Torsion, energy magnetization, and thermal Hall effect

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We study the effective action of hydrostatic response to torsion in the absence of spin connections in gapped $(2+1)$-dimensional topological phases at low temperatures. In previous studies, a torsional Chern-Simons term with a temperature-squared (T^2) coefficient was proposed as an alternative action to describe the thermal Hall effect with the idea of balancing the diffusion of heat by a torsional field. However, the question remains whether this action leads to a local bulk thermal response that is not suppressed by the gap. In our hydrostatic effective action, we show that the T^2 bulk term is invariant under variations up to boundary terms considering the geometry dependence of local temperature, which precisely describes the edge thermal current. Furthermore, there are no local boundary diffeomorphism anomalies or bulk inflow thermal currents at equilibrium, and also there is no edge-to-edge adiabatic thermal current pumping upon changing the gravitational background. These results are consistent with exponentially suppressed thermal current for gapped phases.

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I. INTRODUCTION

The Chern-Simons term originating from external electromagnetic fields is known to be the effective action for the integer quantum Hall effect, where the quantization of Hall conductance is guaranteed by gauge invariance [1] or boundary gauge anomalies [2–4]. Based on Luttinger's seminal work [5], a torsional field has been introduced to balance the diffusion of heat [6–8]. Analogously, torsional Chern-Simons terms have been proposed to describe the torsional viscosity at zero temperature [9–11] and the thermal Hall effect at finite temperature [6,12–15]. However, torsional anomalies are controversial because of their dependence upon ultraviolet (UV) cutoff [16–24]. A clear physical meaning for torsional anomalies is thus highly needed.

Recently, the thermal Hall effect was observed experimentally in gapped topological phases [25–27], and it has attracted much current attention due to the observed large signature from charge-neutral excitations [28–36]. However, in spite of the fast evolving experimental techniques, the fundamental understanding of whether thermal Hall current flows through the bulk of these systems is still incomplete. For gapped topological phases, based on anomaly matching and the generalized Laughlin argument, it was suggested in Refs. [12,13] that there can exist a bulk thermal Hall current. This argument contradicts the results in Refs. [8,37,38], where bulk thermal Hall currents are always exponentially suppressed by the bulk gap. Hence, we aim to resolve this contradiction here, which will add a new perspective to investigate the thermal Hall effect.

In this paper, by coupling matter fields to teleparallel gravity, we study the response of the matter to an inhomogeneous gravitational field at equilibrium. A hydrostatic effective action is derived, which turns out to be the torsional Chern-Simons term, and its coefficient is the energy magnetization [39–41]. The energy magnetization can contain a constant UV-dependent piece at zero temperature for a continuous model such as the massive Dirac fermion [42]. We further show that there can be a temperature-squared energy magnetization, which corresponds to a finite-temperature torsional Chern-Simons term in gapped systems. In sharp contrast with its zero-temperature counterpart, this term can be recast as a topological θ-term in terms of Kaluza-Klein gauge fields, such that it is invariant under variations of background fields, and it manifests itself as boundary currents. Therefore, the presence of the background field cannot induce a bulk thermal current. Also, from the boundary perspective, the resulting boundary energy current does not possess local diffeomorphism anomalies, hence there is no corresponding bulk inflow energy current. However, boundary global gravitational anomalies do quantize the change of the coefficient of this θ-term across the boundary, which reveals the relative topological meaning [43] of the θ-term between adjacent materials. Apart from addressing the described debates, our theory provides a top-down approach for magnetization and energy magnetization: we show that various properties of magnetization and energy magnetization can be obtained from macroscopic effective action with symmetry considerations, and they are independent of details of the microscopic model.

II. OVERVIEW AND SUMMARY OF RESULTS

Although transport is a nonequilibrium phenomenon, it is surprisingly simple that certain topological responses can be characterized from equilibrium aspects. Gaps between equilibrium and nonequilibrium quantities in quantum Hall

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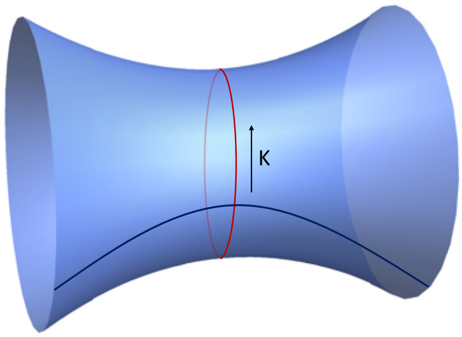

FIG. 1. Illustration of concepts of a Euclidean field theory on a two-dimensional manifold describing an equilibrium state. The red line and the black line stand for time axis and space axis, respectively. The vector K is a timelike Killing vector pointing along the time direction, which originates from the static nature of equilibrium states. The time axis is further compactified to a thermal loop (red line) so as to describe thermal physics, which in turn yields two scalars. The first one is the local temperature $T(x)$, defined from the length of the thermal loop $\beta(x)$, i.e., $T(x) \equiv 1/\beta(x)$. The second one is the local chemical potential $\mu(x)$, defined from the Wilson loop of electromagnetic gauge fields $\oint A_0 dx_E^0$, i.e., $\mu(x) \equiv -\oint A_0 dx_E^0/\beta(x)$.

systems can be bridged by Laughlin's argument [1] as well as the Streda formula [44–46]. To be more specific, Laughlin's argument tells us that the quantum Hall response can be understood as the adiabatic response of a gapped ground state (equilibrium state property at zero temperature). Upon inserting a magnetic flux in a cylinder geometry, the adiabatic charge-pumping process requires the anomalous edges to absorb the charge. This absorbing process is described by the anomaly inflow [2]. For the other aspect, the Streda formula relates the Hall conductance to the thermodynamic property of magnetization in equilibrium. These two aspects are well-established for electric transports, but the validity of the anomaly inflow aspect for thermal transports is still under debate. The problems are twofold: On the one hand, the previously proposed Laughlin argument [12,13] for the thermal Hall effect requires the existence of bulk thermal Hall currents so as to absorb the edge quantum anomaly. On the other hand, for gapped systems there are hardly any bulk excitations, so as an entropy current the nonzero bulk thermal Hall current is questionable.

Motivated by these aspects, we study the Euclidean field theory, which describes the bulk of a quantum Hall system at equilibrium. In particular, to address the anomaly matching problem, we need to couple the system with background gravitational fields while maintaining the system at equilibrium. Involving gravitational fields, equilibrium conditions are more subtle [5,39]: equilibrium is reached only when mechanical forces are balanced by statistical forces, which stem from inhomogeneous distributions of charge or energy. Interestingly, it turns out that these equilibrium conditions can be geometrically visualized in Fig. 1: (i) There exists a timelike Killing vector K due to the static nature of equilibrium states. (ii) The time axis along the K direction is compactified to a thermal loop (red line) so as to capture thermal physics. (iii) The local temperature turns out to be the inverse of thermal loop length, and the local chemical potential is the Wilson loop of electromagnetic fields along the K direction. For equilibrium states satisfying these equilibrium conditions, a generic formalism describing the physics at a long lengthscale is the hydrostatics, or equivalently a Euclidean field theory equipped with a timelike Killing vector [47,48], where the thermodynamic properties of the system are captured by a hydrostatic action from derivative expansions. Built on this setup, we will derive a hydrostatic action to the linear power of derivatives [see Eq. (12)]. This action not only reproduces known results for electric transports, but more importantly it also clarifies the anomaly-inflow aspect of thermal transports. Namely, considering the geometry dependence of local temperature, our hydrostatic effective action can be recast as a topological θ term with no local boundary diffeomorphism anomalies, so there is no bulk thermal Hall current, in consistency with exponential suppression of bulk thermal currents in gapped phases.

The paper is organized as follows: In Sec. III, we derive equilibrium conditions for the external field arising from the balancing between statistical forces and mechanical forces. We also obtain the conserved charge currents and energy currents from charge $U(1)$ and temporal translation symmetry. In Sec. IV, we derive the hydrostatic effective action for magnetization as well as energy magnetization, whose relation to the microscopic linear response theory is studied in details in Sec. V. In Sec. VI, we show that our hydrostatic effective action for the thermal Hall effect can be recast as a topological θ term and thus there is no anomaly inflow or bulk thermal Hall current at low temperature. Finally, in Sec. VI, we show that the thermal Hall conductance is quantized by global anomalies.

III. GENERALIZED EINSTEIN'S RELATION AND CONSERVED CURRENTS

As outlined in the previous section, we will focus on systems under time-independent external fields varying slowly in space, which reach their equilibrium when statistical forces from inhomogeneity of thermodynamic variables are balanced by mechanical forces. The equilibrium conditions for this balancing will be derived in this part from hydrostatics, where the static nature can be rephrased as the existence of translation symmetry along a timelike Killing vector K. For concreteness, let us consider a charge-conserved thermal partition function $Z[A_\mu,\ e_\mu^{*a}]$ with such a Killing vector K on a Euclidean spacetime manifold, where the vielbein e_μ^{*a} and the electric field A_μ vary slowly [49]. Here assuming the Euclidean spacetime coordinates are $x_E^\mu = (x_E^0, x^i)$, we will use the Wick rotated coordinates of $x^\mu = (-ix_E^0, x^i)$ to write the tensorial form of physical observables including the external fields. The benefit of this convention is that definitions of physical observables such as the currents and energy currents share the same form as those of a real-time quantum field theory. Similar notations can be found for instance in Refs. [48,49].

