Charge and spin supercurrents in magnetic Josephson junctions with spin filters and domain walls

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We analyze theoretically the influence of domain walls (DWs) on the DC Josephson current in magnetic superconducting S_m/Fl/F/Fl/S_m junctions. The Josephson junction consists of two "magnetic" superconductors S_m (superconducting film covered by a thin ferromagnetic layer), spin filters Fl, and a ferromagnetic layer F with or without DW(s). The spin filters Fl allow electrons to pass with one specific spin orientation, such that the Josephson coupling is governed by a fully polarized long-range triplet component. In the absence of DW(s), the Josephson and spin currents are nonzero when the right and left filters, $\mathrm{Fl}_{r,l}$, pass electrons with equal spin orientation and differ only by a temperature-independent factor. They become zero when the spin orientation of the triplet Cooper pairs passing through the $\mathrm{Fl}_{r,l}$ have opposite directions. Furthermore, for the different chiralities of the injected triplet Cooper pairs, the spontaneous currents arise in the junction yielding a diode effect. Once a DW is introduced, it reduces the critical Josephson current I_c in the case of equal spin polarization and makes it finite in the case of opposite spin orientation. The critical current I_c is maximal when the DW is in the center of the F film. A deviation of the DW from the center generates a force that pushes the DW to the center of the F film. In addition, we consider the case of an arbitrary number N of DWs, with the case $N = 2$ resembling a magnetic skyrmion.

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I. INTRODUCTION

Over the past decade, there has been a significant interest in studying the properties of superconductor/ferromagnet (S/F) hybrid structures. One of the particular aspects of these heterostructures is related to remarkable phenomena caused by the magnetic interaction of topological textures in the superconductor (Abrikosov and Pearl vortices [1,2]) and in the ferromagnet (domain walls or skyrmions [3–6]). The interaction of vortices with the magnetic field in a ferromagnet may result in a spontaneous generation of vortices in the superconductor S in S/F bilayers [7–11]. This effect occurs in the absence of a direct contact between the electron systems in S and F (no proximity effect) and is caused by the magnetic field generated by vortices or the magnetic textures.

At the same time, the penetration of Cooper pairs into a ferromagnet (the proximity effect) leads to a number of further interesting effects. In particular, the Josephson current in S/F/S junctions may change sign in a certain temperature interval (see [12–17] and also reviews [18,19]). Another interesting effect is the triplet component which arises in S/F hybrid structures with an inhomogeneous magnetization $\mathbf{M}(\mathbf{r})$ in F. If the magnetization is uniform, the Cooper pairs penetrating into the ferromagnet consist of singlet and short-range triplet components, respectively. Both components penetrate into the ferromagnet over a short length scale $\xi_J \approx \sqrt{D_F/J}$ (in the diffusive case), where D_F is the diffusion coefficient and J is the exchange field which, in most ferromagnets, is much larger than the temperature T [19,20]. If the magnetization $\mathbf{M}(\mathbf{r})$ is nonhomogeneous, as occurs, for example, in a S/F_m/F structure, then a long-range triplet component (LRTC) may occur in the system. Here, F_m is a weak ferromagnet with magnetization magnitude $\mathbf{m}$ much less than $\mathbf{M}$ and a direction is noncollinear to $\mathbf{M}$. This component propagates into the F region over a long, compared to ξ_J, length of the order of $\xi_T \approx \sqrt{D_F/\pi T}$ [21,22]. In this case, the superfluid component in most parts of F consists solely of triplet Cooper pairs. For example, the Josephson coupling in a S/F_m/F/F_m/S structure can be realized through the LRTC as was predicted [23–29] (see also reviews [18–20,30–32], and references therein) and observed experimentally [33–48]. Interestingly, the long-range triplet Cooper pairs with spin-up and -down orientations penetrate the ferromagnet F regardless of the magnetization orientation $\mathbf{M}$ [49] so that the spin current I_{sp} in S/F_m/F/F_m/S Josephson junctions is absent, whereas the charge current I_Q is nonzero. Only in the presence of spin filters at the S/F_m interfaces does the current I_{sp} become finite.

In this paper, we calculate the Josephson charge I_Q and spin I_{sp} currents in the S_m/Fl/F/Fl/S_m Josephson junctions under various conditions, where S_m = S/F_m is a conventional superconductor covered by a thin ferromagnetic layer. First we consider the system without DWs and calculate the currents $I_{Q,sp}$: (a) in the absence or presence of spin filters at the S/F_m interfaces and (b) for equal or different polarizations or chiralities of the triplet Cooper pairs injected into F from the left and right superconductors S. Most importantly, we also study the influence of the domain walls in F (DWs) on the I_Q and I_{sp} in the dirty case when the condensate Green's functions $\hat{f}$ obey the Usadel equation. Within this approximation the Green's functions $\hat{f}$ do not depend on the momentum direction $\mathbf{p}/|\mathbf{p}|$. Therefore, according to the Pauli principle, the functions for the LRTC $\hat{f}(t, t') \sim \langle c_\uparrow(t)c_\uparrow(t')\rangle$ are zero at coinciding times $t = t'$. In other words, these are odd functions of the Matsubara frequency ω, $\hat{f}(\omega) = -\hat{f}(-\omega)$, so that summing over all ω gives zero: $\hat{f}(t, t) \sim \sum_\omega \hat{f}(\omega) = 0$. The triplet odd-frequency Cooper pairs exist in any superconducting system if there is a Zeeman interaction of electron spins and a magnetic or exchange field. This case was studied long ago [50–55]. Unlike homogeneous superconductors with Zeeman interaction, where the triplet component coexists with the singlet one, the recently studied hybrid S/F systems allow the separation of triplet and singlet Cooper pairs. In addition, we assume a weak proximity effect allowing linearization of the necessary equations and the boundary conditions yielding simple analytical expressions for $\hat{f}(r)$ and the currents $I_{Q,sp}$.

Although the Josephson effect has been studied for similar structures in various limiting cases (see references above as well as [56–61]), there is no systematic study of the dependence of the $I_{Q,sp}$ on spin polarization, chiralities, and the presence of the spin filters and DWs. In particular, we show that although the current I_Q is zero for opposite polarization directions and different chiralities of injected Cooper pairs in the presence of spin filters, it becomes finite in the presence of DWs. We will consider an arbitrary number of DWs and pay special attention to the case of two DWs. The latter case, to some extent, may be regarded as a model of magnetic texture such as a skyrmion with $N = 1$ winding number (like a Bloch or Néel skyrmion) when the magnetization profile $\mathbf{M}$ has the same orientation outside the DWs and the opposite orientation between DWs [3–6].

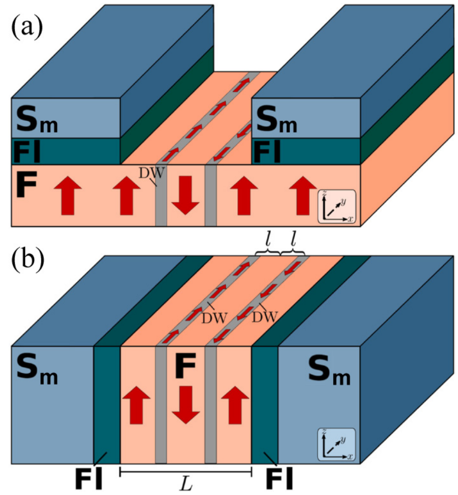

FIG. 1. Possible S_m/Fl/F/Fl/S_m junction setups for the system under consideration shown exemplarily for two domain walls (DW). Here, $S_m = S/F_m$ is a conventional superconductor covered by a thin ferromagnetic layer, Fl are spin filters, and F is a ferromagnetic layer. The red arrows indicate the orientation of the magnetization; the current flows from left to right.

