for the bosonic Josephson junction

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We investigate a bosonic Josephson junction by using the path-integral formalism with relative phase and population imbalance as dynamical variables. We derive an effective only-phase action performing functional integration over the population imbalance. We then analyze the quantum effective only-phase action, which formally contains all the quantum corrections. To the second order in the derivative expansion and to the lowest order in $\hbar$, we obtain the quantum correction to the Josephson frequency of oscillation. Finally, the same quantum correction is found by adopting an alternative approach. Our predictions are a useful theoretical tool for experiments with atomic or superconducting Josephson junctions.

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I. INTRODUCTION

Two superconductors or superfluids separated by a tunneling barrier give rise to the so-called Josephson junction [1–3]. In contrast to superconducting Josephson junctions, it is possible to have a huge population imbalance with atomic Josephson junctions due to the appearance of the self-trapping phenomena [4]. The phase model [5] is often used to describe the quantum behavior of Josephson junctions. This model is based on the quantum commutation rule between the number-difference operator and the phase-difference operator [6]. Because the phase-number commutation rule is approximately correct for systems with a large number of condensed electronic Cooper pairs or bosonic atoms, the phase model is considered a reasonable starting point to then get beyond-mean-field quantum effects [7–10].

In this paper, we study a Josephson junction by using the Feynman path-integral approach [11,12]. In particular, we consider a system of interacting bosons which are tunneling between two sites. At the mean-field (saddle-point) level we recover the classical phase-imbalance model [4]. Performing path integration over the population imbalance we obtain the only-phase effective action of the system. The quantum effective only-phase action, which formally comprises all the quantum corrections, is then examined. The quantum correction to the Josephson frequency of oscillation is obtained to the second order in the derivative expansion and to the lowest order in $\hbar$. Finally, by using a different strategy based on the quantum average of the equation of motion, the same quantum correction is recovered. We also discuss the possible experimental detection of this quantum correction with atomic or superconducting Josephson junctions.

II. TWO-SITE MODEL

The macroscopic quantum tunneling of bosonic particles or Cooper pairs in a Josephson junction made of two superfluids or two superconductors separated by a potential barrier can be described within a quantum field theory formalism.

The simplest Lagrangian of a system made of bosonic particles which are tunneling between two sites ($j = 1, 2$) is given by

$$L = \sum_{j=1,2} \left[i\hbar\, \psi_j^* \dot{\psi}_j - \frac{U}{2}|\psi_j|^4 \right] + \frac{J}{2}(\psi_1^*\psi_2 + \psi_2^*\psi_1), \quad (1)$$

where $\psi_j(t)$ is the adimensional complex field of bosons in the j site at real time t, U is the on-site interaction strength of particles, J is the tunneling energy, $\hbar$ is the reduced Planck constant, i is the imaginary unit, and dot means the derivative with respect to time t.

To make clear the crucial role of the hopping term, which contains the tunneling energy J, we set

$$\psi_j(t) = \sqrt{N_j(t)}\, e^{i\phi_j(t)}, \quad (2)$$

where $N_j(t)$ is the number of the bosons in the j site and $\phi_j(t)$ is the phase angle. We also introduce the total number

$$N = N_1(t) + N_2(t), \quad (3)$$

which is a constant of motion, and the relative phase

$$\phi(t) = \phi_2(t) - \phi_1(t), \quad (4)$$

which is not a constant of motion, similarly to the total phase

$$\bar{\phi}(t) = \phi_1(t) + \phi_2(t). \quad (5)$$

We can then define the population imbalance as

$$z(t) = \frac{N_1(t) - N_2(t)}{N}. \quad (6)$$

In this way the Lagrangian becomes

$$\bar{L}(\phi, z, \bar{\phi}, N) = \frac{i\hbar}{2}\dot{N} - \frac{U}{4}N^2 + \frac{N\hbar}{2}(z\dot{\phi} - \dot{\bar{\phi}}) - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos(\phi). \quad (7)$$

The last term in this Lagrangian is the one that makes possible the periodic oscillation of a macroscopic number of particles between the two sites. Note that the constant term $-UN^2/4$ and the term $(N\hbar/2)\dot{\bar{\phi}}$ containing the exact differential $\dot{\bar{\phi}}$ can be safely removed.

