Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas

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We study thermal correlation functions of the one-dimensional impenetrable lattice anyons. These correlation functions can be presented as a difference of two Fredholm determinants. To describe the large-time and long-distance behavior of these objects, we use the effective form-factor approach. The asymptotic behavior is different in the spacelike and timelike regions. In particular, in the timelike region we observe the additional power factor on top of the exponential decay. We argue that this result is universal as it is related to the discontinuous behavior of the phase shift function of the effective fermions. At particular values of the anyonic parameter, we recover the asymptotics of the spin-spin correlation functions in the XX quantum chain.

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I. INTRODUCTION

Quantum one-dimensional models have always attracted a lot of attention due to the rich structures of their correlation functions and the possibility to address nonperturbative phenomena [1–3]. For low temperatures, the culmination of these developments resulted in the formulation of effective field theories (the Luttinger model) [3,4]. With the advancement of experimental techniques in cold-atom experiments [5–8], the interest in the nonequilibrium dynamics or dynamics of highly excited states motivated a lot of theoretical research resulting in new concepts such as generalized Gibbs ensembles, quench action [9,10], generalized hydrodynamics [11–13] (GHD), and others. The main approach to the correlation function in integrable models is a direct summation of the form factors in the spectral expansion. The computation of the correlation functions on the finite entropy states is very different from the vacuum case due to the different decay rate of the form factors with the system size (exponential versus power-law). Therefore, different approaches were developed to tackle these kinds of problems, including the quantum transfer-matrix approach [14–20], nonlinear differential equations [21,22], the axiomatic definition of the thermal form factors in the integrable quantum field theories [23–31], adaptation of the GHD methods [32–35], as well as partial summations of the few particle-hole excitations [36–40] and extracting the most singular parts of the form factors [41,42].

Recently, we developed a method to deal with correlation functions in finite entropy states [43]. This method allows one to derive the behavior of the correlation functions in free-fermionic models for the observables that can be expressed as Fredholm determinants of integrable kernels. In Ref. [43] we focused mostly on static correlation functions, and applied the method to the XY quantum chain.

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In this work, we continue the development of the method of effective form factors for dynamical correlation functions. As a model of interest, we choose one-dimensional impenetrable anyons on a lattice [44]. This model describes quantum particles with unusual statistics [44–49], which can be realized experimentally in ultracold quantum gases confined in optical traps [50–58]. Furthermore, this type of model appears after the spin-charge separation in interacting systems of spinful fermions and spin chains (at certain values of the anyonic parameter) [59–67]. Similar determinants can also be obtained as the correlation functions of Wigner strings [68]. Also, they appear in the description of the mobile impurity propagating in the gas of free fermions [69–72]. In the latter case, the anyonic parameter can be identified with the total momentum of the system (at the infinite coupling).

The main idea of the effective form-factor approach is to replace computation of the correlation functions averaged over some ensemble to zero-temperature correlators with the appropriately modified phase shift. The correlation functions for one-dimensional impenetrable anyons can be presented as a linear combination of the Fredholm determinants [44]. Therefore, we may identify the phase shift comparing these determinants to the one that emerges from the summation of the effective form factors. For the spacelike region, we can simplify the corresponding kernels for large time and space separation and find the effective phase shift for all values of the quasimomenta. The timelike region is characterized by the critical points that separate different types of asymptotic behavior. So we can robustly find the effective phase shift only away from these points. Even though the vicinity of critical points where we do not know the solutions vanishes in the large-time limit, we cannot simply combine solutions in the different asymptotic regions into a single phase shift as the latter will be discontinuous. To tackle this problem, we have assumed the existence of the gluing regularization functions. While we have not been able to find them explicitly, we have demonstrated that they only affect the overall constant in the asymptotic expression of the Fredholm determinants.

The structure of the paper is as follows. In Sec. II A, we define the anyonic model and recall its spectrum and the presentation of some correlation functions in terms of Fredholm determinants. In Sec. II B, for the reader's convenience, we collect the main results obtained in this paper. In Sec. III, we recall the effective form-factor approach and give two expressions for the τ function in the thermodynamic limit. In Sec. IV, the effective form-factor approach is applied to the derivation of the large-time and long-distance asymptotics of the dynamical correlation functions. We discuss separately spacelike and timelike regimes. In Sec. V, we summarize the main results of the paper, compare with the known results in the literature, and discuss different possibilities for further research. The Appendix contains technical details of the asymptotic analysis of the form factors with the regularized effective phase shift.

II. MODEL

A. Definition

The one-dimensional impenetrable lattice anyons on L sites can be described by the following Hamiltonian [44]:

$$H = -\sum_{j=1}^{L}\frac{1}{2}(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + h\sum_{j=1}^{L} a_j^\dagger a_j, \tag{1}$$

$$a_{L+1} = a_1, \quad a_{L+1}^\dagger = a_1^\dagger. \tag{2}$$

The operator algebra is specified by the anyonic parameter $0 \leqslant \kappa \leqslant 1$ and reads

$$a_j a_k^\dagger = \delta_{jk} - e^{-i\pi\kappa\epsilon(j-k)} a_k^\dagger a_j, \tag{3a}$$

$$a_j a_k = -e^{i\pi\kappa\epsilon(j-k)} a_k a_j, \tag{3b}$$

$$a_j^\dagger a_k^\dagger = -e^{i\pi\kappa\epsilon(j-k)} a_k^\dagger a_j^\dagger, \tag{3c}$$

where $\epsilon(j) = \mathrm{sgn}(j)$, and we prescribe that $\epsilon(0) = 0$.

The $\kappa = 0$ case corresponds to fermions, and $\kappa = 1$ describes operators in the Hilbert space of the impenetrable bosons. Note also that in the latter case, the Hamiltonian (1) can be identified with the Hamiltonian of the quantum XX spin chain after the mapping $a_j = \sigma_j^+$, $a_j^\dagger = \sigma_j^-$.

The spectrum of the Hamiltonian H can be found by means of the Bethe ansatz. The N-particle states are labeled by N momenta $\{p_1, p_2, \dots, p_N\}$ from the set of L inequivalent solutions of the equation

$$e^{ipL} = e^{-i\pi\kappa(N-1)}. \tag{4}$$

The energies of such states are

$$E(\{p_1, p_2, \dots, p_N\}) = \sum_{j=1}^{N}\varepsilon(p_j), \tag{5}$$

$$\varepsilon(p) = h - \cos p. \tag{6}$$

An interesting and nontrivial problem in the considered model is to analyze two-point correlation functions

$$G_-(x,t) = \frac{\mathrm{Tr}[e^{-\beta H} a_{x+1}^\dagger(t) a_1(0)]}{\mathrm{Tr}[e^{-\beta H}]}, \tag{7}$$

$$G_+(x,t) = \frac{\mathrm{Tr}[e^{-\beta H} a_{x+1}(t) a_1^\dagger(0)]}{\mathrm{Tr}[e^{-\beta H}]}. \tag{8}$$