II. BASIC EQUATIONS

We consider an S_m/Fl/F/Fl/S_m Josephson junction with one or multiple DWs in the F film (wire). Schematically the considered system is shown in Fig. 1. The junction consists of two "magnetic" superconductors S_m and of two filters (Fl) which allow only electrons with a single spin polarization, parallel or antiparallel to the z axis, to pass through. The magnetic superconductors may be made of conventional superconductors covered by ferromagnetic thin films with the magnetization aligned parallel to the x or y axis. The magnetization vector $\mathbf{M} = (0, 0, M)$ is supposed to be oriented along the z axis. The filters may be magnetic insulators selecting electrons with a spin collinear to the z axis. The Cooper pairs penetrating into the F film due to proximity effect consist of triplet long-range components only. We assume that the proximity effect is weak as is the case in most experimental setups. The Cooper pairs are described by a matrix quasiclassical Green's function $\hat{f}(x)$, which is supposed to be small, $|\hat{f}| \ll 1$. The function $\hat{f}(x)$ in the F film obeys the linearized Usadel equation [18–20,30,32]

$$-\partial^2_{xx}\hat{f} + \kappa^2_\omega \hat{f} + \frac{i\kappa^2_J}{2}(n_z(x)[\hat{X}_{03}, \hat{f}]_+ + n_k(x)[\hat{X}_{0k}, \hat{f}]_+) = 0, \tag{1}$$

where $\kappa^2_\omega = 2|\omega|/D_F$, $\kappa^2_{J\omega} = 2J\ \mathrm{sgn}\omega)/D_F$, J is an exchange field, and D_F is the diffusion coefficient in the F film which is assumed to be spin independent. Note that the quasiclassical equation with a spin-dependent D_F has been derived previously in various models [62,63]. The 4×4 matrix $\hat{X}_{ik}(x) = \hat{\tau}_i \cdot \hat{\sigma}_k$ is a tensor product of the Pauli matrices in the Gor'kov-Nambu, $\hat{\tau}_i$, and spin space, $\hat{\sigma}_k$, respectively. The square brackets are anticommutators $[\hat{X}_{03}, \hat{f}]_+ = \hat{X}_{03} \cdot \hat{f} + \hat{f} \cdot\ \hat{X}_{03}$. The DW is assumed to be of the Bloch type ($k = y, z$) and is described by a unit vector $\mathbf{n}(x) = (0, n_y(x), n_z(x))$, where $n_y = \sin\alpha(x)$, $n_z(x) = \cos\alpha(x)$. The angle $\alpha(x)$ describes the DW profile: it is equal to 0 (left from DW) and to π (right from DW) far away from DW. The characteristic size of the DW is

$$d_W = \int dx\, \sin\alpha(x). \tag{2}$$

In a general case Eq. (1) can be solved only numerically. However, an exact solution can also be obtained under some assumptions like, for example, a piecewise linear form of the DW [21,64]. Here we will use a simple model assuming that the width of the DW is small. Then, the last term in Eq. (1) can be written in the form $\delta(x - l_i)d_{DW}[\hat{X}_{02}, \hat{f}]_+$, where l_i is a position of the DW. This approximation is valid for any dependence $\alpha(x)$. Thus, the Usadel equation reduces to

$$-\partial^2_{xx}\hat{f} + \kappa^2_\omega \hat{f} + i(\kappa^2_J/2)[\hat{X}_{03}, \hat{f}]_+ = 0, \tag{3}$$

with matching conditions at $x = l_i$,

$$\hat{f}|_{l_i+0} - \hat{f}|_{l_i-0} = 0, \tag{4}$$

$$\partial_x\hat{f}|_{l_i+0} - \partial_x\hat{f}|_{l_i-0} = i(\kappa_{DW}/2)[\hat{X}_{02}, \hat{f}]_+, \tag{5}$$

where $\kappa_{DW} = \kappa^2_J d_{DW}$.

Although we assume that the DW thickness d_{DW} is small, its finite value in the right-hand side of Eq. (5) allows one to take into account noncollinearity of the magnetization vectors inside ($\mathbf{M}_{\rm in}$) and outside ($\mathbf{M}_{\rm out}$) of the DW: $\mathbf{M}_{\rm in} \parallel \mathbf{e}_y$ and $\mathbf{M}_{\rm out} \parallel \mathbf{e}_z$. Therefore, a conversion of the LRTC into the short-range component and vice versa occurs at the DW. Thus, the problem considered here differs drastically from the problem with collinear magnetizations $\mathbf{M}_{\rm out}$ and zero width d_{DW} [65] where there is no such conversion.

A solution of Eq. (3) consists of a short-range and a long-range component, respectively. The first one decays on a short distance of the order of $\xi_J \approx \sqrt{D_F/J}$, while the second varies on a much longer characteristic length of the order of $\xi_{lr} \approx \sqrt{D_F/\pi T}$. Observe that the condensate matrix Green's function $\hat{f}$ is off-diagonal in the Gor'kov-Nambu space, i.e., $\hat{f}$ is proportional to $\hat{\tau}_1$, $\hat{\tau}_2$ matrices. In addition, the LRTC, $\hat{f}_{lr}$, is also off-diagonal in the spin space, i.e., $\hat{f}$ is proportional $\hat{\sigma}_1, \hat{\sigma}_2$ matrices such that the third term in Eq. (3) for this component vanishes. This means that in a general case the matrix LRTC $\hat{f}_{lr}$ obeying Eq. (3) can be written in the form

$$\hat{f}_{lr}(x) = \sum_{\{i,k\}} a_{ik}(x)\hat{X}_{ik}, \tag{6}$$

where $\{i;k\} = \{1,2;1,2\}$. A concrete form of the LRTC is determined by the boundary conditions at $x = \pm L$. These boundary conditions, originally employed in Refs. [66–70], can be represented in a simple form:

$$\partial_x \hat{f}|_{x=\pm L} = \pm\kappa_b \hat{X}_{r,l} F_{S-}, \tag{7}$$

$$F_{S-} = i\Delta\, \mathrm{Im}(1/\zeta_{\omega+}), \tag{8}$$

where $\hat{X}_{r,l} = \hat{T}\cdot\hat{X}_m\cdot\hat{T}^\dagger$, $\kappa_b = 1/(R_b\sigma_F)$, R_b is an interface (barrier) resistance per unit area, σ_F is conductivity of the F film, $\zeta_{\omega\pm} = \sqrt{(\omega \pm iJ_m)^2 + \Delta^2}$, and J_m is an exchange field in the F_m which can be weaker than the exchange field in F. As follows from Eq. (8) the function $F_{S-}(\omega)$ is an odd function of ω. All condensate functions injected into the ferromagnet F are proportional to F_{S-}, i.e., they describe the odd-frequency long-range triplet component. The functions $\hat{X}_{r,l}F_{S-} = (\hat{T}\cdot\hat{X}_m\cdot\hat{T}^\dagger)F_{S-}$ are the Green's functions of the Cooper pairs passing through the right (left) filters. The matrix coefficient $\hat{T}$ describes the tunneling of Cooper pairs through the filters and is defined as [66]

$$\hat{T} = (\mathcal{T} + \mathcal{U}\hat{X}_{33})/\sqrt{2}. \tag{9}$$

The term $\hat{X}_m F_{S-} \equiv \hat{f}_m$ in Eq. (8) is a matrix Green's function in the weak ferromagnet F_m where the function F_{S-} is an odd function of the Matsubara frequencies $\omega = \pi T(2n+1)$ and describes the triplet component. The form of the matrix $\hat{X}_m(\pm L) \equiv \hat{X}_m|_{l,r}$ depends on the chirality of the LRTC, i.e., on the orientation of the magnetization $\mathbf{m}$ in the F_m ferromagnetic film [71],

$$\hat{X}_m^{(\mathrm{x})} = \hat{X}_{11},\ \mathbf{m} \parallel \mathbf{e}_x, \tag{10}$$

$$\hat{X}_m^{(\mathrm{y})} = \hat{X}_{12},\ \mathbf{m} \parallel \mathbf{e}_y. \tag{11}$$

The matrices $\hat{X}_m^{(\mathrm{x})}, \hat{X}_m^{(\mathrm{y})}$ describe triplet Cooper pairs with spin up and down, which have different chirality. The filter action converts the $\{\hat{X}_m^{(\mathrm{x})}, \hat{X}_m^{(\mathrm{y})}\}$ matrices to $\{\hat{X}_{l,r}^{(\mathrm{x})}, \hat{X}_{l,r}^{(\mathrm{y})}\}$, where

$$\hat{X}_{l,r}^{(\mathrm{x})} = \hat{X}_{11} - s_{l,r}\hat{X}_{22},\ \mathbf{m} \parallel \mathbf{e}_x, \tag{12}$$

$$\hat{X}_{l,r}^{(\mathrm{y})} = \hat{X}_{12} + s_{l,r}\hat{X}_{21},\ \mathbf{m} \parallel \mathbf{e}_y. \tag{13}$$

The parameter $s = 2\,\mathrm{Re}(\mathcal{T}\mathcal{U}^*)/(|\mathcal{T}|^2 + |\mathcal{U}|^2)$ characterizes the degree of spin-up and spin-down polarization of the triplet Cooper pairs injected into the film F. If $\mathcal{U} = 0$, Cooper pairs with up and down spins penetrate into the F film with equal probabilities, and therefore the number of the triplet pairs with both spin orientations in the F is the same. This case has been called nematic LRTC in Ref. [49]. If $|\mathcal{T}| = |\mathcal{U}| = 1$, then $s = \pm 1$, and the triplet Cooper pairs are fully polarized with total spin parallel or antiparallel to the z axis. Note that a magnetic half-metal can be used as a spin filter. The case of $s = 0$ corresponds to the absence of filters at the F_m/F interfaces.

Equations (7)–(12) are a generalization of the Kupriyanov-Lukichev boundary conditions [72], which in turn were obtained from Zaitsev's boundary conditions [73] (see also Ref. [74], where the applicability of the Kupriyanov-Lukichev boundary conditions is discussed).