III. MEAN-FIELD DYNAMICS

The quantum mechanics of the Josephson junction can be derived from the Feynman path integral [11,12]

$$\begin{aligned}\int \mathcal{D}[\psi_1(t)]\mathcal{D}[\psi_1^*(t)]\mathcal{D}[\psi_2(t)]\mathcal{D}[\psi_2^*(t)]e^{i\int \bar{L}(\phi,z,\bar{\phi},N)dt/\hbar} \\ = \int \mathcal{D}[\phi(t)]\mathcal{D}[z(t)]\int \mathcal{D}[N(t)]e^{i\int[\bar{L}(\phi,z,\bar{\phi},N)-\hbar\dot{N}\bar{\phi}/2]dt/\hbar} \\ \times \underbrace{\int \mathcal{D}[\bar{\phi}(t)]e^{i\int \dot{N}\bar{\phi}dt/2}}_{2\delta[\dot{N}]} \\ = \int \mathcal{D}[\phi(t)]\mathcal{D}[z(t)]e^{i\int L(\phi,z)dt/\hbar},\end{aligned} \tag{8}$$

integrating over all the configurations of the dynamical variables $\phi(t)$ and $z(t)$ with

$$L(\phi,z) = \frac{N\hbar}{2}z\dot{\phi} - \frac{UN^2}{4}z^2 + \frac{JN}{2}\sqrt{1-z^2}\cos(\phi), \tag{9}$$

where $N(t) = N$ is a constant as a consequence of integrating out $\bar{\phi}$. Here, we have omitted a constant originating from the Jacobian for the Madelung transformation in Eq. (2) and the transformation into relative coordinates given in Eqs. (4) and (6):

$$\begin{aligned}\mathcal{D}[\psi_1]\mathcal{D}[\psi_1^*]\mathcal{D}[\psi_2]\mathcal{D}[\psi_2^*] &= \mathcal{D}[N_1]\mathcal{D}[\phi_1]\mathcal{D}[N_2]\mathcal{D}[\phi_2] \\ &= N\mathcal{D}[z]\mathcal{D}[\phi]\mathcal{D}[N]\mathcal{D}[\bar{\phi}].\end{aligned} \tag{10}$$

Unfortunately, the exact calculation of these path integrals is an extremely difficult task also numerically, and consequently some approximation scheme is needed.

The simplest approximation scheme to treat our quantum problem is the so-called mean-field (or saddle-point) approximation [11,12], where one takes into account only the configurations which extremize the action functional

$$S = \int L(\phi,z)dt. \tag{11}$$

These configurations are the ones which satisfy the Euler-Lagrange equations. In our case, they are

$$\dot{\phi} = J\frac{z}{\sqrt{1-z^2}}\cos(\phi) + UNz, \tag{12}$$

$$\dot{z} = -J\sqrt{1-z^2}\sin(\phi). \tag{13}$$

These equations describe the mean-field dynamics of the macroscopic quantum tunneling in a Josephson junction, where $\phi(t)$ is the relative phase angle of the complex field of the superfluid (or superconductor) between the two junctions at time t and $z(t)$ is the corresponding relative population imbalance of the Bose condensed particles (or Cooper pairs).

