Quasiconserved quantities in the perturbed spin-$\frac{1}{2}$ XXX model

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We consider the isotropic spin-$\frac{1}{2}$ Heisenberg spin chain weakly perturbed by a local translationally and SU(2)-invariant perturbation. Starting from the local integrals of motion of the unperturbed model, we modify them in order to obtain quasiconserved integrals of motion (charges) for the perturbed model. Such quasiconserved quantities are believed to be responsible for the existence of the prethermalization phase at intermediate timescales. We find that for a sufficiently local perturbation the quasiconserved quantities indeed exist, and we construct an explicit form for the first few of them.

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I. INTRODUCTION

In classical mechanics there is a well understood distinction between integrable and nonintegrable systems, as well as between their long-time dynamics. Namely, a generic nonintegrable system typically exhibits an ergodic behavior, leading to a chaos, whereas integrable systems are nonergodic and their phase space trajectories are confined to some subregions of the phase space due to the existence of many conserved quantities. Moreover, there is a result of tremendous importance, the Kolmogorov-Arnold-Moser (KAM) theorem, which ensures that classical integrable systems under a weak integrability-breaking perturbation are stable for a sufficiently long time [1–5].

Extending the KAM theorem to the quantum case is a long-standing problem. Although recent findings demonstrate some progress in this direction [6], a complete understanding is missing and there are numerous open questions. In part this is due to the fact that in the quantum case even the very definition of integrability is subtle [7].

A widely accepted criterion for quantum integrability is that a Hamiltonian H_0 is integrable, if there exists a large number of extensive, functionally independent, and mutually commuting conserved quantities (charges):

$$[H_0, \mathcal{Q}_j] = [\mathcal{Q}_k, \mathcal{Q}_j] = 0. \tag{1}$$

Importantly, the conserved charges $\mathcal{Q}_j$ are assumed to be local, in a sense that they are given by the sums of operators with a finite support.

Just like in the classical case, the long time dynamics is very different for integrable and nonintegrable systems (to be precise, we do not consider here systems that exhibit Anderson or many-body localization). Nonintegrable systems thermalize according to the eigenstate thermalization hypothesis (ETH), which (at least according to one of the interpretations) means that the total isolated system acts as a heath bath for its own subsystems. This leads to the spread of entanglement over the whole system, such that in the long time limit it is impossible to retrieve any information about the initial state using only local measurements [8,9]. On the contrary, in integrable systems, thermalization is very different. It is described by the generalized Gibbs ensemble (GGE), which takes into account that there are many other conserved quantities apart from the total energy and the number of particles, as it is for the standard grand-canonical ensemble [10–12]. Moreover, it has been shown that to accurately describe thermalization of integrable systems one should extend the GGE by including not only the local conserved charges, as in Eq. (1), but also the *quasilocal* ones [13–16].

Then, in the spirit of the KAM theorem, one may ask what will happen if a quantum integrable system is slightly perturbed away from integrability. What kind of thermalization will it exhibit? Naively, one would expect that nearly integrable quantum systems simply thermalize following the ETH. However, it is widely believed that such systems also exhibit a different, the so-called prethermal, behavior at intermediate times [17–19]. Different studies [20–22] suggest that the eventual thermalization occurs at much later times $t_{\text{th}} \sim \lambda^{-2}$, where $\lambda \ll 1$ is the strength of the perturbation, and the scaling can be understood from the golden rule. Moreover, it is believed that this prethermal phase should be described by some effective GGE [23]. Therefore, it is natural to ask what are the charges that define this effective GGE in the prethermal phase. Clearly, since this effective description is only valid at times $t \lesssim \lambda^{-2}$, these charges can only be *quasiconserved* with the accuracy $O(\lambda^2)$. In other words, since the exact conservation laws of the unperturbed system constrain the dynamics of an integrable system, one can expect that the dynamics of a perturbed system should be restricted by the quasiconserved charges.

This idea is also supported by the developments in the context of the slowest operators [24,25]. Indeed, for an operator O that commutes with a Hamiltonian H, the time evolution

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$e^{iHt}Oe^{-iHt}$ is trivial. In terms of the quantum information language this means that the information encoded in $O(0)$ does not spread. On the contrary, if $[O(0), H] \neq 0$, the typical timescale of information spreading is inversely proportional to the norm of $[O(0), H]$, as follows from the Baker-Campbell-Hausdorff formula. Thus, to slow down the spread of the quantum information one needs to suppress (at least) the first order term in the Baker-Campbell-Hausdorff expansion.

Let us mention that the search for quasiconserved charges can be linked with an old problem in functional analysis, related to almost commuting matrices [26] and explicitly stated by Halmos in [27]. This long-standing question "when two almost commuting matrices are close to matrices that exactly commute" was answered eventually by Lin [28]. More precisely, Lin showed that given $\epsilon > 0$ there exists $\delta > 0$ such that if $N \times N$ matrices A, B are Hermitian, with $\| AB - BA \| < \delta$ and $\| A \|, \| B \| \leqslant 1$, then there exists commuting Hermitian $N \times N$ matrices X, Y such that $\| A - X \| + \| B - Y \| < \epsilon$, where $\| \cdot \|$ is a matrix norm. Importantly, $\delta = \delta(\epsilon)$ *does not* depend on the dimension N. Recently, Hastings obtained an explicit estimate $\epsilon(\delta) \sim \delta^{1/5}$, where the exponent may depend on the choice of the operator norm [29]. Quite remarkably, the question whether one can find triples of almost commuting matrices has generically a negative answer [30]. A similar story about *unitary* matrices is more involved [31,32]. There, the existence of almost commuting unitary matrices have some topological obstructions given by the so-called Bott indices. There is an extensive mathematical literature on the subject, see, e.g., Ref. [33]. Finally, we would like to mention that there is a somewhat related research direction in the context of AdS/CFT correspondence, which deals with the so-called long-range deformed spin chains [34,35] and $T\bar{T}$ deformations [36,37]. However, these studies deal with the deformations that preserve integrability to all orders in the perturbation strength and thus differ from the present work.