So far, we have assumed that the phases of the order parameter in the superconductors S are chosen to be equal to zero. The presence of the phases $\pm\varphi/2$ at $\mathrm{S}_{r,l}$ can be easily introduced via a gauge transformation $\hat{S}_\varphi = \exp(\pm i\hat{X}_{30}\varphi/4)$: $\hat{g}_{S,\varphi} = \hat{S}_\varphi \cdot\ \hat{g}_S \cdot \hat{S}_\varphi^\dagger$ (see, e.g., [75]) so that the boundary condition (7) can be written as

$$\partial_x \hat{f}|_{\,x=\pm L} = \pm\kappa_b[\cos(\varphi/2) \pm i\hat{X}_{30}\sin(\varphi/2)]\hat{X}_{l,r}F_{S-}. \tag{14}$$

The matrix condensate function $\hat{X}_m^{(\mathrm{x,y})}F_{S-}$ describes a short-range triplet component in the film F_m, but it becomes a long-range one in the F film because of the noncollinearity of the magnetization vectors $\mathbf{m}$ and $\mathbf{M}$. Note that the functions $\hat{X}_{12}F_{S-}$ and $\hat{X}_{21}F_{S-}$, written explicitly, consist of triplet components with up and down spins $\hat{X}_{12}F_{S-} \sim \langle c_\uparrow(t)c_\uparrow(0)\rangle + \langle c_\downarrow(t)c_\downarrow(0)\rangle$, $\hat{X}_{21}F_{S-} \sim \langle c_\uparrow(t)c_\uparrow(0)\rangle - \langle c_\downarrow(t)c_\downarrow(0)\rangle$ so that the function $\hat{X}F_{S-} = (\hat{X}_{12} \pm \hat{X}_{21})F_{S-}$ describes the Cooper pairs polarized in one direction (see Appendix A for further details).

Knowing the Green's functions $\hat{f}$, we can readily calculate the charge $I_Q = \mathbf{I}_Q\cdot\mathbf{e}_x$ and the spin currents $I_{sp} = \mathbf{I}_{sp}^{(z)}\cdot\mathbf{e}_x$ using the following expressions:

$$I_Q = \frac{\sigma_F}{e}2\pi T\sum_{\omega\geqslant 0} I_{Q,\omega}, \tag{15}$$

$$I_{sp} = \mu_B\frac{\sigma_F}{e^2}2\pi T\sum_{\omega\geqslant 0} I_{sp,\omega}, \tag{16}$$

where the "spectral" currents $I_{Q,\omega}$ and $I_{sp,\omega}$ are defined as

$$I_{Q,\omega} = (i/4)\mathrm{Tr}\{(\hat{\tau}_3\cdot\hat{\sigma}_0)\hat{f}_{LR,0}\partial_x\hat{f}\} \equiv i\{\hat{f}\partial_x\hat{f}\}_{30}, \tag{17}$$

$$I_{sp,\omega} = (i/4)\mathrm{Tr}\{(\hat{\tau}_0\cdot\hat{\sigma}_3)\hat{f}\partial_x\hat{f}\} \equiv i\{\hat{f}\partial_x\hat{f}\}_{03} \tag{18}$$

and further details are given in Appendix B. Similar expressions for the charge and spin current are obtained in Refs. [64,71,76,77]. Observe that the traces in the Nambu space for charge and spin currents are actually different. In the literature one could also find the expression for the Josephson spin current in a different form $I_{sp,\omega} = (i/4)\mathbf{Tr}\{(\hat{\tau}_3\cdot\hat{\sigma}_3)\hat{f}\partial_x\hat{f}\}$, which might be related to the different choice of basis in the definition of the Green's functions. Here we follow the same basis as in Ref. [20] complemented by the Ivanov-Fominov transformation, Eq. (A9). In this representation, the Usadel equation has the standard form (B5) with the boundary conditions Eq. (14).

In order to find the Josephson current, we need to solve Eq. (3) with the matching conditions (4) and boundary conditions (8).

We first consider the case of the F film with a uniform magnetization, $\mathbf{M} = (0, 0, M)$, without DWs. Although such magnetic Josephson junctions have been already studied previously in different limiting cases (ballistic and diffusive) using various mostly numerical techniques [23–29,56–61,78,79] we will discuss the main results in the dirty case and in the limit of the weak proximity effect. Then the formulas for currents acquire a simple analytical form, not known previously, that allows for a straightforward physical interpretation. In addition, we will focus our study on the case of fully polarized triplet Cooper pairs of different chiralities.

In particular, the solution of Eq. (3), $\hat{f}_{LR,0}$, which obeys the boundary conditions (8) has the form

$$\hat{f}_{lr,0} = \hat{C}\frac{\cosh(\kappa_\omega x)}{\sinh(L\kappa_\omega)} + \hat{S}\frac{\sinh(\kappa_\omega x)}{\cosh(L\kappa_\omega)} \tag{19}$$

with

$$\hat{C} = \frac{\kappa_b}{2\kappa_\omega}\Big[\hat{X}_+ \cos\Big(\frac{\varphi}{2}\Big) + i\hat{X}_{30}\cdot\hat{X}_- \sin\Big(\frac{\varphi}{2}\Big)\Big]F_{S-}, \tag{20}$$

$$\hat{S} = \frac{\kappa_b}{2\kappa_\omega}\Big[\hat{X}_- \cos\Big(\frac{\varphi}{2}\Big) + i\hat{X}_{30}\cdot\hat{X}_+ \sin\Big(\frac{\varphi}{2}\Big)\Big]F_{S-}, \tag{21}$$

where we have defined $\hat{X}_\pm = \hat{X}_r \pm \hat{X}_l$. Substituting $\hat{f}_{LR,0}$ from Eq. (19) into Eqs. (17) and (18), we obtain

$$I_{Q,\omega} = \tilde{I}_\omega[-i\{\hat{X}_r\cdot\hat{X}_l\}_{30}\cos\varphi + \{\hat{X}_r\cdot\hat{X}_l\}_{00}\sin\varphi], \tag{22}$$

$$I_{sp,\omega} = \tilde{I}_\omega[-i\{\hat{X}_r\cdot\hat{X}_l\}_{03}\cos\varphi + \{\hat{X}_r\cdot\hat{X}_l\}_{33}\sin\varphi], \tag{23}$$

$$\tilde{I}_\omega = \frac{(\kappa_b F_{S-})^2}{\kappa_\omega \sinh(2L\kappa_\omega)}, \tag{24}$$

and $\kappa_\omega = \sqrt{2|\omega|/D_F}$. Observe that the matrices $\hat{C}, \hat{S}$ and $\hat{X}_r, \hat{X}_l$ anticommute with matrices $\hat{X}_{30}, \hat{X}_{03}$ so that the traces $\{\hat{C}^2\}_{30}, \{\hat{C}^2\}_{03}$, etc., are equal to zero. In the following we calculate the charge and spin currents for different cases in detail.

A. Currents in the absence of filters

For the case of equal chiralities of the triplet Cooper pairs injected from the right (left) S/Fl interfaces ($\mathbf{m}_l \parallel \mathbf{m}_r || \mathbf{e}_x$ or $\mathbf{m}_l || \mathbf{m}_r \parallel \mathbf{e}_y$) and defining $\hat{X}_r = \hat{X}_l = \hat{X}_{11} \equiv \hat{X}_m^{(x)}$ or $\hat{X}_r = \hat{X}_l = \hat{X}_{12} \equiv \hat{X}_m^{(y)}$, the charge and spin spectral currents are

$$I_{Q,\omega}^{(x,x)} = I_{Q,\omega}^{(y,y)} = \tilde{I}_\omega \sin\varphi, \tag{25}$$

$$I_{sp,\omega}^{(x,x)} = I_{sp,\omega}^{(y,y)} = 0, \tag{26}$$

i.e., the charge current has the usual form $I_{Q,\omega} = \tilde{I}\sin\varphi$ whereas the spin current is zero. For the case of different chiralities ($\hat{X}_r = \hat{X}_{12}, \hat{X}_l = \hat{X}_{11}$) the currents are given by

$$I_{Q,\omega}^{(x,y)} = 0, \tag{27}$$

$$I_{sp,\omega}^{(x,y)} = \tilde{I}_\omega \cos\varphi, \tag{28}$$

where indices (x,x) and (x,y) refer to the chirality of the Cooper pairs penetrating the film F on the right and on the left, that is, $I_{Q,\omega}^{(x,y)} \sim \{\hat{X}_r^{(x)}\cdot\hat{X}_l^{(y)}\}$. We also note an important feature of the obtained currents. In particular, the critical spectral current $\tilde{I}_\omega$ in the considered S_m/Fl/F/Fl/S_m junction has the sign opposite to that in the S/N/S Josephson junction since in the latter case the critical current $\tilde{I}_{S/N/S} \sim F_S^2 > 0$, while in the system under consideration $\tilde{I}_\omega \sim F_{S-}^2 < 0$ [see Eq. (8)], here $F_S = \Delta/\sqrt{\omega^2+\Delta^2}$. This is a simple representation of the fact that the LRTC leads to a π-Josephson coupling. Observe that the negative F_{S-}^2 means a negative local pair density [80,81]. It leads not only to the change of sign of the critical Josephson current but also to an enhanced density of states at zero energy [82].