It is important to stress that, due to the term $(N\hbar/2)z\dot{\phi}$ in the Lagrangian (9), the dynamical variables $\phi(t)$ and $z(t)$ are canonically conjugated. This means that one can introduce the new dynamical variable

$$p_\phi(t) = \frac{N\hbar}{2}z(t), \tag{14}$$

which is the generalized momentum conjugated to the Lagrangian coordinate $\phi(t)$. Moreover, with the Legendre transformation $H = p_\phi\dot{\phi} - L$, one obtains the Hamiltonian [4]

$$H(\phi,p_\phi) = \frac{Up_\phi^2}{\hbar^2} - \frac{JN}{2}\sqrt{1-\frac{4p_\phi^2}{N^2\hbar^2}}\cos(\phi) \tag{15}$$

of a nonrigid pendulum [4]. The Hamilton's equations of motion obtained with $H(\phi,p_\phi)$ are merely Eqs. (12) and (13).

A. Linearized equations and Josephson oscillation

Assuming that both $\phi(t)$ and $z(t)$ are small, i.e., $|\phi(t)| \ll 1$ and $|z(t)| \ll 1$, the Lagrangian (9) can be approximated as

$$L^{(2)} = \frac{N\hbar}{2}z\dot{\phi} - \frac{JN}{4}\phi^2 - \frac{(JN+UN^2)}{4}z^2, \tag{16}$$

removing a constant term. The Euler-Lagrange equations of this quadratic Lagrangian are the linearized Josephson junction equations

$$\hbar\dot{\phi} = (J+UN)z, \tag{17}$$

$$\hbar\dot{z} = -J\phi, \tag{18}$$

which can be rewritten as a single equation for the harmonic oscillation of $\phi(t)$ and the harmonic oscillation of $z(t)$, given by

$$\ddot{\phi} + \Omega^2\phi = 0, \tag{19}$$

$$\ddot{z} + \Omega^2 z = 0, \tag{20}$$

both with frequency

$$\Omega = \frac{1}{\hbar}\sqrt{J^2+NUJ}, \tag{21}$$

that is, the familiar mean-field frequency of macroscopic quantum oscillation in terms of tunneling energy J, interaction strength U, and number N of particles [4]. In the regime $NU/J \ll 1$ the frequency Ω becomes the Rabi frequency

$$\Omega_R = \frac{J}{\hbar}, \tag{22}$$

while in the regime $NU/J \gg 1$ the frequency Ω becomes the Josephson frequency

$$\Omega_J = \frac{\sqrt{NUJ}}{\hbar}. \tag{23}$$

IV. EFFECTIVE ONLY-PHASE ACTION

Fixing the initial and final points for the $\phi(t)$ paths, while still summing over all $z(t)$ paths, yields the path-integral propagator for the phase,

$$K(\phi_T,T|\phi_0,0) = \int_{\{\phi_0,0\}}^{\{\phi_T,T\}}\mathcal{D}[\phi]\int\mathcal{D}[z]e^{iS[\phi,z]/\hbar}. \tag{24}$$

The quantum-mechanical way to derive an effective only-phase action S_0 for $\phi(t)$, starting from the full action $S[\phi(t), z(t)]$, is to trace out the dynamical variable $z(t)$ with a path integral over it [11,12], namely

$$\int \mathcal{D}[z] e^{iS[\phi,z]/\hbar} \propto e^{iS_0[\phi]/\hbar}. \quad (25)$$

In our case, the action functional S is determined by the Lagrangian L given by expression (7), containing both $\phi(t)$ and $z(t)$. However, the complete Lagrangian L of Eq. (7) cannot be used to extract analytically this effective only-phase Lagrangian because one can explicitly calculate only quadratic integrals. To perform these calculations we can use quadratic expansions, i.e., the Gaussian approximation [11,12]. Expanding the Lagrangian L of Eq. (7) at the Gaussian level with respect to $z(t)$ we obtain

$$L(\phi, z) = \frac{N\hbar}{2} z\dot{\phi} - \frac{UN^2 + JN\cos(\phi)}{4} z^2 + \frac{JN}{2}\cos(\phi). \quad (26)$$