This paper is devoted to the search for quasiconserved quantities in a quantum spin chain weakly perturbed away from integrability. The rest of the paper is organized as follows. In Sec. II we describe the model and discuss some general properties of the exact conserved charges that are present in the absence of the perturbation. In Sec. III we present an ansatz for the quasiconserved charges and discuss the requirements that the ansatz must satisfy. Finally, in Sec. IV we demonstrate our findings, conclude, and formulate some open questions for future research. The paper is supplemented by several technical Appendixes which clarify our derivations using algebraic tools. Similar techniques can also be used for other spin systems.

II. THE MODEL AND CONSERVED CHARGES OF THE INTEGRABLE PART

Let us start by describing the specific model that we are going to deal with. Consider a Hamiltonian

$$H_\lambda = H_0 + \lambda H_1, \tag{2}$$

where H_0 is an integrable part, H_1 is an integrability-breaking perturbation, and $\lambda > 0$ is a numerical parameter characterizing the perturbation strength. We assume $\lambda \ll 1$, such that the perturbation is weak. For the unperturbed system, we take a spin-$\frac{1}{2}$ isotropic Heisenberg spin chain (XXX model) on a one-dimensional lattice:

$$H_0 = J \sum_j \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}, \tag{3}$$

where $\boldsymbol{\sigma}_j$ is the vector of Pauli matrices and J is an exchange constant. It is well known that H_0 is integrable, the exact spectrum and the eigenstates can be found using the Bethe ansatz [38,39], and one has a large number of conserved charges. Let us now break the integrability by a perturbation of the following form:

$$H_1 = J \sum_j \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2}, \tag{4}$$

which is nothing other than the next-to-nearest neighbor Heisenberg interaction. In what follows we put $J = 1$ and assume that the system is in the thermodynamic limit. Let us mention that both the unperturbed Hamiltonian H_0 and the perturbation H_1 are translationally and SU(2) invariant. Also, both are the sums of local operators with the support on two and three sites (for H_0 and H_1, respectively). Many facts are known about the low-energy properties of this model [40]: when $\lambda < \lambda_c \approx 0.241$, the model is gapless and is described by a marginally perturbed $SU(2)_1$ Wess-Zumino-Witten model, while for $\lambda_c < \lambda \leqslant 1/2$ the ground state is dimerized (with exactly known Majumdar-Ghosh state at $\lambda = 1/2$) with a gap in the spectrum. Spontaneous incommensurate order appears for $\lambda > 1/2$. In the limit $\lambda \to \infty$ the Hamiltonian (2) is equivalent to a pair of decoupled XXX models (on even and odd sites), and integrability is restored. However, we do not consider the case of large λ and restrict ourselves to $\lambda \ll 1$. Before we turn to the problem of constructing the quasiconserved quantities for the perturbed Hamiltonian (2), let us briefly summarize the most important properties of the charges conserved by the integrable Hamiltonian H_0.

Local conserved charges [as those given in Eq. (1)] can be generated iteratively starting from $\mathcal{Q}_2 \equiv H_0$ [by convention, $\mathcal{Q}_1$ is the total magnetization] and using the following relation [41–43]:

$$\mathcal{Q}_{n+1} = [B, \mathcal{Q}_n], \tag{5}$$

where B is the so-called boost operator, which reads

$$B = \frac{1}{2i} \sum_j j\, \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}. \tag{6}$$

Thus, the boost operator acts as a ladder operator in the space of conserved charges of the model. Importantly, every next charge has a larger support as compared to the previous one. For the XXX model, the nth charge $\mathcal{Q}_n$ is a sum of operators with a support up to n sites. As discussed in Ref. [41], the conserved charges with $n > 3$ generated as in Eq. (6) contain terms from the charges with smaller n. For the sake of simplicity we work with a different basis $\{Q_n\}$ in which every next charge does not contain any terms that are present in the previous ones. The first two charges coincide in both bases, i.e., $\mathcal{Q}_k = Q_k$ for $k = 1, 2$. For completeness, here we present

expressions for the few higher charges in this basis:

$$\begin{aligned}
Q_3 &= \sum_j (\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \cdot \boldsymbol{\sigma}_{j+2}, \\
Q_4 &= \sum_j \{((\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \times \boldsymbol{\sigma}_{j+2}) \cdot \boldsymbol{\sigma}_{j+3} \\
&\quad + \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2}\}, \\
Q_5 &= \sum_j \{((((\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \times \boldsymbol{\sigma}_{j+2}) \times \boldsymbol{\sigma}_{j+3}) \cdot \boldsymbol{\sigma}_{j+4} \\
&\quad + (\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+2} + \boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \cdot \boldsymbol{\sigma}_{j+3}\}, \\
Q_6 &= \sum_j \{(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+3}) \\
&\quad + ((\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1} + \boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+2}) \times \boldsymbol{\sigma}_{j+3}) \cdot \boldsymbol{\sigma}_{j+4}, \\
&\quad + ((((\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \times \boldsymbol{\sigma}_{j+2}) \times \boldsymbol{\sigma}_{j+3}) \times \boldsymbol{\sigma}_{j+4}) \cdot \boldsymbol{\sigma}_{j+5} \\
&\quad + ((\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1}) \times \boldsymbol{\sigma}_{j+2}) \cdot \boldsymbol{\sigma}_{j+4}\},
\end{aligned} \tag{7}$$

and the general form of Q_n can be found in Ref. [41]. We emphasize once again that B can only generate the *local* charges, whereas the Hamiltonian (3) also possesses the quasilocal ones [13]. To our knowledge, a corresponding boost operator that can generate quasilocal charges has not been found.

Let us now turn on the perturbation (4), such that the total Hamiltonian is H_λ as given by Eq. (2), and $\lambda \ll 1$. The quantities Q_n are no longer conserved, since they do not commute with H_λ. Neither they are quasiconserved, since $\|[H_\lambda, Q_n]\| \propto \lambda$. Hence, they change significantly over times much shorter than $t_{\text{th}} \sim \lambda^{-2}$ and can not govern the dynamics in the prethermal phase.

