

**Lowest Landau level theory of the bosonic Jain states**Hart Goldman  and T. Senthil*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

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Quantum Hall systems offer the most familiar setting where strong interparticle interactions combine with the topology of single-particle states to yield novel phenomena. Despite our mature understanding of these systems, an open challenge has been to develop a microscopic theory capturing both their universal and nonuniversal properties, when the Hamiltonian is restricted to the noncommutative space of the lowest Landau level. Here we develop such a theory for the Jain sequence of bosonic fractional quantum Hall states at fillings  $\nu = \frac{p}{p+1}$ . Building on a lowest Landau level description of a parent composite Fermi liquid at  $\nu = 1$ , we describe how to dope the system to reach the Jain states. Upon doping, the composite fermions fill noncommutative generalizations of Landau levels, and the Jain states correspond to integer composite fermion filling. Using this approach, we obtain an approximate expression for the bosonic Jain sequence gaps with no reference to any long-wavelength approximation. Furthermore, we show that the universal properties, such as Hall conductivity, are encoded in an effective noncommutative Chern-Simons theory, which is obtained on integrating out the composite fermions. This theory has the same topological content as the familiar Abelian Chern-Simons theory on commutative space.

DOI: [10.1103/PhysRevB.105.075130](https://doi.org/10.1103/PhysRevB.105.075130)**I. INTRODUCTION**

Two-dimensional many-particle systems in a strong magnetic field display some of the most famous examples of quantum correlated phenomena. At special, partial fillings  $\nu$  of a Landau level, an incompressible phase is formed with Hall conductivity,  $\sigma_{xy} = \nu e^2/h$ , in what is known as the fractional quantum Hall effect (FQHE). Since Laughlin's original explanation of the basic physics of the FQHE [1], a number of distinct theoretical approaches have been developed which provide a deeper and more versatile understanding of the FQHE. A prominent and successful approach is built on the framework of flux attachment [2], which trades the original theory of charged particles in a magnetic field for a theory of new entities, dubbed composite bosons or composite fermions, interacting with a Chern-Simons gauge field. In particular, the composite fermion construction provides a simple unifying explanation of the vast majority of observed FQH phases as integer quantum Hall (IQH) states of composite fermions [3–5]. It also provides the foundation to understand the metallic states found in electronic systems near even denominator filling fractions [6–11], and enables parent descriptions of non-Abelian quantum Hall states [12,13].

For the gapped quantum Hall states, the universal long distance properties are captured by Chern-Simons topological quantum field theories, enriched by a global U(1) symmetry that corresponds to particle number conservation [14,15]. Within the flux attachment framework, these Chern-Simons theories can be usefully viewed as arising from integrating out Landau levels of composite fermions. This description encapsulates the topological order and symmetry fractionalization data of the quantum Hall state. However, it is not

suitable if we are interested in estimating nonuniversal, microscopic quantities, such as the magnitude of the gaps, given a microscopic Hamiltonian. Especially salient in this regard are Hamiltonians defined by projecting to the lowest Landau level (LLL). Then the electron kinetic energy is quenched and all energy scales are determined by the interaction strength. The interactions must therefore be treated completely non-perturbatively, and the problem is typically only amenable to study through a variety of numerical methods, such as variational wave-function calculations, exact diagonalization, and the density matrix renormalization group. An interesting analytical approach, due to Murthy and Shankar (for a review, see Ref. [16]), used a Hamiltonian description of composite fermions in the LLL to yield good estimates of gaps and other nonuniversal quantities for a variety of filling fractions. Nevertheless, the universal properties of both the composite Fermi liquid and the Jain states seem harder to directly extract from this formalism.

An important open question in quantum Hall physics is to obtain a unified analytic framework that captures both universal and nonuniversal aspects of the physics. This task is particularly challenging when the many-particle Hilbert space is restricted to a single Landau level. Recently, inspired by the earlier efforts of Pasquier, Haldane, and Read [17,18], one of us, together with Zhihuan Dong, made progress on this problem by developing a composite fermion approach to the metallic state of bosons at  $\nu = 1$  that is strictly in the LLL [19,20]. This theory, like earlier approaches, involves composite fermions coupled to a fluctuating U(1) gauge field, although this time the fields live on the noncommutative space of the LLL. Despite this progress for the metallic state, there continues to be no construction of gapped quantum Hall states



FIG. 1. A schematic of the emergence of the bosonic Jain states. Our starting point is a theory of bosons in a magnetic field with interaction energy,  $V_{\text{int}}$ , and cyclotron frequency,  $\omega_c$ . The bosons are projected to the lowest Landau level (LLL) when  $V_{\text{int}} \ll \omega_c$  (we use units  $\hbar = c = 1$ ). By changing variables to composite fermions  $c$ , coupled to a fluctuating gauge field  $a_\mu$ , we consider a composite Fermi liquid (CFL) theory on the noncommutative space of the LLL, which is valid below the interaction scale. We access the Jain states by turning on a background vector potential  $\delta A_i$ , such that the composite fermions fill  $p$  Landau levels with a gap  $\Delta_p$ . Integrating out the composite fermions leads to an effective *noncommutative* Chern-Simons theory, which captures the topological data of the Jain state.

that allows for a derivation of the long wavelength Chern-Simons field theory but also provides estimates of the gaps and other nonuniversal features.

The purpose of this work is to provide such a theory for the specific case of Jain states of bosons at fillings  $\nu = \frac{p}{p+1}$ , with  $p$  a positive integer, which approach the metallic state of bosons at  $\nu = 1$  as  $p \rightarrow \infty$ . An outline of our construction is given in Fig. 1. Starting with the description of the metallic composite Fermi liquid ground state of bosons at  $\nu = 1$  in terms of noncommutative composite fermion field theory, we show that the proximate Jain states can be obtained by filling Landau levels of composite fermions, in a manner directly analogous to the derivation using ordinary composite fermions. In the process, we obtain an approximate mean-field expression for the bosonic Jain sequence gaps,

$$\Delta_p = \frac{2\pi\bar{\rho}}{m_*} \frac{1}{1+p}, \quad (1.1)$$

where  $\bar{\rho}$  is the charge density,  $m_*$  is the composite fermion effective mass, and  $p = 2\pi\bar{\rho}/\delta B$  is the Jain sequence index. This expression is corrected compared to the usual composite fermion mean-field result, in which the gap is proportional to the effective magnetic field felt by the composite fermions,  $m_*\Delta = \delta B = 2\pi\bar{\rho}/p$ . In our result, working on the noncommutative space of the LLL has corrected  $p \rightarrow p+1$ . Using the Hartree-Fock approximation of the effective mass, our result returns values for the gap that are compatible with exact diagonalization estimates [21,22]. In principle, other nonuniversal data, such as collective modes and structure factors, can also be sought using our mean-field framework.

Furthermore, we show that integrating out the composite fermions yields a noncommutative Chern-Simons (NCCS) field theory for these Jain states, which describes the correct topological order. Many years ago, based on hydrodynamic considerations, Susskind and Polychronakos [23,24] proposed the NCCS field theory and its matrix model regularization to describe quantum Hall states at short distances, and it has been found to correctly capture ground-state topological features and wave functions [25–27]. More recently, the quantum Hall matrix models have been generalized to the Jain sequences [28] and to non-Abelian quantum Hall states [29,30]. Their universal geometric response has also been studied [31,32]. However, the connection of the NCCS theory to realistic microscopic Hamiltonians and to composite fermion field theory has been opaque. These relationships are explained in our approach. The NCCS theory appears as

an effective description at scales  $\omega \ll \Delta_p$ , and its noncommutativity is set by the charge density, rather than the total magnetic field, in agreement with the hydrodynamic approach of Susskind [23].

Since the universal properties of fractional quantum Hall phases are well described by the usual commutative Chern-Simons field theory, one may question what is gained by using the noncommutative version. It is therefore interesting that in our work, which is an approximate treatment of microscopic Hamiltonians in the LLL, the noncommutative theory emerges rather naturally, arising as a bridge between a fully microscopic theory and the usual long wavelength topological quantum field theory.

We finally comment that although there is an extensive literature on noncommutative field theories (for reviews, see Refs. [33,34]), the structure of the theories we encounter differs from what is most commonly discussed. In particular, our focus is on noncommutative theories of fermions at a nonzero density of global U(1) charge and in a background magnetic field. Our work contributes to the general understanding of this class of problems. Nevertheless, we will make contact with and use some of the results of the extensive existing literature, as appropriate, throughout the paper.

We proceed as follows. In Sec. II, we review the noncommutative composite fermion approach the problem of bosons at  $\nu = 1$  engineered in Refs. [19]. We then proceed to develop our mean-field approach to the bosonic Jain sequence in Sec. III, in the process obtaining our result for the gaps. In Sec. IV, we integrate out the composite fermions to obtain a NCCS field theory describing the universal aspects of the Jain sequence states. We conclude in Sec. V.

## II. RECAP: THE LLL COMPOSITE FERMION THEORY OF BOSONS AT $\nu = 1$

### A. Basics of the LLL and mean-field theory of the metallic state at $\nu = 1$

We start by considering a 2d system of (bosonic or fermionic) charged particles with density  $\bar{\rho}$  in a strong magnetic field,  $B$ , such that the filling,  $\nu = 2\pi\bar{\rho}/B$ , is less than or equal to one. The particles then form highly degenerate Landau levels, and they can each fill states in the lowest Landau level. In the limit of large magnetic field, we may therefore project the Hilbert space to states in the LLL. On restricting to the LLL, the particles are only described by their guiding center coordinates,

$$R_i = x_i - \ell_B^2 \varepsilon_{ij} \pi_j, \quad (2.1)$$

where  $\ell_B = |B|^{-1/2}$  is the magnetic length and  $\pi_i$  is the gauge invariant momentum. Here and throughout the manuscript repeated indices are summed. Assuming canonical commutation relations, these coordinates do not commute,

$$[R_i, R_j] = -i\ell_B^2 \varepsilon_{ij} \equiv i\Theta \varepsilon_{ij}. \quad (2.2)$$

Therefore the geometry of the LLL is noncommutative, with noncommutativity parameter  $\Theta = -\ell_B^2$ .

On LLL projection, the kinetic energy is quenched, and the Hamiltonian may be expressed in terms of interactions of the projected density operator,<sup>1</sup> which we denote  $\rho_L(\mathbf{q}) = \sum_n e^{i\mathbf{q}\cdot\mathbf{R}_n}$ , where boldface denotes spatial vectors and  $n$  is a particle index. Using Eq. (2.2), the density operator can be seen to satisfy the famous Girvin-MacDonald-Platzman (GMP) algebra [35],

$$[\rho_L(\mathbf{q}), \rho_L(\mathbf{p})] = 2i \sin\left(\frac{\ell_B^2(\mathbf{q} \times \mathbf{p})}{2}\right) \rho_L(\mathbf{q} + \mathbf{p}). \quad (2.3)$$

The GMP algebra contains information on the geometric symmetries of the LLL:  $\rho_L$  can be expanded in terms of the generators of area-preserving diffeomorphisms of space, which in turn satisfy the  $\mathcal{W}_\infty$  algebra [36–38]. While these are not symmetries of the microscopic Hamiltonian they encode the structure of the noncommutative space of the LLL. Our interest is in projected Hamiltonians of the form

$$H = \frac{1}{2} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \tilde{U}(\mathbf{q}) \rho_L(-\mathbf{q}) \rho_L(\mathbf{q}), \quad (2.4)$$

where  $\tilde{U}(\mathbf{q}) = U_0 \exp -\frac{q^2 \ell_B^2}{2}$  is the Fourier transform of a contact interaction augmented with a form factor. We first review the solution of this problem developed in Ref. [19] for the particular case of bosons at  $\nu = 1$ .