To perform the integration over the $z(t)$ paths, it is useful to use the time-sliced representation of the propagator. In that case the paths are subdivided into n time steps, chosen of equal duration $\delta t = T/n$, and at the end of the calculation one lets n tend to infinity. The path integral over $z(t)$ is then performed as an n-fold integral over the variables z_j, with $j = 1, \dots, n$. After performing these integrations, we find that the propagator for the phase can be written as

$$K(\phi_T, T|\phi_0, 0) = \left(\prod_{j=1}^{n-1}\int_0^{2\pi} d\phi_j\right)\prod_{j=1}^{n} K_{\text{inf}}(\phi_j, t_j|\phi_{j-1}, t_{j-1}), \quad (27)$$

where the infinitesimal propagator is given by

$$K_{\text{inf}}(\phi_j, \delta t|\phi_{j-1}, 0) = \sqrt{\frac{N\hbar}{(UN + J\cos\phi_j)4\pi i\delta t}} \times \exp\left\{-\frac{N\hbar(\phi_j - \phi_{j-1})^2}{4i(UN + J\cos\phi_j)\delta t} + \frac{i}{2\hbar} JN\cos(\phi_j)\delta t\right\}. \quad (28)$$

The prefactor before the exponential is a normalization factor that ensures the condition

$$\lim_{\delta t\to 0} K_{\text{inf}}(\phi_j, \delta t|\phi_{j-1}, 0) = \delta(\phi_j - \phi_{j-1}). \quad (29)$$

This prefactor is crucial to remove the divergences resulting from the quantum fluctuations of $\phi(t)$ [13]. Letting the number n of time slices go to infinity, the exponential phase factor for the ϕ path tends to $\exp[iS_0[\phi(t)]/\hbar]$ with

$$S_0[\phi] = \int dt\left(\frac{N\hbar^2\dot{\phi}^2}{4(UN + J\cos\phi)} + \frac{JN}{2}\cos\phi\right). \quad (30)$$

The full propagator can then be written as

$$K(\phi_T, T|\phi_0, 0) = \int_{\{\phi_0,0\}}^{\{\phi_T,T\}} \mathcal{D}[\phi] e^{iS_0[\phi]/\hbar}, \quad (31)$$

where now the path-integral measure is determined by the prefactor in Eq. (28), and given by

$$\int \mathcal{D}[\phi] = \lim_{n\to\infty}\prod_{j=1}^{n}\int_{-\pi}^{\pi}\frac{d\phi_j}{\sqrt{(U + J\cos\phi_j/N)4\pi i\delta t/\hbar}}. \quad (32)$$

Note that the only-phase action $S_0[\phi]$ of Eq. (30) can also be obtained substituting z in the Lagrangian $L(\phi, z)$ of Eq. (26) with the expression

$$z = \frac{\hbar\dot{\phi}}{UN + J\cos(\phi)}, \quad (33)$$

which is the Euler-Lagrange equation of $L(\phi, z)$ for the dynamical variable $z(t)$. The Euler-Lagrange equation of the relative phase $\phi(t)$ derived from the action $S_0[\phi]$ of Eq. (30) is

$$\frac{\hbar^2\ddot{\phi}}{UN + J\cos(\phi)} + \frac{J}{2}\frac{\hbar^2\dot{\phi}^2\sin(\phi)}{[UN + J\cos(\phi)]^2} + J\sin(\phi) = 0. \quad (34)$$

Its linearized version is

$$\frac{\hbar^2}{UN + J}\ddot{\phi} + J\phi = 0, \quad (35)$$

which gives again Eq. (21) for the Josephson frequency.

Schrödinger equation for phase wave function

The Lagrangian of the action (30) is given by

$$L_0 = \frac{N\hbar^2\dot{\phi}^2}{4[UN + J\cos(\phi)]} + \frac{JN}{2}\cos(\phi), \quad (36)$$

and the corresponding generalized momentum p_ϕ reads

$$p_\phi = \frac{\partial L_0}{\partial\dot{\phi}} = \frac{N\hbar^2\dot{\phi}}{2[UN + J\cos(\phi)]}. \quad (37)$$

As expected, p_ϕ is proportional to the population imbalance z given by Eq. (33): $p_\phi = (N\hbar/2)z$. The Legendre transformation $H_0 = p_\phi\dot{\phi} - L_0$ gives the Hamiltonian

$$H_0(\phi, p_\phi) = [UN + J\cos(\phi)]\frac{p_\phi^2}{N\hbar^2} - \frac{JN}{2}\cos(\phi). \quad (38)$$

Clearly, this Hamiltonian can be obtained from the one of Eq. (15) expanding the square root up to the quadratic term with respect to p_ϕ.