It is easy to check the symmetry relations

$$G_\pm(-x,-t) = G_\pm(x,t)^*, \tag{9}$$

and also for $t = 0$

$$G_-(x,0) + e^{-i\pi\kappa\,\mathrm{sgn}(x)} G_+(-x,0) = \delta_{x,0}, \tag{10}$$

which allow us to consider only $t \geqslant 0$. In what follows, we will restrict ourselves to the analysis of the correlator $G_-(x,t)$. An analogous analysis can be done for $G_+(x,t)$. It was shown that these correlators in the thermodynamic limit $L \to \infty$ can be written in terms of Fredholm determinants [44]. We will use the following equivalent representation for $G_-(x,t)$:

$$G_-(x,t) = \det(1 + \hat{W} + \delta\hat{W}) - \det(1 + \hat{W}), \tag{11}$$

where $\hat{W}$ and $\delta\hat{W}$ are integral operators on $[-\pi, \pi]$ with the kernels

$$W(p,q) = \frac{1}{2\pi} e_-(p) e_-(q) e^{\frac{i(p-q)}{2}} \frac{e(p) - e(q)}{\sin\frac{p-q}{2}}, \tag{12}$$

$$\delta W(p,q) = \frac{1}{2\pi} e_-(p) e_-(q), \tag{13}$$

$$n_F(p) = \frac{1}{e^{\beta\varepsilon(p)} + 1}, \tag{14}$$

$$e_-(p) = \sqrt{n_F(p)}\, e^{-ixp/2 + it\varepsilon(p)/2}, \tag{15}$$

$$e(p) = \sin^2\frac{\pi\kappa}{2} \fint_{-\pi}^{\pi}\frac{dq}{2\pi} e^{ixq - it\varepsilon(q)} \cot\frac{q-p}{2} + \frac{1}{2}\sin(\pi\kappa) e^{ixp - it\varepsilon(p)}. \tag{16}$$

Equation (11) allows us to compute the correlation function $G_-(x,t)$ numerically. However, large-time and long-distance asymptotics of the correlation functions are hard to extract by numerical means due to the oscillatory behavior of integral kernels. In the present paper, we analyze these asymptotics analytically by means of the effective form-factor approach [43].

B. Results

Before presenting an application of the effective form-factor method to the problem, for the reader's convenience we collect the main results obtained in this paper: the asymptotic formulas for the correlation function $G_-(x,t)$ for large x and t with a fixed ratio $v = x/t$. To present the answer, we will need the effective phase-shift functions $\nu_\pm(q)$ defined as

$$\nu_\pm(q) = \pm\frac{1}{2\pi i}\ln(1 + n_F(q)(e^{\pm i\pi\kappa} - 1)). \tag{17}$$

The asymptotic behavior of $G_-(x,t)$ depends essentially on v. The spacelike region is specified by the condition $v > 1$, and the asymptotics there reduces to the analysis of a single integral (60). Depending on the velocity, there are two additional regimes within the spacelike region: the so-called saddle-point-dominated regime $1 < v < v_c$, and the pole-dominated regime $v > v_c$. The critical velocity v_c separating these two regimes can be read off from Eq. (69).

The asymptotics for $1 < v < v_c$ reads

$$G_-(x,t) \approx C_1 K(x,t) t^{-1/2} e^{-x\ln z_{\rm sp} + t\sqrt{v^2-1} + ith}, \tag{18}$$

where

$$z_{\rm sp} = iv + i\sqrt{v^2-1}, \tag{19}$$

$$K(x,t) = Z^2[\nu_+]e^{ix\int_{-\pi}^{\pi}\nu_+(q)dq}. \tag{20}$$

For $v \geqslant v_c$ a pole gives the leading contribution

$$G_-(x,t) \approx C_2 K(x,t) e^{-x\ln z_0 + \frac{i\pi}{\beta}(1-\kappa)}, \tag{21}$$

where z_0 is given by

$$z_0 = h_0 + \sqrt{h_0^2-1}, \quad h_0 = h + \frac{i\pi}{\beta}(1-\kappa). \tag{22}$$

The prefactors $Z^2[\nu_+]$, C_1, and C_2 are constants on the rays of fixed v. Their explicit expressions are given by Eqs. (44), (71), and (73). Note, in the case of the saddle-point contribution there is an additional power factor $t^{-1/2}$ correcting the exponential decay of the correlation function. For $v < -1$ there are similar regions, and the asymptotics can be obtained from the above upon the replacement $\nu_+ \to \nu_-$.

For the timelike region, $0 < v < 1$, the asymptotics of the correlation function is given by

$$G_-(x,t) \approx R_\infty t^{-\delta_1^2-\delta_2^2} e^{i\int_{-\pi}^{\pi}[x - t\varepsilon'(q)]\nu(q)dq} \times \left(\frac{a_1 e^{-ixq_1 + it\varepsilon(q_1)}}{t^{\frac{1}{2}+\delta_1}} + \frac{a_2 e^{-ixq_2+it\varepsilon(q_2)}}{t^{\frac{1}{2}+\delta_2}}\right). \tag{23}$$

For a fixed v, constants a_1 and a_2 are given by Eq. (91), while the constant R_∞ still remains unknown. The critical momenta q_1 and q_2 are defined by

$$q_1 = \arcsin v, \quad q_2 = \pi - \arcsin v, \tag{24}$$

the effective phase shift $\nu(q)$ is a piecewise function

$$\nu(q) = \begin{cases} \nu_+(q) & \text{if } -\pi < q < q_1 \text{ or } q_2 < q \leqslant \pi, \\ \nu_-(q) & \text{if } q_1 < q < q_2, \end{cases} \tag{25}$$

and δ_1 and δ_2 are the magnitudes of jumps of $\nu(q)$ at critical momenta

$$\delta_1 = \nu_-(q_1) - \nu_+(q_1), \quad \delta_2 = \nu_+(q_2) - \nu_-(q_2). \tag{26}$$

In addition to the expected exponential decay of the correlation function $G_-(x,t)$, we observe an additional power factor $t^{-\delta_1^2-\delta_2^2}$ depending on the parameters of the model.

III. EFFECTIVE FORM-FACTOR APPROACH

A. Effective form factors and tau function

In this section, we recall the effective form-factor approach initiated in [43]. To specify the effective form factor, we require two smooth periodic functions $\nu(k)$, $g(k)$. The first one is called the effective phase shift and defines the shifted set of momenta as solutions of

$$e^{ikL} = e^{-2\pi i\nu(k)}. \tag{27}$$

Here L is regarded as a system size. Since $\nu(k)$ is periodic, i.e., it has a zero winding number in terms of [43], the largest ordered set of the *shifted* momenta has L terms $\mathbf{k} = \{k_1,\dots,k_L\}$. Each k_i is a solution of (27). The *unshifted* momenta are solutions of

$$e^{iqL} = 1. \tag{28}$$

All momenta are considered up to the equivalence $k \sim k + 2\pi$, and it is convenient to choose them to have real parts in the Brillouin zone $[-\pi,\pi]$.