III. QUASICONSERVED CHARGES

We now proceed with looking for the quasiconserved quantities that survive during the pre-thermal phase up to times $\sim\lambda^{-2}$. This simply means that we are looking for a set of operators $\tilde{Q}_n$ that satisfy

$$\|[H_\lambda, \tilde{Q}_n]\| \propto \lambda^2 \tag{8}$$

and commute with each other with the accuracy $O(\lambda^2)$.

First of all, the Hamiltonian H_λ has the translational and SU(2) symmetries, therefore we require that the quasiconserved charges $\tilde{Q}_n$ possess these symmetries as well. In analogy with the integrable case, we identify the second charge with the Hamiltonian, i.e., $\tilde{Q}_2 = H_\lambda$ (note that $\tilde{Q}_1 = Q_1$, as the total magnetization is conserved by H_λ). This gives us the relation

$$\tilde{Q}_2 = Q_2 + \lambda H_1. \tag{9}$$

Since the perturbation is weak, it is natural to expect that similar relations should hold for higher charges as well. Therefore, we make an ansatz

$$\tilde{Q}_n = Q_n + \lambda \sum_{s=m}^{M} \delta Q_n^{(s)}, \tag{10}$$

where $\delta Q_n^{(s)}$ is a local operator consisting of terms having the support on s sites. The values of m and M will be specified below, at the moment we can only expect that $M > n$, i.e., the maximal support of $\tilde{Q}_n$ is larger than that of Q_n. This is a reasonable assumption because the perturbed Hamiltonian H_λ itself has a greater support than H_0. Let us emphasize that the ansatz (10) fully determines the λ dependence of the quasiconserved charge $\tilde{Q}_n$. This is simply because the perturbed Hamiltonian H_λ has terms at most linear in λ, and thus keeping in $\tilde{Q}_n$ any higher order terms results in the excess of precision. Thus, taking into account Eq. (10), one can clearly see that in order to fulfill the requirement in Eq. (8), the commutator $[H_\lambda, \tilde{Q}_n]$ should not contain terms linear in λ, i.e., the following condition must be fulfilled:

$$[H_1, Q_n] + \sum_{s=m}^{M} \left[H_0, \delta Q_n^{(s)}\right] = 0. \tag{11}$$

In this case the commutator of $\tilde{Q}_n$ and H_λ reads

$$[H_\lambda, \tilde{Q}_n] = \lambda^2 \sum_{s=m}^{M} \left[H_1, \delta Q_n^{(s)}\right] \tag{12}$$

and Eq. (8) is clearly satisfied.

Let us now discuss the structure of $\delta Q_n^{(s)}$ from Eq. (10) in more detail. First of all, translational invariance allows us to express it in the following form:

$$\delta Q_n^{(s)} = \sum_{k=2}^{s} \sum_{\boldsymbol{\ell}_k(s)} c_n(\boldsymbol{\ell}_k(s)) \sum_j O_j(\boldsymbol{\ell}_k(s)), \tag{13}$$

where $c_n(\boldsymbol{\ell}_k(s))$ are real numerical coefficients, and the local operator $O_j(\boldsymbol{\ell}_k(s))$ has a support on s sites and acts nontrivially on k sites ($2 \leqslant k \leqslant s$), specified by the components ℓ_i of the vector $\boldsymbol{\ell}_k(s)$:

$$j + \ell_1, \ \ldots, \ j + \ell_k, \tag{14}$$

where ℓ_i take values from $\{0, \ldots, s-1\}$. Importantly, all ℓ_i are distinct and *not necessarily ordered*. Note that $\boldsymbol{\ell}_k(s)$ always has $\ell_p = 0$ and $\ell_q = s - 1$ for some indices $p, q \in \{1, \ldots, k\}$.

Due to the SU(2) symmetry, $O_j(\boldsymbol{\ell}_k(s))$ in Eq. (13) can be expressed in the basis of SU(2)-invariant tensor products of k spin-$\frac{1}{2}$ operators. As shown in Refs. [44,45], this basis has dimension

$$D_k = \sum_{r=0}^{[k/2]} \frac{n!(3r-k+1)}{(k-2r)!r!(r+1)!}, \tag{15}$$

where $[x]$ is the integer part of x.

To construct this basis we follow Refs. [41–43] and introduce nested cross products of the Pauli vectors, with the nesting going toward the left:

$$\begin{aligned}
\boldsymbol{V}_j^{(1)} &= \boldsymbol{\sigma}_{j+\ell_1}, \\
\boldsymbol{V}_j^{(2)} &= \boldsymbol{\sigma}_{j+\ell_1} \times \boldsymbol{\sigma}_{j+\ell_2}, \\
\boldsymbol{V}_j^{(3)} &= (\boldsymbol{\sigma}_{j+\ell_1} \times \boldsymbol{\sigma}_{j+\ell_2}) \times \boldsymbol{\sigma}_{j+\ell_3}, \\
\boldsymbol{V}_j^{(k)} &= \boldsymbol{V}_j^{(k-1)} \times \boldsymbol{\sigma}_{j+\ell_k}.
\end{aligned} \tag{16}$$

Then we consider the following scalar [and hence SU(2) invariant] spin polynomials:

$$f_j(\ell_1, \ldots, \ell_k) \equiv f_j(\boldsymbol{\ell}_k) = \boldsymbol{V}_j^{(k-1)} \cdot \boldsymbol{\sigma}_{j+\ell_k}. \tag{17}$$

For instance, for the two-spin polynomial we have $f_j(0,l) = \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+l}$, for the three-spin polynomial $f_j(0,l,m) = (\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+l}) \cdot \boldsymbol{\sigma}_{j+m}$, and for the four-spin polynomial the definition yields $f_j(l,m,n,p) = [(\boldsymbol{\sigma}_{j+l} \times \boldsymbol{\sigma}_{j+m}) \times \boldsymbol{\sigma}_{j+n}] \cdot \boldsymbol{\sigma}_{j+p}$.