Promoting p_ϕ to the operator $\hat{p}_\phi = -i\hbar\partial_\phi$ we can immediately write the time-dependent Schrödinger equation $i\hbar\partial_t\Psi = \hat{H}_0\Psi$, namely

$$i\hbar\partial_t\Psi(\phi, t) = \left[-\left[U + \frac{J}{N}\cos(\phi)\right]\partial_\phi^2 - \frac{JN}{2}\cos(\phi)\right]\Psi(\phi, t), \quad (39)$$

for the wave function $\Psi(\phi, t)$ of the relative phase ϕ. Moreover, the quantum Hamiltonian $\hat{H}_0 = H_0(\hat{\phi}, \hat{p}_\phi)$ is such that

$$\langle\phi_b|e^{-i\hat{H}_0(t_b-t_a)/\hbar}|\phi_a\rangle = \int \mathcal{D}[\phi]\mathcal{D}[p_\phi] e^{\frac{i}{\hbar}\int_{t_a}^{t_b}[\dot{\phi}p_\phi - H_0(\phi,p_\phi)]dt} = \int \mathcal{D}[\phi] e^{iS_0[\phi]/\hbar}. \quad (40)$$

Thus, we recover the effective action $S_0[\phi]$ of Eq. (30) after functional integration over the generalized momentum p_ϕ. The integration measure (32) arises here from the presence of an unusual kinetic term in the Hamiltonian $H_0(\phi, p_\phi)$ [13–15].

V. QUANTUM EFFECTIVE ONLY-PHASE ACTION

It is important to stress that under condition $UN \gg J$ (Josephson regime), the integration measure (32) does not depend on $\phi(t)$. Moreover, the action $S_0[\phi]$ becomes

$$S_{\rm J}[\phi] = \int dt\left[\frac{\hbar^2\dot{\phi}^2}{4U} + \frac{JN}{2}\cos(\phi)\right]. \tag{41}$$

In this regime, we can ignore a quartic contribution of ϕ in the kinetic term in Eq. (30). Equation (41) is merely the familiar only-phase action of a capacitively shunted superconducting Josephson junction, where the population imbalance is very small, i.e., $|z(t)| \ll 1$, but the number of bosonlike Cooper pairs N is large. Within the framework of superconducting junctions, $I_0 = eJN/\hbar$ is the critical electric current with e the electric charge of the electron, and $C = 2e^2/U$ is the electric capacitance [11].

Let us work with the action of Eq. (41). We want to determine the corrections to the Josephson frequency due to quantum fluctuations adopting the formalism of the quantum effective action [16–18]. The quantum effective action $\Gamma[\phi]$ is a modified expression of the action $S_{\rm J}[\phi]$ which takes into account quantum corrections. The minimization of $\Gamma[\phi]$ gives the exact equations of motion for the expectation value of the field, which is here denoted with the same symbol $\phi(t)$ of the field [16–18]. First, we rewrite the action (41) as follows,

$$S_{\rm J}[\phi] = \int dt\left[\frac{M_{\rm J}}{2}\dot{\phi}^2 - V(\phi)\right], \tag{42}$$

where $M_{\rm J} = \hbar^2/(2U)$ and the only-phase potential energy is expanded as

$$V(\phi) = \frac{M_{\rm J}\Omega_{\rm J}^2}{2}\phi^2 + \tilde{V}(\phi), \tag{43}$$