The physical underpinnings of the lowest Landau level description is a view of the composite fermions as charge-vortex dipoles [39]. Unlike in ordinary flux attachment, these composite fermions are understood to be neutral, because the vortices deplete a unit of charge at their cores.<sup>2</sup> They possess a dipole moment proportional to their momentum,

$$\mathbf{d} = \ell_B^2 \mathbf{k} \times \hat{z}. \quad (2.5)$$

The composite fermions thus naturally acquire a quadratic dispersion from their dipole energy,  $\epsilon(\mathbf{k}) \sim |\mathbf{d}|^2$ . We will refer back to this basic physical picture at many points in this work.

We begin with a formal representation of the many body bosonic Hilbert space and the density operator introduced by Pasquier and Haldane [17], and developed further by Read [18]. For  $N$  particles, any state in the many-body Hilbert space of bosons at  $\nu = 1$  can be written as

$$|\Psi\rangle = \sum_{\{m_i\}} a_{m_1, \dots, m_N} |m_1, \dots, m_N\rangle, \quad (2.6)$$

<sup>1</sup>The projection of the density operator will also include a form factor  $e^{-q^2 \ell_B^2/4}$ . We will follow the common practice of incorporating this form factor into the interaction so that the Hamiltonian is expressed in terms of the operator  $\rho_L$ , which satisfies the GMP algebra.

<sup>2</sup>For recent applications of this idea to the problem of fermions at  $\nu = 1/2$ , see Refs. [40–44].

where  $m_i = 1, \dots, N$  is an orbital index for the  $N$  single-particle states in the Landau level (thus the filling  $\nu = 1$ ),  $|m_1, \dots, m_N\rangle$  is a product of single-particle orbitals, and  $a_{m_1, \dots, m_N}$  are constants that are symmetric under exchange of indices. One may represent the basis states in terms of composite fermion creation (annihilation) operators,  $c_{mn}^\dagger$  ( $c_{mn}$ ), where  $m$  and  $n$  are again orbital indices,

$$|m_1, \dots, m_N\rangle = \varepsilon^{n_1 \dots n_N} c_{m_1 n_1}^\dagger \dots c_{m_N n_N}^\dagger |0\rangle, \quad (2.7)$$

where  $|0\rangle$  is the vacuum state annihilated by  $c_{mn}$ . We may interpret the left and right indices, respectively, as being associated with (bosonic) charges and vortices, which bind to form the composite fermion. In terms of the fermion operators, the projected density operator is

$$(\rho_L)_{mm'} = c_{mn}^\dagger c_{m'n} \quad (2.8)$$

and is the generator of the physical electromagnetic symmetry.

This fermionic description is redundant. The physical states are invariant under  $SU(N)$  rotations of the  $n_i$  indices, meaning that the theory has a  $SU(N)$  gauge symmetry. The presence of this gauge symmetry implies a constraint, in that the physical states satisfy

$$(\rho_R)_{nn'} |\Psi\rangle = c_{mn}^\dagger c_{m'n'} |\Psi\rangle = \delta_{nn'} |\Psi\rangle. \quad (2.9)$$

Here  $\rho_R$  is the  $SU(N)$  charge density, and we may interpret it as the density of vortices in the LLL. Note that its trace is shared with that of the charge density,  $\rho_L$ , and represents the physical EM charge,  $\text{Tr}[\rho_L] = \text{Tr}[\rho_R] = N$ .

We may construct momentum-space representations of the fermion operators,  $c(\mathbf{q})$ ,  $c^\dagger(\mathbf{q})$  using the LL orbital matrix elements of the magnetic translation operator,  $\tau_{\mathbf{q}} = e^{i\mathbf{q}\cdot\mathbf{R}}$ ,

$$c_{mn} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \langle m | \tau_{\mathbf{q}} | n \rangle c(\mathbf{q}). \quad (2.10)$$

These operators satisfy the standard anticommutation relations

$$\{c(\mathbf{q}), c^\dagger(\mathbf{q}')\} = (2\pi)^2 \delta^{(2)}(\mathbf{q} - \mathbf{q}'). \quad (2.11)$$

In terms of these operators, we may construct  $\rho_L$  and  $\rho_R$  as follows:

$$\begin{aligned} \rho_L(\mathbf{q}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} c^\dagger(\mathbf{k} - \mathbf{q}) c(\mathbf{k}) e^{-i\Theta \mathbf{k} \times \mathbf{q}/2}, \\ \rho_R(\mathbf{q}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} c^\dagger(\mathbf{k} - \mathbf{q}) c(\mathbf{k}) e^{i\Theta \mathbf{k} \times \mathbf{q}/2}, \end{aligned} \quad (2.12)$$

where we recall  $\Theta = -\ell_B^2 = -1/(2\pi\bar{\rho})$  at  $\nu = 1$ . In this representation, the constraint in Eq. (2.9) becomes

$$\rho_R(\mathbf{q}) = (2\pi)^2 \bar{\rho} \delta^{(2)}(\mathbf{q}). \quad (2.13)$$

Using the anticommutation relation in Eq. (2.11), it is immediate that both of these operators furnish representations of the GMP algebra:  $\rho_L$  satisfies Eq. (2.3) while  $\rho_R$  satisfies Eq. (2.3) but with a minus sign on the right-hand side. Using the constraint in Eq. (2.13) and expanding  $\rho_L$  in powers of  $\mathbf{q}$ , it is straightforward to see that the composite fermions possess

the dipole moment<sup>3</sup> in Eq. (2.5),

$$d_i = \ell_B^2 \varepsilon_{ij} \mathcal{P}^j, \quad (2.14)$$

where  $\mathcal{P}$  is the composite fermion momentum operator.

Equipped with the fermionic representation of the density operator, we may revisit the Hamiltonian in Eq. (2.4), which is now a four-fermion interaction plus the constraint (2.13). In Refs. [18–20], this problem was attacked using a Hartree-Fock mean-field approach, in which the authors sought a ground state of fermions at finite density, consistent with the expectation of a composite Fermi liquid at  $\nu = 1$ ,

$$\langle c^\dagger(\mathbf{q})c(\mathbf{q}) \rangle \neq 0. \quad (2.15)$$

This leads to a Hartree-Fock Hamiltonian of the form

$$H_{\text{HF}} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \epsilon(\mathbf{q}) c^\dagger(\mathbf{q})c(\mathbf{q}), \quad (2.16)$$

which describes a metallic state of composite fermions, with effective mass of order the interaction strength. The quadratic dispersion,  $\epsilon(\mathbf{q}) \sim |\mathbf{q}|^2$  at small  $\mathbf{q}$ , is naturally understood as the dipole energy of the composite fermion.

The assumption of a nonvanishing expectation value for the Hartree-Fock order parameter,  $c^\dagger(\mathbf{q})c(\mathbf{q})$ , spontaneously breaks the  $SU(N)$  gauge symmetry: it only commutes with  $\rho_R(\mathbf{q})$  at  $\mathbf{q} = 0$ . The important fluctuations about the Hartree-Fock ground state are therefore those near  $\mathbf{q} = 0$ , which can be described by coupling to a  $U(1)$  gauge field,  $a_\mu$ . However, this is not an ordinary  $U(1)$  gauge symmetry, since the fields under consideration live on the noncommutative space of the LLL. Therefore one must consider a noncommutative composite fermion field theory [19,20]. This theory will anchor our analysis through the rest of this work, and we are now prepared to introduce it.

### B. Noncommutative composite fermion field theory and the Seiberg-Witten map

To build a noncommutative field theory, it is necessary to understand how to multiply functions on noncommutative space. These requires the introduction of the Moyal star product. For functions of noncommutative coordinates,  $x_i$ , the Moyal star product is

$$f(x) \star g(x) = \lim_{x' \rightarrow x} \exp\left(i \frac{\Theta}{2} \varepsilon^{ij} \frac{\partial}{\partial x'^i} \frac{\partial}{\partial x^j}\right) f(x')g(x). \quad (2.17)$$

Forming products in this way, the effective composite fermion action [19,20], including gauge field fluctuations, is

$$S = \int d\tau d^2\mathbf{x} \left[ c^\dagger \star D_\tau c + \frac{1}{2m_*} (D_i c)^\dagger \star (D^i c) + i a_0 \bar{\rho} \right], \quad (2.18)$$

$$D_\mu c = \partial_\mu c - i \delta A_\mu \star c - i c \star a_\mu. \quad (2.19)$$

<sup>3</sup>Each term in the expansion of  $\rho_L$  corresponds to a generator of diffeomorphisms in the LLL, which satisfy the  $\mathcal{W}_\infty$  algebra. The dependence of the dipole moment on momentum is natural from this point of view: it is the generator of translations.

Here  $\delta A_\mu$  is the background *probe* electromagnetic vector potential, such as, for instance, that needed to describe the deviation of the theory from  $\nu = 1$ . On the other hand,  $a_\mu$  is an emergent, fluctuating  $U(1)$  gauge field. The equation of motion of  $a_\tau$  enforces the constraint,

$$c^\dagger \star c = \bar{\rho} = \frac{B}{2\pi}, \quad (2.20)$$

where  $B$  is the magnetic field felt by the underlying bosonic charges, which are at filling  $\nu = 1$  when  $\delta A = 0$ . The theory in Eq. (2.18) therefore describes a metallic state of composite fermions at density set by the background magnetic field.

Unlike in ordinary commutative field theory, on noncommutative space even  $U(1)$  gauge symmetries have non-Abelian representations. The physical electromagnetic  $U(1)$  symmetry acts to the left on the noncommutative composite fermions,

$$c \rightarrow U_L \star c, \quad (2.21)$$

$$\delta A \rightarrow U_L \star \delta A \star U_L^\dagger - i \partial U_L \star U_L^\dagger, \quad (2.22)$$

$$a \rightarrow a, \quad (2.23)$$

while the emergent  $U(1)$  gauge symmetry associated with the constraint acts to the right,

$$c \rightarrow c \star U_R, \quad (2.24)$$

$$\delta A \rightarrow \delta A, \quad (2.25)$$

$$a \rightarrow U_R^\dagger \star a \star U_R - i U_R^\dagger \star \partial U_R. \quad (2.26)$$

Respectively, these gauge symmetries correspond to the conserved charge densities,

$$\rho_L = -c \star c^\dagger, \quad \rho_R = c^\dagger \star c. \quad (2.27)$$

On quantization, each of these density operators can be seen to satisfy their own GMP algebra, Eq. (2.3), as they correspond to the right and left densities introduced in the previous section.