Note that the Josephson current $I_{Q,\omega}^{(x,x)}$ is finite for collinear orientations of the magnetic moments $\mathbf{m}$ in the left and right films F_m and is zero ($I_{Q,\omega}^{(x,y)} = 0$) for the orthogonal orientations of the vectors $\mathbf{m}_{l,r}$. The opposite is true for the spin current. It is zero in the case of vectors $\mathbf{m}_l = \mathbf{m}_r$ and is finite if $\mathbf{m}_l\cdot\mathbf{m}_r = 0$, i.e., when the vectors $\mathbf{m}_{r,l}$ are orthogonal. Moreover, in the second case a spontaneous spin current arises in the system even when the phase difference φ is zero.

The formulas for the currents (25)–(28) are derived for the case when the vectors $\mathbf{m}_{r,l}$ lie in the plane perpendicular to the z axis so that $\mathbf{m}_{r,l}\cdot\mathbf{e}_z = \cos\alpha_{r,l} = 0$. They can be easily generalized for the arbitrary angles $\alpha_{r,l}$. Taking into account that only the components $\mathbf{m}_{r,l}\cdot\mathbf{e}_{x,y}$ contribute to the LRTC, in a more general case the currents $\tilde{I}$ are equal to

$$\tilde{I}_\alpha = \tilde{I}\sin\alpha_r \sin\alpha_l. \tag{29}$$

The formulas for the charge and spin currents $I_{Q,sp,\omega}$ are represented in Table I. The angles $\alpha_{r,l}$ are chosen to be equal to $\pi/2$ so that $\sin\alpha_r = \sin\alpha_l = 1$.

B. Currents in the presence of filters

Now we calculate the currents for the case of a uniform $\mathbf{M}$ in F and in the presence of filters at the interfaces F/F_m. Recall that in the absence of spin filters, the currents are spin independent. As we show below, the presence of spin filters makes both currents spin dependent. For the case of equal chiralities, i.e., $\hat{X}_r = \hat{X}_l = \hat{X}^{(\nu)}$ ($\nu = x$ or y), the charge and spin currents can be found by using formulas for $\hat{X}^{(x)}$ and $\hat{X}^{(y)}$, Eq. (10),

$$I_{Q,\omega}^{(x,x)} = I_{Q,\omega}^{(y,y)} = \tilde{I}_\omega(1+s_r s_l)\sin\varphi, \tag{30}$$

$$I_{sp,\omega}^{(x,x)} = I_{sp,\omega}^{(y,y)} = \tilde{I}_\omega(s_r+s_l)\sin\varphi. \tag{31}$$

These formulas show that for parallel spin orientations of fully polarized triplet Cooper pairs injected from the right and left superconductors, the values of the coefficients $1 + s_r s_l = 2$ and $s_r + s_l = 2\,\mathrm{sgn}(s)$ are the same, but the direction of the spin current depends on the sign of s. In the case of opposite spin polarization both currents are zero.

For different chiralities $\hat{X}_r = \hat{X}^{(x)}, \hat{X}_l = \hat{X}^{(y)}$ we find

$$I_{Q,\omega}^{(x,y)} = \tilde{I}(s_r+s_l)\cos\varphi, \tag{32}$$

$$I_{sp,\omega}^{(x,y)} = \tilde{I}(1+s_r s_l)\cos\varphi. \tag{33}$$

TABLE I. Summary of the charge and spin supercurrents in magnetic Josephson junctions with spin filters and their modifications due to domain walls for different ferromagnetic filter orientations.

CURRENTS				
	XX(YY) no FI	**XY no FI**	**XX(YY) + FI**	**XY + FI**
I_Q	$\widetilde{I}\sin\varphi$	0	$\widetilde{I}(1+s_r s_l)\sin\varphi$	$-\widetilde{I}(s_r+s_l)\cos\varphi$
I_{sp}	0	$\widetilde{I}\cos\varphi$	$\widetilde{I}s_r s_l\sin\varphi$	$-\widetilde{I}(1+s_r s_l)\cos\varphi$

Corrections to the Currents due to DWs				
	XX(YY) + FI, P case	**XX(YY) + FI, AP case**		
I_Q	$(\widetilde{I}-\delta I)\sin\varphi$	$\delta\widetilde{I}\sin\varphi$		

With equal spin polarizations ($s_r = s_l$), the spontaneous charge and spin currents occur in this case even in the absence of a phase difference. Interestingly, the direction of the spontaneous charge current depends on the sign of spins s of injected triplet Cooper pairs. In the case of opposite spin polarization, these currents disappear. Note that the spontaneous currents may lead to the Josephson diode effect (see Ref. [83], and references therein). The conclusion about the possibility of spontaneous currents in different models of superconducting magnetic systems (with or without spin-orbit interaction) have been obtained earlier [29,49,68,84,85] (see also recent papers [86,87], and references therein). For convenience we summarize the results for the charge and spin currents in Table I.

III. MODIFICATIONS OF THE CURRENTS DUE TO DWs

Next we consider the modifications of the currents, obtained above, for the case of the domain wall in the F film. We restrict our analysis to the case of equal chiralities (the generalization to the case of different chiralities is straightforward) and also assume that the spacing between the nearest DWs is much larger than the decay length of the short-range component $\hat{f}_{sr}$, i.e., $|l_1 - l_2| \gg \xi_J \approx \sqrt{D_F/J}$. The main effect of the domain wall is the creation of a short-range triplet component, which results in a correction $\delta\hat{f}_{lr}$ to the long-range component $\hat{f}_{lr,0}$ defined by Eq. (19). While the short-range component exists only near each DW, the LRTC extends over a larger distance, which can be of the order of L. In particular, the correction $\delta\hat{f}_{lr}$ arises due to matching conditions for the function $\delta\hat{f}_{lr}(x)$ at $x = l_i$, where l_i is the coordinate of a DW. These conditions for $\delta\hat{f}_{lr,0}(x)$ and its partial derivative are

$$[\delta\hat{f}_{lr}]|_{l_i} = 0, \tag{34}$$

$$[\partial_x\delta\hat{f}_{lr}]|_{l_i} = i(\kappa_{DW}/2)[\hat{X}_{02}, \hat{f}_{sr}(l)]_+. \tag{35}$$

As usual, these are complemented by the boundary conditions

$$\partial_x\delta\hat{f}_{lr}|_{\pm L} = 0. \tag{36}$$

In the presence of several DWs, the solution for $\delta\hat{f}_{lr}$ can be represented in the form

$$\delta\hat{f}_{lr}(x) = \sum_i \delta\hat{f}_{lr}^{(i)}(x), \tag{37}$$

where the $\delta\hat{f}_{lr}^{(i)}(x)$ is a perturbation of the LRTC generated by the ith DW. In order to find this function, one needs to determine a short-range component $\hat{f}_{sr}^{(i)}$ produced by the ith DW, which we do in the next section.

A. Short-range component generated by the domain wall

The short-range component obeys Eq. (3) and matching conditions (4) and (5) that can be written as

$$[\hat{f}_{sr}]|_l = 0, \tag{38}$$

$$[\partial_x\hat{f}_{sr}]|_l = i\frac{\kappa_{DW}}{2}[\hat{X}_{02}, \hat{f}_{lr,0}]_+, \tag{39}$$

where we also dropped the subindex i in l_i for simplicity. Taking into account Eqs. (5), (20), and (21), we can rewrite Eq. (39) as follows:

$$[\partial_x\hat{f}_{sr}]|_l = i\kappa_{DW}\left(\hat{C}_2\frac{\cosh\tilde{l}}{\sinh\tilde{L}} + \hat{S}_2\frac{\sinh\tilde{l}}{\cosh\tilde{L}}\right), \tag{40}$$

where $\tilde{l} = \kappa_\omega l$, $\tilde{L} = \kappa_\omega L$ and $\hat{C}_2 = [\hat{X}_{02}, \hat{C}]_+$, $\hat{S}_2 = [\hat{X}_{02}, \hat{S}]_+$. A solution for the short-range component, Eq. (3), obeying the matching conditions (38) and (39) in the vicinity of ith DW can written in the form

$$\hat{f}_{sr} = \hat{f}_{sr}^{(\mathrm{A})}\cos(\varphi/2) + \hat{f}_{sr}^{(\mathrm{B})}\sin(\varphi/2), \tag{41}$$

where the matrices $\hat{f}_{sr}^{(A,B)}$ Green's functions contain exponentially decaying functions

$$\hat{f}_{sr}^{(A)}(x) = -i\begin{cases} A_+\hat{X}_{n+}\exp\left[K_+(x-l)\right] + A_-\hat{X}_{n-}\exp\left[K_-(x-l)\right], & x < l \\ \bar{A}_+\hat{X}_{n+}\exp\left[-\bar{K}_+(x-l)\right] + \bar{A}_-\hat{X}_{n-}\exp\left[-\bar{K}_-(x-l)\right], & l < x, \end{cases} \tag{42}$$

where $K_\pm^2 = \kappa_\omega^2 \pm i\kappa_J^2$ and $n = 1, 2$ for y, x chiralities, $\bar{K}_\pm = K_\mp$ and $\hat{X}_{n\pm} = (\hat{X}_{n0} - \hat{X}_{n3})$. The matrix $\hat{f}_{sr}^{(B)}$ is equal to

$$\hat{f}_{sr}^{(B)}(x) = i\hat{X}_{30}\cdot\hat{f}_{sr}^{(A)}(x), \tag{43}$$

with the replacement $A \Rightarrow B$. The matching condition (38) yields

$$A_+ = A_- = \bar{A}_- = \bar{A}_+ \equiv A, \tag{44}$$

$$B_+ = B_- = \bar{B}_- = \bar{B}_+ \equiv B. \tag{45}$$

The coefficients A and B are determined from Eq. (39). In what follows we consider several cases.