In this case, the corresponding effective action $S_{\text{eff}}[A_\mu,\ e_\mu^{*a}] = \ln Z$ can be organized in terms of derivative expansion. In doing so, we can first write down all possible scalars invariant under symmetries of the system, which can be constructed from external fields and the Killing vector. For clarity, we can denote these scalars as $s_i^{(n)}$, with the subscript i labeling its power of derivative and the superscript (n) for different scalars. From these scalars, the effective action can be organized as follows [48]:

$$S_{\rm eff} = \int \sqrt{|\det g|}\left\{P[s_0] + \sum_n \sum_{i=1} \alpha_i^{(n)}[s_0]s_i^{(n)}\right\}, \tag{1}$$

where we have used the symbol P for the zeroth-order term known as the internal pressure. As we shall show later, local temperature and chemical potential manifest themselves as zeroth-order scalars due to the temporal translational symmetry, whose equilibrium values are determined by the balancing between statistical forces and mechanical forces, and this yields equilibrium conditions. Combined with the $U(1)$ symmetry and the temporal translational symmetry, one can further define conserved charge currents as well as conserved energy currents.

Let us now construct the zeroth-order scalars and relate them to local temperature and local chemical potential. We first explicitly write down the metric

$$g_{\mu\nu} = e_\mu^{*a} e_\nu^{*b}\eta_{ab}, \tag{2}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, \dots)$ takes the form of a Minkowski metric as a result of the Wick rotated coordinate $x^\mu = (-ix_E^0, x^i)$. Greek letters μ, ν and Latin letters a, b stand for Einstein indices and Lorentz indices, i.e., $\mu = 0, 1, \dots$ and $a = 0, 1, \dots$ We use i, j and I, J for spatial indices of μ and a, respectively. If we recast e_μ^{*0} as $e_\mu^{*0} = (1 + \phi_g, A_{gi})$, then ϕ_g is Luttinger's fictitious gravitational field [5], and A_{gi} can be regarded as the gravitomagnetic field. The imaginary time axis is compactified to a circle known as the thermal loop so as to describe thermal effects. In equilibrium, the partition function should be time-independent, so we have a timelike Killing vector $K^\mu = (1, 0, \dots)$ in the basis of $\partial_\mu = (i\partial_{E0}, \partial_i)$ in our convention (see Fig. 1), and its normalized counterpart is

$$u^\mu = \frac{1}{\sqrt{K^2}}(1, 0, \dots), \tag{3}$$

where $K^2 \equiv K^\mu K_\mu$, and u^μ thus points along the tangential direction of the thermal loop. For later convenience, we shall align e_μ^{*0} with u_μ, which implies that $e_\mu^{*I}u^\mu = 0$ and $e_0^{*0} = \sqrt{K^2}$.

Then, due to compactification of the temporal axis, in the presence of the background vielbein e_μ^{*0} and $U(1)$ gauge field A_μ, we can define two scalars [47,48]—the length of the thermal loop, i.e., $\beta(x)$, and the Wilson loop along the time direction—which yield the local temperature as well as the chemical potential. The local temperature $T(x)$ is defined as the inverse of $\beta(x)$, i.e.,

$$T(x) \equiv \frac{1}{\int_0^{\frac{1}{T_0}} dx_E^0\sqrt{K^2}} = \frac{T_0}{\sqrt{K^2}} = \frac{T_0}{e_0^{*0}}, \tag{4}$$

where $\sqrt{K^2}$ is the induced metric of the thermal loop, $x_E^0 \in [0,\ 1/T_0]$ is the parametrization of thermal loops [50], and $T(x)$ satisfies the Tolman-Ehrenfest relation [51] $T(x)\sqrt{K^2} = $ const.

The local chemical potential is defined as the temporal Wilson loop divided by $\beta(x)$, i.e.,

$$\mu(x) \equiv -T(x)\int_0^{\frac{1}{T_0}} A_0 dx_E^0 = -\frac{A_0}{\sqrt{K^2}} = -\frac{A_0}{e_0^{*0}}, \tag{5}$$

where $\int_0^{\beta_0} A_0 dx_E^0$ is the temporal Wilson loop, and the second equality is from the transverse gauge [47,52]: $\partial_{E0}A_0 = 0$. One can also define the spin chemical potential as the Wilson loop for spin connection $\omega_{ab\mu}$, i.e., $\frac{K^\mu\omega_{ab\mu}}{\sqrt{K^2}}$, but as we shall show later, the spin chemical potential should be set to zero if we want to have a conserved energy current.

Equations (4) and (5) relate $T(x)$ and $\mu(x)$ to the gravitational potential and electric potential, respectively. These equilibrium conditions can be further appreciated by deriving the generalized Einstein relations. In equilibrium, currents arising from inhomogeneous particle (energy) distribution are compensated by those from external electric fields (torsional electric fields), which are encoded in the time-independent conditions, i.e., $0 = \mathcal{L}_K e_\mu^{*a} = \mathcal{L}_K A_\mu$, and they yield the generalized Einstein relations (for details, please refer to Appendix B)

$$T\nabla_\nu \frac{\mu}{T} - u^\mu F_{\mu\nu} = 0 \tag{6}$$

and

$$\frac{1}{T}\nabla_\mu T - T^a{}_{\sigma\mu}u_a u^\sigma = 0, \tag{7}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, $T^a{}_{\sigma\mu} = \partial_\sigma e_\mu^{*a} + \omega^a{}_{b\sigma}e_\mu^{*b} - (\mu \leftrightarrow \sigma)$ is the torsion tensor, and the spin chemical potential is set to zero. These generalized Einstein equations are valid even when we relax the transverse gauge condition.

After obtaining these generalized Einstein relations, we turn to define the conserved charge current $\mathcal{J}^\mu$ as well as the conserved energy current $\mathcal{J}_E^\mu$, which are from $U(1)$ symmetry and the temporal translational symmetry, respectively. From $U(1)$ symmetry,

$$\mathcal{J}^\mu = \sqrt{|\det g|}\,j^\mu, \quad \partial_\mu\mathcal{J}^\mu = 0, \tag{8}$$

where $j^\mu \equiv -\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta A_\mu}$ satisfies $\frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}\,j^\mu) = 0$. From temporal translational invariance induced by K^μ (see Appendix B 3 b for details), one can define the energy current as

$$\mathcal{J}_E^\mu = \sqrt{|\det g|}\left[K^a\tau_a^\mu + (A_\nu K^\nu)j^\mu\right], \tag{9}$$

where $\tau_a^\mu \equiv -\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta e_\mu^{*a}}$ is the energy-momentum tensor. This current is conserved if $K^\mu\omega_{ab\mu}$ vanishes [53], i.e., $\partial_\mu\mathcal{J}_E^\mu = 0$. For later convenience, we shall set spin connections to zero hereafter so as to have a conserved energy current, and this corresponds to the teleparallel gravity [54]. In the absence of external electromagnetic fields and vielbeins, we have $\mathcal{J}_E^\mu = \tau_a^\mu K^a - \mu j^\mu$. Notice that the conserved energy current here in the Euclidean theory is essentially the thermal current in the equilibrium states (up to some regularization subtleties as can be seen in e.g., Appendix C). This is special to the Euclidean theory at equilibrium and does *not* imply that the thermal current is conserved in real-time evolution. However, in our paper, for consistency, we will keep the Euclidean theory terminology. Hence our later result of the energy current and energy magnetization will correspond to the thermal current and heat magnetization in the literature of the real-time formalism.

IV. HYDROSTATIC EFFECTIVE ACTION

For the current response, we can look for the derivative expansion of the first order and write the general covariant form of the action containing one derivative as

$$S_{\text{eff}}^{(1)} = -\int m_{g,0}\epsilon^{\mu\nu\rho} e_{\mu}^{*a}\partial_\nu e_{\rho}^{*b}\eta_{ab} - \int m_{N,0}\epsilon^{\mu\nu\rho}u_\mu\partial_\nu A_\rho, \tag{10}$$

where $\int$ is for $\int d^3x_E$, and both $m_{g,0}$ and $m_{N,0}$ are functions of zeroth-order scalars, i.e., $m_{g,0} = m_{g,0}(\mu, T)$ and $m_{N,0} = m_{N,0}(\mu, T)$. For simplicity, we assume the system has emergent Lorentz symmetry such as in a Chern insulator, while it is straightforward to generalize to nonrelativistic electrons. For the nonrelativistic case, we need to treat space indices and the time index differently, but the main discussion of charge response and thermal response remains valid and only requires charge $U(1)$ and temporal translation symmetry. It is also worth pointing out that the celebrated Chern-Simons term $\frac{\nu_H}{4\pi}\int \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$ ($\nu_H \in \mathbb{Z}$) is contained in the second term of the action above, i.e., $-\int m_{N,0}\epsilon^{\mu\nu\rho}u_\mu\partial_\nu A_\rho$ [55].