The effective form factors are defined for the subsets of momenta $\mathbf{q}$ of the size $L-1$. Such subsets can be parametrized by the position of the "hole,"

$$\mathbf{q}^{(a)} = \{q_1,\dots,q_{a-1},q_{a+1},\dots,q_L\}, \quad a = 1,\dots,L. \tag{29}$$

The effective form factor then reads

$$|\langle\mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 = L^{1-2L}\prod_{j=1}^{L}\frac{e^{g(k_j)-g(q_j)}\sin^2\pi\nu(k_j)}{1+\frac{2\pi}{L}\nu'(k_j)} \times e^{g(q_a)}\det{}^2 D^a, \tag{30}$$

where $\det D^a$ is defined for $\mathbf{q}^{(a)}$ and is merely a trigonometric variation of the Cauchy determinant, in which the row corresponding to q_a is omitted and replaced with the line of 1,

$$\det D^a = \begin{vmatrix} \cot\frac{k_1-q_1}{2} & \cdots & \cot\frac{k_L-q_1}{2} \\ \vdots & \ddots & \vdots \\ \cot\frac{k_1-q_L}{2} & \cdots & \cot\frac{k_L-q_L}{2} \\ 1 & \cdots & 1 \end{vmatrix}. \tag{31}$$

As we deal only with the square of the determinant, we can set this line as the last one.

The tau (correlation) function is defined as a series over these form factors,

$$\tau(x,t) = \sum_{\mathbf{q}^a}|\langle\mathbf{k}|\mathbf{q}^a\rangle|^2 e^{-ix[P(\mathbf{k})-P(\mathbf{q}^a)]+it[E(\mathbf{k})-E(\mathbf{q}^a)]}. \tag{32}$$

Here we use notations for the momentum and energy of many-particle state $|\mathbf{q}\rangle$,

$$P(\mathbf{q}) = \sum_{q\in\mathbf{q}} q, \quad E(\mathbf{q}) = \sum_{q\in\mathbf{q}}\varepsilon(q). \tag{33}$$

In Ref. [43] we have demonstrated that in the thermodynamic limit $L\to\infty$, the tau function can be presented as a difference of two Fredholm determinants,

$$\tau(x,t) = \det(1+\hat{V}+\delta\hat{V}) - \det(1+\hat{V}), \tag{34}$$

where $\hat{V}$ and $\delta\hat{V}$ are integral operators on $[-\pi,\pi]$ with kernels

$$V(p,q) = \frac{1}{2\pi}c_-(p)c_-(q)e^{\frac{i(p-q)}{2}}\frac{c(p)-c(q)}{\sin\frac{p-q}{2}}, \tag{35}$$

$$\delta V(p,q) = \frac{1}{2\pi}c_-(p)c_-(q), \tag{36}$$

$$c_-(p) = \sin\pi\nu(p)e^{-ixp/2+it\varepsilon(p)/2+g(p)/2}, \tag{37}$$

$$c(p) = \int_{-\pi}^{\pi}\frac{dq}{2\pi}e^{ixq-it\varepsilon(q)-g(q)}\cot\frac{q-p}{2} + \cot\pi\nu(p)e^{ixp-it\varepsilon(p)-g(p)}. \tag{38}$$

This form allows us to relate the correlation function of anyons with the tau function for a special choice of $\nu(k)$ and $g(k)$. This relation will be described in the next section.

B. Finite-size scaling

In this subsection, we give an alternative formula for the tau function based on first taking the thermodynamic limit of the form factors and then performing the summation. The obtained expressions will have a simple form convenient for asymptotic analysis [43].

We start by representing $\det D^a$ in a factorized form,

$$\prod_{i=1}^{L} \frac{\sin^2 \pi\nu(k_i)}{L^2} \det{}^2 D^{(a)} = Z^2 \mathcal{Z}_a, \tag{39}$$

$$Z = \prod_{i=1}^{L}\prod_{j=1}^{i-1} \frac{\sin\frac{k_i-k_j}{2}}{\sin\frac{q_i-q_j}{2}}, \tag{40}$$

$$\mathcal{Z}_a = \sin^2\frac{\pi\nu(k_a)}{L}\prod_{j\neq a}^{L}\frac{\sin^2\frac{k_j-q_a}{2}}{\sin^2\frac{q_j-q_a}{2}}. \tag{41}$$

Extracting the hole-dependent factors, the tau function (32) can be rewritten as

$$\tau(x,t) = LK(x,t)\sum_{a=1}^{L} e^{g(q_a)}\mathcal{Z}_a e^{-ixq_a+it\varepsilon(q_a)}, \tag{42}$$

where $K(x,t)$ is an a-independent part given by

$$K(x,t) = Z^2 e^{-ix[P(\mathbf{k})-P(\mathbf{q})]+it[E(\mathbf{k})-E(\mathbf{q})]}\prod_{j=1}^{L}\frac{e^{g(k_j)-g(q_j)}}{1+\frac{2\pi}{L}\nu'(k_j)}. \tag{43}$$

The expressions Z^2 and $K(x,t)$ have a finite thermodynamic limit [43],

$$\ln Z = -\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)}{4\sin\frac{q-k}{2}}\right]^2, \tag{44}$$

$$\ln K(x,t) = 2\ln Z - \int_{-\pi}^{\pi}\nu(q)g'(q)dq + i\int_{-\pi}^{\pi}[x-\varepsilon'(q)t]\nu(q)dq. \tag{45}$$

The hole-dependent factors are suppressed in the thermodynamic limit,

$$\mathcal{Z}_a \approx L^{-2}\sin^2\pi\nu(q_a)\exp\left(-\fint_{-\pi}^{\pi} dq\,\nu(q)\cot\frac{q-q_a}{2}\right), \tag{46}$$

but the whole tau function (42) has a finite thermodynamic limit and can be presented as an integral,

$$\tau(x,t) = K(x,t)\int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{g(k)}\sin^2\pi\nu(k)e^{-ixk+it\varepsilon(k)} \times \exp\left(-\fint_{-\pi}^{\pi} dq\,\nu(q)\cot\frac{q-k}{2}\right). \tag{47}$$

Thus we have two alternative presentations of the tau function in the thermodynamic limit: Eq. (34) as a difference of Fredholm determinants, and Eq. (47) in terms of integrals. The first form is convenient for the identification with other models, and the second form is convenient for large x and t analysis.

IV. ASYMPTOTIC BEHAVIOR OF ANYONIC CORRELATION FUNCTION

A. Anyons and effective fermions

To apply the method of effective form factors for the large x and t asymptotics of the correlation function $G_-(x,t)$ given by (11), we have to find suitable functions $\nu(k)$ and $g(k)$. This can be done after the identification of the kernels in (11) and in (34). In this section, we focus on the case of $h>0$; the case $h<0$ can be considered similarly. Also, we restrict the value of the parameter of anyonic statistics to $0\leqslant\kappa<1$. The peculiarities with the limiting case $\kappa=1$ corresponding to the quantum XX spin chain are briefly discussed in Sec. V.