Let us now comment on the size of the space V_k spanned by the k-spin polynomials (17) and its relation to the dimensionality of the complete basis given by Eq. (15). Taking into account that the mixed product $(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \times \ldots \boldsymbol{A}_{n-1}) \cdot \boldsymbol{A}_n$ can be written in a number of equivalent ways [41–43], it is clear that some permutations of the set $\{\ell_1, \ldots, \ell_k\}$ result in the same polynomial $f_j(\ell_1, \ldots, \ell_k)$ up to a sign. Moreover, not all polynomials are linearly independent. For instance, for $k = 2$ and 3 there is clearly only one unique polynomial, in agreement with Eq. (15). Likewise, for $k = 4$ there are three inequivalent permutations of $\{\ell_1, \ldots, \ell_4\} = \{0, l, m, n\}$, and only two of the resulting polynomials are linearly independent. Namely, we have

$$f_j(0,l,m,n) - f_j(0,m,l,n) + f_j(0,n,l,m) = 0, \tag{18}$$

which one can easily check by a direct calculation using Eqs. (20) and (21). Thus, for $k = 4$ the basis is two dimensional. However, this differs from Eq. (15) that yields $D_4 = 3$. The reason is that for even $k > 2$, the basis of SU(2)-invariant operators also contains elements of the form $(\boldsymbol{\sigma}_{j+\ell_1} \cdot \boldsymbol{\sigma}_{j+\ell_2}) \cdots (\boldsymbol{\sigma}_{j+\ell_{k-1}} \cdot \boldsymbol{\sigma}_{j+\ell_k})$. Interestingly, it turns out that the latter are not needed for our purposes. At the same time, for odd k the number of linearly independent k-spin polynomials equals D_k and they form a complete basis.

We thus see that the operator $O_j(\boldsymbol{\ell}_k(s))$ in Eq. (13) can be taken as

$$\begin{aligned} O_j\big(\boldsymbol{\ell}_k(s)\big) &= f_j(\ell_1, \ldots, \ell_k) \\ &= [((\boldsymbol{\sigma}_{j+\ell_1} \times \boldsymbol{\sigma}_{j+\ell_2}) \times \cdots) \times \boldsymbol{\sigma}_{j+\ell_{k-1}}] \cdot \boldsymbol{\sigma}_{j+\ell_k}, \end{aligned} \tag{19}$$

where ℓ_j are the same as those discussed after Eq. (14). Obviously the charges Q_n of the unperturbed model H_0 can also be expressed in terms of $f_j(\ell_1, \ldots, \ell_k)$ from Eq. (17). Let us note that the corrections $\delta Q_n^{(s)}$ in Eq. (10) are allowed to have terms that are present in the unperturbed charge Q_n. Explicitly, the spin polynomial in Eqs. (17) and (19) can be written as

$$f_j(\ell_1, \ldots, \ell_k) = \sum_{\beta_1 \cdots \beta_k} C_{\beta_1 \cdots \beta_k} \bigotimes_{p=1}^{k} \sigma_{j+\ell_p}^{\beta_p}, \tag{20}$$

where the coefficients $C_{\beta_1 \ldots \beta_k}$ arise from expanding the scalar product in the spin polynomial $f_j(\ell_1, \ldots, \ell_k)$. For $k = 2$ and 3, one obviously has $C_{\beta_1\beta_2} = 1$ and $C_{\beta_1\beta_2\beta_3} = \varepsilon_{\beta_1\beta_2\beta_3}$, correspondingly, where ε_{ijk} is the Levi-Civita symbol. For $k > 3$ the coefficients are given by

$$C_{\beta_1 \cdots \beta_k} = \sum_{\alpha_1 \cdots \alpha_{k-3}} \varepsilon_{\alpha_1 \beta_{k-1} \beta_k} \varepsilon_{\alpha_{k-3} \beta_1 \beta_2} \prod_{q=1}^{k-4} \varepsilon_{\alpha_q \alpha_{q+1} \beta_{k-q-1}}. \tag{21}$$

Thus, to fully understand the operator content of the quasiconserved charges $\tilde{Q}_n$, it remains to determine the support of the corrections $\delta Q_n^{(s)}$, i.e., the range of summation over s in Eq. (10). Let us consider the the spin polynomial $f_i(m_1, \ldots, m_k)$ with the support on s sites, i.e., for some p and q we have $m_p = 0$ and $m_q = s - 1$. Then, as shown in Appendix A, the commutator $[f_j(0,l),\ f_i(m_1, \ldots, m_k)]$ contains terms with the support up to $s + l$. Thus, since in general the unperturbed charge Q_n may contain K terms with the support on $\{n_1, \ldots, n_{K-1}, n\}$ sites, with $n_1 < \cdots < n$, the commutator $[H_1, Q_n]$ consists of terms with the support $\{n_1 + 2, \ldots, n_{K-1} + 2, n + 2\}$. Therefore, we immediately see that in order to satisfy Eq. (11) the correction $\sum_{s=m}^{M} \delta Q_n^{(s)}$ must include terms having the support on $\{n_1 + 1, \ldots, n_{K-1} + 1, n + 1\}$ sites, such that the maximal support of the commutator $[H_0, \sum_{s=m}^{M} \delta Q_n^{(s)}]$ matches that of $[H_1, Q_n]$ and the two commutators cancel each other. Therefore, for the quasiconserved charges we finally have

$$\tilde{Q}_n = Q_n + \lambda \sum_{s=n_1}^{n+1} \delta Q_n^{(s)}, \tag{22}$$

where n_1 and n are, correspondingly, the smallest and the largest support of the terms in Q_n. Note that some of the terms in Eq. (22) may have zero coefficient. For instance, as shown in the next section, in $\tilde{Q}_3$ the only corrections that are present have the support $s = n + 1 = 4$. With this, we now proceed to investigating the possibility of satisfying Eqs. (8) and (11).