To bring more physical insights, we will justify the underlying physics of magnetization and energy magnetization for these coefficients in Eq. (10) by deriving this effective action from conserved currents and making connection with results in the Cooper-Halperin-Ruzin transport theory [39]. The effective action is derived by coupling $\mathcal{J}^i$ and $\mathcal{J}_E^i$ to their probe fields, A_i and e_i^{*0}/e_0^{*0}, i.e.,

$$S_{\text{eff}}^{(1)} = -\int \left(\mathcal{J}_E^i - A_0\mathcal{J}^i\right)\left(\frac{1}{e_0^{*0}}e_i^{*0}\right) - \int \mathcal{J}^i A_i, \tag{11}$$

where $\mathcal{J}_E^i - A_0\mathcal{J}^i \equiv \sqrt{|\det g|}K^a\tau_a^i$ couples to e_i^{*0}/e_0^{*0}, and the zeroth components of currents are not written down due to the time-independent condition. This time-independent condition implies that $\partial_\mu\mathcal{J}_{(E)}^\mu = \partial_i\mathcal{J}_{(E)}^i = 0$, so these conservation laws are solved by $\mathcal{J}^i = \partial_j m_N^{ij}$ and $\mathcal{J}_E^i = \partial_j m_g^{ij}$, with skew symmetric m_N^{ij} and m_g^{ij} known as the magnetization and energy magnetization [39–41], respectively. As we can see from our hydrostatic theory, (energy) magnetization currents are equilibrium currents in the presence of inhomogeneous background fields that do not participate in transport [39,56]. In particular, it is important to subtract the energy magnetization current to give the correct thermal Hall response theory [40]. In $(2+1)$ dimensions, the magnetization and energy magnetization can be further recast as $m_N^{ij} = \epsilon^{ij0}m_N$ and $m_g^{ij} = \epsilon^{ij0}m_g$. These solutions of currents $\mathcal{J}^i$ and $\mathcal{J}_E^i$ are of at least first-order dependence in the derivative expansion Eq. (1), so they should be encoded in the action in Eq. (10). This can be straightforwardly appreciated by recasting the action in Eq. (11) in terms of magnetization and energy magnetization,

$$S_{\text{eff}}^{(1)} = -\int m_{g,0}\epsilon^{\mu\nu\rho}e_\mu^{*0}\partial_\nu e_\rho^{*0} - \int m_{N,0}\epsilon^{\mu\nu\rho}e_\mu^{*0}\partial_\nu A_\rho, \tag{12}$$

where $m_{N,0}$ and $m_{g,0}$ are defined as

$$m_N \equiv \sqrt{K^2}m_{N,0}, \tag{13a}$$

$$m_g \equiv K^2 m_{g,0} + \sqrt{K^2}(i_K A)m_{N,0} = K^2(m_{g,0} - \mu m_{N,0}), \tag{13b}$$

and this is one of our main results. Notice that $K^2 = (1+\phi_g)^2$, so Eq. (13) reproduces scaling relations suggested in Ref. [39]. It is worth pointing out that the action in Eq. (12) does match the one in Eq. (10), because in our choice of vielbein and coordinates, $e_0^{*I} = 0$, $e_\mu^{*I}u^\mu = 0$, and thus $\int \epsilon^{\mu\nu\rho}e_\mu^{*I}\partial_\nu e_\rho^{*I} = 0$. In a general choice of vielbein, the effective action Eq. (12) is covariantly generalized to Eq. (10). We will use Eq. (12) as our hydrostatic effective action for later discussions.

Finally, let us highlight two comments about our effective action: (i) Our results of magnetization and energy magnetization reproduce those in Ref. [39]. Namely, in terms of ϕ_g, we can determine the functional form of the magnetization and energy magnetization to be $m_N^{ij} = \epsilon^{0ij}(1+\phi_g)m_{N,0}(\mu, T)$ and $m_g^{ij} = \epsilon^{0ij}(1+\phi_g)[(1+\phi_g)m_{g,0}(\mu, T) + A_0 m_{N,0}]$, which are the scaling relations suggested in Ref. [39]. (ii) The discussion from symmetry perspective so far can be applied beyond gapped systems. For example, for gapless systems such as a Fermi liquid at finite temperature, due to a finite correlation length of fermions, we can still describe the long wavelength physics by a local hydrostatic effective action and its gradient expansion.

V. EFFECTIVE ACTION AND LINEAR-RESPONSE THEORY

Our effective action describes the macroscopic property of a system at a generically inhomogeneous equilibrium state. Now in this section, we obtain the equations for the (energy) magnetization by connecting our effective action to microscopics, and we highlight general constraints of (energy) magnetization for gapped systems. Starting with our effective action, these equations do *not* depend on microscopic details other than the symmetry of the system. More specifically, the (energy) magnetization currents must match in calculations by (i) varying our hydrostatic effective action, and (ii) applying microscopic linear response theory. For the former, our hydrostatic action yields

$$\mathcal{J}^i = \epsilon^{ij0}\left(-\frac{\partial m_{N,0}}{\partial\mu}\right)\partial_j A_0 + \epsilon^{ij0}\left(m_{N,0} - \mu\frac{\partial m_{N,0}}{\partial\mu} - T\frac{\partial m_{N,0}}{\partial T}\right)\partial_j e_0^{*0}, \tag{14a}$$

$$\mathcal{J}_E^i = \epsilon^{ij0}\left(-\frac{\partial m_{Q,0}}{\partial\mu}\right)\partial_j A_0 + \epsilon^{ij0}\left(2m_{Q,0} - \mu\frac{\partial m_{Q,0}}{\partial\mu} - T\frac{\partial m_{Q,0}}{\partial T}\right)\partial_j e_0^{*0}, \tag{14b}$$

where $m_{Q,0} \equiv m_{g,0} - \mu m_{N,0}$, and we have used Eqs. (4) and (5) to rewrite the gradient of local temperature and the chemical potential gradient in terms of the gradient of e_0^{*0} and A_0.

Equations (14) can also be derived from the definition of magnetization currents, i.e., $\mathcal{J}^i = \epsilon^{ij0}\partial_j m_N$ and $\mathcal{J}_E^i = \epsilon^{ij0}\partial_j m_g$.

For the latter, perturbative expansions, or equivalently Feynman diagrams in momentum space, lead to

$$\mathcal{J}^i(q) = -\langle \mathcal{J}^i(q)\mathcal{J}^0(-q)\rangle \delta A_0(q) - \big\langle \mathcal{J}^i(q)\sqrt{|\det g|}\tau_0^0(-q)\big\rangle \delta e_0^{*0}(q), \quad (15a)$$

$$\mathcal{J}_E^i(q) = -\big\langle \mathcal{J}_E^i(q)\mathcal{J}^0(-q)\big\rangle \delta A_0(q) - \big\langle \mathcal{J}_E^i(q)\sqrt{\big|\det g\big|}\tau_0^0(-q)\big\rangle \delta e_0^{*0}(q), \quad (15b)$$

where the expectation values and variations are taken with respect to the local microscopic state and it is sufficient to only keep terms from linear perturbations. By comparing results from these two approaches, we can obtain a set of equations for $m_{N,0}$ and $m_{Q,0}$,

$$\frac{\partial m_{N,0}}{\partial \mu} = \frac{i}{2}\epsilon_{ki0}\partial_{q_k}\langle \mathcal{J}^i(q)\mathcal{J}^0(-q)\rangle, \quad (16a)$$

$$\left(m_{N,0} - T\frac{\partial m_{N,0}}{\partial T}\right) = -\frac{i}{2}\epsilon_{ki0}\partial_{q_k}\big\langle \mathcal{J}^i(q)\mathcal{J}_E^0(-q)\big\rangle, \quad (16b)$$

$$\frac{\partial m_{Q,0}}{\partial \mu} = \frac{i}{2}\epsilon_{ki0}\partial_{q_k}\big\langle \mathcal{J}_E^i(q)\mathcal{J}^0(-q)\big\rangle, \quad (16c)$$

$$\left(2m_{Q,0} - T\frac{\partial m_{Q,0}}{\partial T}\right) = -\frac{i}{2}\epsilon_{ki0}\partial_{q_k}\big\langle \mathcal{J}_E^i(q)\mathcal{J}_E^0(-q)\big\rangle, \quad (16d)$$

which reproduce results in Ref. [40]. These are first-order differential equations, so we can obtain $m_{g,0}$ and $m_{N,0}$ unambiguously only when reference states are provided. Still, these differential equations provide valuable insights on constraints for magnetization in a gapped system. Most importantly, in a system with a gap Δ, $\mathcal{J}_E$ is expected to be exponentially suppressed, i.e., $e^{-\beta\Delta}$, because there are hardly any excitations in the bulk, and thus entropy is exponentially suppressed at temperatures low compared to the gap. When combining this exponential suppression with Eqs. (16a) and (16b), we have $m_{N,0} = \frac{\nu_H\mu}{2\pi} + c_2 T$, where c_2 is a constant and $\nu_H \in \mathbb{Z}$. The $\frac{\nu_H\mu}{2\pi}$ term is a well-known result from the integer quantum Hall effect, and the $c_2 T$ term is from Eq. (16b) by setting terms on the right-hand side to zero. As for Eqs. (16c) and (16d) with terms on the right-hand side equal to zero, their solution is $m_{Q,0} = c_1 T^2$, and c_1 is a constant. By putting these results together, we have $m_{g,0} = m_{Q,0} + \mu m_{N,0} = c_1 T^2 + c_2\mu T + \frac{\nu_H\mu^2}{2\pi}$. Three comments are in order: First, c_1 and c_2 cannot be determined perturbatively from Eqs. (16) given above, which, as we shall show in the next section, is because they are rooted in boundary modes. Second, the μ^2 term in $m_{g,0}$ can give rise to another torsional Chern-Simons term, i.e., $S = -\int \frac{\mu^2\nu_H}{2\pi}\epsilon^{\mu\nu\rho}e_\mu^{*0}\partial_\nu e_\rho^{*0}$. It is interesting to notice that cutoffs in the Hughes-Leigh-Fradkin parity-odd action [9] are replaced by the chemical potential, and its quantization is inherited from the integer quantum Hall effect. Finally, it is worth pointing out that in experimental systems, there can exist gapless phonons that yield finite contributions to $\mathcal{J}_E$ [57,58].