Noncommutative field theories like Eq. (2.18) are exceptionally challenging to work with, especially in the presence of a constraint like Eq. (2.20). A common approach to studying them is to map them to a corresponding field theory on commutative space, using a mapping developed by Seiberg and Witten [45]. This map relates the fields  $c$ ,  $a$ , and  $\delta A$  in Eq. (2.18) to new fields,  $\psi$ ,  $\hat{a}$ , and  $\hat{\delta A}$ , defined on commutative space,<sup>4</sup> in an expansion in powers of the noncommutative parameter,  $\Theta = -\ell_B^2$ . Remarkably, applying the Seiberg-Witten map to the noncommutative CFL theory in Eq. (2.18), one obtains [19] the Halperin-Lee-Read (HLR) theory [6] for bosons at  $\nu = 1$ , but with additional short-ranged corrections,

$$\hat{\mathcal{L}} = \psi^\dagger (\partial_\tau - i \hat{a}_\tau - i \delta \hat{A}_\tau) \psi + \frac{1}{2m_*} |(\partial_i - i \hat{a}_i - i \delta \hat{A}_i) \psi|^2 - i \frac{1}{4\pi} \hat{a} d \hat{a} + \Theta \mathcal{L}_{\text{corr}}. \quad (2.28)$$

<sup>4</sup>Frequently in the literature on noncommutative field theory, the hat notation actually denotes the *noncommutative* fields. For consistency with Refs. [19,20], we continue to use the inverted notation.

Via the Seiberg-Witten map, the constraint term,  $ia_\tau \bar{\rho}$ , in the noncommutative CFL theory becomes a Chern-Simons term in the commutative theory, since  $2\pi \bar{\rho} \Theta = -\nu = -1$ . Flux attachment in the HLR theory can therefore be thought of as “emerging” from the constraint on  $\rho_R$ , Eq. (2.20), in the noncommutative CFL theory! The terms in  $\Theta \mathcal{L}_{\text{corr}}$  are fixed by the Seiberg-Witten map but are sub-leading in powers of  $\Theta$ , meaning that they are irrelevant at long wavelengths. For more details on this calculation, see Ref. [19].

The Seiberg-Witten map thus exchanges a theory of composite fermions on noncommutative space, whose density has a nontrivial form factor, for a theory of ordinary composite fermions augmented by additional short-ranged terms. This theory is therefore easily amenable to doping away from  $\nu = 1$ , resulting in the usual Jain sequence of bosonic FQH states, in for which the composite fermions fill an integer number of Landau levels,

$$\nu = \frac{p}{p+1}. \quad (2.29)$$

Because this theory corresponds to a noncommutative CFL theory in the LLL, one can also calculate estimates for LLL dynamical features at large  $p$  (weak  $\delta \hat{B} \equiv \delta \hat{F}_{xy}$ ). For example, using this theory, one can calculate the gaps for the Jain sequence states. One finds<sup>5</sup>

$$\begin{aligned} \Delta(\delta \hat{B}) &= \frac{|\delta \hat{B}|}{m_*} (1 + \Theta \delta \hat{B}) + \mathcal{O}((\Theta \delta \hat{B})^2) \\ &= \frac{2\pi \bar{\rho}}{m_*} \left( \frac{1}{p} - \frac{1}{p^2} \right) + \mathcal{O}(p^{-3}), \end{aligned} \quad (2.30)$$

where we have used the fact that in the Jain state the filling fraction of the composite fermions is  $\nu_{\text{CF}} = 2\pi \bar{\rho} / \delta \hat{B} = p$ . In this work we will go much further by doping the noncommutative CFL theory, Eq. (2.18), directly. This will enable us to compute dynamical properties of the LLL bosonic Jain states, such as their gaps, in a mean-field approximation valid to all orders in  $\Theta$ . We further show that our results match the long wavelength estimate of Eq. (2.30). In addition, we will obtain the low energy effective theory for gauge fluctuations about the mean field. This will enable us to provide a microscopic derivation of the noncommutative Chern-Simons theory description of quantum Hall states and make contact with prior work that proposed such a description based on hydrodynamic arguments [23,24].

### III. DOPING AWAY FROM $\nu = 1$

#### A. A physical picture: quantum mechanics of charge-vortex dipoles

Before turning on a noncommutative magnetic field in the composite fermion field theory of Eq. (2.18), a clear physical picture can be formed by considering the single-particle problem of charges and vortices in the LLL, with the composite fermions viewed as charge-vortex dipoles. The Jain states are then obtained by filling Landau levels of the composite

fermions. Remarkably, we will show later on that this framework corresponds precisely to the single-particle limit of the noncommutative composite fermion field theory.

Technically, our analysis of the charge-vortex dipole problem will rely on the classic work of Nair and Polychronakos [46], which solved the Landau level problem for particles moving in a noncommutative space. This work has since seen considerable followup, e.g., in Refs. [47–54]. Although physical quantum Hall systems and composite fermion theory are occasionally mentioned in this literature, the precise connection to this physics has not been made before as far as we know. We explain this connection below.

Consider bosonic charges and vortices in the  $\nu = 1$  system, respectively characterized by guiding center coordinates,  $R_i$  and  $R_i^v$ . Because the system has a finite charge density, the vortices see the same magnetic field but with opposite sign to the charges. Hence their coordinates satisfy an algebra with opposite sign,

$$[R_i, R_j] = -i\ell_B^2 \varepsilon_{ij}, \quad [R_i^v, R_j^v] = +i\ell_B^2 \varepsilon_{ij}, \quad [R_i, R_j^v] = 0. \quad (3.1)$$

We also define “composite fermion” coordinates,  $r_i^{\text{CF}} = (R_i + R_i^v)/2$ , which commute

$$[r_i^{\text{CF}}, r_j^{\text{CF}}] = 0. \quad (3.2)$$

As expected, the composite fermion feels no magnetic field. Since the composite fermion is a dipole of a charge and a vortex, we may define a dipole moment,  $d_i = R_i^v - R_i$ , which is proportional to the canonical momentum of the composite fermion,

$$p_i^{\text{CF}} = \frac{1}{\ell_B^2} \varepsilon_{ij} d^j, \quad [r_i^{\text{CF}}, p_j^{\text{CF}}] = i\delta_{ij}, \quad [p_i^{\text{CF}}, p_j^{\text{CF}}] = 0. \quad (3.3)$$

It is thus natural to take the effective Hamiltonian to be the dipole energy,

$$H = \frac{1}{2m_*} (p^{\text{CF}})^2. \quad (3.4)$$

Indeed, this expectation is borne out in the Hartree-Fock analysis of Ref. [19].

We now investigate how this picture changes on varying the background magnetic field,  $B \rightarrow B + \delta B$ , while keeping the charge density fixed so that  $\nu_{\text{CF}} = 2\pi \frac{\bar{\rho}}{\delta B} = \frac{B}{\delta B} = p$ , where  $p$  is an integer. The filling fraction of the bosonic charges is therefore

$$\nu = 2\pi \frac{\bar{\rho}}{B + \delta B} = \frac{p}{1+p}, \quad (3.5)$$

which is the bosonic Jain sequence. Tuning the magnetic field alters the guiding center algebra of the charges while preserving that of the vortices,

$$[R_i, R_j] = -i\ell_{B+\delta B}^2 \varepsilon_{ij}, \quad [R_i^v, R_j^v] = +i\ell_B^2 \varepsilon_{ij}, \quad [R_i, R_j^v] = 0, \quad (3.6)$$

where  $\ell_{B+\delta B}^2 = 1/(B + \delta B)$ . Now the composite fermions see a magnetic field, and their coordinates no longer commute,

$$[r_i^{\text{CF}}, r_j^{\text{CF}}] = i\frac{1}{4}(\ell_B^2 - \ell_{B+\delta B}^2) \varepsilon_{ij} \equiv i\theta \varepsilon_{ij}. \quad (3.7)$$

<sup>5</sup>We note that Ref. [19] contains a sign error in its final expression for the gaps. We correct it here.

The composite fermion momentum defined in Eq. (3.3) now satisfies

$$\begin{aligned} [r_i^{\text{CF}}, p_j^{\text{CF}}] &= \frac{\ell_{B+\delta B}^2 + \ell_B^2}{2\ell_B^2} i\delta_{ij}, \\ [p_i^{\text{CF}}, p_j^{\text{CF}}] &= i \frac{\ell_B^2 - \ell_{B+\delta B}^2}{\ell_B^4} \varepsilon_{ij} = i \frac{\delta B}{1 + \delta B/B} \varepsilon_{ij}. \end{aligned} \quad (3.8)$$

Notice that the momentum commutator no longer vanishes, corresponding to the fact that the composite fermions are experiencing a magnetic field. This commutation algebra, together with the Hamiltonian in Eq. (3.4), is precisely the noncommutative Landau level problem studied by Nair and Polychronakos [55] (after a trivial rescaling of the momentum, which we will perform below, to make it canonically conjugate to the composite fermion coordinate).

As in the ordinary Landau problem, we may diagonalize the Hamiltonian in Eq. (3.4) by constructing creation and annihilation operators out of the momentum, proportional to  $p_x^{\text{CF}} \pm i p_y^{\text{CF}}$ . This allows one to easily compute the gaps, which are

$$\Delta_p = \frac{\delta B}{m_*} \frac{1}{1 + \delta B/B} = \frac{2\pi \bar{\rho}}{m_*} \frac{1}{1 + p}. \quad (3.9)$$

Amazingly, we will find that analysis and the resulting gaps precisely match that of the noncommutative composite fermion mean-field theory we will develop in the next section.

Implicit in the above analysis is the identification of  $\delta B$  with the variation of the physical magnetic field. Hence we expect the density of states of the noncommutative Landau level to be  $\delta B$ . This can be checked by constructing an operator,  $\hat{\mathcal{D}}_i$ , which commutes with the Hamiltonian and generates magnetic translations of  $r^{\text{CF}}$ , the composite fermion guiding center coordinate [55]. To do this, we start by defining a new momentum that satisfies canonical commutation relations with  $r^{\text{CF}}$ ,

$$\tilde{p}^{\text{CF}} = \frac{2\ell_B^2}{\ell_{B+\delta B}^2 + \ell_B^2} p^{\text{CF}}. \quad (3.10)$$

Using  $\tilde{p}^{\text{CF}}$ , the theory is now defined by the algebra,

$$\begin{aligned} [r_i^{\text{CF}}, r_j^{\text{CF}}] &= i\theta \varepsilon_{ij}, \quad [r_i^{\text{CF}}, \tilde{p}_j^{\text{CF}}] = i\delta_{ij}, \\ [\tilde{p}_i^{\text{CF}}, \tilde{p}_j^{\text{CF}}] &= 4i \frac{(\ell_B^2 - \ell_{B+\delta B}^2)}{(\ell_{B+\delta B}^2 + \ell_B^2)^2} \varepsilon_{ij} \equiv i\delta B_{\text{CF}} \varepsilon_{ij}. \end{aligned} \quad (3.11)$$

It is now easy to see that the operators,  $\mathcal{D}_i = (\tilde{p}_i^{\text{CF}} - \delta B_{\text{CF}} \varepsilon_{ij} r_j^{\text{CF}})/(1 - \theta \delta B_{\text{CF}})$ , commute with the Hamiltonian and satisfy canonical commutation relations with  $r_i^{\text{CF}}$  (this leads to the chosen normalization). These operators satisfy the algebra,

$$[\mathcal{D}_x, \mathcal{D}_y] = -i \frac{\delta B_{\text{CF}}}{1 - \theta \delta B_{\text{CF}}} = -i\delta B. \quad (3.12)$$

This is a strong indication that the density of states of the Landau level is  $\delta B$ . Further discussion of the degeneracy of Landau levels on the noncommutative torus can be found in Refs. [48,56–58], and we sketch the derivation in Appendix A. Furthermore, we will observe below that the change in the physical magnetic field here,  $\delta B$ , can be identified with the

magnetic field obtained by mapping a noncommutative background field to an ordinary one via the Seiberg-Witten map.