Equating $G_-(x,t)=\tau(x,t)$, we see that their integral kernels coincide if we choose $\nu(p)$ and $g(p)$ to satisfy the equations

$$c_-(p) = e_-(p), \quad c(p) = e(p). \tag{48}$$

The first equation gives a relation between $g(p)$ and $\nu(p)$,

$$e^{-g(p)} = \frac{\sin^2\pi\nu(p)}{n_F(p)}. \tag{49}$$

The second equation allows us to obtain an integral equation for $\nu(p)$,

$$\int_{-\pi}^{\pi}\frac{dq}{2\pi}\left(\frac{\lambda_+(q)}{\tan\frac{q-p-i0}{2}}+\frac{\lambda_-(q)}{\tan\frac{q-p+i0}{2}}\right)e^{ixq-it\varepsilon(q)} = 0, \tag{50}$$

where we have denoted

$$\lambda_\pm(q) = \frac{e^{\pm2\pi i\nu_\pm(q)}-e^{\pm2\pi i\nu(q)}}{n_F(q)}, \tag{51}$$

$$e^{\pm2\pi i\nu_\pm(q)} = 1+n_F(q)(e^{\pm i\pi\kappa}-1). \tag{52}$$

We can solve Eq. (50) asymptotically for large x and t. The solution has different forms for two different values of $v\equiv x/t$. We call $|v|>1$ the spacelike region and $|v|<1$ the timelike region. These names should not be confused with similar terms in the relativistic theory—there the spectrum is linear for all momenta. In our case, the names come from the condition in which the function

$$\Phi(q) \equiv vq+\cos q \tag{53}$$

has (timelike) or does not have (spacelike) a critical point for $q\in[-\pi,\pi]$. This function is merely the phase $xq-\varepsilon(q)t$ up to rescaling by time and shift by the constant h.

B. Asymptotic behavior of the correlation function in the spacelike region

To treat Eq. (50), we first have to look at each of the integrals separately. It is useful to present them as

$$\int_{-\pi}^{\pi}\frac{dq}{2\pi}\frac{\lambda_\pm(q)e^{it\Phi(q)}}{\tan\frac{q-p\mp i0}{2}} = \lambda_\pm(p)\int_{-\pi}^{\pi}\frac{dq}{2\pi}\frac{e^{it\Phi(q)}}{\tan\frac{q-p\mp i0}{2}} + \int_{-\pi}^{\pi}\frac{dq}{2\pi}\frac{[\lambda_\pm(q)-\lambda_\pm(p)]e^{it\Phi(q)}}{\tan\frac{q-p}{2}}. \tag{54}$$

If we assume that $\nu(q)$ does not become singular even in the asymptotic region, then in the spacelike region the second term on the right-hand side of Eq. (54) becomes exponentially small for large x and t. The remaining integral in (54) can be rewritten as

$$\int_{-\pi}^{\pi} \frac{dq}{2\pi i} \frac{e^{it\Phi(q)}}{\tan\frac{q-p\mp i0}{2}} = e^{it\Phi(p)}[F(p) \pm 1], \tag{55}$$

where

$$F(p) = e^{-it\Phi(p)} \fint_{-\pi}^{\pi} \frac{dq}{2\pi i} \frac{e^{it\Phi(q)}}{\tan\frac{q-p}{2}}. \tag{56}$$

For large $t > 0$, the function $F(p)$ can be approximated up to exponentially small terms as

$$F(p) \approx \mathrm{sgn}(\Phi'(p)). \tag{57}$$

As the spacelike region is characterized by the absence of critical points of $\Phi(p)$ for $p \in [-\pi, \pi)$, we can set $\mathrm{sgn}(\Phi'(p)) = \mathrm{sgn}\, v$. This way, one of the two integrals in Eq. (50) is exponentially small due to (54) and (55), while the other allows us to find the effective phase shift for large $t > 0$,

$$\nu(p) \approx \nu_{\mathrm{sgn}\, v}(p), \tag{58}$$

where $\nu_{\pm}(p)$ are defined by Eqs. (52). We use this asymptotic solution and the relation (49) in (47) to obtain

$$\tau(x,t) = K(x,t)T(x,t)e^{ith}, \tag{59}$$

where $K(x,t)$ is given by Eq. (45) and $T(x,t)$ corresponds to the integral in (47), which, after the change of variables $z = e^{ik}$, takes the following form:

$$T(x,t) = \frac{1}{2\pi i}\oint_{C_>} \frac{dz}{z} \frac{e^{t\theta(z)}S(z)}{J(z) + e^{i\pi\kappa}}. \tag{60}$$

Here $C_>$ is a counterclockwise circle with a radius slightly larger than 1, and

$$\theta(z) = -v \ln z - \frac{i}{2}(z + z^{-1}), \tag{61}$$

$$S(z) = \exp\left(i\int_{-\pi}^{\pi} dq\, \nu(q)\frac{z+e^{iq}}{z-e^{iq}}\right), \tag{62}$$

$$J(z) = \exp\left(\beta\left(h - \frac{z+z^{-1}}{2}\right)\right). \tag{63}$$

In what follows, we consider $v > 1$; the other case, $v < -1$, can be considered in the same manner. To find large x and t asymptotics of $T(x,t)$, we deform the contour $C_>$ to the steepest-descent curve C_1 defined by

$$\mathrm{Im}\,\theta(z) = \mathrm{Im}\,\theta(z_{\mathrm{sp}}) = -\pi v/2 \tag{64}$$

going through the saddle point z_{sp},

$$z_{\mathrm{sp}} = iv + i\sqrt{v^2-1}. \tag{65}$$

Deforming the contour, we might cross the poles of the integrand, which can only appear from the denominator, since

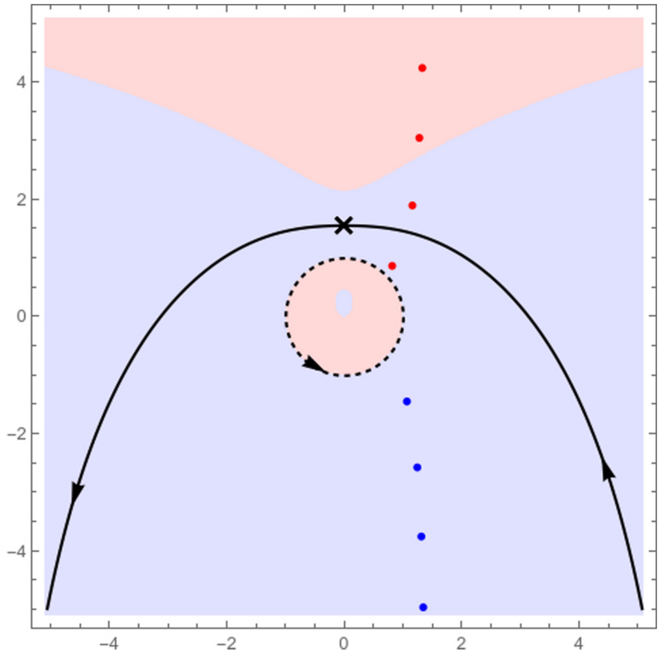

FIG. 1. The integration contours. The dashed circle corresponds to the initial contour of integration $C_>$. The black solid line represents the steepest-descent contour C_1. The cross marks the position of the saddle point. Red and blue dots correspond to the poles z_n defined by Eq. (67) for non-negative and negative indices, respectively. The shaded areas show the regions of positive (pink) and negative (light blue) values of $\mathrm{Re}\,\theta(z)$; see Eq. (61).