(a) The x chirality, parallel $s_{r,l}$ orientations. In this case, $\hat{X}_r = \hat{X}_l = (\hat{X}_{11} - s\hat{X}_{22})$. The coefficients $A_P^{(x)}, B_P^{(x)}$ are equal to

$$A_P^{(x)} = -sr_\omega\frac{\cosh\tilde{l}}{\sinh\tilde{L}}F_{S-}, \tag{46}$$

$$B_P^{(x)} = -sr_\omega\frac{\sinh\tilde{l}}{\cosh\tilde{L}}F_{S-}, \tag{47}$$

where $r_\omega = (\kappa_{DW}\kappa_b)/(K_0\kappa_\omega)$ and $K_0 = 4\,\mathrm{Re}K_+$.

(b) The x chirality, antiparallel spin orientations, i.e., $s_r = s = -s_l$. In this case $\hat{X}_{r,l} = \hat{X}_{11} \mp s\hat{X}_{22}$). The coefficients $A_{AP}^{(x)}, B_{AP}^{(x)}$ are given by $A_{AP}^{(x)} = B_P^{(x)}$, $B_{AP}^{(x)} = A_P^{(x)}$.

(c) The y chirality, parallel (antiparallel) $s_{r,l}$ orientations. Then, $\hat{X}_{l,r} = (\hat{X}_{12} - s_{l,r}\hat{X}_{21})$ and the coefficients $A^{(y)}, B^{(y)}$ are equal to

$$A_P^{(y)} = A_{AP}^{(y)} = A_P^{(x)}/s, \tag{48}$$

$$B_P^{(y)} = B_{AP}^{(y)} = B_P^{(x)}/s. \tag{49}$$

In the next section, we calculate the function $\delta\hat{f}_{lr,0}(x)$.

B. Correction to the LRTC due to a domain wall

Finally, the correction $\delta\hat{f}_{lr}(x)$ obeys the equation

$$-\partial_{xx}^2\delta\hat{f}_{lr} + \kappa_\omega^2\delta\hat{f}_{lr} = 0, \tag{50}$$

complemented by the conditions (34)–(36). The solution of Eq. (50), which obeys the boundary conditions (36), is

$$\delta\hat{f}_{lr}(x) = \begin{cases} \hat{C}_<\cosh(\tilde{x}+\tilde{L}), & -L < x < l \\ \hat{C}_>\cosh(\tilde{x}-\tilde{L}), & l < x < L. \end{cases} \tag{51}$$

The matrices $\hat{C}_\lessgtr$ are found from the matching conditions (34) and (35):

$$\hat{C}_\lessgtr = \hat{C}_\lessgtr^{(A)}\cos(\varphi/2) + i\hat{X}_{30}\hat{C}_\lessgtr^{(B)}\sin(\varphi/2) \tag{52}$$

and find for $\hat{C}_\lessgtr^{(A,B)}$,

$$\hat{C}_\lessgtr^{(A)} \equiv \hat{a}_\lessgtr = -4\frac{\kappa_{DW}}{\kappa_\omega}\frac{\cosh(\tilde{L}\mp\tilde{l})}{\sinh(2\tilde{L})}A\hat{X}_{n2}, \tag{53}$$

$$\hat{C}_\lessgtr^{(B)} \equiv \hat{b}_\lessgtr = -4\frac{\kappa_{DW}}{\kappa_\omega}\frac{\cosh(\tilde{L}\mp\tilde{l})}{\sinh(2\tilde{L})}B\hat{X}_{n2}, \tag{54}$$

where the signs $\pm$ correspond to $x \gtrless l$ and $n = 1$ for y chirality and $n = 2$ for x chirality.

Having known the long-range Green's function $\hat{f}_{lr} = \hat{f}_{lr,0} + \delta\hat{f}_{lr}$, we can find a change of the current in the presence of a DW.

C. Change of the currents due to domain wall

The corrections to the currents are

$$\delta I_Q = (\sigma_F/e)2\pi T\sum_{\omega\geqslant 0}\delta I_{Q,\omega}, \tag{55}$$

$$\delta I_{sp} = \mu_B(\sigma_F/e^2)2\pi T\sum_{\omega\geqslant 0}\delta I_{sp,\omega} \tag{56}$$

and the partial currents $\delta I_{Q,\omega}$ and $\delta I_{sp,\omega}$ are given by

$$\delta I_{Q,\omega} = i\{\delta\hat{f}_{lr}\partial_x\hat{f}_{lr,0} + \hat{f}_{lr,0}\partial_x\delta\hat{f}_{lr}\}_{30}, \tag{57}$$

$$\delta I_{sp,\omega} = i\{\delta\hat{f}_{lr}\partial_x\hat{f}_{lr,0} + \hat{f}_{lr,0}\partial_x\delta\hat{f}_{lr}\}_{03}. \tag{58}$$

We find

$$\delta I_{Q,\omega} = \kappa_\omega\{(\hat{C}^{(B)} + \hat{S}^{(B)})\cdot\hat{a} - (\hat{C}^{(A)} + \hat{S}^{(A)})\cdot\hat{b}\}_{00}, \tag{59}$$

$$\delta I_{sp,\omega} = \kappa_\omega\{(\hat{C}^{(B)} + \hat{S}^{(B)})\cdot\hat{a} - (\hat{C}^{(A)} + \hat{S}^{(A)})\cdot\hat{b}\}_{33}. \tag{60}$$

Here, the matrices $\hat{C}^{(A,B)}$ and $\hat{S}^{(A,B)}$ are presented in Appendix C [Eqs. (C1)–(C4)], and the matrices $\hat{a} \equiv \hat{a}_>$, $\hat{b} \equiv \hat{b}_>$ are defined in Eqs. (53) and (54).

Then, we find for the currents of Cooper pairs injected from the right and left S_m reservoirs with equal chiralities and arbitrary spin polarizations

$$\delta I_Q = \delta I_{Q,\omega}\sin\varphi, \tag{61}$$

$$\delta I_{sp} = \delta I_{sp,\omega}\sin\varphi. \tag{62}$$

The critical currents $\delta I_{Q,\omega}$ and $\delta I_{sp,\omega}$ depend on the chiralities and polarizations of Cooper pairs propagating from the right and from the left. For the case *(a) (xx) chiralities, P case ($s = s_r = s_l$)*,

$$\delta I_{Q,\omega;P}^{(xx)} = -2\frac{\kappa_{DW}^2}{K_0}\left(\frac{\kappa_b}{\kappa_\omega}\right)^2 F_{S-}^2\frac{\cosh(\tilde{L}+\tilde{l})\cosh(\tilde{L}-\tilde{l})}{\sinh^2(2\tilde{L})}, \tag{63}$$

$$\delta I_{sp,\omega;P}^{(xx)} = s\delta I_{Q,\omega;P}^{(xx)}. \tag{64}$$

For the case *(b) (xx) chiralities, AP case ($s \equiv s_r = -s_l$)* we find

$$\delta I_{Q,\omega;AP}^{(xx)} = -\,\delta I_{Q,\omega;P}^{(xx)},\ \delta I_{sp,AP}^{(xx)} = -\delta I_{sp,P}^{(xx)}. \tag{65}$$

Comparing this equation and Eqs. (63) and (64), we see that the signs of the currents $\delta I_{Q,AP}^{(xx)}$ and $\delta I_{sp,AP}^{(xx)}$ are changed.