Using this single-particle picture of charge-vortex dipoles to physically ground us, we will now derive these same results using the noncommutative composite fermion theory in Eq. (2.18), which we do using a noncommutative vector potential. In doing so, we will demonstrate how the introduction of a uniform noncommutative magnetic field in the field theory picture corresponds to the deformation of the noncommutativity of the charge degrees of freedom presented above.

## B. Mean-field theory of the composite fermion Landau problem

### 1. Mean-field Hamiltonian and Landau level spectrum

We now construct the bosonic Jain sequence by doping the noncommutative field theory, Eq. (2.18) and studying the resulting problem in mean-field theory. While it is not possible to alter the density of the composite fermions given the constraint, Eq. (2.20), one can still turn on a background magnetic field associated with  $\delta A_i$ . Indeed, if we neglect fluctuations of  $a_i$ , the noncommutative composite fermion theory in a uniform magnetic field is remarkably similar to the analogous problem of electrons on an ordinary space. The composite fermions continue to form Landau levels, with gaps set by the cyclotron frequency. Furthering the analogy, when small fluctuations of  $a_i$  are introduced, we will show in Sec. IV that integrating out the filled Landau levels leads to a *noncommutative* Chern-Simons theory. This will ultimately lead us to the bosonic Jain sequence states.

We start by passing to a Hamiltonian formulation, fixing the density via the constraint, Eq. (2.20). Neglecting fluctuations of  $a_i$ , i.e.,  $\langle a_i \rangle = 0$ , the mean-field Hamiltonian can be written as

$$\mathcal{H} = \int d^2\mathbf{x} \frac{1}{2m_*} (D_i c)^\dagger \star D_i c, \quad D_i c = \partial_i c - i\delta A_i \star c, \quad (3.13)$$

where we have used the Moyal star product defined in Eq. (2.17). The equation of motion of this theory leads to the single-particle Schrödinger equation

$$Hc = -\frac{1}{2m_*} D_i (D_i c) = Ec. \quad (3.14)$$

Rather than fixing to a specific gauge and computing the spectrum by solving the resulting differential equation, it is more conceptually straightforward in this noncommutative context to solve the theory simply by considering its algebraic properties, as in Ref. [55] and the analysis of the previous section. The commutator of the  $D_i$  operators is the noncommutative field strength,

$$\delta F_{xy} = i[D_x, D_y]_\star = \partial_x(\delta A_y) - \partial_y(\delta A_x) - i[A_x, A_y]_\star, \quad (3.15)$$

where we have defined the star commutator,  $[A, B]_\star = A \star B - B \star A$ . To solve the Schrödinger equation, Eq. (3.14), for uniform  $\delta F_{xy}$ , we can again construct creation and annihilation operators using  $D_x \pm iD_y$ , leading to Landau levels with energy set by  $\delta F_{xy}$ ,

$$\epsilon_n = \omega_c \left(n + \frac{1}{2}\right), \quad (3.16)$$

where

$$\omega_c(\delta F_{xy}) = \frac{\delta F_{xy}}{m_*} \quad (3.17)$$

is the Landau level gap. This expression superficially differs from the result for Landau level gaps in Eq. (2.30). However, we stress that the noncommutative field strength  $\delta F_{xy}$  is not equal to the physical magnetic field, which sets composite fermion Landau level degeneracy and corresponds to the physical magnetic field  $\delta B$  in Sec. III A. We now describe how to relate these two quantities.

### 2. Some formalism: covariant coordinates

To define the filling fraction for the composite fermion Landau levels, it is necessary to determine their degeneracy. While naïvely one may expect that the degeneracy is given by  $\delta F_{xy} \times \text{Area}$ , this is in fact not the case. Indeed, in the analysis of the dipole model in Sec. III A, the field setting the energy gap was found to differ from the physical magnetic field. A hint of this can be seen in the fact that the commutator of the ordinary coordinate operators,  $x_i$ , with the gauge covariant momenta  $D_i$  is not gauge invariant:  $D_i$  transforms as an adjoint, while  $x_i$  does not transform. Noncommutative gauge invariance therefore dictates that we work with coordinate operators that transform under gauge transformations. These are the so-called covariant coordinates,

$$Y^i[\delta A] = x^i + \Theta \varepsilon^{ij} \delta A_j, \quad (3.18)$$

which transform as adjoints under the left-acting gauge transformations, Eq. (2.22),

$$Y^i \rightarrow U_L \star Y^i \star U_L^\dagger, \quad (3.19)$$

ensuring gauge invariant commutation relations with  $D_i$ .

The transformation law follows from the close relationship between noncommutative gauge transformations and area-preserving diffeomorphisms (APDs) of the noncommutative space. For a more detailed review of this topic, see Ref. [33]. Consider a gauge transformation,  $U_L = e^{i\lambda}$ . Then for any function  $f(\mathbf{x})$ ,

$$\begin{aligned} U_L \star f(\mathbf{x}) \star U_L^\dagger &= f(\mathbf{x}) - \Theta \varepsilon^{ij} \partial_i \lambda \partial_j f(\mathbf{x}) + \mathcal{O}(\Theta^2) \\ &= f(x^i + \Theta \varepsilon^{ij} \partial_j \lambda + \dots). \end{aligned} \quad (3.20)$$

This means that a combination of left and right gauge transformations is infinitesimally equivalent to a translation by a vector,  $\Theta \varepsilon^{ij} \partial_j \lambda$ , which leaves the area element invariant. Therefore, under a left-acting gauge transformation, the covariant coordinate,  $Y^i$ , transforms as an adjoint,

$$\begin{aligned} Y^i &\rightarrow x^i + \Theta \varepsilon^{ij} U_L \star \delta A_j \star U_L^\dagger + \Theta \varepsilon^{ij} \partial_j \lambda + \mathcal{O}(\Theta^2) \\ &= U_L \star Y^i \star U_L^\dagger. \end{aligned} \quad (3.21)$$

A useful consequence of this transformation law is that functions of covariant coordinates also transform as adjoints,

$$\begin{aligned} f[Y(\mathbf{x})] &\rightarrow f[U_L(\mathbf{x}) \star Y(\mathbf{x}) \star U_L^\dagger(\mathbf{x})] \\ &= U_L(\mathbf{x}) \star f[Y(\mathbf{x})] \star U_L^\dagger(\mathbf{x}). \end{aligned} \quad (3.22)$$

This property follows immediately from the equivalence of adjoint gauge transformations with APDs, Eq. (3.20), and it will figure heavily in Sec. IV.

### 3. Bosonic Jain sequence and gaps

We are now prepared to extract the Landau level degeneracy and physical magnetic field, as well as make contact with the dipole quantum mechanics problem in Sec. III A. We start by replacing operators that act with a Moyal product to the right with operators defined to act with left multiplication, such that all operators only act to the left on states [55]. We may then define new coordinate operators, which we (suggestively) name  $R^i$  and  $R_i^v$ , such that

$$R_i c = Y_i \star c, \quad R_i^v c = c \star x_i. \quad (3.23)$$

These operators furnish two different, mutually commuting guiding center algebras, which may be naturally expressed in terms of a new magnetic field,

$$\delta \hat{B} = \frac{\delta F_{xy}}{1 + \Theta \delta F_{xy}}, \quad (3.24)$$

such that

$$\begin{aligned} [R_i, R_j] &= i\Theta(1 + \Theta \delta F_{xy}) \varepsilon_{ij} = -\frac{i}{B + \delta \hat{B}} \varepsilon_{ij} = -i\ell_{B+\delta \hat{B}}^2 \varepsilon_{ij}, \\ [R_i^v, R_j^v] &= i\ell_B^2 \varepsilon_{ij}, \quad [R_i, R_j^v] = 0, \end{aligned} \quad (3.25)$$

where we recall  $\Theta = -\ell_B^2$ . This is the same guiding center algebra as in the dipole quantum mechanics model, Eq. (3.1), with  $\delta \hat{B}$  identified with the physical magnetic field, which we had earlier denoted  $\delta B$ .

The mean-field Hamiltonian, Eq. (3.14), can also be expressed in terms of  $R_i$  and  $R_i^v$ , since in noncommutative field theory commutators with  $x^i$  are derivatives,

$$[x^i, c(\mathbf{x})]_\star = i\Theta \varepsilon^{ij} \partial_j c(\mathbf{x}). \quad (3.26)$$

Therefore the covariant derivative can be written as

$$D^i c = \frac{i}{\Theta} \varepsilon^{ij} (R_j - R_j^v) c, \quad (3.27)$$

meaning that it is proportional to the composite fermion dipole moment, as anticipated in Sec. III A. In mean-field theory, the single-particle Hamiltonian of the composite fermions is therefore identical to the Hamiltonian in Eq. (3.4),

$$H = \frac{1}{2m_*} \frac{1}{\ell_B^4} (R_i - R_i^v)^2. \quad (3.28)$$

The mean-field composite fermion theory and the charge-vortex dipole problem are equivalent! We can therefore immediately apply the result [55] for the Landau level degeneracy in Sec. III A to find that it is indeed set by  $\delta \hat{B}$ ,

$$d_{LL} = \frac{\delta \hat{B}}{2\pi} \times \text{Area}. \quad (3.29)$$

See Refs. [48,56–58] and Appendix A for a more detailed account of how one arrives at this result for the case of the theory on a torus. We note that the field strength  $\delta \hat{B}$  is also the value of the field strength obtained from the Seiberg-Witten map, meaning that this result is consistent with the requirement of flux quantization in the commutative approximation of the theory [59,60].

To summarize, by using the proper gauge covariant coordinates, we have shown that the introduction of the noncommutative vector potential in the composite fermion

mean-field theory corresponds precisely to the deformation of the noncommutative parameter of the charges in the single-dipole model, with Eq. (3.24) as the physical magnetic field. With this result, we can define the composite fermion filling fraction,

$$\nu_{\text{CF}} = 2\pi \frac{\langle \rho_L \rangle}{\delta \hat{B}} = 2\pi \frac{\bar{\rho}}{\delta \hat{B}}. \quad (3.30)$$

The bosonic Jain states occur when the composite fermions form integer quantum Hall states, with  $\nu_{\text{CF}} = p$ ,  $p$  an integer.