VI. $m_{Q,0} = c_1 T^2$, BULK-EDGE CORRESPONDENCE AND ITS TOPOLOGICAL MEANING

As we have discussed above, $m_{Q,0}$ is expected to be $c_1 T^2$ for an insulator and thus $m_{g,0} = c_1 T^2 + c_2\mu T + \frac{\nu_H\mu^2}{2\pi}$, where both c_1 and c_2 cannot be fixed from bulk perturbative calculations. In this part, we shall turn to the torsional Chern-Simons term with $m_{g,0} = c_1 T^2 + c_2\mu T$ and explore its topological meaning as well as bulk-edge correspondence. The c_2 term can be recast as a boundary term [59], but by direct calculation for edge chiral fermions, one can find that $c_2 = 0$, so we shall focus on the c_1 term hereafter [60]. As we will see, this term can be recast as a topological θ term and thus endow the boundary thermal current with topological meaning.

We shall first reveal the topological nature of this $c_1 T^2$ and then show how to extract its physical information. To this end, we first study its robustness under small perturbations: under variations of e_μ^{*0}, the $c_1 T^2$ term is invariant up to a boundary term, i.e., $-\int \epsilon^{\mu\nu\rho}\partial_\nu(c_1 T^2 e_\mu^{*0}\delta e_\rho^{*0})$, so this term is robust against bulk perturbations, and it cannot be obtained from bulk perturbative calculations. This is because this $c_1 T^2$ term is secretly a topological θ term, and we can rewrite it as [61]

$$-\int c_1 T_0^2 \epsilon^{0ij}\partial_i\big(e_j^{*0}/e_0^{*0}\big), \quad (17)$$

where e_j^{*0}/e_0^{*0} is the emergent Kaluza-Klein gauge field associated with temporal translation, as we can see: (i) Under local spatial translation, (e_j^{*0}/e_0^{*0}) transforms like a conventional spatial 1-form. (ii) Under local temporal translation, i.e., $\delta x^\mu = \xi^0(\boldsymbol{x})\delta_0^\mu$, we have $\delta(e_i^{*0}/e_0^{*0}) = -\partial_i\xi^0$, which is an effective $U(1)$ gauge transformation due to the identification $\xi^0 \simeq \xi^0 + \frac{1}{T_0}$. Despite the robustness of this topological theta term against bulk variations, we can still extract its physical information by considering two adjacent materials with different values of c_1. Around the boundary, the system is inhomogeneous and c_1 can develop a dependence on the coordinate across the boundary. To be more concrete, we assume the two materials are located at $y > 0$ and $y < 0$ with a smooth boundary, and we model c_1 as a function of y interpolating between these two materials. The corresponding effective edge theory is thus given as

$$-\int d^2x_E\, dy\, c_1(y)T_0^2\epsilon^{0ij}\partial_i\big(e_j^{*0}/e_0^{*0}\big) = \int \tilde{c}_1 T^2 e_0^{*0}e_1^{*0}d^2x_E, \quad (18)$$

where $\tilde{c}_1 \equiv -[c_1(+\infty) - c_1(-\infty)]$. The lowest-order approximation of this edge action near $e_0^{*0} = 1$ and $e_i^{*0} = 0$ reproduces results in Refs. [12,13]. However, the corresponding physical meaning is different in a significant way. Compared to the results in Refs. [12,13], our bulk effective action is invariant under local temporal coordinate transformations, so its effective edge action cannot be obtained from the usual anomaly matching approach, and there are no bulk energy currents derived from our bulk action. In the rest of the paper, we will focus on the effective edge theory. From here on, our Einstein indices and Lorentz indices will both take {0, 1}, and the inner product will be taken with the induced metric $\tilde{g}$ on the boundary. We will keep the Killing vector notation for the boundary discussion: $K^\mu = (1, 0)$.

One can read off a boundary energy-momentum tensor from our effective edge theory and then study the edge conservation laws. Namely, $\tau_0^1 = -\tilde{c}_1 T^2 \frac{1}{\sqrt{|\det \tilde{g}|}} e_0^{*0}$ and $\tau_0^0 = \tilde{c}_1 T^2 \frac{1}{\sqrt{|\det \tilde{g}|}} e_1^{*0}$, and it is rather interesting to find that it satisfies the corresponding Noether theorem in the presence of torsion (see Appendix A), i.e., $\frac{1}{\sqrt{|\det \tilde{g}|}} \partial_\nu(\sqrt{|\det \tilde{g}|}\tau_a^\nu) - e_a^\mu(\tau_b^\nu T^b{}_{\mu\nu}) = 0$. Namely, there is no perturbative diffeomorphism anomaly at the edge. As for the edge energy current, by definition, it is $\mathcal{J}^i_{E,\text{boundary}} = \sqrt{|\det \tilde{g}|} K^a \tau_a^i = -\tilde{c}_1 T_0^2$, so it is clearly conserved. Compared to the Chern-Simons action for the integer quantum Hall effect, this $c_1 T^2$ torsional Chern-Simons term fails to cause thermal currents flowing from the bulk to the edge. It was claimed in Ref. [12] that a bulk thermal Hall current is needed to compensate for boundary anomalies. We would like to stress that the energy current $\mathcal{J}_E^\mu$ is different from τ_0^μ. Hence, one cannot interpret $\frac{1}{\sqrt{|\det \tilde{g}|}} \partial_\nu(\sqrt{|\det \tilde{g}|}\tau_a^\nu) = e_a^\mu(\tau_b^\nu T^b{}_{\mu\nu})$ as nonconservation of the energy current. Furthermore, in the lab frame where the system is at rest, the edge energy density (not to be confused with $\mathcal{J}^0_{E,\text{boundary}}$) is $\frac{K_\mu}{\sqrt{K^2}} \mathcal{J}^\mu_{E,\text{boundary}} = 0$, which does not depend on the background gravitational field. This indicates that no thermal energy is pumped in or out of the boundary upon adiabatic change of the gravitational background. Still, energy pumping between boundaries through the bulk is possible if there exist gapless modes, e.g., phonons [57,58].

Due to the robustness of the temperature-squared term in our effective action, a natural question is how we can calculate this term from a microscopic model and fix the coefficient c_1 in the bulk of a homogeneous material. The answer is that $m_{g,0}$ can be uniquely fixed only when reference states are given, which is because Eqs. (16) for $m_{g,0}$ are first-order differential equations. For example, we can take the condition $m_{g,0}|_{\mu\to-\infty} = 0$ with a physical meaning of setting energy magnetization of a state with no electrons to zero. Alternatively, we can impose the condition $\frac{\partial m_{g,0}}{\partial T}|_{T\to\infty} = 0$ because all states are excited in the $T \to \infty$ limit (temperatures much higher than the width of energy spectrum). For a demonstration of the approach on $(2+1)$-dimensional massive Dirac fermions, interested readers are referred to Appendix C for details.

In reality, we can implement the comparison to reference states by putting two different materials with different μ or T adjacent to each other, for example, $\mu = 0$ and $\mu = -\infty$, respectively. Assuming μ smoothly interpolates between these two materials, in the same spirit as Eq. (18), our effective action $\int c_1 T^2 \epsilon^{\mu\nu\rho} e_\mu^{*0} \partial_\nu e_\rho^{*0}$ manifests itself as energy currents flowing in the interface determined by the difference of c_1 in the bulk of two materials. Since the variation of the action in the bulk is zero, only the edge physics determined by the difference of c_1 is observable, so the torsional effective action $\int c_1 T^2 \epsilon^{\mu\nu\rho} e_\mu^{*0} \partial_\nu e_\rho^{*0}$ is topological in a relative sense [43].

VII. $m_{Q,0} = c_1 T^2$ AND GLOBAL ANOMALIES

After revealing the topological meaning of torsional Chern-Simons terms, a natural question is whether c_1 is quantized after considering the scale invariance of the edge theory similarly as in Ref. [13] and fixing the energy magnetization of a trivial insulator to be 0. We shall explore this by studying boundary energy-momentum tensors from the point of view of perturbative calculations and nonperturbative global anomalies.

To this end, consider the boundary between a topologically nontrivial material at $y > 0$ and a trivial insulator at $y < 0$, which traps right-handed chiral fermions at $y = 0$ if the Chern number equals 1 for the topological material. Now we demonstrate that we can fix the c_1 for the topological material by studying edge chiral fermions. For example, we can directly compare the boundary action Eq. (18) with microscopic calculation of the energy-momentum tensor of right-handed chiral fermions (see Appendix D for details):

$$\langle \tau_0^1 \rangle = \left(\tilde{\Lambda}^2 + \frac{\pi}{12} T_0^2 \right) + O\left[\left(\delta e_\mu^{*a} \right)^2 \right]. \quad (19)$$

Here, $\tilde{\Lambda}^2$ is the UV cutoff, and its specific value depends on regularization schemes. For example, by using the dimensional regularization so that we have, $\tilde{\Lambda}^2 = 0$, and the corresponding conserved energy current is the thermal current. From Eq. (19), and compare with our edge action Eq. (18), we can further conclude that $c_1 = \frac{\pi}{12}$ for this topological material.