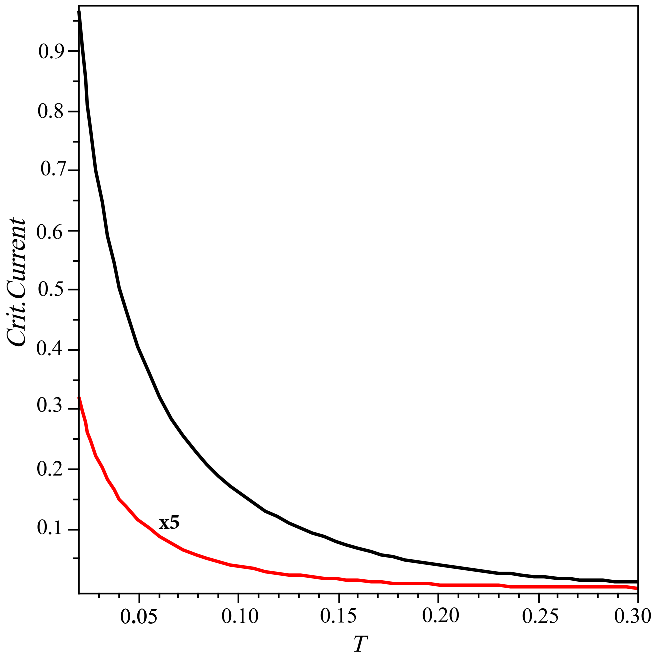

FIG. 2. Temperature dependence of the normalized critical current $I_{c,0}$ in the absence of DW (black curve) and a change of the normalized critical current δI_c due to DW (red curve), which is subtracted from $I_{c,0}$ to obtain the total current. For the sake of the presentation the magnitude of the latter is multiplied by the factor of 5. The change δI_c decreases the Josephson critical current I_c if $I_{c,0}$ is not zero and makes I_c finite if $I_{c,0} = 0$ (antiparallel spin orientations of triplet Cooper pairs injected from the left and from the right). The temperature T and the exchange J_m energy in F_m are normalized to $\Delta(0)$. The parameter $J \equiv J_m/\Delta(0)$ is chosen to be equal to 3 (see Appendix D).

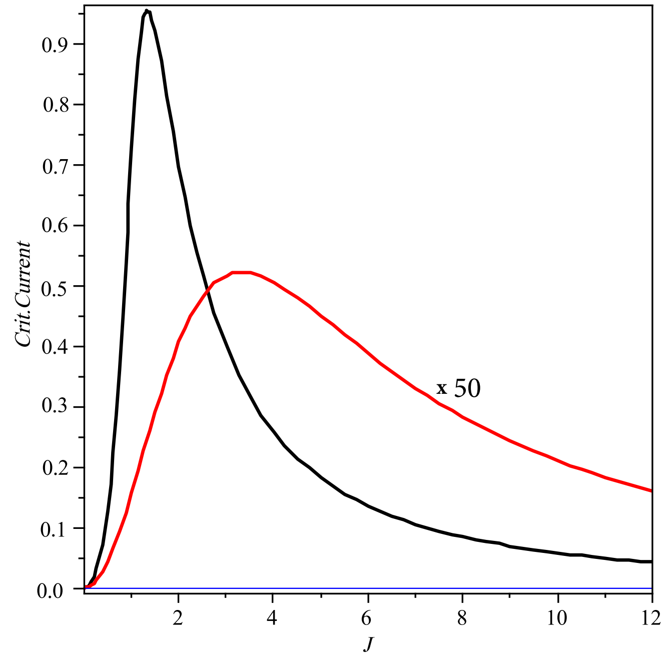

FIG. 3. The dependence of the critical current $I_{c,0}(J_m)$ on the exchange field in F_m film J_m for two values of the normalized temperature and $L/\xi_\Delta = 1$: $T/\Delta(0) = 0.05$ (black) and $T/\Delta(0) = 0.3$ (red) (in the latter case, we multiplied this dependence by the factor 50 because the critical current decreases drastically with increasing T).

Finally, in the case *(c) (yy) chiralities, P(AP) cases*, the currents are

$$\delta I_{Q,P}^{(yy)} = \delta I_{Q,P}^{(xx)} = \delta I_{Q,AP}^{(yy)}, \tag{66}$$

$$\delta I_{sp,P}^{(yy)} = -\delta I_{sp,P}^{(xx)} = \delta I_{sp,AP}^{(yy)}. \tag{67}$$

In the case of the *yy* chirality, the coefficients $A^{(y,y)}$ and $B^{(y,y)}$ do not depend on the polarization s. That is, the currents are equal for different spin orientations: $\delta I_{Q,P}^{(yy)} = \delta I_{Q,AP}^{(yy)}$ and $\delta I_{sp,P}^{(yy)} = \delta I_{sp,AP}^{(yy)}$.

The analysis of the obtained results shows that the DW reduces the Josephson charge and spin currents if Cooper pairs injected from the right and left superconductors have parallel spin orientation. Thus, the action of the DW on the critical current in this case is analogous to the action of paramagnetic impurities, which decrease the penetration length of the LRTC [20,88]. In the case of antiparallel orientations, the DW makes the Josephson critical current finite. It is interesting to note that the maximum magnitude of the total Josephson current $I_Q = |I_{Q,0} + \delta I_Q|$ is achieved at $l = 0$. This means that the Josephson energy has a minimum if the DW is located in the center of the junction for the case of parallel spin polarized Cooper pairs. Note also that the correction to the current $\delta I_{\omega,0}$ is proportional to the square of κ_{DW}: $\delta I_{\omega,0} \sim K_0 r_\omega^2 = (\kappa_{DW}\kappa_b)^2/(K_0\kappa_\omega^2)$. Thus, the contribution to the current due to DW does not depend on whether the magnetization vector **M** in the Bloch DW rotates clockwise or counterclockwise. The results of the change of the Josephson currents due to a single domain wall are also summarized in Table I.

In Fig. 2 we plot the temperature dependence of the Josephson critical current $I_{Q,0}(T)$ in the absence of a DW and a correction to the current due to a DW located in the center of the F film ($l = 0$) (see also Appendix D for details of the numerics). One can see that the critical current $I_{Q,0}(T)$ and the correction due to a DW decay monotonously with increasing the temperature. For completeness we show in Fig. 3 the dependence $I_{Q,0}(J)$ for two temperatures. A similar dependence shows the correction to the Josephson current due to a DW.

To estimate how large are the corrections δI_Q to the Josephson critical current due to a DW observe that Eq. (63) yields

$$\delta I_{Q,\omega} \sim \frac{\kappa_{DW}^2}{K_0}\frac{\kappa_b^2}{\kappa_\omega^2} F_{S-}^2 \cosh^{-2}(\kappa_\omega L), \tag{68}$$

where $K_0 \approx \kappa_{J\omega}$ and we set $l = 0$. On the other hand, the Josephson critical current in the absence of DWs is given by [see Eq. (24)]

$$\tilde{I}_\omega \sim \frac{\kappa_b^2}{\kappa_\omega} F_{S-}^2 \sinh^{-1}(2\kappa_\omega L). \tag{69}$$

Thus, for the ratio $\delta I_{Q,\omega}/\tilde{I}_\omega$ we obtain

$$\delta I_{Q,\omega}/\tilde{I}_\omega \sim \left(\frac{\kappa_{DW}}{\kappa_{J\omega}}\right)^2 \frac{\kappa_{J\omega}}{\kappa_\omega}. \tag{70}$$

The term in brackets is assumed to be small, but the ratio $\kappa_{J\omega}/\kappa_{\omega}$ is large. Therefore, the correction $\delta I_{Q,\omega}$ can be comparable with $\tilde{I}_{\omega}$ and the suppression of the critical current can be significant.

D. Change of the currents due to two domain walls

We assume that the spacing between the nearest DWs is much larger than κ_J. In this case each DW contributes to the Josephson current independently from others. Thus, the correction to the Josephson current due to, for example, two DWs, is given by

$$\delta I^{(xx)}_{Q,\omega} = -2\frac{(\kappa_b\kappa_{DW}F_{S-})^2}{K_0\kappa_{\omega}^2\sinh^2(2\tilde{L})}[\cosh(2\tilde{L}) + \cosh(2\bar{l}) + \cosh(2\delta l)]\sin\varphi, \tag{71}$$

where $\bar{l} = (\tilde{l}_1 + \tilde{l}_2)/2, \delta\tilde{l} = \tilde{l}_1 - \tilde{l}_2$. According to the assumption above, $(\delta\tilde{l}\kappa_J) \gg 1$. This formula means that the DW reduces the critical current $I_c = I_{c0} + \delta I_c$ in the P case and makes it finite in the AP case. The decrease of the current $I_c = I_{c0} - |\delta I_c|$ in the P case would be minimal if $\bar{l} = 0$, i.e., the two DWs are located in the center of the F film.

IV. CONCLUSIONS

We have calculated the Josephson charge I_Q and spin I_{sp} currents in an S/F$_m$/Fl/F/Fl/F$_m$/S Josephson junction when the Josephson coupling is realized via different types of a LRTC. The superconducting condensate in a thin magnetic layer F$_m$ consists of singlet and triplet Cooper pairs penetrating from the S banks into the F$_m$ film. The spin filters Fl pass only the triplet Cooper pairs which are long range in F because the magnetization vector $\mathbf{m}$ in F$_m$ is perpendicular to the magnetization vector $\mathbf{M} \parallel \mathbf{e}_z$ in the F film. The long-range triplet Cooper pairs, penetrating into the F film, differ in chiralities, i.e., by orientation of the vector $\mathbf{m}(\mathbf{m} \parallel \mathbf{e}_x$ or $\mathbf{m} \parallel \mathbf{e}_y)$, and in polarization of the total spin of the triplet Cooper pairs. First, we considered the case of a uniform magnetization in F, $\mathbf{M}(x) = \text{const}$, and of the absence of spin filters. Then, the LRTC consists of equal numbers of fully polarized triplet pairs with opposite directions of the total spin $s_{l,r}$ (the nematic case in the terminology of Ref. [49]). In this case, the spin current $I^{(xx)}_{sp} = I^{(yy)}_{sp}$ is zero and the Josephson current $I^{(xx)}_{Q} = I^{(yy)}_{Q}$ is finite.