with $\Omega_{\rm J}$ given by Eq. (23) and

$$\begin{aligned}\tilde{V}(\phi) &= M_{\rm J}\Omega_{\rm J}^2[1-\cos(\phi)] - \frac{M_{\rm J}\Omega_{\rm J}^2}{2}\phi^2 \\ &= \lambda\phi^4 + O(\phi^6),\end{aligned} \tag{44}$$

with $\lambda = -JN/48$. As discussed in Ref. [19], it is the potential $\tilde{V}(\phi)$ which encodes quantum fluctuations within the formalism of the quantum effective action, where the action $S_{\rm J}[\phi]$ is substituted by the quantum effective action $\Gamma_{\rm J}[\phi]$ given by

$$\Gamma_{\rm J}[\phi] = \int dt\left[\frac{Z(\phi)}{2}\dot{\phi}^2 - V_{\rm e}(\phi)\right], \tag{45}$$

to the second order in the derivative expansion [17–19]. In this quantum effective action, the term

$$Z(\phi) = M_{\rm J} + \hbar Z_1(\phi) + \hbar^2 Z_2(\phi) + \cdots \tag{46}$$

is the effective mass, and

$$V_{\rm e}(\phi) = V(\phi) + \hbar V_{\rm e1}(\phi) + \hbar^2 V_{\rm e2}(\phi) + \cdots \tag{47}$$

is the effective potential, written in terms of a $\hbar$ expansion [19]. In particular, to the first order of this $\hbar$ expansion one finds [19]

$$Z_1(\phi) = \frac{1}{32M_{\rm J}^2}\frac{\left[\partial_\phi^3\tilde{V}(\phi)\right]^2}{\left[\Omega_{\rm J}^2 + \frac{1}{M_{\rm J}}\partial_\phi^2\tilde{V}(\phi)\right]^{5/2}}, \tag{48}$$

and

$$V_{\rm e1}(\phi) = \frac{1}{2}\left(\sqrt{\Omega_{\rm J} + \frac{1}{M_{\rm J}}\partial_\phi^2\tilde{V}(\phi)} - \Omega_{\rm J}\right). \tag{49}$$

Note that the potential $V(\phi) = M_{\rm J}\Omega_{\rm J}^2/2 + \tilde{V}(\phi)$ must be convex so that Eqs. (48) and (49) are real. This is realized in the domain $|\phi| < \sqrt{2}$, which has already been satisfied under $|\phi| \ll 1$ that makes the approximation from Eq. (30) to Eq. (41) valid. The corresponding equation of motion with first-order quantum corrections, obtained extremizing $\Gamma_{\rm J}[\phi]$, is given by [19]

$$[M_{\rm J} + \hbar Z_1(\phi)]\ddot{\phi} + \frac{\hbar}{2}\partial_\phi Z_1(\phi)\dot{\phi}^2 = -\partial_\phi[V(\phi) + \hbar V_{\rm e1}(\phi)]. \tag{50}$$

The equation of motion (50) is derived through the second-order derivative expansion and neglecting higher-order contributions of $\hbar$. This approximation is valid under [19]

$$\left|\frac{\lambda\phi^4}{M_{\rm J}\Omega_{\rm J}^2\phi^2/2}\right| \ll 1, \tag{51}$$

and, by inserting the oscillator length $\phi = \sqrt{\hbar/(M_{\rm J}\Omega_{\rm J})}$,

$$\left|\frac{2\hbar\lambda}{M_{\rm J}^2\Omega_{\rm J}^3}\right| \ll 1. \tag{52}$$

Inequality (51) reads $|\phi| \ll 2\sqrt{3}$, which should have already been satisfied under $|\phi| \ll 1$. The latter one (52) reads $UN/J \ll 36N^2$. This inequality indicates that the approximation is valid under a sufficiently small ratio between the interaction energy and the tunneling energy within the Josephson regime $1 \ll UN/J \ll 36N^2$.