$S(z)$ is a holomorphic function for $|z| > 1$. This way, we get

$$T(x,t) = \frac{1}{2\pi i}\oint_{C_1} \frac{dz}{z} \frac{e^{t\theta(z)}S(z)}{J(z)+e^{i\pi\kappa}} - \sum_{n=-\infty}^{n_0} \mathrm{res}_{z=z_n} \frac{e^{t\theta(z)}S(z)}{z[J(z)+e^{i\pi\kappa}]}, \tag{66}$$

where the points z_n are defined as

$$z_n = h_n + \sqrt{h_n^2 - 1}, \quad h_n = h + \frac{i\pi}{\beta}(2n+1-\kappa), \tag{67}$$

and n_0 is the maximal number of a pole, which was crossed in the deformation process. This number depends on the velocity v and can be found from the inequality

$$\arg z_{n_0} < \frac{\pi}{2} - \frac{h}{v} < \arg z_{n_0+1}. \tag{68}$$

Schematically, the contours $C_>$, C_1, and the positions of poles z_n are shown in Fig. 1.

The formula (66) allows one immediately to read off the asymptotic behavior. The residues produce exponentially decaying terms; the leading contribution is given by the smallest real part $\mathrm{Re}\,\theta(z_n)$. For a wide range of the parameters of the model, we observed that this was achieved for the pole at z_0. Another type of contribution to the asymptotics comes from the saddle-point evaluation of the integral in (66). To find the overall leading contribution, we need to compare $\mathrm{Re}\,\theta(z_0)$ and $\mathrm{Re}\,\theta(z_{\mathrm{sp}})$. This leads to the equation for the critical velocity v_c

separating two regimes,

$$v_c \ln\left(v_c + \sqrt{v_c^2 - 1}\right) - \sqrt{v_c^2 - 1} = v_c \ln|z_0| - \frac{\pi}{\beta}(1-\kappa). \tag{69}$$

For $v < v_c$, the saddle point is dominating and $T(x,t)$ is given by

$$T(x,t) \approx C_1 t^{-1/2} e^{-x\ln z_{\rm sp} - \frac{it}{2}(z_{\rm sp} + z_{\rm sp}^{-1})}, \tag{70}$$

$$C_1 = \frac{S(z_{\rm sp})}{J(z_{\rm sp}) + e^{i\pi\kappa}} \frac{1}{\sqrt{2\pi\sqrt{v^2-1}}}. \tag{71}$$

For $v \geqslant v_c$, the pole gives the leading contribution

$$T(x,t) \approx C_2 e^{-x\ln z_0 - \frac{it}{2}(z_0 + z_0^{-1})}, \tag{72}$$

$$C_2 = -\frac{2}{\beta}\frac{e^{-i\pi\kappa}}{z_0 - z_0^{-1}} S(z_0). \tag{73}$$

We also provide the simplified expression for $K(x,t)$,

$$\ln K(x,t) \approx 2\ln Z[\nu] + ix\int_{-\pi}^{\pi} \nu(q)dq, \tag{74}$$

where $\nu(q)$ is given by Eqs. (58) and (52), and $Z[\nu]$ is defined by Eq. (44).

Using the identification $G_-(x,t) = \tau(x,t)$, the asymptotic behavior of the correlation function $G_-(x,t)$ in the spacelike region can be found from Eq. (59),

$$G_-(x,t) \approx K(x,t)T(x,t)e^{ith}, \tag{75}$$

where $K(x,t)$ is given by Eq. (74), and $T(x,t)$ is given by one of Eqs. (70) and (72) depending on the value of v. We compare these asymptotic expressions for the correlation functions with numerical evaluation of Fredholm determinants (11) in Fig. 2. We see that the asymptotics given by the integral (the red solid line), i.e., by the tau function, is hardly distinguishable from the true correlation function even for small x.

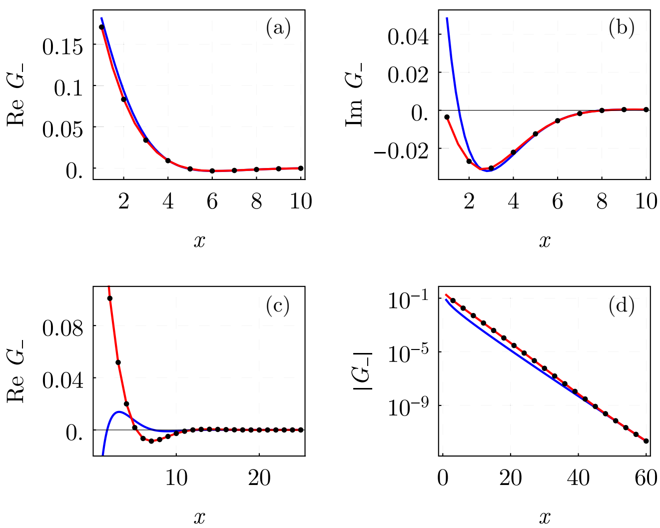

FIG. 2. Asymptotic behavior of $G_-(x,t)$ for $\kappa = 0.6$, $h = 0.7$, and $\beta = 2.3$. These parameters correspond to critical velocity $v_c \approx 1.676$. Black dots present $G_-(x,t)$ computed numerically from (11). Red lines present the effective τ function (59) computed with (58). Blue lines present asymptotics of integrals in (59) given by Eqs. (70) and (72). Panels (a) and (b) correspond to the overcritical region $v = 2.5$. Panels (c) and (d) show the real part and the absolute value of $G_-(x,t)$ in the subcritical region $v = 1.3$, respectively.

C. Asymptotic behavior of correlation function in the timelike region

Now let us try to apply the same reasoning for the timelike region, $|v| < 1$. In this case, there are two critical points q_1 and q_2,

$$\Phi'(q_i) = 0, \quad q_i \in [-\pi, \pi), \tag{76}$$

therefore the approximation (57) naively gives rise to the solution

$$\nu(p) \approx \nu_{\mathrm{sgn}\,\Phi'(p)}(p), \tag{77}$$

where $\nu_\pm(p)$ are defined by Eqs. (52). This is valid for all p lying far enough from the critical points. Indeed, the approximation (57) holds everywhere outside small vicinities of width $\sim t^{-1/2}$ around critical points q_1 and q_2.