In the presence of the spin filters, both currents, $I^{(xx)}_{Q} = I^{(yy)}_{Q} = \tilde{I}_Q(1 + s_r s_l)\sin\varphi$ and $I^{(xx)}_{sp} = I^{(yy)}_{sp} = \tilde{I}_{sp}(s_r + s_l)\sin\varphi$, are finite. If the chiralities and spin directions of the LRTC are equal, the currents are finite and differ only by a prefactor. In the case of antiparallel spin orientations and $s_r = -s_l = s$, both currents are zero. If the triplet Cooper pairs injected on the right and on the left have different chiralities, spontaneous currents may arise: $I^{(xy)}_{Q} = -\tilde{I}_Q(s_r + s_l)\cos\varphi, I^{(xy)}_{sp} = -\tilde{I}_{sp}(1 + s_r s_l)\cos\varphi$. This means that the currents may occur in the absence of the phase difference and the direction of the charge current depends on spins. The spontaneous currents may be the reason for the Josephson diode effect, discussed recently [83]. All these results are summarized in Table I.

We have studied the change of the charge and spin currents in the presence of an arbitrary number of DWs in the F film. It turns out that a DW reduces the critical Josephson current if the spin directions of the Cooper pairs injected from the right and left superconductors S$_m$ are parallel ($s_r = s_l$) and the suppression of the critical current may be significant. The critical current reaches a maximum if the DW is located in the center of the F film. In the case of an antiparallel spin, $s_r = -s_l$, the critical current $I^{(\mathrm{AP})}_{c,0}$ in the absence of a DW is zero, but becomes finite in the presence of a DW.

The case of two DWs, which may be considered as a model of a skyrmion, is particularly interesting. The dependence of the change of the critical current $\delta I^{(xx)}_{Q,\omega}$ due to two DWs is given by Eq. (71). In the case of parallel spins ($s_r = s_l$), the critical current $I_{Q,c}$ has a maximum if the DWs are located in the center of the F film. In the case of antiparallel spins ($s_r = -s_l$), the maximum $I_{Q,c}$ corresponds to the location of two DWs at the edges of the F film.

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APPENDIX A: GREEN'S FUNCTIONS $\hat{G}_{ik}$

First we calculate the exact Green's functions $\hat{G}_{ik}$ and show that they and the quasiclassical Green's functions $\hat{g}^{(x)}(s)$ and $\hat{g}^{(y)}(s)$ describe the fully polarized triplet Cooper pairs with spin $s = \pm 1$. We use the Nambu indices defined in Ref. [20], so that $c_{n,s} = c_s$ and $c_{n,s} = c^{\dagger}_{\bar{s}}$ for $n = 2$; $c_s = c_{\uparrow}$ for $s = 1$ ($\bar{s} = 2$) and $c_s = c_{\downarrow}$ for $s = 2$ ($\bar{s} = 1$). The Green's function $\hat{G}_{12}$ is

$$\begin{aligned}\hat{G}_{12}(t,t') &= -i\langle c_{ns}(t)\hat{X}_{12}c^{\dagger}_{n's'}(t*)\rangle\\ &= -i\langle c_{ns}(t)\hat{\tau}_1\otimes\hat{\sigma}_2c^{\dagger}_{n's'}(t*)\rangle\\ &= -i\langle c_s(t)\hat{\sigma}_2c^{\dagger}_{\bar{s}'}(t') + c^{\dagger}_{\bar{s}}(t)\hat{\sigma}_2c_{s'}(t')\rangle\\ &= -i\langle -c_{\uparrow}(z)c_{\uparrow}(t') + c_{\downarrow}(z)c_{\downarrow}(t') - c^{\dagger}_{\uparrow}(z)c^{\dagger}_{\uparrow}(t')\\ &\quad + c^{\dagger}_{\downarrow}(z)c^{\dagger}_{\downarrow}(t')\rangle\end{aligned} \tag{A1}$$

and

$$\begin{aligned}\hat{G}_{21}(t,t') &= -i\langle c_{ns}(t)\hat{X}_{21}c^{\dagger}_{n's'}(t*)\rangle\\ &= -i\langle c_{\uparrow}(z)c_{\uparrow}(t') + c_{\downarrow}(z)c_{\downarrow}(t') - c^{\dagger}_{\uparrow}(z)c^{\dagger}_{\uparrow}(t')\\ &\quad - c^{\dagger}_{\downarrow}(z)c^{\dagger}_{\downarrow}(t')\rangle.\end{aligned} \tag{A2}$$

Analogously, we obtain for $\hat{G}_{11}$ and $\hat{G}_{22}$

$$\begin{aligned}\hat{G}_{11}(t,t') = -i\langle c_{\uparrow}(z)c_{\uparrow}(t') + c_{\downarrow}(z)c_{\downarrow}(t') + c^{\dagger}_{\uparrow}(z)c^{\dagger}_{\uparrow}(t')\\ + c^{\dagger}_{\downarrow}(z)c^{\dagger}_{\downarrow}(t')\rangle,\end{aligned} \tag{A3}$$

$$\begin{aligned}\hat{G}_{22}(t,t') = -i\langle c_{\uparrow}(z)c_{\uparrow}(t') - c_{\downarrow}(z)c_{\downarrow}(t') - c^{\dagger}_{\uparrow}(z)c^{\dagger}_{\uparrow}(t')\\ + c^{\dagger}_{\downarrow}(z)c^{\dagger}_{\downarrow}(t')\rangle.\end{aligned} \tag{A4}$$

Combining Eqs. (A1)–(A4), one can write

$$\hat{G}^{(x)}(t,t') \equiv \hat{G}_{11} - s\hat{G}_{22} = -2i\langle c_s(t)c_s(t') + c^{\dagger}_s(t)c^{\dagger}_s(t')\rangle, \tag{A5}$$

$$\hat{G}^{(y)}(t,t') \equiv \hat{G}_{12} + s\hat{G}_{21} = 2i(-1)^s\langle c_s(t)c_s(t') + c^{\dagger}_s(t)c^{\dagger}_s(t')\rangle. \tag{A6}$$

Equations (A5) and (A6) show that both Green's functions $\hat{G}^{(x)}$ and $\hat{G}^{(y)}$ are off-diagonal in the Nambu space and define triplet Cooper pairs with spin up ($s = 1$) and down ($s = -1$), which describe a fully polarized triplet component. Since the matrix structure of the Green's functions does not change upon going over to the quasiclassical functions,

$$\hat{g}_{BVE} = \frac{i}{\pi}\nu \int d\xi\, \hat{G}, \tag{A7}$$

the same statement is true for matrix functions $\hat{g}_{BVE}$.

Note the transformation suggested by Ivanov-Fominov:

$$\hat{g} = \hat{U} \cdot \hat{g}_{BVE} \cdot \hat{U}^{\dagger}, \tag{A8}$$

$$\hat{U} = \tfrac{1}{2}(\hat{X}_{00} + i\hat{X}_{33}) \cdot (\hat{X}_{00} - i\hat{X}_{33}) \tag{A9}$$

transform the function $\hat{g}_{BVE}$ introduced in Ref. [20] into the functions $\hat{g}$, employed here. It does not change the relations because the matrix $\hat{U}$ commutes with the matrices $\hat{X}_{00}$ and $\hat{X}_{33}$.