Alternatively, we can fix the value of c_1 nonperturbatively by compactifying the spatial dimension and considering the global anomaly of the edge theory on a torus. One reason for doing so is to compare with Ref. [13]. In addition, this approach will not refer to microscopic details of the edge theory, and therefore it is a more general argument. Now the idea is to connect the global anomaly of the partition function under the modular transformation of the torus to the field theory response to an inserted gravitomagnetic flux, as can be described by our boundary action. The compactification to a torus is done by identifying spacetime coordinates in the following way: $(x_E^0, x^1) \sim (x_E^0 + \beta_0, x^1) \sim (x_E^0, x^1 + L)$, where $\beta_0 \equiv \frac{1}{T_0}$. Boundary conditions for fermions are (anti)periodic along the (temporal) spatial direction.

We then insert a gravitomagnetic flux to deform this spacetime torus and mimic the modular transformation, as shown in Fig. 2. This process is implemented as the insertion of a series of infinitesimal gravitomagnetic fluxes (e.g., $\oint \delta e_1^{*0} dx^1 = -i\frac{\beta_0}{N}$, $N \to \infty$, with the imaginary number i from our convention of writing tensors in the Wick rotated basis of $\partial_\mu = (i\partial_{E0}, \partial_i)$), where each step can be geometrically represented as an infinitesimal deformation of the spacetime torus (see Fig. 2). The infinitesimal deformation changes our boundary action by $\delta S = -\int \tau_0^1 \delta e_1^{*0}$ with $\delta e_1^{*0} = -i\frac{\beta_0}{NL}$ and we have used $|\det \tilde{g}| = 1$. After this process, the torus is mapped to itself (see Fig. 2), but with the coordinate basis changed, which is known as the modular transformation [62]. The ensuing action transformation is $\delta S = -\int \tau_0^1 \delta e_1^{*0}$, with $\delta e_1^{*0} = -i\frac{\beta_0}{L}$ and $\tau_0^1 = c_1 T_0^2$ from our effective boundary action. Notice that τ_0^1 does not change during the described process, and we have used this property.

Now we are ready to fix the coefficient c_1 by considering the global gravitational anomaly. As a result of the global gravitational anomaly, the modular transformation changes

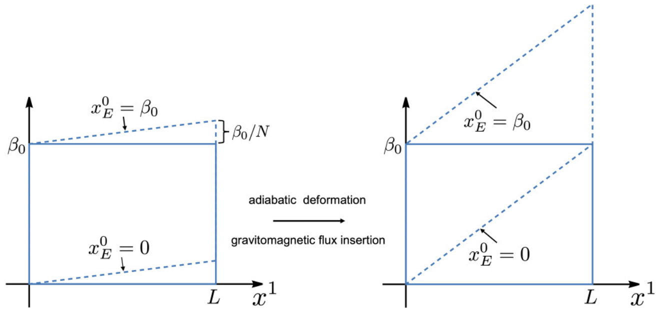

FIG. 2. Gravitomagnetic flux insertion and modular transformation. Left panel: an infinitesimal deformation of a torus corresponding to $\delta e_1^{*0} = -i\frac{\beta_0}{NL}$, where the imaginary number comes from our notation of writing tensors in the Wick rotated basis of $\partial_\mu = (i\partial_{E0}, \partial_i)$. We start with the vielbein: $e_1^{*0} = 0$ and $e_0^{*0} = 1$. The compactification directions (equal x_E^0 and x^1) of the deformed torus are along the dashed lines. Right panel: a modular transformation that maps a torus onto itself after the gravitomagnetic flux insertion. This corresponds to $\delta e_1^{*0} = -i\frac{\beta_0}{L}$. We have $|\det \tilde{g}| = 1$ through the process, i.e, the deformation preserves area.

the partition function for chiral fermions by a phase factor [13,62,63], i.e., $Z \to e^{i\frac{\pi}{12}}Z$. Comparing to the change of our effective boundary action upon the described gravitomagnetic flux insertion, we find $e^{i\frac{\pi}{12}} = e^{i\int c_1 T_0^2 \frac{\beta_0}{L} d^2x_E}$, and thus again we conclude $c_1 = \frac{\pi}{12}$, which shows that c_1 is quantized by global anomalies. Since the c_1 is from the topological state of Chern number $+1$, we can conclude that for the topological state of Chern number $+1$, the energy magnetization is $\frac{\pi}{12}T^2$, and from the thermal generalization of the Streda formula [64], the thermal Hall conductivity is $\kappa_H = -\frac{\pi}{6}T$.

VIII. CONCLUSIONS

In summary, we have derived the general effective hydrostatic action for gapped quantum matter coupled to teleparallel gravity. The action up to linear order of the derivative expansion captures the static response of charge and energy currents. The linear order in derivative terms include a torsional Chern-Simons term, with its physical meaning as the energy magnetization. For a gapped system, there can exist a temperature-squared torsional Chern-Simons term in our hydrostatic effective action, which is topological, and its quantization is inherited from boundary global gravitational anomalies. In contrast to previous literature discussing the torsional Chern-Simons term, in our theory there are no local boundary diffeomorphism anomalies or bulk inflow thermal currents. In addition, we have derived various properties for the magnetization as well as the energy magnetization from our effective action.

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APPENDIX A: DERIVATION OF NOETHER IDENTITY FROM DIFFEOMORPHISM

In this Appendix, we shall derive the Noether identity arising from general coordinate invariance. For a given action S, we define the charge current as

$$j^\mu \equiv -\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta A_\mu}, \tag{A1}$$

the energy-momentum tensor as

$$\tau_a^\mu \equiv -\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta e_\mu^{*a}}, \tag{A2}$$

and the spin current as

$$S^{ab\mu} \equiv \frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta \omega_{ab\mu}}, \tag{A3}$$

where A_μ, e_μ^{*a}, and $\omega_{ab\mu}$ are the $U(1)$ gauge field, the vielbein, and the spin connection, respectively. We use e_a^μ to denote the inverse vielbein, which is defined through $e_a^\mu e_\mu^{*b} = \delta_a^b$. In addition, we define ∇_μ as the total covariant derivative acting on both Einstein indices μ and Lorentz indices a, which contains both the spin connection $\omega_{ab\mu}$ and the affine connection $\Gamma^\mu{}_{\nu\rho}$. D_μ is used for the covariant derivative, and it only contains spin connections.

We consider a coordinate transformation generated by vector ξ^μ. The variations of fields e_μ^{*a}, $\omega_{ab\mu}$, and A_μ are

$$\delta e^{*a} = \mathcal{L}_\xi e^{*a} = i_\xi T^a + D\xi^a - \left(i_\xi\omega^a{}_b\right)e^{*b}, \tag{A4}$$

$$\delta\omega^a{}_b = \mathcal{L}_\xi\omega^a{}_b = i_\xi\Omega^a{}_b + D\left(i_\xi\omega^a{}_b\right), \tag{A5}$$

and

$$\delta A = \mathcal{L}_\xi A = i_\xi F + d i_\xi A, \tag{A6}$$

where $A = A_\mu dx^\mu$, $e^{*a} = e_\mu^{*a}dx^\mu$, $\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu$, i_ξ denotes interior products, and $\Omega^a{}_b \equiv d\omega^a{}_b + (\omega\wedge\omega)^a{}_b$ is the curvature. The variation of action is

$$\begin{aligned}\delta S=&\int\sqrt{|\det g|}\left[-\left(-\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta e^{*a}_{\nu}}\right)\delta e^{*a}_{\nu}+\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta\omega_{ab\nu}}\delta\omega_{ab\nu}-\left(-\frac{1}{\sqrt{|\det g|}}\frac{\delta S}{\delta A_{\nu}}\right)\delta A_{\nu}\right]\\ =&\int\sqrt{|\det g|}\left[\xi^{\mu}\left(-\tau^{\nu}_{a}T^{a}{}_{\mu\nu}+S^{ab\nu}\Omega_{ab\mu\nu}-j^{\nu}F_{\mu\nu}\right)+(\nabla_{\nu}+T^{\rho}{}_{\nu\rho})\tau^{\nu}_{a}\xi^{a}\right]\\ &+\int\sqrt{|\det g|}(i_{\xi}\omega^{a}{}_{b})\left[e^{*b}_{\nu}\tau^{\nu}_{a}-\frac{1}{\sqrt{|\det g|}}D_{\mu}\left(\sqrt{|\det g|}S_{a}{}^{b\mu}\right)\right]+\int\sqrt{|\det g|}(i_{\xi}A)\left(\frac{1}{\sqrt{|\det g|}}\partial_{\mu}\sqrt{|\det g|}j^{\mu}\right).\end{aligned}\tag{A7}$$

Because of $U(1)$ symmetry, we have $\frac{1}{\sqrt{|\det g|}}\partial_{\mu}\sqrt{|\det g|}j^{\mu}=0$, and the Noether identity from general coordinate invariance is

$$(\nabla_{\nu}+T^{\rho}{}_{\nu\rho})\tau^{\nu}_{a}-e^{\mu}_{a}\left(\tau^{\nu}_{b}T^{b}{}_{\mu\nu}-S^{cd\nu}\Omega_{cd\mu\nu}+j^{\nu}F_{\mu\nu}\right)=-\omega^{c}{}_{da}\left[e^{*d}_{\nu}\tau^{\nu}_{c}-\frac{1}{\sqrt{|\det g|}}D_{\mu}\left(\sqrt{|\det g|}S_{c}{}^{d\mu}\right)\right].\tag{A8}$$

If there exists internal rotational symmetry among indices a, then one can prove that

$$\omega^{a}{}_{bc}\left[e^{*b}_{\nu}\tau^{\nu}_{a}-\frac{1}{\sqrt{|\det g|}}D_{\mu}\left(\sqrt{|\det g|}S_{a}{}^{b\mu}\right)\right]=0,\tag{A9}$$

and the Noether identity in Eq. (A8) becomes

$$(\nabla_{\nu}+T^{\rho}{}_{\nu\rho})\tau^{\nu}_{a}-e^{\mu}_{a}\left(\tau^{\nu}_{b}T^{b}{}_{\mu\nu}-S^{cd\nu}\Omega_{cd\mu\nu}+j^{\nu}F_{\mu\nu}\right)=0,\tag{A10}$$

which matches the results in Refs. [8,11].