Because  $\delta \hat{B}$  sets the degeneracy of the composite fermion Landau level, we interpret it as shift in the magnetic field felt by the underlying bosonic charges from  $B = 2\pi \bar{\rho}$ . Therefore one is naturally led to the conclusion that the filling fraction of the physical bosons is

$$\nu = 2\pi \frac{\bar{\rho}}{B + \delta \hat{B}} = \frac{p}{1 + p}. \quad (3.31)$$

This is precisely the expected form of the bosonic Jain sequence [21,22]! In terms of the density  $\bar{\rho}$  and composite fermion Landau level index  $p$ , one finds that the gap for each Jain sequence state is thus

$$\Delta_p = \frac{2\pi \bar{\rho}}{m_*} \frac{1}{1 + p} \approx \frac{2\pi \bar{\rho}}{m_*} \left( \frac{1}{p} - \frac{1}{p^2} \right) + \mathcal{O}(p^{-3}). \quad (3.32)$$

This expression, for which we have not invoked any long wavelength approximation, is one of the main results of this work. It matches the result from the Seiberg-Witten map, Eq. (2.30), but involves no long wavelength approximation and is thus valid even at small values of  $p$ .

Notably, for the  $\nu = 1/2$  Laughlin state, we find

$$\Delta_1 = \frac{\pi \bar{\rho}}{m_*} \approx (0.65) \pi U_0 \bar{\rho}. \quad (3.33)$$

This result uses the Hartree-Fock effective mass from Ref. [19] for the case of a local contact potential and relies on mean-field theory,  $\langle a \rangle = 0$ , but nevertheless incorporates the noncommutativity of the LLL exactly. It is comparable to the value obtained with exact diagonalization, which is  $[0.095(5)] 2\pi U_0 \bar{\rho} / \nu \approx (0.38) \pi U_0 \bar{\rho}$ . [21,22]. We note, however, that the effective mass can in principle depend on the magnetic field,  $m_*(\delta \hat{B})$ , which will lead to corrections to our result, particularly for states far away from the compressible state (small  $p$ ) and very close to the compressible state (large  $p$ ), where gauge fluctuations likely play an important role. In Sec. V, we will comment further on the physics leading to such a magnetic field dependent mass and how they could be handled theoretically to improve our result.

We now briefly comment on the states with negative composite fermion filling,  $p < 0$ , which correspond to  $\nu > 1$ . For  $p = -1$ , which in HLR theory corresponds to a superfluid,  $\delta F_{xy}$  diverges, and so too does the gap in Eq. (3.31). This indicates a singularity of the noncommutative field theory, which presupposes  $|\Theta|$  as the minimum uncertainty of the  $x$  and  $y$  coordinates [55,61]. In other words, all field strengths are cut off by the noncommutativity of space at scales of  $1/|\Theta|$ . The noncommutative field strength,  $|\delta F_{xy}|$ , continues to exceed  $1/|\Theta|$  until  $p = -2$ , which in HLR corresponds to the  $\nu = 2$  bosonic integer quantum Hall state [62], and so the

noncommutative composite fermion field theory is problematic for  $-2 < \nu_{\text{CF}} \leq 0$ .

Before concluding this section, we note that it would have been quite challenging to directly confirm that the magnetic field felt by the underlying bosonic charges is indeed  $B + \delta \hat{B}$  using the Pasquier-Haldane-Read formalism, as the composite fermion operators become rectangular matrices when the filling deviates from  $\nu = 1$ . However, the correspondence with the simple charge-vortex dipole model makes it clear that this is the only valid option. Moreover, we will demonstrate that the topological orders associated with the states at filling  $\nu_{\text{CF}} = p$  correspond precisely to that of the  $p^{\text{th}}$  bosonic Jain state. This is the topic we now turn to.

#### IV. THE BOSONIC JAIN SEQUENCE: UNIVERSAL FEATURES

##### A. Fluctuations and Hall response

Having concluded that doping the noncommutative composite fermion field theory, Eq. (2.18), leads to the bosonic Jain sequence of fractional quantum Hall states, we are now prepared to assess their universal properties, which for ordinary composite fermions are encoded in a Chern-Simons effective field theory for the gauge fluctuations. In the original work of Lopez and Fradkin [4] (based on the usual flux attachment transformation to composite fermions without restricting to the LLL), the Chern-Simons effective action was directly calculated by integrating out the composite fermions and expanding the resulting functional determinant. We follow the same logic for the composite fermions obtained in the present LLL construction. In this section, we present a physically transparent derivation, in which we consider linear response starting from the first quantized dipole Hamiltonian, Eq. (3.28), and match the result to an effective *noncommutative* Chern-Simons theory. In Appendix B, we present a more formal derivation by calculating the polarization tensor that determines the quadratic part of the effective action for the gauge fields when the composite fermions are integrated out. We perform our calculations without fixing to a particular gauge, so noncommutative gauge invariance is manifest throughout.

We begin by introducing fluctuations into the Hamiltonian in Eq. (3.28). This amounts to replacing the vortex guiding center coordinates  $R_i^y$  with their covariant counterparts

$$R_i^y c = c \star X_i[a], \quad (4.1)$$

where

$$X^i[a] = x^i - \Theta \varepsilon^{ij} a_j \quad (4.2)$$

are the covariant coordinates for the fluctuating gauge field. Under a right-acting gauge transformation, Eq. (2.26), they transform as

$$X^i \rightarrow U_R^\dagger \star X^i \star U_R, \quad (4.3)$$

and their commutator is

$$[X_i, X_j]_\star = i\Theta(\varepsilon_{ij} - \Theta f_{ij}), \quad (4.4)$$

although we will be primarily interested in situations where  $f_{ij} = \partial_i a_j - \partial_j a_i + i[a_i, a_j]_\star = 0$ . With this new definition



for  $R_i^v$ , the covariant derivative continues to take the form of Eq. (3.27). Therefore the single-particle Hamiltonian retains the form of Eq. (3.28), but now it is understood to transform in the adjoint representation under both left and right-acting gauge transformations.

We now study the response of the charges and vortices to the physical and emergent electric fields, using the first quantized description. If  $V_b[R]$  and  $V_v[R^v]$  are the scalar potentials felt by the physical charges and the vortices respectively, the Hamiltonian is

$$H = \frac{1}{2m_*} \frac{1}{\ell_B^4} (R_i - R_i^v)^2 + V_b[R_i] + V_v[R_i^v]. \quad (4.5)$$

Because both  $R_i$  and  $R_i^v$  are defined in terms of the covariant coordinates, the scalar potentials are each adjoints under gauge transformations, as in Eq. (3.22). In terms of the field theory representation, we may therefore consider them as background values of  $\delta A_0$  and  $a_0$ . We emphasize that the inclusion of scalar potentials for both the charges and vortices is essential, since we will see that in order to satisfy the constraints implemented by the gauge fluctuations, as in Eq. (2.20), the vortices and the charges will simultaneously exhibit a Hall effect. The same phenomenon occurs in ordinary flux attachment: within the FQH state, establishing an electric field for the physical charges leads to an average electric field for the emergent (statistical) gauge field.

The introduction of the scalar potentials leads to currents of boson charges and vortices. In the Heisenberg picture, the world-line of a boson (vortex) is  $R_i(t)$  ( $R_i^v(t)$ ), so we define the current densities as

$$j_i^b = \bar{\rho} \left\langle \frac{dR_i}{dt} \right\rangle, \quad j_i^v = \bar{\rho} \left\langle \frac{dR_i^v}{dt} \right\rangle. \quad (4.6)$$

In particular, we consider  $j_i^b$  (which is an adjoint under left-acting gauge transformations), to be the physical current density. We also define the physical and emergent electric fields as

$$E^i = -\frac{i}{\ell_{B+\delta\hat{B}}^2} \langle \varepsilon^{ij} [R_j, V_b[R]] \rangle \equiv -\langle \partial_{R_i} V_b[R] \rangle, \quad (4.7)$$

$$e^i = \frac{i}{\ell_B^2} \langle \varepsilon^{ij} [R_j^v, V_v[R^v]] \rangle \equiv -\langle \partial_{R_i^v} V_v[R^v] \rangle, \quad (4.8)$$

For constant magnetic fields (as in the case of interest), these are simply the gradients of the scalar potentials. One can confirm the physically intuitive conclusion that  $E$  is the physical electric field by noticing that, if we write  $V_b[Y] = \delta A_0(\mathbf{x})$ , where  $Y(\mathbf{x})$  is the covariant coordinate and  $\delta A_0(\mathbf{x})$  is a static scalar potential, then we can use the star commutator with  $Y$  to relate

$$-\ell_{B+\delta\hat{B}}^2 \partial_{Y_i} V_b[Y] = \ell_B^2 \delta F_{0i}, \quad (4.9)$$

since  $\delta A_i$  is time-independent. For a uniform electric field, this leads to

$$\delta F_{0i} = \frac{1}{1 + \delta\hat{B}/B} E_i. \quad (4.10)$$

One can check using the formulas in, e.g., Ref. [19] that this is the relation between the noncommutative electric field,  $\delta F_{0i}$ , and the ordinary Abelian electric field of the Seiberg-Witten map. Since we have already commented on how the

Seiberg-Witten gauge field satisfies proper flux quantization, this means that  $E$  can be considered the physical electric field.

We compute the DC response to uniform electric fields by solving the Heisenberg equations of motion. To make the equations more compact, we reintroduce the notation,  $p_i^{\text{CF}} = \frac{1}{\ell_B^2} \varepsilon^{ij} (R_j^v - R_j)$ , and we drop terms which vanish for  $\langle f_{ij} \rangle = 0$ . Then we obtain

$$\begin{aligned} \frac{dp_i^{\text{CF}}}{dt} &= \frac{1}{i} [p_i^{\text{CF}}, H] \\ &= \delta F_{xy} \varepsilon_{ij} \frac{p_j^{\text{CF}}}{m_*} - \frac{1}{\ell_B^2} (\ell_{B+\delta\hat{B}}^2 \partial_{R_i} V_b[R] + \ell_B^2 \partial_{R_i^v} V_v[R^v]), \end{aligned} \quad (4.11)$$

$$\frac{dR_i}{dt} = \frac{1}{i} [R_i, H] = \frac{\ell_{B+\delta\hat{B}}^2 p_i^{\text{CF}}}{\ell_B^2 m_*} - \ell_{B+\delta\hat{B}}^2 \varepsilon_{ij} \partial_{R_j} V_b[R], \quad (4.12)$$

$$\frac{dR_i^v}{dt} = \frac{1}{i} [R_i^v, H] = \frac{p_i^{\text{CF}}}{m_*} + \ell_B^2 \varepsilon_{ij} \partial_{R_j^v} V_v[R^v]. \quad (4.13)$$

The first equation determines the composite fermion drift velocity,  $p^{\text{CF}}/m_*$ , while the latter two equations determine the individual charge and vortex responses.