These functions arise as a result of the action of the spin filters (we set $T = 1, U = \pm 1$).

$$\hat{G}^{(x)} = T_{33} \cdot \hat{G}_{11} \cdot T_{33}^{\dagger}, \tag{A10}$$

$$\hat{G}^{(y)} = T_{33} \cdot \hat{G}_{12} \cdot T_{33}^{\dagger}, \tag{A11}$$

$$T_{33} = \frac{1}{\sqrt{2}}(T + U\hat{X}_{33}). \tag{A12}$$

APPENDIX B: CHARGE AND SPIN CURRENTS

The charge density ρ is equal to

$$\begin{aligned}\rho(r,t') &= C_Q \sum_p \langle c_{ns}^{\dagger}(t)\hat{X}_{30}c_{n's'}(t')\rangle \\ &= C_Q \sum_p \langle c_{ns}^{\dagger}(t)\hat{\tau}_3 c_{n's'}(t')\rangle = \langle c_s^{\dagger}(t)c_s(t') - c_{\bar{s}}(t)c_{\bar{s}}^{\dagger}(t')\rangle \\ &= C_Q \sum_p \langle c_{\uparrow}^{\dagger}(t)c_{\uparrow}(t') + c_{\downarrow}^{\dagger}(t)c_{\downarrow}(t') - c_{\uparrow}(t)c_{\uparrow}^{\dagger}(t') \\ &\quad - c_{\downarrow}(t)c_{\downarrow}^{\dagger}(t')\rangle,\end{aligned} \tag{B1}$$

where C_Q is a constant which will be defined below. The operators $c_{ns}^{\dagger}(t')$, $c_{n's'}(t)$, as before, depend on times t, t'. For equal times $t = t'$, we obtain

$$\begin{aligned}\rho(t) &= 2C_Q \sum_p \langle c_{\uparrow}^{\dagger}c_{\uparrow} + c_{\downarrow}^{\dagger}c_{\downarrow}\rangle \\ &= -2iC_Q \sum_p \{\hat{G}\}_{30} = \frac{2}{\pi}C_Q\nu(0)\{\hat{g}_{BVE}\}_{00}.\end{aligned} \tag{B2}$$

Here, $\hat{g}_{BVE}$ is the quasiclassical Green's function derived in [20]. The magnetic moment is

$$\begin{aligned}M &= C_M \sum_p \langle c_{ns}^{\dagger}(t)\hat{X}_{03}c_{n's'}(t')\rangle \\ &= C_M \sum_p \langle c_s^{\dagger}\sigma_3 c_{s'} + c_{\bar{s}}\hat{\sigma}_3 c_{\bar{s}'}^{\dagger}\rangle \\ &= C_M \sum_p \langle c_{\uparrow}^{\dagger}(t)c_{\uparrow}(t') + c_{\downarrow}^{\dagger}(t)c_{\downarrow}(t')\rangle.\end{aligned} \tag{B3}$$

For equal times $t = t'$, we obtain

$$\begin{aligned}M(t) &= 2C_M \sum_p \langle c_{\uparrow}^{\dagger}(t)c_{\uparrow}(t) + c_{\downarrow}^{\dagger}(t)c_{\downarrow}(t)\rangle \\ &\Rightarrow 2iC_M \sum_p \{\hat{G}\}_{03} \Rightarrow \frac{2}{\pi}C_M\{\hat{g}_{BVE}\}_{33}.\end{aligned} \tag{B4}$$

To find the formula for the charge (spin) current, consider the Usadel equation for the Keldysh function

$$\hat{\tau}_3 \cdot \partial_t \hat{g} + \partial_{t*}\hat{g} \cdot \tau_3 = D_{\rm F}\partial_x(\hat{g} \cdot \partial_x \hat{g}) + iJ[\hat{X}_{33}, \hat{g}]. \tag{B5}$$

Introducing $\bar{t} = (t + t')/2$ and $\tau = t - t'$, Eq. (B5) can be written as

$$\tfrac{1}{2}\partial_{\bar{t}}[\hat{\tau}_3, \hat{g}]_+ + \partial_{\tau}[\hat{\tau}_3, \hat{g}] = D_F\partial_x(\hat{g} \cdot \partial_x\hat{g}) + iJ[\hat{X}_{33}, \hat{g}]. \tag{B6}$$

We multiply Eq. (B6) first by $\hat{X}_{30}$, then by $\hat{X}_{03}$, and calculate the trace. We get the law of conservation of the charge and the magnetization

$$\frac{\partial\rho}{\partial\bar{t}} = -\partial_x j_Q, \quad \frac{\partial M}{\partial\bar{t}} = -\partial j_{sp}, \tag{B7}$$

where the charge current j_Q is equal to

$$\begin{aligned}j_Q &= -\frac{\sigma_n}{e}2\pi T \sum_{\omega\geqslant 0}\frac{1}{4}{\rm Tr}\hat{\tau}_3\hat{g}\nabla\hat{g} \\ &= -\frac{\sigma_n}{e}2\pi T \sum_{\omega\geqslant 0}\{\hat{g}\nabla\hat{g}\}_{30}\end{aligned} \tag{B8}$$

and the spin current j_{sp} is given by

$$\mathbf{j}_{sp} = -\mu_B\frac{\sigma_n}{e^2}(i\pi T)\sum_{\omega\geqslant 0}\nabla\{\hat{g}\}_{03}. \tag{B9}$$

The charge density ρ is

$$\rho = e\nu(0)(i2\pi T)\sum_{\omega\geqslant 0}\{\hat{g}\}_{00}. \tag{B10}$$

The Drude conductivity σ_n is

$$\sigma_n = 2\nu(0)D_n e^2. \tag{B11}$$

The magnetic moment M_z is [see, e.g., [20], Eq. (A28)]

$$M_z = \mu_B\nu(0)(i2\pi T)\sum_{\omega\geqslant 0}\{\hat{g}\}_{33}. \tag{B12}$$

APPENDIX C: COEFFICIENTS IN THE CHANGE OF THE CURRENTS DUE TO TWO DWs

The coefficients $\hat{C}^{(\rm A,B)}$ and $\hat{S}^{(\rm A,B)}$ in Eqs. (59) and 60) are determined by Eqs. (20) and (21). They are equal to

$$\hat{C}^{(\rm A)} = \frac{\kappa_b}{2\kappa_\omega}(\hat{X}_r + \hat{X}_l)\cos(\varphi/2)F_{S-}, \tag{C1}$$

$$\hat{C}^{(\rm B)} = \frac{\kappa_b}{2\kappa_\omega}(\hat{X}_r - \hat{X}_l)\sin(\varphi/2)F_{S-}, \tag{C2}$$

$$\hat{S}^{(\rm A)} = \frac{\kappa_b}{2\kappa_\omega}(\hat{X}_r - \hat{X}_l)\cos(\varphi/2)F_{S-}, \tag{C3}$$

$$\hat{S}^{(\rm B)} = \frac{\kappa_b}{2\kappa_\omega}(\hat{X}_r + \hat{X}_l)\sin(\varphi/2)F_{S-}. \tag{C4}$$

The matrices $\hat{a}$, $\hat{b}$ equal

$$\hat{a} = -4r_\omega \frac{\cosh(\tilde{L} \pm \tilde{l})}{\sinh(2\tilde{L})} A \cos(\varphi/2)\hat{X}_{n2}; \tag{C5}$$

$$\hat{b} = -4r_\omega \frac{\cosh(\tilde{L} \pm \tilde{l})}{\sinh(2\tilde{L})} B \sin(\varphi/2)\hat{X}_{n2} \tag{C6}$$

with $n = 1, 2$ for y and x chiralities.

APPENDIX D: DETAILS OF THE NUMERICS

The critical current $\tilde{I}_0(t)$ of the considered Josephson junction without DWs is

$$\tilde{I}_0(t) = I_0 N(t), \tag{D1}$$

$$I_0 = \frac{\sigma_F \Delta}{e} \xi_\Delta \kappa_b^2, \tag{D2}$$

$$N(t) = 2\pi t \sum_{n\geqslant 0} \left[\mathrm{Im} \frac{\tilde{\Delta}(t)}{\sqrt{(t_n + i\tilde{J}_m)^2 + \tilde{\Delta}(t)^2}} \right]^2 \times \frac{1}{\sqrt{t_n} \sinh(2\tilde{L}\sqrt{t_n})}, \tag{D3}$$

where $\xi_\Delta = \sqrt{D_F/2\Delta}$, $\tilde{L} = L/\xi_\Delta$, $t_n = \pi t(2n + 1)$, $t = T/\Delta(0)$.

The correction to the current due to a single DW is

$$\delta\tilde{I}(t) = -I_{DW} N_{DW}(t), \tag{D4}$$

$$I_{DW} = \frac{\sigma_F \Delta}{e} (\xi_\Delta \kappa_{DW} \kappa_b)^2 \xi_\Delta, \tag{D5}$$

$$N_{DW}(t) = 2\pi t \sum_{n\geqslant 0} \left[\mathrm{Im} \frac{\tilde{\Delta}(t)}{\sqrt{(t_n + i\tilde{J}_m)^2 + \tilde{\Delta}(t)^2}} \right]^2 \times \frac{\cosh[\sqrt{t_n}(\tilde{L} + \tilde{l})] \cosh([\sqrt{t_n}(\tilde{L} - \tilde{l})])}{t_n[\sinh(2\tilde{L}\sqrt{t_n})]^2}. \tag{D6}$$

The temperature dependence of $\tilde{\Delta}(t) \equiv \Delta(T)/\Delta(0)$ can be approximated as

$$\Delta(T) \cong \Delta(0) \tanh[1.74\sqrt{(T_c/T - 1)}]. \tag{D7}$$

In the limits of $T = 0$ and $T \Rightarrow T_c$ it reproduces the limiting expressions (see, e.g., [89])

$$2\Delta(0) \cong 3.5T_c, \tag{D8}$$

$$\Delta(T)|_{T\Rightarrow T_c} \cong 3.06\sqrt{(T_c - T)T_c}. \tag{D9}$$

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