It is very tempting to ignore these domains and approximate $\nu(p)$ as a truly discontinuous function, since we are interested in the large-t behavior. This procedure, however, is not consistent with the approximations made in Eq. (54) where we have discarded critical point contributions (the last integral). But even bigger problems appear when one tries to use discontinuous $\nu(p)$ for the asymptotic expression. For instance, the double integral (44) is divergent for such a choice.

Therefore, we expect that the solution of Eq. (50) will have the following "regularized" form:

$$\nu(p) = A(p) + B(p)s(p), \tag{78}$$

where

$$A(p) = \frac{\nu_+(p) + \nu_-(p)}{2}, \quad B(k) = \frac{\nu_+(p) - \nu_-(p)}{2}, \tag{79}$$

and the function $s(p)$ is a regularization of the sgn function,

$$s(p) = f(\sqrt{t}\Phi'(p)), \tag{80}$$

with f being a smooth function satisfying

$$f(\pm\infty) = \pm 1. \tag{81}$$

So away from the critical points on a distance bigger than $O(1/\sqrt{t})$, we recover the solution (77). We demonstrate this schematically in Fig. 3. Notice that the regularization is needed only for the imaginary parts, and the real parts of $\nu_+(p)$ and $\nu_-(p)$ coincide. Now for the smooth $\nu(p)$ we can use all the results from the previous sections. In particular, we can integrate Eq. (44) by parts to obtain

$$\ln Z = \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk\, \nu'(q)\nu'(k) \ln\left|\sin\frac{q-k}{2}\right|. \tag{82}$$

We can perform asymptotic analysis of this expression for large t and obtain

$$Z \approx t^{-\frac{1}{2}(\delta_1^2 + \delta_2^2)} Z_{\rm reg}, \tag{83}$$

where $Z_{\rm reg}$ is a t-independent factor depending on $s(p)$, and

$$\delta_1 = \nu_-(q_1) - \nu_+(q_1), \quad \delta_2 = \nu_+(q_2) - \nu_-(q_2). \tag{84}$$

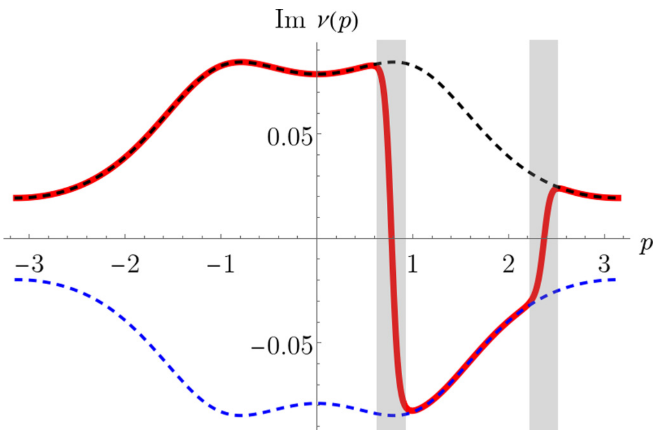

FIG. 3. The schematic dependence of the effective phase shift $\nu(p)$. The black and blue dotted lines represent $\nu_+(q)$ and $\nu_-(q)$, respectively. The red lines shows the regularized expression for $\nu(p)$. The shaded rectangles show the regions where the transition between ν_+ and ν_- happens and the regularization is required to approximate $\nu(p)$. These regions are located near critical points q_1, q_2 and their widths are $O(t^{-1/2})$. We show only the imaginary part, as the real part is continuous and does not require regularization.

Therefore, the only regularization dependence remains in the overall constant prefactor. It is remarkable that the exponent of the power-law t-dependence of Z is universal [it does not depend on the regularization $s(p)$ for any f satisfying (81)]. These computations and the exact form for Z_{reg} are given in the Appendix.

Let us also discuss the asymptotic behavior of the remaining part of the tau function. In there we substitute already discontinuous $\nu(q)$. Namely, we analyze the integral

$$T(x,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n_F(k) e^{-it\Phi(k)} e^{-Y(k)}, \tag{85}$$

where

$$Y(k) = \fint_{-\pi}^{\pi} dq\, \nu(q) \cot\frac{q-k}{2}. \tag{86}$$

The function $Y(k)$ is logarithmically divergent at q_1 and q_2 because of the discontinuity of $\nu(q)$. It leads to powerlike singularities in the integrand of (85) which are integrable if $\text{Re}\,\delta_j > -\frac{1}{2}$. In our case, $\nu_+(k)$ and $\nu_-(k)$ are conjugate to each other, rendering the real part of the effective phase shift continuous, $\text{Re}\,\delta_j = 0$.

We separate a regular part $\tilde{Y}(k)$ of $Y(k)$ as

$$Y(k) = \tilde{Y}(k) + [\nu_-(k) - \nu_+(k)] \ln\left(\frac{\sin\frac{q_1-k}{2}}{\sin\frac{q_2-k}{2}}\right)^2, \tag{87}$$

$$\begin{aligned}\tilde{Y}(k) = &\int_{-\pi}^{q_1} dq[\nu_+(q) - \nu_+(k)] \cot\frac{q-k}{2}\\ &+ \int_{q_1}^{q_2} dq[\nu_-(q) - \nu_-(k)] \cot\frac{q-k}{2}\\ &+ \int_{q_2}^{\pi} dq[\nu_+(q) - \nu_+(k)] \cot\frac{q-k}{2}.\end{aligned} \tag{88}$$

Now all is prepared to find the asymptotic behavior of $T(x,t)$ for large x and t coming from the contributions of two critical points q_1 and q_2,

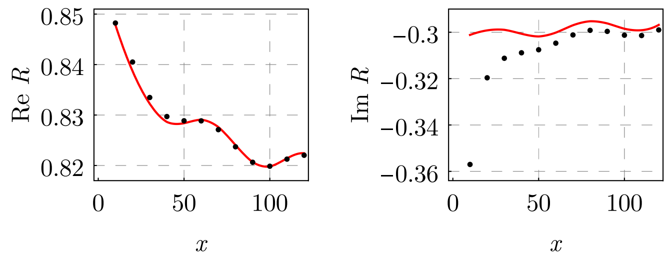

FIG. 4. Real and imaginary part of $R(x,t)$ with $v = x/t = 0.5$, $\kappa = 0.6$, $h = 0.7$, and $\beta = 2.3$. The red line presents $R(x,t)$ for which the integral $T(x,t)$ in Eq. (85) is computed exactly. Black dots present $R(x,t)$ for which we use asymptotics of integral $T(x,t)$ given by Eq. (89),

$$T(x,t) \approx T_1 + T_2, \tag{89}$$

where

$$T_j = a_j e^{-it\Phi(q_j)}, \tag{90}$$

$$\begin{aligned}a_j = &\frac{n_F(q_j)}{2\pi} e^{-\tilde{Y}(q_j)} \left(2\sin\frac{q_2-q_1}{2}\right)^{-2\delta_j}\\ &\times \left(\frac{it\Phi''(q_j)}{2}\right)^{-\frac{1}{2}-\delta_j} \Gamma\left(\frac{1}{2}+\delta_j\right).\end{aligned} \tag{91}$$