In addition to the equations of motion, the theory also has the constraint in Eq. (2.20), along with the equation of motion for  $a_i$ , which in the field theory sets  $\langle J_R \rangle = \frac{i}{2m} \langle c^\dagger \star Dc - (Dc)^\dagger \star c \rangle = 0$ . Physically, we can understand these constraints as the requirement that the vortices are fixed to have filling  $\nu = -1$ . In the Pasquier-Haldane-Read language, this is the requirement that the number of vortex orbitals is fixed to the number of physical bosons, even on tuning the physical filling away from  $\nu = 1$ . The constraint can therefore be recast as the requirement that the vortices have unit Hall conductivity,

$$j_v^i = -\frac{1}{2\pi} \varepsilon^{ij} e_j. \quad (4.14)$$

Plugging this back into Eq. (4.13), we see that this is equivalent to the statement

$$\langle p^{\text{CF}} \rangle = 0. \quad (4.15)$$

Because  $p^{\text{CF}}$  is proportional to the (gauge covariant) composite fermion dipole moment, the boson and the vortex coordinates sit on top of each other. With this constraint, Eq. (4.11) leads to a relation between the physical and emergent electric fields:

$$E_i = -\frac{\ell_B^2}{\ell_{B+\delta\hat{B}}^2} e_i = -\frac{p+1}{p} e_i. \quad (4.16)$$

Looking to Eq. (4.12), we immediately obtain the physical Hall conductivity

$$j_b^i = \frac{1}{2\pi} \bar{\rho} \ell_{B+\delta\hat{B}}^2 \varepsilon^{ij} E_j = \frac{1}{2\pi} \frac{p}{p+1} \varepsilon^{ij} E_j. \quad (4.17)$$

Introducing units, the Hall conductivity is

$$\sigma_{xy} = \nu \frac{e^2}{h} = \frac{p}{p+1} \frac{e^2}{h}. \quad (4.18)$$

Hence the noncommutative composite fermion theory indeed leads to the correct Hall conductivity for the bosonic Jain sequence states!

### B. Noncommutative Chern-Simons theory

Equipped with the result for the Hall conductivity in Eq. (4.18) and the relation between the electric fields in Eq. (4.16), we can construct an effective Chern-Simons action at long wavelengths that reproduces them as the equations of motion,

$$S_{\text{eff}} = -i \int d^2\mathbf{x} d\tau \left[ \frac{p+1}{4\pi} a da + \frac{p}{2\pi} a dA' + \frac{p}{4\pi} A' dA' + \mathcal{O}(\Theta) \right], \quad (4.19)$$

where we use the notation  $AdB = \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu B_\lambda$ . To connect with the discussion above,  $a_\mu$  is simply the fluctuating gauge field, while  $A' = (A'_0(\mathbf{x}), 0)$  is a probe field on top of the background field  $\delta A$  that gives rise to the (physical) electric field,  $E_i = -\partial_i A'_0$ . The equation of motion for  $a_i$  reproduces the relation in Eq. (4.16), and integrating out  $a$  altogether returns the Hall response in Eq. (4.18). For an explicit derivation of the Chern-Simons effective action for  $a_\mu$  from integrating out  $p$  noncommutative composite fermion Landau levels, see Appendix B.

Importantly, at no point have we actually invoked the Seiberg-Witten map to an Abelian gauge theory. Indeed, the electric fields used in the analysis of Sec. IV A transform as adjoints under the *noncommutative* gauge symmetries, Eqs. (2.22) and (2.26). Therefore the true effective action should display full noncommutative gauge invariance. Attempting to construct such an action leads to significant complications: because the left and right-acting gauge transformations are non-Abelian, there appears to be no gauge invariant mutual Chern-Simons term that may be represented in terms of star products of local operators. We will comment more on the pursuit of a noncommutative mutual Chern-Simons term in Sec. IV C.

For the purposes of diagnosing the topological order,<sup>6</sup> we may simply set  $A' = 0$  and construct a gauge invariant action for the fluctuating gauge field,  $a$ . The only such action that can be represented in terms of star products of local operators is the noncommutative Chern-Simons (NCCS) theory [23,24],

$$S_{\text{NCCS}} = - \int d^2\mathbf{x} d\tau \frac{i(p+1)}{4\pi} \varepsilon^{\mu\nu\lambda} \left[ a_\mu \star \partial_\nu a_\lambda + \frac{2i}{3} a_\mu \star a_\nu \star a_\lambda \right]. \quad (4.20)$$

As in non-Abelian gauge theories, gauge invariance dictates that this theory has a cubic interaction term even though the gauge group is  $U(1)$ . In fact, we can motivate the presence of

<sup>6</sup>Statements about topological order here have the caveat that  $a$  couples to a fermion field in the fundamental representation. For ordinary commutative gauge theories, this can be formally captured by viewing  $a$  as a  $\text{spin}_c$  connection rather than as an ordinary  $U(1)$  gauge field. For an explanation of this concept in a condensed matter context, see Refs. [13,63–65]. For the topological order, this means that the quasiparticle statistics are shifted by  $\pi$  compared to the usual formulas (equivalently, there is understood to be an additional spin-1/2 Wilson line).

the cubic term using the relation in Eq. (4.16). If we turn off  $E$ , this equation becomes (dropping the brackets)

$$0 = -\partial_{R_i} V_v[R^v] = \frac{i}{\ell_B^2} \varepsilon^{ij} [R_j^v, V_v[R^v]]. \quad (4.21)$$

Now if we identify  $V_v[X]$  with a fluctuation of  $a_0(\mathbf{x})$  and take  $a_i(\mathbf{x})$  to be static, the commutator with the covariant coordinate,  $X_i$ , gives the field strength,

$$0 = -\partial_i a_0 + i[a_0, a_i]_\star = f_{0i}. \quad (4.22)$$

This matches the equation of motion of the NCCS theory, where the commutator originates from differentiating the cubic term.

The NCCS theory, particularly in its representation of a matrix model [23,24], has been extensively discussed as a short-wavelength description of fractional quantum Hall phases. However, the connection of these models to realistic microscopic Hamiltonians has been obscure. What is unique here is that we have obtained this theory as a *long-wavelength* effective field theory of the bosonic Jain states that incorporates the noncommutativity of the lowest Landau level. Our result thus explains the connection between the noncommutative Chern-Simons theory and realistic microscopic models of quantum Hall phases, which until now was poorly understood. Furthermore, we have found that the noncommutativity of the Chern-Simons theory is set by the charge density,  $\bar{\rho}$  (since  $\Theta = -1/2\pi\bar{\rho}$ ), rather than the total magnetic field,  $B + \delta\hat{B}$ , in agreement with Susskind's original proposal [23].

The topological ground-state properties, such as anyons and their braiding, of NCCS theory are the same as the familiar Abelian Chern-Simons theory on commutative space. Indeed, it has been shown that at the classical level the NCCS action is equivalent to ordinary Chern-Simons action<sup>7</sup> under the Seiberg-Witten map [61,67] (a similar result was derived for the corresponding Wess-Zumino-Witten models [68]), and perturbative calculations have suggested that this correspondence extends to the quantum level as well [69]. Furthermore, the quantum Hall matrix models can be seen to reflect the correct topological order: for example, for the Laughlin states, Polychronakos demonstrated the existence of quasihole states with the correct fractional charge [24].

We note also that another argument for the emergence of NCCS theory was made in Ref. [70], which obtained a cubic interaction with Moyal phase factors using ordinary composite fermions (as in Ref. [4]) and proposed an emergent noncommutative gauge symmetry. Our conclusion contrasts with the result in Ref. [70], since the noncommutative gauge symmetry we consider is incorporated *a priori* into the parent microscopic theory. Indeed, we do not believe there is any reason for noncommutative gauge symmetry to emerge unless it is inherited from a short distance, LLL theory.

There is also an interesting parallel between the first quantized composite fermion Hamiltonian we considered in

<sup>7</sup>The noncommutative theory actually contains gauge transformations that are singular in the corresponding commutative, Seiberg-Witten mapped theory, which lead to quantization of the level even on the plane [46,61,66].

Eq. (4.5), which is stated in terms of covariant world-line coordinates, and the quantum Hall matrix models. Indeed, covariant worldline coordinates are also the basic variables in the matrix model description of noncommutative Chern-Simons theory, in which the definition of the covariant coordinates as noncommuting coordinates plus gauge fields is implemented dynamically [23,24]. However, unlike the usual analysis of the matrix models, we did not introduce a regulator at long distances in order to convert the covariant coordinate operators to finite-dimensional matrices. It would be interesting in the future to explore what can be learned from applying such an approach to our composite fermion Hamiltonian.

Before moving on, we pause to make a technical comment regarding the specification of the topological order described by Eq. (4.20). In discussing ordinary non-Abelian Chern-Simons theory, one must be careful to specify a regularization. Due to the cubic interaction, the choice of regularization at short distances can lead to a one-loop exact shift in the non-Abelian Chern-Simons level [71–73], which is matched by an analogous quantum shift in the corresponding Wess-Zumino-Witten model, where it appears in the computation of the central charge. The NCCS theory is no different. Perturbative calculations using a Maxwell regulator have found a level shift of  $k \rightarrow k + \text{sgn}(k)$  in U(1) NCCS theory with level  $k$  [74], although this does not affect the topological order [75]. The same shift<sup>8</sup> arises in the matrix models, where it comes from normal ordering a constraint [24], and in this case it does affect the topological order (and shift the filling fraction accordingly). For our purposes, the NCCS level in Eq. (4.20) is meant to reflect the full quantum Chern-Simons level, i.e. we implicitly choose a regulator in which no such shift appears.

**C. Toward a noncommutative mutual Chern-Simons theory**

We now revisit the question of how to construct an effective Chern-Simons action for both the background ( $\delta A$ ) and fluctuating ( $a$ ) gauge fields. As we commented above, because the left and right-handed noncommutative U(1) gauge symmetries are not Abelian, there is no *local* gauge invariant mutual Chern-Simons term. However, it should be possible to construct a nonlocal mutual Chern-Simons term, one which leverages the inherent nonlocality of field theories on noncommutative space. Unfortunately, despite much effort, particularly in Refs. [76,77], this term has proven elusive. While we will not completely solve this problem here, we will propose an action [which is not necessarily the generalization of Eq. (4.19)] that is at least gauge invariant to  $\mathcal{O}(\Theta)$ . We expect that the intuition underlying this construction may prove useful in the pursuit of a full solution to this problem.

The basic problem with constructing a mutual Chern-Simons theory on noncommutative space is the same as in ordinary non-Abelian gauge theory: gauge invariance would

necessitate that both participating gauge fields transform simultaneously, which does not appear possible by definition. However, with covariant coordinates, it is possible to *induce* left-acting gauge transformations on the right-handed gauge field and *vice versa*. For example, right-acting gauge transformations act as APDs on  $\delta A[X]$ ,

$$\delta A[X] \rightarrow \delta A[U(\mathbf{x}) \star X \star U^\dagger(\mathbf{x})] = U(\mathbf{x}) \star \delta A[X] \star U^\dagger(\mathbf{x}), \tag{4.23}$$

and analogously for  $a[Y]$ .

Leveraging this property, we can define a noncommutative Chern-Simons action that is gauge invariant to  $\mathcal{O}(\Theta)$  under both left- and right-acting gauge transformations by using the combination  $a[Y] + \delta A[X]$ ,

$$S_{\text{NCCS}}[a[Y] + \delta A[X]], \tag{4.24}$$

where  $S_{\text{NCCS}}$  is defined in Eq. (4.20). Notice that such an action is fully gauge invariant under right (left) gauge transformations if  $Y$  ( $X$ ) is replaced with the coordinate  $\mathbf{x}$ , at the cost of breaking invariance under the other gauge group.