Because we want to determine the first-order quantum correction to the Josephson frequency, we consider $\tilde{V}(\phi)$ of Eq. (44) up to the quadratic term of ϕ. In this way, up to the quadratic term of ϕ, we obtain

$$Z_1(\phi) = 0, \tag{53}$$

and

$$\begin{aligned}V_{\rm e1}(\phi) &= \frac{\Omega_{\rm J}}{2}(\sqrt{\cos\phi} - 1) \\ &\simeq \frac{3\lambda}{M_{\rm J}\Omega_{\rm J}}\phi^2 = -\frac{\sqrt{NUJ}}{8\hbar}\phi^2.\end{aligned} \tag{54}$$

It follows that, to the first order of $\hbar$, the modified frequency of oscillation is given by

$$\tilde{\Omega}_{\rm J} = \frac{\sqrt{NUJ}}{\hbar}\sqrt{1 - \frac{1}{2}\sqrt{\frac{U}{JN}}}. \tag{55}$$

Clearly, for $N \gg 1$ one recovers Eq. (23). In the framework of the the capacitively shunted superconducting Josephson junction, where $U = 2e^2/C$ and $JN = (\hbar/e)I_0$, the frequency can be written as

$$\tilde{\Omega}_{\rm J} = \frac{1}{\hbar}\sqrt{\frac{2e\hbar I_0}{C}}\sqrt{1 - \frac{1}{2}\sqrt{\frac{2e^3}{\hbar C I_0}}}. \quad (56)$$

Reference [20] reports the Josephson dynamics with two one-dimensional quasicondensates of ^{87}Rb atoms trapped in a double-well potential. The number of atoms is typically $N = 2500$ and the ratio between the interaction energy and the tunneling energy is $NU/(2J) \sim 10^2$ [20]. These experimental data give, for the relative difference between the mean-field Josephson frequency $\Omega_{\rm J}$ and the beyond-mean-field one $\tilde{\Omega}_{\rm J}$, the value $(\Omega_{\rm J} - \tilde{\Omega}_{\rm J})/\Omega_{\rm J} \simeq 0.1\%$, which indicates that the quantum correction slightly reduces the Josephson frequency. One could observe the effect of this quantum correction more significantly in the deep Josephson regime by increasing the ratio NU/J at fixed N, namely, by increasing the interparticle interaction strength U or decreasing the Josephson coupling J while satisfying Eq. (52) because $\tilde{\Omega}_{\rm J}/\Omega_{\rm J} = [1 - \Omega_{\rm J}/(2N\Omega_{\rm R})]^{1/2}$ with $\Omega_{\rm J}/\Omega_{\rm R} = \sqrt{NU/J}$. Considering, instead, superconducting Josephson junctions, in Ref. [21] the experimental value $2e\Omega_{\rm J}/I_0 = 2[2e^3/(\hbar C I_0)]^{1/2} \simeq 2.3 \times 10^{-3}$ results in $(\Omega_{\rm J} - \tilde{\Omega}_{\rm J})/\Omega_J \simeq 0.03\%$, which reduces the Josephson frequency due to the quantum correction in the order of 10^{-4}. A larger value of $2e\Omega_{\rm J}/I_0$ would make the quantum correction more significant.

Alternative derivation of quantum corrections

In this section, we discuss a different approach to derive the quantum correction of the only-phase dynamics. This approach is similar to the one recently developed to determine beyond-mean-field corrections to the critical temperature in two-band superconductors [22]. Given the nonlinear only-phase Josephson equation

$$\frac{\hbar^2}{UN}\ddot{\phi} + J\sin(\phi) = 0, \quad (57)$$

derived from Eq. (41), we set

$$\phi(t) = \phi_0(t) + \tilde{\phi}(t), \quad (58)$$

where $\phi_0(t)$ is the mean-field solution and $\tilde{\phi}(t)$ encodes quantum fluctuations, which are assumed to be small. At the quadratic level with respect to $\tilde{\phi}(t)$ we get

$$\frac{\hbar^2}{UN}\ddot{\phi}_0 + \frac{\hbar^2}{UN}\ddot{\tilde{\phi}} + J\sin(\phi_0) + J\cos(\phi_0)\tilde{\phi} - \frac{J}{2}\sin(\phi_0)\tilde{\phi}^2 = 0. \quad (59)$$