The final formula for the asymptotics of the correlation function $G_-(x,t)$ is

$$\begin{aligned}G_-(x,t) \approx &R_\infty T(x,t) t^{-\delta_1^2-\delta_2^2} e^{iht}\\ &\times \exp\left(i\int_{-\pi}^{\pi}(x - t\sin q)\nu(q)dq\right),\end{aligned} \tag{92}$$

where R_∞ is a constant different on each ray $v = x/t$ that additionally depends on the parameters κ, h, and inverse temperature β. To check this asymptotics, we plot in Fig. 4 the ratio $R(x,t)$ of $G_-(x,t)$ calculated numerically from (11) to the asymptotics from the right-hand side of Eq. (92) without R_∞. We observe that it approaches a constant value. The possible deviations are of order $O(1/\sqrt{t})$, which is consistent with our approximations made for the $\nu(k)$. It would be interesting to see if these corrections can be interpreted in terms of the nonlinear Luttinger liquid paradigm [73,74].

V. SUMMARY AND OUTLOOK

In this paper, we found the asymptotics of dynamical correlation functions of anyonic gas with the parameter of anyonic statistics $0 \leqslant \kappa < 1$ using a recently introduced [43] effective form-factor approach. The main difficulty of this method is to find the phase-shift function $\nu(q)$ for effective fermions solving an integral equation. For large x and t we found approximate solutions for this integral equation that depend on the ratio $v = x/t$. For the spacelike region, $v > 1$, the solution $\nu(q)$ can be approximated by the smooth function $\nu_+(q)$. In this case, the asymptotics of the correlation function is given by asymptotic analysis of integrals producing the leading contribution either from a pole or from a saddle point. In the case of a saddle-point contribution, there is an additional power factor correcting the exponential decay of the correlation function.

For the timelike region, $|v| < 1$, we approximate the solution $\nu(q)$ for a large finite t by a function having discontinuities at critical points and corresponding to the solution of the integral equation at $t = \infty$. Unfortunately, this approximate solution cannot be used directly to find the asymptotics of the correlation function by the methods of [43], since the latter requires a smooth $\nu(q)$. For large finite t we consider a class of regularized $\nu(q)$ having the same limit at $t = \infty$ as the genuine solution. It is remarkable that the regularized $\nu(q)$ lead to the same asymptotics up to a prefactor independent of t. This universal time dependence of asymptotics has an additional powerlike factor to the exponential decay of the correlation function. The exponent of this powerlike factor is related directly to the jumps of $\nu(q)$ at critical points. We hope that the use of a better approximation to $\nu(q)$ as a solution of the integral equation for a large finite t will fix the exact form of the constant prefactor. Further analysis of the correlation functions in the timelike region by the method of effective form factors will be presented in future publications.

We believe that the appearance of the power-law corrections is universal and takes place in all dynamical correlation functions of quantum one-dimensional models at finite temperature (entropy) in the timelike region. Recently, one of us observed similar behavior for a continuum model [75]. Equivalent phenomena are present in the XX spin chain [19,20,76]. Finally, quite unexpectedly, similar asymptotics appear also while describing large x and t behavior of the classical integrable systems [77–79]. Perhaps, it is related to the fact that the tau functions in such systems can be presented as Fredholm determinants, and the role of momentum distribution $n_F(q)$ is played by the reflection coefficient [80,81]. We plan to investigate these models in the future.

The limiting case $\kappa = 1$ of the model corresponds to the quantum XX spin chain model studied intensively in the literature. Therefore, it is interesting to look at the limits of our results as $\kappa \to 1$ and compare with the known formulas. For the paramagnetic phase, $h > 1$, in the timelike region the results for the asymptotics were obtained in [82] up to an overall constant depending on β and h. Our results have the same structure as a function of t. The ferromagnetic phase, $h < 1$, was studied in [19,20,76] in the spacelike region and [76] in the timelike region. Unfortunately, the direct application of our approach is not possible due to the appearance of singularities of $\nu_\pm(q)$ at $q = \pm \arccos h$, where $\varepsilon(q) = 0$. We believe that these singularities can be properly resolved. But one needs to develop a more delicate limiting procedure, on which we hope to report in the near future.

An important ingredient in the derivation of asymptotics in [1,76,82] is the use of the fact that the correlation function satisfies differential-difference equations of an Ablowitz-Ladik integrable system. It would be interesting to generalize this approach to the correlation functions with arbitrary anyonic parameter κ and determine the precise v dependence of R_∞ in Eq. (92).

Another important application of our approach is to use it to describe the scaling behavior of the correlation functions of the anyonic gas. One has to be able to reproduce results for the asymptotics obtained in [83–85]. Recently, using effective form factors, the finite-temperature tau function for the continuum case was investigated in [75].

Finally, an important generalization would be to the interacting case. Recently, the asymptotic behavior of the static one-body correlation function at zero temperature was derived for the interacting anyonic gas via the Luttinger liquid approach [86]. To reproduce this result, at least in the Tonks-Girardeau limit, we would have to take into account next to leading asymptotics. Indeed, the only way to reproduce zero-temperature power-law behavior is first to recover finite T CFT predictions, which, roughly speaking, replace power law as $1/x^\Delta \to 1/[\sinh(Tx)/T]^\Delta$ [87]. In the expansion of this expression at large x one obtains not only the leading exponential but also a bunch of the subleading ones. One way to capture this could be in a more precise identification between integral kernels. Right now, we do not know how to generalize our methods to the fully interacting model, i.e., to the case when Fredholm presentation is not available. A perspective direction could be to derive it directly from the form-factor series [42].

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APPENDIX: REGULARIZATION OF THE PREFACTOR AND POWERLIKE BEHAVIOR

In this Appendix, we describe a regularization of the divergent integral

$$\mathcal{A} = \ln Z = \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, \nu'(q)\nu'(k) \ln\left|\sin\frac{q-k}{2}\right| \tag{A1}$$

for the case of discontinuous $\nu(k)$. We use the regularization described in Eqs. (78)–(81) and find the asymptotics of this integral for large times.