IV. RESULTS AND DISCUSSION

Taking into account Eqs. (10), (13), (17), and (19) to construct the ansatz for the quasiconserved charges $\tilde{Q}_n$, we fix the coefficients $c_n(\boldsymbol{\ell}_k(s))$ in Eq. (13) such that the criterion (11) is satisfied, which guarantees the required scaling behavior of the commutator norm in Eq. (8). In order to evaluate the commutators in Eq. (11) we used the results of Appendix B. Then, for the lowest order quasiconserved charge $\tilde{Q}_3$ we obtain

$$\begin{aligned} \tilde{Q}_3 &= Q_3 + \lambda\, \delta Q_3^{(4)}, \\ \delta Q_3^{(4)} &= \sum_j (\boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+1} + \boldsymbol{\sigma}_j \times \boldsymbol{\sigma}_{j+2}) \cdot \boldsymbol{\sigma}_{j+3}. \end{aligned} \tag{23}$$

From Eqs. (7) and (23) we see that the correction $\delta Q_3^{(4)}$ has a structure resembling that of Q_3, i.e., every term in $\delta Q_3^{(4)}$ has the same number of spins, but the support is increased by one site. Let us now look at the norm of $[\tilde{Q}_3, H_\lambda]$. For computational convenience we use Frobenius norm defined as

$$\|X\|_F = \sqrt{\mathrm{tr}(X^\dagger X)}. \tag{24}$$

Then, from Eqs. (12) and (24) we have

$$\frac{1}{\|H_0\|_F} \|[H_\lambda, \tilde{Q}_3]\|_F = 8\,\lambda^2, \tag{25}$$

where $\|H_0\|_F = \sqrt{3N}\, 2^{N/2}$ is introduced for normalization. One can try to further minimize the commutator norm (25) by adding to the quasiconserved charge $\tilde{Q}_3$ a term $a\lambda Q_3$, where a is an arbitrary real constant. Since $[H_0, Q_3] = 0$, this does not affect the condition (11). However, the new term contributes to the right-hand side of Eq. (12). Consequently, Eq. (25) transforms into

$$\frac{\|[H_\lambda, \tilde{Q}_3 + a\,\lambda Q_3]\|_F}{\|H_0\|_F} = 4\sqrt{2}\,\lambda^2 \sqrt{(2 + a + a^2)}, \tag{26}$$

which is minimal at $a=-1/2$ and equals $2\sqrt{14}\approx 7.48$, which differs from the value at $a=0$ by $\delta\approx 0.6\%$. Naively, one could think that the norm in Eq. (26) can be made zero if we allow for complex a (which would mean $\tilde{Q}_3$ is non-Hermitian). This is not the case since in the latter case the right-hand side of Eq. (26) becomes $4\sqrt{2}\lambda^2\sqrt{2+\mathrm{Re}(a)+|a|^2}$.

Repeating the steps from the beginning of this section, we obtained the quasiconserved charges $\tilde{Q}_n$ with $n=4$, 5, and 6. The latter two, $\tilde{Q}_5$ and $\tilde{Q}_6$, contain a large number of terms and for the sake of readability, we present their explicit form in Appendix C. Here we only give the expression for $\tilde{Q}_4$, which reads

$$\begin{aligned}
\tilde{Q}_4 &= Q_4+\lambda\sum_{s=3}^{5}\delta Q_4^{(s)},\\
\delta Q_4^{(3)} &= -2\sum_j \boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_{j+2},\\
\delta Q_4^{(4)} &= \sum_j\{\boldsymbol{\sigma}_j-((\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_{j+2})\times\boldsymbol{\sigma}_{j+1})\}\cdot\boldsymbol{\sigma}_{j+3},\\
\delta Q_4^{(5)} &= \sum_j\{((\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_{j+1})\times\boldsymbol{\sigma}_{j+2})\cdot\boldsymbol{\sigma}_{j+4}\\
&\quad+((\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_{j+1})\times\boldsymbol{\sigma}_{j+3})\cdot\boldsymbol{\sigma}_{j+4}\\
&\quad+((\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_{j+2})\times\boldsymbol{\sigma}_{j+3})\cdot\boldsymbol{\sigma}_{j+4}\}.
\end{aligned}\tag{27}$$

Then, similarly to the case of $\tilde{Q}_3$, we calculate the commutator norm

$$\frac{\|[H_\lambda,\tilde{Q}_4+a\lambda Q_4]\|_F}{\|H_0\|_F}=4\sqrt{2}\,\lambda^2\,(4a^2+a+10)^{1/2}.\tag{28}$$

Equation (28) is minimal at $a=-1/8$ and equals $\sqrt{318}\lambda^2$, whereas for $a=0$ the squared norm (28) becomes $\sqrt{320}\lambda^2$. The difference is approximately $\delta=0.6\%$, just like for the previous quasiconserved charge. However, the minimization procedure gives much better results for higher order charges. As we show in Appendix C, for $n=5$ and 6 one has $\delta\approx 6\%$ and $\delta\approx 19\%$, correspondingly. Thus, the role of the correction $a\lambda Q_n$ to the quasiconserved charge $\tilde{Q}_n$ is more pronounced for larger values of n and increases rapidly.

One can check by a lengthy but straightforward direct calculation that the commutator of the quasiconserved charges $[\tilde{Q}_n,\tilde{Q}_m]$ does not contain terms linear in λ. We expect that this is correct for any n and m and have checked this explicitly for $\tilde{Q}_n$ with $3\geqslant n\geqslant 6$. It is also straightforward to see that any linear combination of the quasiconserved charges is a quasiconserved charge itself.

To summarize, in this paper we have shown that an isotropic Heisenberg spin chain, weakly perturbed away from integrability by a next to nearest neighbor interaction of strength λ, possesses quasiconserved charges $\tilde{Q}_n$ with $3\leqslant n\leqslant 6$, which are approximately conserved up to times of the order λ^{-2}. We conjecture that the perturbed model H_λ from Eq. (2) has as many quasiconserved charges $\tilde{Q}_n$ as there are conserved charges Q_n for the integrable model (3), but the proof of our conjecture is beyond the scope of the present paper. We also expect that our results can be extended to the case of other perturbations and open boundary conditions, as well as to other one-dimensional models with SU(2) symmetry, e.g., the Hubbard model, which we leave for future studies. The presence of these quasiconserved charges could affect some transport properties, see, e.g., [46,47].