The reason the action in Eq. (4.24) is only invariant to  $\mathcal{O}(\Theta)$  stems from the fact that it is nonlocal. Full gauge invariance under, e.g., right-acting gauge transformations would require

$$\begin{aligned} \delta A[X] &\rightarrow U^\dagger[Y] \star \delta A[X] \star U[Y] \\ &= U^\dagger(\mathbf{x}) \star \delta A[X] \star U(\mathbf{x}) + \mathcal{O}(\Theta^2), \end{aligned} \tag{4.25}$$

where the  $\mathcal{O}(\Theta^2)$  term is nonvanishing. Resolving this issue would require introducing new Wilson linelike operators  $V, W$  which transform under both left and right gauge transformations as follows:

$$\begin{aligned} U(1)_L : V &\rightarrow U^\dagger[X] \star V \star U(\mathbf{x}), \\ W &\rightarrow U[X] \star W \star U^\dagger[X], \end{aligned} \tag{4.26}$$

$$\begin{aligned} U(1)_R : V &\rightarrow U[Y] \star W \star U^\dagger[Y], \\ W &\rightarrow U^\dagger[Y] \star W \star U(\mathbf{x}). \end{aligned} \tag{4.27}$$

The gauge invariant Chern-Simons action would then be

$$S_{\text{NCCS}}[V^\dagger \star a[Y] \star V + W \star \delta A[X] \star W^\dagger]. \tag{4.28}$$

Unfortunately, we have not been able to construct explicit expressions for the operators,  $V$  and  $W$ , and we leave this for future work. It is also not clear to us how to construct more general  $K$  matrices than  $(p, p, p)$  using this approach. We finally note that a discussion of noncommutative Chern-Simons theories with such  $K$  matrices can be found in Ref. [76]. However, the authors of that work fiat the mixed transformation laws for each gauge field, instead of attempting to induce them using covariant coordinates. The theories discussed in Ref. [76] therefore cannot be obtained using the composite fermion approach outlined in our work, in which left and right gauge transformations do not mix.

**V. DISCUSSION**

A major challenge in quantum Hall physics has been to develop a microscopic theory that is defined in the lowest Landau level and is capable of capturing both universal and nonuniversal physics. In this work, we have met this challenge for the specific case of bosonic Jain sequences at fillings

<sup>8</sup>We note that Ref. [66] found the one-loop shift due to the Maxwell regulator to be  $k \rightarrow k + 2 \text{sgn}(k)$ , based on some differences in normalization with Ref. [74]. This difference can be properly settled by computing the free energy in the large- $k$  limit using the background field formalism, as in Ref. [72]. We leave this for future work.

$\nu = \frac{p}{p+1}$  using a composite fermion construction [17–20] that, unlike the standard flux attachment, explicitly lives in the lowest Landau level. Previous work [18,19] employed this construction to discuss the metallic composite Fermi liquid state for bosons at  $\nu = 1$ . An effective field theory description for this state consists of composite fermions coupled to a U(1) gauge field on the noncommutative space of the LLL. Starting from this description, we doped the theory away from  $\nu = 1$  to access the Jain sequence states. This is achieved by subjecting the composite fermions to a background, noncommutative magnetic field while holding their density fixed. Integrating out the composite fermions, we obtained a noncommutative Chern-Simons field theory, which encodes the topological features of the ground state. This conclusion significantly clarifies long-standing questions about the role of noncommutative Chern-Simons theory in the study of the fractional quantum Hall effect at short distances. It is a low-energy effective theory arising from integrating out composite fermions in the LLL. It captures the correct topological order of the ground state but does not contain any dynamical information on its own. Its noncommutativity is set by the charge density, as in the original proposal of Susskind [23].

Our microscopic approach incorporates both the universal and nonuniversal data of quantum Hall states within a single theoretical framework. As an important demonstration, we presented an elegant, closed-form expression for the Jain sequence gaps, invoking only a mean-field approximation. It should also be possible to extract other dynamical features as well, such as the dispersion of the GMP mode or the momentum dependence of the static structure factor. In approaching such calculations, it is important to note that in noncommutative field theory the density and current operators are not gauge invariant at finite momentum, meaning that some care will be necessary to ensure that results are gauge invariant. Another related problem that will be important to attack in the future is the structure of the mutual Chern-Simons term in noncommutative field theory, which we expect to lack a representation in terms of star products of local operators. Another problem that could be tackled within our description is to study the evolution between the Jain states and a bosonic superfluid state by turning on a periodic potential in the LLL (along the lines of what was done at filling  $\nu = 1$  in Ref. [20]).

The success of our approach for the bosonic Jain sequence invites the question of how to extend our framework to the fermionic Jain sequences. This would require a fully LLL theory of the composite Fermi liquid states at even denominator fillings in fermionic systems. Constructing such a theory is of great importance. For example, it would shed light on the emergence of a particle-hole symmetric composite Fermi liquid theory at  $\nu = 1/2$ , like the Dirac theory proposed by Son [9] (an analogous “reflection symmetry” was proposed by one of us for the states at  $\nu = 1/2n$  [10], but the status of that symmetry on LLL projection in clean systems is an open question; for alternate proposals in the LLL limit, see Ref. [40]). A noncommutative, LLL field theory of the  $\nu = 1/2$  state was proposed recently [78], but further work is needed in this direction.

We now comment on how our approximate result for the bosonic Jain sequence gaps in Eq. (3.32) can be improved. In

obtaining Eq. (3.32), we took the composite fermion effective mass,  $m_*$ , to be given by the result of the Hartree-Fock calculation at  $\nu = 1$ . A better approximation would be to calculate the effective mass directly at the filling of the Jain state, which would reveal if the effective mass has a field dependence,  $m_* = m_*(\delta B)$ . This could alter the dependence of the gaps  $\Delta_p$  on the Jain state index  $p$  from the form in Eq. (3.32). Such a calculation of  $m_*(\delta B)$  can conceivably be performed within the Hamiltonian theory of Murthy and Shankar [16]. Indeed, we may regard the Hartree-Fock calculation within the Hamiltonian theory as providing an improved mean-field ansatz on top of which fluctuation effects can be included using the noncommutative field theory.

Gauge fluctuations can also lead to field dependence of the effective mass. In the composite Fermi liquid itself, these fluctuations lead to a diverging effective mass. As emphasized by HLR in Ref. [6], on moving to proximate Jain fractions, this divergence will be cut off at an energy scale given by the Jain gap  $\Delta_p$ . This leads to a  $\Delta_p$  that behaves (for small  $\delta B$  and short-ranged interactions) as  $\Delta_p \sim |\delta B|^{\frac{3}{2}} \sim |p|^{-3/2}$ . However, this asymptotic form is likely to only be relevant for very small  $|\delta B|$ , i.e., very large  $p$ . In the present problem of the bosonic Jain states, describing the region of large  $p$  using the composite Fermi liquid will be additionally problematic, as the true ground state at  $\nu = 1$  is the paired Pfaffian state. The large  $p$  region will then involve competition between pairing and Landau level formation of composite fermions, and this will determine the details of the gap sizes and other nonuniversal characteristics. Therefore, for  $p$  that is not too large, we expect that the mean-field description used in this work will be adequate.

Finally, a rich subject that we have not yet touched on is the response of quantum Hall systems to spatial curvature, which straddles universal and dynamical data [38,79,80]. The Hall viscosity, or the parity-odd response to shears, is not universal *a priori*, but is determined by the Wen-Zee shift—a universal quantity—in Galilean invariant quantum Hall states [79,81,82]. Indeed, the matrix model regularization of noncommutative Chern-Simons theory has been shown to yield the correct Hall viscosity (up to an orbital contribution) for the Laughlin states and some non-Abelian quantum Hall states [31,32]. These arguments indicate that the noncommutative Chern-Simons theories we obtain at low energies will encode the Hall viscosity for the bosonic Jain states. This can also be checked by adapting the response calculations in Sec. IV A to finite wave vector, as the Hall viscosity appears in the coefficient of the  $\mathcal{O}(q^2)$  contribution to the Hall conductivity [83,84].

A major open problem in this area is to obtain a microscopic derivation of the coupling of composite fermions to geometry, the so-called “orbital spin” of the composite fermion. A framework like ours that can bridge the gap between short and long-wavelength physics is an ideal platform on which to solve this problem. However, an obstacle to such a construction is inherent to noncommutative gauge theories: because of the relationship between gauge symmetry and area-preserving diffeomorphisms, it is generally not possible to construct a gauge invariant (or covariant) stress tensor that satisfies a local continuity equation [85] (a noncommutative equivalent to the Belinfante procedure is not known to us), and

the construction of a Wen-Zee term suffers from similar challenges to the ordinary mutual Chern-Simons term. Resolving these questions and developing a noncommutative composite fermion field theory including a coupling to curvature will be an important direction for future studies.

*Note added.* Recently, we were made aware of an alternate perspective on noncommutative Chern-Simons field theory by Du, Mehta, and Son in Ref. [86].

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### APPENDIX A: LANDAU LEVEL DEGENERACY ON THE nonCOMMUTATIVE TORUS

In this Appendix, we sketch the derivation of the degeneracy of Landau levels on the noncommutative torus, which proceeds in analogy to the derivation on the ordinary torus. For more details, see Refs. [48,56–58], as well as the extensive literature on field theory on the noncommutative torus and  $T$  duality, which is reviewed in Ref. [33].

Without loss of generality, we choose to work on the square torus and identify,

$$x_i \sim x_i + 2\pi R, \quad (\text{A1})$$

where  $R$  is the compactification radius. We define the physical position operators by exponentiation,

$$V_i = e^{ix_i/R}, \quad (\text{A2})$$

which are invariant under shifts of  $2\pi R$ . They satisfy the algebra

$$V_1 V_2 = V_2 V_1 e^{-i\Theta/R^2}. \quad (\text{A3})$$

We work with the single-particle Hamiltonian in Eq. (3.13), but we now choose to work in Landau gauge,

$$(\delta A_x, \delta A_y) = (0, \delta F_{xy} x), \quad (\text{A4})$$

The covariant derivatives,  $D_i = \partial_i - i\delta A_i$ , satisfy

$$[D_i, D_j] = -i\delta F_{xy}. \quad (\text{A5})$$

Because the covariant derivatives involve  $x$  rather than  $U_2 = e^{ix/R}$ , they transform under shifts of  $x$  as

$$D_y \rightarrow D_y - i\delta F_{xy}(2\pi R). \quad (\text{A6})$$

As in the case of the ordinary torus, this shift can be eliminated by a suitable gauge transformation, but now such gauge transformations are *noncommutative*. The noncommutativity of the gauge group will ultimately be what alters the degeneracy from the noncommutative gauge flux.