APPENDIX B: DERIVATIONS OF EQ. (6), EQ. (7), AND CONSERVED ENERGY CURRENT

In this Appendix, we shall give a detailed derivation of the generalized Einstein relation from equilibrium conditions, i.e., $\mathcal{L}_K(\cdots)=0$ and $K=K^{\mu}\partial_{\mu}$, where $(\cdots)$ stands for external fields, including A_{μ}, $g_{\mu\nu}$, and so on. In addition, the derivations of conserved energy currents are also presented in detail.

1. Derivation of Eq. (6)

We impose the following equilibrium condition:

$$\mathcal{L}_K(A+d\theta)=0,\tag{B1}$$

where $d\theta$ is a gauge transformation, and this says that A satisfies $\mathcal{L}_KA=0$ up to a gauge transformation. Equation (B1) can be recast as

$$0=\mathcal{L}_K(A+d\theta)=i_KdA+d(i_KA+i_Kd\theta).\tag{B2}$$

Following conventions in Ref. [52], we define $\Lambda_K\equiv i_Kd\theta$ and chemical potential $-T_0\frac{\mu}{T(x)}=i_KA+\Lambda_K$. Correspondingly, we find

$$0=K^{\mu}F_{\mu\nu}-T_0\partial_{\nu}\frac{\mu}{T}=u^{\mu}F_{\mu\nu}-T\partial_{\nu}\frac{\mu}{T}.\tag{B3}$$

For simplicity, one usually uses the transverse gauge condition, i.e., $i_Kd\theta=0$. This says that a gauge-fixing condition is imposed to get rid of the time dependence in the gauge transformation parameter θ.

2. Derivation of Eq. (7)

Similar to Eq. (6), we can derive Eq. (7) by imposing the following condition:

$$\mathcal{L}_Ku_{\mu}=0.\tag{B4}$$

To be more specific, $\mathcal{L}_Ku_{\mu}$ can be calculated as follows:

$$\begin{aligned}0&=\frac{1}{\sqrt{K^2}}\mathcal{L}_Ku_{\mu}\\&=u^{\nu}\nabla_{\nu}u_{\mu}+\frac{1}{\sqrt{K^2}}\nabla_{\mu}K^{\nu}u_{\nu}-\frac{1}{\sqrt{-K^2}}T^{\sigma}{}_{\mu\rho}u_{\sigma}K^{\rho}\\&=u^{\nu}\nabla_{\nu}u_{\mu}-\frac{1}{T}\nabla_{\mu}T-T^{\sigma}{}_{\mu\rho}u_{\sigma}u^{\rho},\end{aligned}\tag{B5}$$

where we have used

$$\begin{aligned}\mathcal{L}_Ku_{\mu}&=K^{\nu}(\partial_{\nu}u_{\mu}-\Gamma^{\alpha}{}_{\mu\nu}u_{\alpha})\\&\quad+(\partial_{\mu}K^{\nu}+\Gamma^{\nu}{}_{\alpha\mu}K^{\alpha})u_{\nu}-T^{\nu}{}_{\mu\alpha}K^{\alpha}u_{\nu}\\&=K^{\nu}\nabla_{\nu}u_{\mu}+\nabla_{\mu}K^{\nu}u_{\nu}-T^{\nu}{}_{\mu\alpha}u_{\nu}K^{\alpha}.\end{aligned}\tag{B6}$$

Notice that $u_{\mu}dx^{\mu}=e^{*0}_{\mu}dx^{\mu}$, so we have

$$\nabla_{\nu}u_{\mu}=\partial_{\nu}e^{*0}_{\mu}-\Gamma^{\alpha}{}_{\mu\nu}e^{*0}_{\alpha}=-\omega^{0}{}_{b\nu}e^{*b}_{\mu}\tag{B7}$$

and

$$K^{\nu}\nabla_{\nu}u_{\mu}=-\omega^{0}{}_{b\nu}e^{*b}_{\mu}K^{\nu},\tag{B8}$$

where we have used the following identity:

$$\partial_{\mu}e^{*a}_{\nu}-\Gamma^{\alpha}{}_{\nu\mu}e^{*a}_{\alpha}+\omega^{a}{}_{b\mu}e^{*b}_{\nu}=0.\tag{B9}$$

For a metric

$$ds^2=\eta_{ab}e^{*a}_{\mu}e^{*b}_{\nu}dx^{\mu}dx^{\nu},\tag{B10}$$

with $e^{*0}_{\mu}=u_{\mu}$, we have $e^{*I}_{\mu}u^{\mu}=0$. This means that $K^{\nu}\nabla_{\nu}u_{\mu}=-(i_K\omega)^{0}{}_{\mu}$, and thus

$$\frac{1}{T}\nabla_{\mu}T-T^{a}{}_{\sigma\mu}u_au^{\sigma}+(i_u\omega)_{0\mu}=0.\tag{B11}$$

If the spin chemical potential is set to zero, the equation above becomes

$$\frac{1}{T}\nabla_{\mu}T-T^{a}{}_{\sigma\mu}u_au^{\sigma}=0.\tag{B12}$$

3. Conserved energy current

a. Conserved energy currents from diffeomorphism and temporal translation symmetry

The Noether identity from diffeomorphism is (see Appendix A for details)

$$\frac{1}{\sqrt{|\det g|}}D_\nu\big(\sqrt{|\det g|}\tau_a^\nu\big)-e_a^\mu\big(\tau_b^\nu T^b{}_{\mu\nu}-S^{cd\nu}\Omega_{cd\mu\nu}+j^\nu F_{\mu\nu}\big)=-\omega^c{}_{da}\left[e_\nu^{*d}\tau_c^\nu-\frac{1}{\sqrt{|\det g|}}D_\mu\big(\sqrt{|\det g|}S_c{}^{d\mu}\big)\right],\tag{B13}$$

where D_μ is the covariant derivative with only spin connections, but not Γ. By using $0=\mathcal{L}_K e_\mu^{*a}=\mathcal{L}_K A_\mu=\mathcal{L}_K\omega_{ab\mu}$, we have the following identities:

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu[\sqrt{|\det g|}(i_K e^{*b})\tau_a^\mu]=-(i_K T^a)_\mu\tau_a^\mu+(i_K e^{*a})\frac{1}{\sqrt{|\det g|}}D_\mu\big(\sqrt{|\det g|}\tau_a^\mu\big),\tag{B14}$$

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu[\sqrt{|\det g|}(i_K A)j^\mu]=-F_{\nu\mu}K^\nu j^\mu,\tag{B15}$$

and

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu[\sqrt{|\det g|}(i_K\omega)_{ab}S^{ab\mu}]=-(i_K\Omega_{ab})_\mu S^{ab\mu}+(i_K\omega)_{ab}\frac{1}{\sqrt{|\det g|}}D_\mu(\sqrt{|\det g|}S^{ab\mu}).\tag{B16}$$