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APPENDIX A

Let us consider the commutator of $f_i(0,l)$ and $f_j(m_1,\ldots,m_k)$:

$$[f_j(0,l),\ f_i(m_1,\ldots,m_k)]\ =\sum_{i,j}\sum_{\alpha}\sum_{\beta_1\cdots\beta_k}C_{\beta_1\ldots\beta_k}\left[\sigma_j^\alpha\sigma_{j+l}^\alpha,\bigotimes_{p=1}^{k}\sigma_{i+m_p}^{\beta_p}\right],\tag{A1}$$

where the polynomial $f_i(m_1,\ldots,m_k)$ contains terms with k Pauli matrices and is assumed to have the support on s sites. Then we rewrite the commutator on the right-hand side of Eq. (A1) as

$$\sum_{i,j}\left[\sigma_i^\alpha\sigma_{i+l}^\alpha,\bigotimes_{p=1}^{k}\sigma_{j+m_p}^{\beta_p}\right]=\sum_j\sum_{p=1}^{k}\left(\mathcal{A}_j(p)+(1-\delta_{l,m_p})\prod_{q=1\ q\neq p}^{k}\left(1-\delta_{l,m_p-m_q}\right)\mathcal{B}_j(p)\right),\tag{A2}$$

where $\mathcal{A}_j(p)$ and $\mathcal{B}_j(p)$ arise from the terms in Eq. (A1) with $i=j+m_p$ and $i+l=j+m_p$, correspondingly. Explicitly, we have

$$\mathcal{A}_j(p)=\left[\sigma_j^\alpha\sigma_{j+l}^\alpha,\sigma_j^{\beta_p}\bigotimes_{q=1\ q\neq p}^{k}\sigma_{j+m_q-m_p}^{\beta_q}\right],\quad \mathcal{B}_j(p)=\sigma_j^\alpha\left[\sigma_{j+l}^\alpha,\sigma_{j+l}^{\beta_p}\right]\bigotimes_{r=1\ r\neq p}^{k}\sigma_{j+l+m_r-m_p}^{\beta_r}.\tag{A3}$$

It is straightforward to see that

$$\mathcal{A}_j(p) = \sum_{q=1\ q\neq p}^{k} \delta_{l,m_q-m_p}\mathcal{A}_j^{(1)}(p,q) + \prod_{q=1\ q\neq p}^{k} \big(1-\delta_{l,m_q-m_p}\big)\mathcal{A}_j^{(2)}(p), \tag{A4}$$

where

$$\mathcal{A}_j^{(1)}(p,q) = \big[\sigma_j^\alpha\sigma_{j+l}^\alpha, \sigma_j^{\beta_p}\sigma_{j+l}^{\beta_q}\big] \bigotimes_{r=1\ r\neq p,q}^{k} \sigma_{j+m_r-m_p}^{\beta_r}, \quad \mathcal{A}_j^{(2)}(p) = \big[\sigma_j^\alpha, \sigma_j^{\beta_p}\big]\sigma_{j+l}^\alpha \bigotimes_{r=1\ r\neq p}^{k} \sigma_{j+m_r-m_p}^{\beta_r}. \tag{A5}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{A}_j^{(1)}(p,q) &= 2i\left(\delta_{\alpha,\beta_q}\sum_{\gamma_p}\varepsilon_{\alpha,\beta_p,\gamma_p}\sigma_j^{\gamma_p} + \delta_{\alpha,\beta_p}\sum_{\gamma_q}\varepsilon_{\alpha,\beta_q,\gamma_q}\sigma_{j+l}^{\gamma_q}\right)\bigotimes_{r=1\ r\neq p,q}^{k}\sigma_{j+m_r-m_p}^{\beta_r},\\
\mathcal{A}_j^{(2)}(p) &= 2i\sum_{\gamma_p}\varepsilon_{\alpha,\beta_p,\gamma_p}\sigma_j^{\gamma_p}\sigma_{j+l}^{\alpha}\bigotimes_{r=1\ r\neq p}^{k}\sigma_{j+m_r-m_p}^{\beta_r}, \quad \mathcal{B}_j(p) = 2i\,\sigma_j^\alpha\sum_{\gamma_p}\varepsilon_{\alpha,\beta_p,\gamma_p}\sigma_{j+l}^{\gamma_p}\bigotimes_{r=1\ r\neq p}^{k}\sigma_{j+l+m_r-m_p}^{\beta_r}.
\end{aligned} \tag{A6}$$

It is now easy to see that the commutator in Eq. (A1) consists of terms with the support not greater than $s+l$. Indeed, let us consider, e.g., $\mathcal{B}_j(p)$ from Eq. (A6) and take $m_p = 0$. Then, for some $r = r_*$ we have $m_{r_*} = s-1$ and the Pauli matrix with the largest index is $\sigma_{j+l+s-1}^{\beta_{r_*}}$, whereas the Pauli matrix with the smallest index is σ_j^α. Clearly such a term has a support on $s+l$ sites.

APPENDIX B

In this Appendix we explicitly calculate the commutator of 2-spin polynomial $f_j(0,l)$ with n-spin polynomials for $2 \leqslant n \leqslant 4$. We use the following identities:

$$\begin{aligned}
&[\boldsymbol{A}\cdot\boldsymbol{X}, \boldsymbol{A}\cdot\boldsymbol{Y}] = 2i(\boldsymbol{X}\times\boldsymbol{Y})\cdot\boldsymbol{A},\\
&[\boldsymbol{A}\cdot\boldsymbol{B}, (\boldsymbol{B}\times\boldsymbol{L})\cdot\boldsymbol{R}] = -2i((\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{L})\cdot\boldsymbol{R},\\
&[\boldsymbol{A}\cdot\boldsymbol{B}, (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{R}] = 4i(\boldsymbol{B}\cdot\boldsymbol{R} - \boldsymbol{A}\cdot\boldsymbol{R}),\\
&[\boldsymbol{A}\cdot\boldsymbol{B}, ((\boldsymbol{L}\times\boldsymbol{A})\times\boldsymbol{B})\cdot\boldsymbol{R}] = 2i((\boldsymbol{L}\times\boldsymbol{B})\cdot\boldsymbol{R} - (\boldsymbol{L}\times\boldsymbol{A})\cdot\boldsymbol{R}),
\end{aligned} \tag{B1}$$