Performing the quantum average $\langle \cdots \rangle$ with the condition $\langle \phi(t)\rangle = \phi_0(t)$, i.e., $\langle \tilde{\phi}(t)\rangle = 0$, we obtain

$$\frac{\hbar^2}{UN}\ddot{\phi}_0 + J\left(1 - \frac{1}{2}\langle\tilde{\phi}^2\rangle\right)\sin(\phi_0) = 0. \quad (60)$$

The quantum average $\langle\tilde{\phi}^2\rangle$ can be calculated as follows,

$$\langle\tilde{\phi}^2\rangle = \frac{1}{\mathcal{Z}}\int D[\tilde{\phi}]\tilde{\phi}^2\, e^{iS_{\rm J}^{(2)}[\tilde{\phi}]/\hbar}, \quad (61)$$

where

$$S_{\rm J}^{(2)}[\tilde{\phi}] = \int\left[\frac{\hbar^2}{4U}\dot{\tilde{\phi}}^2 - \frac{JN}{4}\tilde{\phi}^2\right]dt \quad (62)$$

is the quadratic action for the fluctuations and

$$\mathcal{Z} = \int D[\tilde{\phi}]e^{iS_{\rm J}^{(2)}[\tilde{\phi}]/\hbar} \quad (63)$$

is the corresponding real-time partition function. Then, one easily finds the zero-temperature result

$$\langle\tilde{\phi}^2\rangle = \sqrt{\frac{U}{JN}}. \quad (64)$$

Thus, the only-phase Josephson equation corrected by quantum fluctuations reads

$$\frac{\hbar^2}{UN}\ddot{\phi}_0 + J\left(1 - \frac{1}{2}\sqrt{\frac{U}{JN}}\right)\sin(\phi_0) = 0. \quad (65)$$

It follows that the frequency of oscillation with the inclusion of quantum fluctuations is given by Eq. (55). Thus, we have recovered the same result obtained with the quantum effective action formalism.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have adopted a quantum field theory formalism based on the path integral to study the role of quantum fluctuations in a Josephson junction. From the bosonic action of the relative phase and population imbalance, by performing Gaussian integration over the population imbalance, under the condition of taking only up to quadratic terms in the population imbalance (i.e., with an approximated phase-imbalance action), we have derived the effective only-phase action. Quite remarkably, this effective action is highly nonlinear with respect to the phase variable but it gives rise to the same mean-field equation of the approximated phase-imbalance action. We have then examined the quantum effective only-phase action, which formally comprises all the quantum corrections for the dynamics of the expectation value of the relative phase. In this way, we have obtained, with two independent but similar procedures, the quantum-corrected Josephson frequency $\tilde{\Omega}_{\rm J}$, Eqs. (55) and (56). As we have discussed in the last section, the estimated quantum corrections to the Josephson frequencies in a Bose-Josephson junction and in a superconducting Josephson junction are relatively small based on the current experiments [20,21]. Reference [23] is an earlier work that has also considered quantum corrections to the Josephson effects. They started from a spatially three-dimensional Bose gas to obtain the correction to the Josephson energy. A different point between Ref. [23] and our work is that we are treating a spatially zero-dimensional system for simplicity. In Ref. [23], instead, they integrated out the noncondensed field after separating the field operators into a condensed field and a noncondensed field following the Bogoliubov prescription. This approach has led to corrections to the Josephson energy originating from the interparticle interaction and the temperature in Bose condensates. Our result of modified Josephson frequencies, however, can be verified not only in an atomic Josephson junction but also in a superconducting Josephson circuit. We expect that tuning the experimental parameters such as the interaction strength or capacitance enables us to observe more prominent quantum corrections.

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