They lead to

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu\sqrt{|\det g|}\big[(i_K e^{*a})\tau_a^\mu+\sqrt{|\det g|}(i_K A)j^\mu\big]=-S^{\mu ab}(i_K\Omega_{ab})_\mu-\omega^c{}_{da}\left[e_\mu^{*d}\tau_c^\mu-\frac{1}{\sqrt{|\det g|}}D_\mu\big(\sqrt{|\det g|}S_c{}^{d\mu}\big)\right]\tag{B17}$$

or

$$\frac{1}{\sqrt{|\det g|}}\partial_\mu\sqrt{|\det g|}\big[(i_K e^{*b})\tau_b^\mu+(i_K A)j^\mu-(i_K\omega)_{ab}S^{ab\mu}\big]=-\big(i_K\omega^c{}_d\big)e_\mu^{*d}\tau_c^\mu,\tag{B18}$$

which has the form of current conservation if we set the background spin connections to zero. We thus define the conserved energy current as

$$\mathcal{J}_E^\mu=\sqrt{|\det g|}\big[(i_K e^{*a})\tau_a^\mu+(i_K A)j^\mu\big].\tag{B19}$$

b. Conserved energy currents from temporal translation symmetry

The energy current $\mathcal{J}_E^\mu$ defined above can be understood from global translation symmetry directly. In equilibrium, under temporal translations, we have

$$0=\delta e_\mu^{*a}=\delta A_\mu\tag{B20}$$

and

$$\delta\psi=K^\mu\partial_\mu\psi,\tag{B21}$$

where we have set $\omega_{ab\mu}=0$. Correspondingly, the Noether current associated with temporal translation is

$$\mathcal{J}_E^\mu=-\left(\frac{\partial\sqrt{|\det g|}\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi+\text{H.c.}-K^\mu\sqrt{|\det g|}\mathcal{L}\right),\tag{B22}$$

where $\mathcal{L}$ is the Lagrangian density and g is a metric. Assuming that action S depends on vielbeins through $\det g$ and $D_a=e_a^\mu(\partial_\mu+A_\mu)$, we can recast $\mathcal{J}_E^\mu$ as

$$\begin{aligned}\mathcal{J}_E^\mu&=-\left[\frac{\partial\sqrt{|\det g|}\mathcal{L}}{\partial\partial_\mu\psi}K^\alpha(\partial_\alpha+iA_\alpha)\psi+\text{H.c.}\right]\\&\quad+K^\mu\sqrt{|\det g|}\mathcal{L}+(K^\alpha A_\alpha)\left(i\frac{\partial\sqrt{|\det g|}\mathcal{L}}{\partial\partial_\mu\psi}\psi+\text{H.c.}\right)\\&=-\left[\frac{\partial\sqrt{|\det g|}\mathcal{L}}{\partial D_\alpha\psi}\frac{\partial D_\alpha\psi}{\partial e_\mu^{*a}}K^a+\text{H.c.}\right]\\&\quad+K^\mu\sqrt{|\det g|}\mathcal{L}+\sqrt{|\det g|}(K^\alpha A_\alpha)j^\mu\\&=-\frac{\delta S}{\delta e_\mu^{*a}}K^a+\sqrt{|\det g|}(K^\alpha A_\alpha)j^\mu\\&=\sqrt{|\det g|}\big[(i_K e^{*a})\tau_a^\mu+(i_K A)j^\mu\big],\end{aligned}\tag{B23}$$

where the $U(1)$ Noether current is defined as $j^\mu\equiv\frac{1}{\sqrt{|\det g|}}(i\frac{\partial S}{\partial\partial_\mu\psi}\psi+\text{H.c.})$. This matches with our results obtained before.

APPENDIX C: ENERGY MAGNETIZATION FOR (2 + 1)-DIMENSIONAL DIRAC FERMIONS

In this Appendix, we shall provide a detailed derivation of the energy magnetization for (2 + 1)-dimensional massive Dirac fermions.

We consider the following action for (2 + 1)-dimensional Dirac fermions in the Minkowski spacetime:

$$S=\int d^3x\sqrt{|\det g|}\left[\frac{1}{2}\big(\bar\psi\gamma^a ie_a^\mu\partial_\mu\psi-\bar\psi i\overleftarrow{\partial}_\mu\gamma^a e_a^\mu\psi\big)-m\bar\psi\psi\right],\tag{C1}$$

where both the chemical potential and the external electromagnetic field are set to zero. The energy-momentum tensor is

$$\tau_b^\nu=\tfrac{1}{2}(\bar\psi\gamma^\nu i\partial_b\psi+\text{H.c.})-\delta_b^\nu\mathcal{L},\tag{C2}$$

and we have set the spin connection to zero. In addition, we are most interested in the (thermal) Hall effect, so the $\delta_b^\nu\mathcal{L}$ term is neglected hereafter, and we define $\tilde\tau_b^\nu=\tau_b^\nu+\delta_b^\nu\mathcal{L}$. Then, values of the Feynman diagram in Fig. 3 in a homogeneous equilibrium state of temperature T_0 imply that the response energy current is

$$\mathcal{J}^{\mu}_{E,\text{equ}} = -\frac{C_{\text{equ}} m}{4\pi}\epsilon^{\mu\alpha\nu}\partial_\alpha e^{*0}_\nu, \tag{C3}$$

where

$$C_{\text{equ}} = T_0\left[\frac{|m|}{T_0}\tanh\left(\frac{1}{2}\frac{|m|}{T_0}\right) - \frac{\Lambda}{T_0}\tanh\left(\frac{1}{2}\frac{\Lambda}{T_0}\right)\right], \tag{C4}$$

with the UV cut-off Λ large compared to T_0. Here $\mathcal{J}^{\mu}_E$ consists of two parts, one from the zero-point energy depending on the UV cut-off, and the other from thermal excitations. In particular, as far as thermal responses are concerned, terms from the zero-point energy can be neglected, for the following reasons: (i) Eqs. (16) in the main text are linear differential equations of energy magnetization, tracing back to the fact that energy (magnetization) currents depend on energy magnetization linearly. In turn, energy magnetization comprises of two pieces from the zero-point energy and thermal excitations, respectively. (ii) Only the latter piece contributes to the thermal Hall effect. Motivated by this, $\mathcal{J}^{\mu}_E$ in the main text stands for the thermal current with the zero-point energy part subtracted. Let us proceed by setting the term on the right-hand side of Eq. (16d) to $-\frac{m}{4\pi}C_{\text{equ}}$. By solving Eq. (16d), one can obtain the energy magnetization, i.e.,

$$m_{g,\,0} = c_1 T_0^2 - \frac{mT_0}{8\pi}\left\{\left(\frac{|m|}{T_0} - \frac{\Lambda}{T_0}\right) + 4\left[\frac{T_0}{\Lambda}\text{Li}_2(-e^{-\frac{\Lambda}{T_0}}) - \frac{T_0}{|m|}\text{Li}_2(-e^{-\frac{|m|}{T_0}})\right] + 4\ln\left(\frac{1+e^{-|m|/T_0}}{1+e^{-\Lambda/T_0}}\right)\right\}, \tag{C5}$$

where c_1 cannot be determined by solving Eq. (16d). In the zero-temperature limit, i.e., $T_0 \to 0$, we have $m_{g,0} \simeq -\frac{1}{8\pi}(m|m| - m\Lambda)$, so the effective action in Eq. (12) becomes $-\frac{m(|m|-\Lambda)}{8\pi}\int \epsilon^{\mu\nu\rho}e^{*0}_\mu\partial_\nu e^{*0}_\rho$, which matches the torsional Chern-Simons term obtained in Refs. [9,10]. Similarly, at finite temperature, the $c_1 T_0^2$ term suggests that there exists a thermal torsional Chern-Simons term, i.e., $-c_1\int T_0^2\epsilon^{\mu\nu\rho}e^{*0}_\mu\partial_\nu e^{*0}_\rho$.

Now let us determine the value of c_1 by taking the high-temperature limit as a reference state. Consider a UV complete model at high temperature. We expect all quasiparticles to be excited, so $m_{g,0}$ should be temperature-independent. In this limit, $m_{g,0} = c_1 T_0^2 - [\frac{m(|m|-\Lambda)}{8\pi} + \frac{\pi\,\text{sgn}(m)}{24}T_0^2]$ with $m \ll \Lambda$, which, combined with the temperature-independent condition, yields

$$c_1 = \frac{\pi\,\text{sgn}(m)}{24}. \tag{C6}$$

This means that we have fixed c_1 by imposing physical conditions, even though it cannot be determined from perturbative calculations of Feynman diagrams.

In summary, $m_{g,0}$ in the low-temperature limit is

$$m_{g,0} = \text{sgn}(m)\frac{\pi}{24}T_0^2 - \frac{m(|m|-\Lambda)}{8\pi}, \tag{C7}$$

and the ensuing effective action is

$$S_{\text{eff}} = -\int\left[\frac{\pi\,\text{sgn}(m)}{24}T_0^2 + \frac{m(\Lambda-|m|)}{8\pi}\right]\epsilon^{\mu\nu\rho}e^{*0}_\mu\partial_\nu e^{*0}_\rho + O[(\phi_g)^2], \tag{C8}$$

where the first term in the square bracket originates from thermal excitations, while the second term is from the zero-point energy and depends on the UV cut-off.

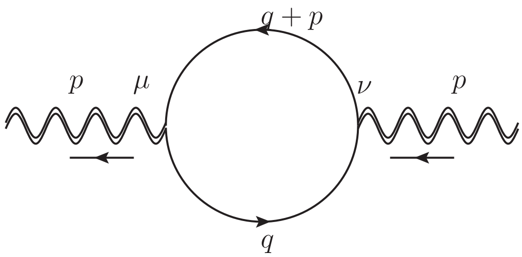

FIG. 3. Feynman diagram for $\langle\tilde{\tau}^{\mu}_a(-p)\tilde{\tau}^{\nu}_b(p)\rangle$. $\tilde{\tau}^{\mu}_a$ is defined as $\tilde{\tau}^{\mu}_a \equiv \tau^{\mu}_a + \delta^{\mu}_a\mathcal{L}$, where the $-\delta^{\mu}_a\mathcal{L}$ term in τ^{μ}_a is subtracted. Double wavy lines stand for external vielbeins.