where $\boldsymbol{A} = \{\sigma_{j_1}^x, \sigma_{j_1}^y, \sigma_{j_1}^z\}$, $\boldsymbol{B} = \{\sigma_{j_2}^x, \sigma_{j_2}^y, \sigma_{j_2}^z\}$, and $\boldsymbol{L}, \boldsymbol{R}$ are arbitrary tensor products that commute with both $\boldsymbol{A}$ and $\boldsymbol{B}$. Then one can show that

$$\sum_{j,k}[f_j(0,l), f_k(0,m)] = 2i\sum_j(f_j(0,l,m) + f_j(0,m,l+m) - f_j(0,l,l+m) - f_j(0,m-l,m)), \quad l\neq m, \tag{B2}$$

and zero if $l = m$.

$$\begin{aligned}
\sum_{j,k}[f_j(0,l), f_k(0,m,n)] = 2i\sum_j\{&(1-\delta_{l,m}-\delta_{l,n})f_j(0,l,m,n) + 2\delta_{l,m}[f_j(0,n-l) - f_j(0,n)]\\
&- 2\delta_{l,n}[f_j(0,m-l) - f_j(0,m)] - (1-\delta_{l,n-m})f_j(m,l+m,0,n)\\
&+ 2\delta_{l,n-m}[f_j(0,l+m) - f_j(0,m)] + f_j(n,l+n,0,m) - f_j(0,l,l+m,l+n)\\
&+ (1-\delta_{l,m})f_j(0,l,l-m,l-m+n) - (1-\delta_{l,n})(1-\delta_{l,n-m})f_j(0,l,l-n,l+m-n)\}
\end{aligned} \tag{B3}$$

and

$$\sum_{j,k}[f_j(0,l), f_k(m,n,p,q)] = 2i\sum_j\{K(l,m,n,p,q) - K(l,n,m,p,q) + K(l,p,q,m,n) - K(l,q,p,m,n)\}, \tag{B4}$$

where we denoted

$$\begin{aligned}
K(l,m,n,p,q) = &4\delta_{l,n-m}[f(0,-l-m+p,-l-m+q) - f(0,p-m,q-m)]\\
&- 2\delta_{l,p-m}[f(-l-m+n,0,-l-m+q) - f(n-m,0,q-m)]\\
&- 2\delta_{l,q-m}[f(-l-m+p,0,-l-m+n) - f(p-m,0,n-m)]\\
&+ 2(1-\delta_{l,n-m}-\delta_{l,p-m}-\delta_{l,q-m})f(0,l,n-m,p-m,q-m)\\
&- 2(1-\delta_{l,m})(1-\delta_{l,m-n})(1-\delta_{l,m-p})(1-\delta_{l,m-q})f(0,l,l-m+n,l-m+p,l-m+q).
\end{aligned} \tag{B5}$$

APPENDIX C

In this Appendix we present the explicit forms of the quasiconserved charges $\tilde{Q}_5$ and $\tilde{Q}_6$. We also calculate the norms of the commutators $[H_\lambda, \tilde{Q}_n + a\lambda Q_5]$, in the same way it is done for $n = 3$ and 4 in Sec. IV of the main text. We then minimize the commutator norm with respect to a.

1. Explicit form of $\tilde{Q}_5$

The basis of SU(2)-invariant operators for $k = 5$ spins is six dimensional [see Eq. (15) in the main text] and one has the following relations between linearly dependent 5-spin polynomials:

$$\begin{pmatrix} f_j(l,q,m,n,p) \\ f_j(l,p,q,m,n) \\ f_j(l,p,m,n,q) \\ f_j(l,n,q,m,p) \\ f_j(l,n,p,m,q) \\ f_j(l,n,m,p,q) \\ f_j(l,m,q,n,p) \\ f_j(l,m,p,n,q) \\ f_j(l,m,n,p,q) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} f_j(l,p,n,m,q) \\ f_j(l,q,n,m,p) \\ f_j(l,q,p,m,n) \\ f_j(m,n,l,p,q) \\ f_j(m,p,l,n,q) \\ f_j(m,q,l,n,p) \end{pmatrix}. \tag{C1}$$

For the quasiconserved charge $\tilde{Q}_5$ we have

$$\begin{aligned} \tilde{Q}_5 &= Q_5 + \lambda \sum_{s=5}^{6} \delta Q_5^{(s)}, \\ \delta Q_5^{(5)} &= \sum_j [f_j(0,1,4) + 2f_j(0,2,4) + f_j(0,3,4) + f_j(1,3,0,2,4) + f_j(0,4,3,1,2) - f_j(0,3,2,1,4)], \\ \delta Q_5^{(6)} &= \sum_j [f_j(0,3,2,1,5) - f_j(0,5,2,1,3) - f_j(1,2,0,3,5) - f_j(1,3,0,2,5) + f_j(1,5,0,2,3)] \\ &\quad + \sum_j [f_j(0,4,2,1,5) - f_j(0,5,2,1,4) - f_j(1,2,0,4,5) - f_j(1,4,0,2,5) + f_j(1,5,0,2,4)] \\ &\quad + \sum_j [f(_j0,4,3,1,5) - f_j(0,5,3,1,4) - f_j(1,3,0,4,5) - f_j(1,4,0,3,5) + f_j(1,5,0,3,4)] \\ &\quad + \sum_j [f_j(0,4,3,2,5) - f_j(0,5,3,2,4) - f_j(2,3,0,4,5) - f_j(2,4,0,3,5) + f_j(2,5,0,3,4)]. \end{aligned} \tag{C2}$$

Using Eq. (24) for the norm of $[H_\lambda, \tilde{Q}_5]$ we obtain

$$\frac{1}{\|H_0\|_F} \|[H_\lambda, \tilde{Q}_5 + a\lambda Q_5]\|_F = 8\,\lambda^2\,(5a^2 - 7a + 20)^{1/2}. \tag{C3}$$

At $a = 0.7$ the norm is minimal and the right-hand side of Eq. (C3) becomes $\approx 33.5\lambda^2$, whereas at $a = 0$ it is equal to $16\sqrt{5}\lambda^2 \approx 35.8\lambda^2$ and the difference is around 6%.