It is natural to divide the derivative of ν into two parts,

$$\nu'(k) = \nu_0'(k) + \nu_1'(k), \tag{A2}$$

where

$$\nu_0'(k) = A'(k) + B'(k)s(k), \quad \nu_1'(k) = B(k)s'(k). \tag{A3}$$

In the large-t limit, $\nu_0'(k)$ is a bounded function while $\nu_1'(k)$ becomes proportional to a δ-function. The double integral $\mathcal{A}$ can be presented as a sum of four parts,

$$\mathcal{A} = \mathcal{A}_{00} + \mathcal{A}_{01} + \mathcal{A}_{10} + \mathcal{A}_{11}, \tag{A4}$$

where

$$\mathcal{A}_{ij} = \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, \nu_i'(q)\nu_j'(k) \ln\left|\sin\frac{q-k}{2}\right|. \quad \text{(A5)}$$

Note, only the $\mathcal{A}_{11}$ part is responsible for the divergence of $\mathcal{A}$ at large t. The parts $\mathcal{A}_{00}$, $\mathcal{A}_{01}$, and $\mathcal{A}_{10}$ have nonsingular limiting values at $t \to \infty$ which do not depend on regularization of $\nu(k)$. We have

$$\mathcal{A}_{00} \approx \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, [\nu'](q)[\nu'](k) \ln\left|\sin\frac{q-k}{2}\right| \quad \text{(A6)}$$

with

$$[\nu'](k) = A'(k) + B'(k)\,\mathrm{sgn}\,\Phi'(k). \quad \text{(A7)}$$

Due to $k \leftrightarrow q$ symmetry, we have $\mathcal{A}_{01} = \mathcal{A}_{10}$. In the limit $t \to \infty$, the function $\nu_1'(k)$ becomes a sum of two δ functions, and therefore

$$\mathcal{A}_{01} = \mathcal{A}_{10} \approx B_1 r_1 \int_{-\pi}^{\pi} dq\, [\nu'](q) \ln\left|\sin\frac{q-q_1}{2}\right| + B_2 r_2 \int_{-\pi}^{\pi} dq\, [\nu'](q) \ln\left|\sin\frac{q-q_2}{2}\right|, \quad \text{(A8)}$$

where

$$B_i = B(q_i), \quad r_i = \mathrm{sgn}\,(\Phi''(q_i)). \quad \text{(A9)}$$

To evaluate $\mathcal{A}_{11}$, we divide the integration region $[-\pi, \pi]$ into two pieces $\Lambda_1 = [-\pi, p]$ and $\Lambda_2 = (p, \pi]$, where point p lies between critical points $q_1 < p < q_2$. This way, the double integral $\mathcal{A}_{11}$ is divided into four parts,

$$\mathcal{A}_{11} = a_{11} + a_{12} + a_{21} + a_{22}, \quad \text{(A10)}$$

where

$$a_{ij} = \frac{1}{2}\int_{\Lambda_i} dq \int_{\Lambda_j} dk\, B(q)B(k)s'(q)s'(k) \ln\left|\sin\frac{q-k}{2}\right|. \quad \text{(A11)}$$

The integrals a_{21} and a_{12} have finite limits at $t \to \infty$,

$$a_{12} = a_{21} \approx 2B_1B_2r_1r_2 \ln\sin\frac{q_2-q_1}{2}. \quad \text{(A12)}$$

The remaining parts of $\mathcal{A}_{11}$ contain singularities. Let us show how they emerge in an example of a_{11}. It is natural to present a_{11} as a sum of two integrals (regular and singular),

$$a_{11} = a_{11}^{(r)} + a_{11}^{(s)}, \quad \text{(A13)}$$

where

$$a_{11}^{(r)} = \frac{1}{2}\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, B(q)B(k)s'(q)s'(k) \times \ln\left|\frac{\sin\frac{q-k}{2}}{\Phi'(q)-\Phi'(k)}\right|, \quad \text{(A14)}$$

$$a_{11}^{(s)} = \frac{1}{2}\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, B(q)B(k)s'(q)s'(k) \times \ln|\Phi'(q)-\Phi'(k)|. \quad \text{(A15)}$$

The first integral can be found using L'Hôpital's rule,

$$a_{11}^{(r)} = 2\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, B(q)B(k)\delta(q-q_1)\delta(k-q_1) \times \ln\left|\frac{\sin\frac{q-k}{2}}{\Phi'(q)-\Phi'(k)}\right| = -2B_1^2 \ln|2\Phi''(q_1)|, \quad \text{(A16)}$$

where we used $r_1^2 = 1$. The second integral can be presented as

$$a_{11}^{(s)} = u_1 + v_1 \ln\sqrt{t}, \quad \text{(A17)}$$

where

$$u_1 = \frac{1}{2}\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, B(q)B(k) \times s'(q)s'(k) \ln|\sqrt{t}\Phi'(q) - \sqrt{t}\Phi'(k)|, \quad \text{(A18)}$$

$$v_1 = -\frac{1}{2}\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, B(q)B(k)s'(q)s'(k). \quad \text{(A19)}$$

Performing rescaling of the integration variables, one can persuade oneself that under the last integrals $B(q)$ can be replaced to B_1, which leads to

$$v_1 = -\frac{B_1^2}{2}\int_{\Lambda_1} dq \int_{\Lambda_1} dk\, s'(q)s'(k) = -2B_1^2. \quad \text{(A20)}$$

Here we have used (81), and all the traces of the regularization have disappeared. With u_1 this will not be the same. Indeed, using $s(k) = f(\sqrt{t}\Phi'(k))$ and changing the variables of integration q and k by $\lambda = \sqrt{t}\Phi'(q)$ and $\mu = \sqrt{t}\Phi'(k)$, we get

$$u_1 = \frac{1}{2}\int_{\tilde{\Lambda}_1} d\lambda \int_{\tilde{\Lambda}_1} d\mu\, b(\lambda)b(\mu)f'(\lambda)f'(\mu) \ln|\lambda-\mu|, \quad \text{(A21)}$$

where the function $b(\lambda)$ is defined as

$$b(\sqrt{t}\Phi'(q)) = B(q) \quad \text{(A22)}$$

and region $\tilde{\Lambda}_1$ is the segment $[\sqrt{t}\Phi'(-\pi), \sqrt{t}\Phi'(p)]$, which becomes the real line when t goes to infinity. Also $b(\lambda)$ goes to B_1 at $t \to \infty$. Therefore, we get

$$u_1 \approx \frac{B_1^2}{2}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu\, f'(\lambda)f'(\mu) \ln|\lambda-\mu|. \quad \text{(A23)}$$

Finally, using $B_1 = -\delta_1/2$, $B_2 = \delta_2/2$, $r_1 = -1$, and $r_2 = 1$, we obtain the following large t asymptotics of $\mathcal{A}$:

$$\mathcal{A} \approx d_0 + d_1 \ln\sqrt{t}, \quad \text{(A24)}$$

where the constant d_1 is universal, i.e., it is independent of a regularizing function f,

$$d_1 = -2\left(B_1^2 + B_2^2\right) = -\tfrac{1}{2}\left(\delta_1^2 + \delta_2^2\right), \quad \text{(A25)}$$

and d_0 depends on a regularizing function f only in summands u_1 and u_2,

$$d_0 = \mathcal{A}_{00} + 2\mathcal{A}_{01} + 2a_{12} + a_{11}^{(r)} + a_{22}^{(r)} + u_1 + u_2. \quad \text{(A26)}$$

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