APPENDIX D: CALCULATIONS OF ENERGY-MOMENTUM TENSOR $\langle\tau^1_0\rangle$ IN (1 + 1)-DIMENSIONAL SPACETIME

In this Appendix, we shall present calculations of the edge energy-momentum tensor in flat spacetime. For chiral fermions with chirality s, the energy-momentum tensor is

$$\begin{aligned}\langle\tau^1_0\rangle &= \left\langle\bar{\psi}\gamma^1\left(\frac{1+s\gamma_5}{2}\right)i\partial_0\psi\right\rangle \\ &= \frac{1}{2}\int\frac{dp_1}{2\pi}\left(\sum_n\frac{1}{\beta_0}\right)\frac{(i\omega_n)^2}{(i\omega_n)^2 - p_1^2}\text{tr}(s\gamma^0\gamma^1\gamma_5) \\ &= s\int\frac{dp_1}{2\pi}\left(\sum_n\frac{1}{\beta_0}\right)\frac{(i\omega_n)^2}{(i\omega_n)^2 - p_1^2},\end{aligned} \tag{D1}$$

where $s = \pm 1$ is for chiralities of Weyl fermions, $\omega_n = \frac{(2n+1)\pi}{\beta_0}$ is the Matsubara frequency, and gamma matrices are defined as $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, and $\gamma^5 = -\sigma^3$.

1. Hard-cutoff regularization

Now we shall calculate the integral above by using a hard-cutoff regularization, i.e.,

$$\begin{aligned}&\int\frac{dp_1}{2\pi}\left(\sum_n\frac{1}{\beta_0}\right)\frac{(i\omega_n)^2}{(i\omega_n)^2 - p_1^2} \\ &= \int\frac{dp_1}{2\pi}\left[-\frac{1}{2}\epsilon n_F(-\epsilon) + \frac{1}{2}\epsilon n_F(\epsilon)\right] \\ &= 2\pi\int\frac{d\epsilon}{2\pi}\frac{\epsilon}{2\pi}n_F(\epsilon) \\ &= 2\pi\left\{\int_{-\infty}^{+\infty}\frac{d\epsilon}{2\pi}\frac{\epsilon}{2\pi}[n_F(\epsilon) - \theta(-\epsilon)] + \int_{-4\pi\tilde{\Lambda}}^{+\infty}\frac{d\epsilon}{2\pi}\frac{\epsilon}{2\pi}\theta(-\epsilon)\right\} \\ &= \left(\frac{\pi}{12\beta_0^2} + \tilde{\Lambda}^2\right),\end{aligned} \tag{D2}$$

where $\epsilon \equiv |p_1|$ is the energy for Weyl fermions, $n_F \equiv \frac{1}{e^{\beta_0 \epsilon}+1}$ is the Fermi-Dirac distribution function, and $\tilde{\Lambda}$ is a cutoff.

2. Dimensional regularization

If we use dimensional regularization instead, the $\tilde{\Lambda}^2$ term vanishes, i.e.,

$$\begin{aligned} &\int \frac{dp_1}{2\pi}\left(\sum_n \frac{1}{\beta_0}\right)\frac{(i\omega_n)^2}{(i\omega_n)^2 - p_1^2} \\ &= \frac{1}{2\beta_0}\sum_n |\omega_n| \\ &= \pi\beta_0^{-2}(1+3+5+\cdots) = \frac{\pi}{12\beta_0^2}, \end{aligned} \tag{D3}$$

where in the second line, we have integrated over p_1 by using dimensional regularization. In the last line, we have used $(1+3+5\cdots) = \frac{1}{12}$, which is because $\sum_{n=1}^{+\infty} n = -\frac{1}{12}$ and $2\sum_{n=1}^{+\infty} n + (1+3+\cdots) = -\frac{1}{12}$. Note that $\sum_n |\omega_n|$ is the vacuum energy of fermions.

3. Results

In summary, we have derived

$$\langle \tau_0^1 \rangle = s\left(\frac{\pi}{12\beta_0^2} + \tilde{\Lambda}^2\right). \tag{D4}$$

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[49] We consider charge-conserved systems for completeness. For systems without charge conservation, the discussion is similar by considering partition function with Killing vector K and external vielbein e^{*a}_μ.

[50] Generally speaking, we shall parametrize them by $\sigma \in [0, 1/\tilde{T}_0]$, where $\tilde{T}_0$ is equal to T_0 when $\sigma = x^0_E$. The Killing vector K becomes $\frac{d}{d\sigma}$, so the length of thermal loops becomes $\int_0^{1/\tilde{T}_0} d\sigma \sqrt{\frac{dx^\mu}{d\sigma} g_{\mu\nu} \frac{dx^\nu}{d\sigma}}$. However, for simplicity, we have assumed that $x^0_E = \sigma$. Because of this choice, we have $\mathcal{L}_K g_{\mu\nu} = \partial_{E0} g_{\mu\nu} = 0$. That is, $g_{\mu\nu}$ (and hence also e^{*a}_μ) must be time-independent.

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[53] If the spin chemical potential is not zero, then this current satisfies

$$\frac{1}{\sqrt{|\det g|}} \partial_\mu \sqrt{|\det g|} [(i_K e^{*b}) \tau^\mu_b + (i_K A) j^\mu - (i_K \omega)_{ab} S^{ab\mu}] = -(i_K \omega^c{}_d) e^{*d}_\mu \tau^\mu_c,$$

where $S^{ab\mu}$ is the spin current defined as the variation of action with respect to the spin connection, i.e., $S^{ab\mu} \equiv \frac{1}{\sqrt{|\det g|}} \frac{\delta S}{\delta \omega_{ab\mu}}$

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[55] We decompose the gauge field as $A = (i_u A) u + A_{\rm trans}$, where i_u is for the interior product with vector u. This implies that $i_u A_{\rm trans} = i_K A_{\rm trans} = 0$. Then, we have

$$\int A \wedge dA = \int [(i_u A) u + A_{\rm trans}] \wedge dA = \int [(i_u A) u \wedge dA + A \wedge dA_{\rm trans}],$$

where in the last line, we have used integration by parts. Because $i_K A_{\rm trans} = 0$ and $\mathcal{L}_K A_{\rm trans} = 0$, we have $i_u(dA_{\rm trans}) = 0$, which means that $dA_{\rm trans}$ is on the plane normal to u and thus $A \wedge dA_{\rm trans} = (i_u A) u \wedge dA$. This shows that the Chern-Simons term can be recast as $\int m_{N,0} u \wedge dA$ with $m_{N,0} = -\frac{\nu_H (i_u A)}{2\pi} = \frac{\nu_H \mu}{2\pi}$.

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[59] The corresponding derivation is given as follows:

$$\begin{aligned} &\int (c_2 \mu T) \epsilon^{\mu\nu\rho} e^{*0}_\mu \partial_\nu e^{*0}_\rho + \int (c_2 T) \epsilon^{\mu\nu\rho} e^{*0}_\mu \partial_\nu A_\rho \\ &= -c_2 \int \frac{A_0 T_0}{e^{*0}_0} \epsilon^{0ij} \partial_i e^{*0}_j - c_2 \int \frac{A_0 T_0}{(e^{*0}_0)^2} \epsilon^{ij0} e^{*0}_i \partial_j e^{*0}_0 \\ &\quad + c_2 \int T_0 \epsilon^{0ij} \partial_i A_j + c_2 \int T_0 \epsilon^{ij0} \frac{e^{*0}_i}{e^{*0}_0} \partial_j A_0 \\ &= +c_2 \int T_0 \epsilon^{0ij} \partial_i \left(A_j - \frac{e^{*0}_j}{e^{*0}_0} A_0 \right), \end{aligned}$$

which turns out to be a topological theta term as well.

[60] This can be appreciated as follows. Consider edge chiral fermions with dispersion $\mathcal{E} = \mathcal{E}(p)$ and velocity $v = \frac{\partial \mathcal{E}}{\partial p}$, where $\mathcal{E}$ takes values from $-\infty$ to $+\infty$ due to its chiral nature. The corresponding current can be calculated as follows:

$$\begin{aligned} j^1 &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{\partial \mathcal{E}}{\partial p} \left[\frac{1}{e^{\beta_0(\mathcal{E}-\mu)} + 1} - \theta(-\mathcal{E}) \right] \\ &= s \int_{-\infty}^{+\infty} \frac{d\mathcal{E}}{2\pi} \left[\frac{1}{e^{\beta_0(\mathcal{E}-\mu)} + 1} - \theta(-\mathcal{E}) \right] = s \frac{\mu}{2\pi}, \end{aligned}$$

where $s = \pm 1$ is from chiralities and $\theta(-\mathcal{E})$ is a regulator used to subtract contributions from the Dirac sea. This shows that the edge currents are independent of temperature, so we have $c_2 = 0$.

[61] The corresponding derivation is given as follows:

$$\begin{aligned} \int T^2 \epsilon^{\mu\nu\rho} e^{*0}_\mu \partial_\nu e^{*0}_\rho &= \int T_0^2 \epsilon^{0ij} \left(\frac{1}{e^{*0}_0} \right)^2 \left(e^{*0}_0 \partial_i e^{*0}_j + e^{*0}_i \partial_j e^{*0}_0 \right) \\ &= \int T_0^2 \epsilon^{0ij} \left[\frac{1}{e^{*0}_0} \partial_i e^{*0}_j - \frac{1}{(e^{*0}_0)^2} e^{*0}_j \partial_i e^{*0}_0 \right] \\ &= \int T_0^2 \epsilon^{0ij} \partial_i \left(e^{*0}_j / e^{*0}_0 \right). \end{aligned}$$

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