2. Explicit form of $\tilde{Q}_6$

According to Eq. (15) in the main text, the basis of SU(2)-invariant operators for $k = 6$ spins is 15 dimensional. We find that only its 14-dimensional subspace V_6 spanned by the 6-spin polynomials is needed. The latter can be chosen as

$$\begin{aligned} V_6 = \{ & f_j(2,4,0,3,1,5),\ f_j(2,4,1,0,3,5),\ f_j(2,4,1,3,0,5),\ f_j(2,4,3,0,1,5),\ f_j(2,4,3,1,0,5), \\ & f_j(3,4,0,1,2,5),\ f_j(3,4,0,2,1,5),\ f_j(3,4,1,0,2,5),\ f_j(3,4,1,2,0,5),\ f_j(3,4,2,0,1,5), \\ & f_j(3,4,2,1,0,5),\ f_j(1,4,3,2,0,5),\ f_j(1,4,3,0,2,5),\ f_j(0,4,3,2,1,5)\}. \end{aligned} \tag{C4}$$

The quasiconserved charge $\tilde{Q}_6$ has the following form:

$$\begin{aligned}
\tilde{Q}_6 &= Q_6 + \lambda \sum_{s=3}^{7} \delta Q_6^{(s)}, \\
\delta Q_6^{(3)} &= -\sum_j f_j(0,2), \quad \delta Q_6^{(4)} = 3\sum_j f_j(0,3), \\
\delta Q_6^{(5)} &= \sum_j \{f_j(0,4) + 2f_j(0,3,1,4) - f_j(0,4,1,2) - 2f_j(0,4,1,3) - f_j(0,4,2,3)\}, \\
\delta Q_6^{(6)} &= \sum_j \{f_j(0,2,1,5) + 2f_j(0,3,1,5) + 2f_j(0,3,2,5) + f_j(0,4,1,5) + 2f_j(0,4,2,5) + f(0,4,3,5) \\
&\quad - f_j(0,5,1,2) - 2f_j(0,5,1,3) - f_j(0,5,1,4) - 2f_j(0,5,2,3) - 2f_j(0,5,2,4) - f_j(0,5,3,4) \\
&\quad - f_j(0,4,3,2,1,5) + f_j(1,4,3,2,0,5) + f_j(3,4,0,2,1,5) - f_j(3,4,2,1,0,5)\},
\end{aligned} \tag{C5}$$

$$\begin{aligned}
\delta Q_6^{(7)} = \sum_j \Big\{ & f_j(0,4,3,2,1,6) + f_j(0,5,3,2,1,6) + f_j(0,5,4,2,1,6) + f_j(0,5,4,3,1,6) + f_j(0,5,4,3,2,6) \\
& - f_j(1,4,3,0,2,5) - f_j(1,4,3,2,0,6) - f_j(1,5,3,2,0,6) - f_j(1,5,4,2,0,6) - f_j(1,5,4,3,0,6) \\
& + f_j(2,4,0,3,1,5) - f_j(2,4,1,3,0,5) + f_j(2,4,3,1,0,5) - f(2,5,4,3,0,6) \\
& - \frac{1}{2}f_j(3,4,0,1,2,6) - \frac{1}{2}f_j(3,4,0,2,1,6) + \frac{1}{2}f_j(3,4,1,0,2,6) + \frac{1}{2}f_j(3,4,1,2,0,6) + \frac{1}{2}f_j(3,4,2,0,1,6) \\
& - \frac{1}{2}f_j(3,4,2,1,0,6) - \frac{1}{2}f_j(3,5,0,1,2,6) - \frac{1}{2}f_j(3,5,0,2,1,6) + \frac{1}{2}f_j(3,5,1,0,2,6) + \frac{1}{2}f(3,5,1,2,0,6) \\
& + \frac{1}{2}f(3,5,2,0,1,6) - \frac{1}{2}f(3,5,2,1,0,6) - \frac{1}{2}f(4,5,0,1,2,6) - \frac{1}{2}f(4,5,0,1,3,6) - \frac{1}{2}f(4,5,0,2,1,6) \\
& - \frac{1}{2}f(4,5,0,2,3,6) - \frac{1}{2}f(4,5,0,3,1,6) - \frac{1}{2}f(4,5,0,3,2,6) + \frac{1}{2}f(4,5,1,0,2,6) + \frac{1}{2}f(4,5,1,0,3,6) \\
& + \frac{1}{2}f(4,5,1,2,0,6) + \frac{1}{2}f(4,5,1,3,0,6) + \frac{1}{2}f(4,5,2,0,1,6) + \frac{1}{2}f(4,5,2,0,3,6) - \frac{1}{2}f(4,5,2,1,0,6) \\
& + \frac{1}{2}f(4,5,2,3,0,6) + \frac{1}{2}f(4,5,3,0,1,6) + \frac{1}{2}f(4,5,3,0,2,6) - \frac{1}{2}f(4,5,3,1,0,6) - \frac{1}{2}f(4,5,3,2,0,6)\Big\}.
\end{aligned} \tag{C6}$$

Then, using Eq. (24), for the commutator norm we obtain

$$\frac{1}{\|H_0\|_F}\|[H_\lambda, \tilde{Q}_6 + a\lambda Q_6]\|_F = 8\,\lambda^2\,(66a^2 + 102a + 113)^{1/2}, \tag{C7}$$

the minimum is achieved at $a = -17/22 \approx 0.77$ and we have $\approx 68.6\lambda^2$, whereas for $a = 0$ we have $\approx 85\lambda^2$, and the difference is close to 19%.

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