The covariant derivative transforms as an adjoint under noncommutative (left-acting) gauge transformations. In Landau gauge, it changes as

$$D_y \rightarrow U \star D_y \star U^\dagger = \partial_y + U \star \partial_y U^\dagger - i\delta F_{xy} U \star x \star U^\dagger. \quad (\text{A7})$$

To determine the gauge transformation that cancels the shift in Eq. (A6), let  $U = e^{-i\alpha y/R}$ . Then

$$D_y \rightarrow \partial_y - i\delta F_{xy} x + i(1 + \Theta\delta F_{xy}) \frac{\alpha}{R}. \quad (\text{A8})$$

To cancel the shift in Eq. (A6), we therefore must have

$$\alpha = \frac{1}{2\pi} \frac{\delta F_{xy}}{1 + \Theta\delta F_{xy}} \times (2\pi R)^2 = \frac{\delta \hat{B}}{2\pi} \times (\text{Area}). \quad (\text{A9})$$

$\alpha$  is therefore the flux of the ‘‘physical’’ magnetic field defined in Eq. (3.24) through the torus! Requiring that the gauge transformation  $U$  itself be periodic on the torus therefore yields the flux quantization condition,

$$\oint_{T^2} \frac{\delta \hat{B}}{2\pi} = \frac{1}{2\pi} \frac{\delta F_{xy}}{1 + \Theta\delta F_{xy}} \times (2\pi R)^2 = n \in \mathbb{Z}. \quad (\text{A10})$$

Now, under the transformation  $U$ , the wave function on noncommutative space,  $\Psi[x, y]$ , transforms as  $\Psi \rightarrow U \star \Psi$ , so we seek a complete set of wave functions with the following properties:

$$\Psi[x, y + 2\pi R] = \Psi[x, y], \quad (\text{A11})$$

$$\Psi[x + 2\pi R, y] = U^\dagger \star \Psi[x, y]. \quad (\text{A12})$$

The space of such wave functions constitutes the space of degenerate ground states on the torus. In ordinary, commutative space, the ground states correspond to the set of theta functions, and the dimension of the space of ground states (the LL degeneracy) is given by the flux piercing the torus. In this case as well, the space of ground states (the fundamental sections of the noncommutative gauge theory on the torus) has dimension set by the number of flux quanta,  $|n|$  [56–58]. This completes the argument that the Landau level degeneracy is set by the magnetic field in Eq. (3.24).

### APPENDIX B: CHERN-SIMONS EFFECTIVE ACTION

In this Appendix, we show explicitly that integrating out  $p$  noncommutative composite fermion Landau levels leads to an effective Chern-Simons action with level  $p + 1$  for the fluctuating gauge field,  $a_\mu$ , as in Eq. (4.19). We do this by computing the polarization tensor,  $\pi_{xy}(\omega)$ , in the uniform ( $\mathbf{q} \rightarrow 0$ ) limit. The other components of the polarization tensor are then fixed by gauge invariance. Additionally, while we do not compute the cubic term in the noncommutative Chern-Simons action here (it is only nonzero at finite wave vector), it is also required to appear by gauge invariance.

For spatially uniform fluctuations,  $a_i(\tau)$ , the coupling to the composite fermions is

$$\Gamma_{ac^\dagger c} = \int d^2 \mathbf{x} d\tau \frac{i}{2m_*} a_i(\tau) [c^\dagger \star D_i^{(0)} c - (D_i^{(0)} c)^\dagger \star c],$$

$$D_i^{(0)} c = \partial_i c - i\delta A_i \star c, \quad (\text{B1})$$

plus a diamagnetic term, which will not play a role here. We may rewrite this coupling in terms of the momentum operator,  $p_i^{\text{CF}} = -iD_i^{(0)}$ , which is the same as the operator we introduced in Sec. III A. Then we may write

$$\Gamma_{ac^\dagger c} = - \int d^2\mathbf{x}d\tau a_i(\tau) c^\dagger \frac{p_i^{\text{CF}}}{m_*} c. \quad (\text{B2})$$

We work with a complete basis of noncommutative Landau level eigenstates, which we denote  $\{|M, \alpha\rangle\}$ , where  $M$  is a Landau level index, and  $\alpha$  parameterizes the Landau level degeneracy. The fermion propagator may then be written as

$$G_M(\omega) = \sum_{\alpha} \frac{|M, \alpha\rangle \langle M, \alpha|}{i\omega - E_M}, \quad (\text{B3})$$

where  $E_M = \epsilon_M - \mu$ ,  $\mu$  is a chemical potential that fixes the density to  $\bar{\rho}$ , and  $\epsilon_M$  are the Landau level energies, Eq. (3.16). Using this form for the propagator, along with the vertex in Eq. (B2), the polarization tensor may be expressed as

$$\begin{aligned} \pi_{xy}(\omega) &= \frac{1}{\text{Area}} \int \frac{d\Omega}{2\pi} \sum_{M,N} \sum_{\alpha,\beta} \frac{\langle M, \alpha | \frac{p_x^{\text{CF}}}{m_*} | N, \beta \rangle \langle N, \beta | \frac{p_y^{\text{CF}}}{m_*} | M, \alpha \rangle}{[i(\omega + \Omega) - E_M][i\Omega - E_N]} \\ & \quad (\text{B4}) \end{aligned}$$

$$\begin{aligned} &= \frac{i\omega}{\text{Area}} \sum_{M \neq N} \sum_{\alpha,\beta} \frac{\langle M, \alpha | \frac{p_x^{\text{CF}}}{m_*} | N, \beta \rangle \langle N, \beta | \frac{p_x^{\text{CF}}}{m_*} | M, \alpha \rangle}{(E_M - E_N)^2} \\ & \quad \times (\Theta(-E_M) - \Theta(-E_N)) + \mathcal{O}(\omega^2). \quad (\text{B5}) \end{aligned}$$

We now use the fact

$$\frac{p_i^{\text{CF}}}{m_*} = \frac{1}{i} [R_i^v, H], \quad (\text{B6})$$

where here  $R_i^v$  is defined (in mean-field theory) to act as  $R_i^v c = c \star x_i$ . This allows us to rewrite

$$\begin{aligned} \pi_{xy}(\omega) &= \frac{i\omega}{\text{Area}} \sum_{M \neq N} \sum_{\alpha,\beta} \langle M, \alpha | R_x^v | N, \beta \rangle \langle N, \beta | R_y^v | M, \alpha \rangle \\ & \quad \times (\Theta(-E_M) - \Theta(-E_N)). \quad (\text{B7}) \end{aligned}$$

We now define the Landau level projection operators,

$$\mathcal{P}_M = \sum_{\alpha} |M, \alpha\rangle \langle M, \alpha|, \quad (\text{B8})$$

such that

$$\begin{aligned} \pi_{xy}(\omega) &= \frac{i\omega}{\text{Area}} \sum_M \text{Tr}_{\alpha} [\mathcal{P}_M R_x^v (1 - \mathcal{P}_M) R_y^v \\ & \quad - \mathcal{P}_M R_y^v (1 - \mathcal{P}_M) R_x^v] \Theta(-E_M), \quad (\text{B9}) \end{aligned}$$

$$\begin{aligned} &= \frac{i\omega}{\text{Area}} \sum_M \text{Tr}_{\alpha} [\mathcal{P}_M [R_x^v, R_y^v] + \mathcal{P}_M R_y^v \mathcal{P}_M R_x^v \\ & \quad - \mathcal{P}_M R_x^v \mathcal{P}_M R_y^v]. \quad (\text{B10}) \end{aligned}$$

Here  $\text{Tr}_{\alpha}$  is the trace over the degenerate indices. To compute the projected vortex coordinates  $\mathcal{P}_M R_i^v \mathcal{P}_M$ , we recall  $p_i^{\text{CF}} = \frac{1}{\ell_B^2} \varepsilon_{ij} (R_j^v - R_j)$ , and we introduce operators

$$r_i^{\text{CF}} = \frac{1}{2} (R_i^v + R_i), \quad (\text{B11})$$

$$\tilde{p}_i^{\text{CF}} = \frac{2\ell_B^2}{\ell_{B+\delta\hat{B}}^2 + \ell_B^2} p_i^{\text{CF}}, \quad (\text{B12})$$

$$\mathcal{D}_i = \frac{1}{1 - \theta \delta B_{\text{CF}}} (\tilde{p}_i^{\text{CF}} - \delta B_{\text{CF}} \varepsilon_{ij} r_j^{\text{CF}}), \quad (\text{B13})$$

$$\theta = \frac{1}{4} (\ell_B^2 - \ell_{B+\delta\hat{B}}^2), \quad \delta B_{\text{CF}} = 4 \frac{\ell_B^2 - \ell_{B+\delta\hat{B}}^2}{(\ell_{B+\delta\hat{B}}^2 + \ell_B^2)^2}, \quad (\text{B14})$$

which satisfy the same algebra as the corresponding operators in Sec. III A. Importantly,  $\mathcal{D}_i$  commutes with  $p_i^{\text{CF}}$  and therefore the Hamiltonian, meaning that it survives Landau level projection. As discussed in Sec. III A, it also satisfies the algebra

$$[\mathcal{D}_x, \mathcal{D}_y] = -i\delta\hat{B}. \quad (\text{B15})$$

In terms of these operators, we can express  $R_i^v$  as

$$R_i^v = \varepsilon_{ij} \left[ \frac{1 - \theta \delta B_{\text{CF}}}{\delta B_{\text{CF}}} \mathcal{D}_j - \frac{1}{\delta B_{\text{CF}}} \left( 1 + \frac{\ell_B^2 - \ell_{B+\delta\hat{B}}^2}{\ell_B^2 + \ell_{B+\delta\hat{B}}^2} \right) \tilde{p}_j^{\text{CF}} \right]. \quad (\text{B16})$$

Because the creation and annihilation operators are built out of  $p^{\text{CF}}$ , the second term vanishes on projection, i.e.,

$$\mathcal{P}_M R_i^v \mathcal{P}_M = \frac{1 - \theta \delta B_{\text{CF}}}{\delta B_{\text{CF}}} \varepsilon_{ij} \mathcal{D}_j = \frac{1}{\delta\hat{B}} \varepsilon_{ij} \mathcal{D}_j. \quad (\text{B17})$$

Thus, since  $\mathcal{D}$  commutes with  $p^{\text{CF}}$  and the degeneracy of each Landau level is  $\frac{\delta\hat{B} \times \text{Area}}{2\pi}$ , if  $p$  Landau levels are filled,

$$\begin{aligned} \pi_{xy}(\omega) &= \frac{i\omega}{\text{Area}} \sum_M \text{Tr}_{\alpha} \left[ [R_x^v, R_y^v] - \frac{1}{\delta\hat{B}^2} [\mathcal{D}_x, \mathcal{D}_y] \right] \Theta(-E_M) \\ & \quad (\text{B18}) \end{aligned}$$

$$= -\frac{\omega}{\text{Area}} \times p \times \frac{\delta\hat{B} \times \text{Area}}{2\pi} \times \left( \ell_B^2 + \frac{1}{\delta\hat{B}} \right). \quad (\text{B19})$$

However, due to the constraint, the composite fermion density is fixed to  $\bar{\rho} = \frac{B}{2\pi}$ , so  $\delta\hat{B} \ell_B^2 = 1/p$ , and this result becomes

$$\pi_{xy}(\omega) = -\frac{\omega}{2\pi} (p + 1). \quad (\text{B20})$$

The resulting Chern-Simons effective action is therefore

$$S_{\text{eff}} = - \int d^2\mathbf{x}d\tau \frac{i(p+1)}{4\pi} a da + \mathcal{O}(\Theta), \quad (\text{B21})$$

reflecting the correct topological order for the  $\nu = p/(p+1)$  Jain state. Notably, the constraint has played an essential role in generating a properly quantized Chern-Simons level.

We now comment on the background and mutual Chern-Simons terms. For a spatially uniform background probe field  $A_i(\tau)$ , the coupling to the composite fermions is the same as for  $a_i(\tau)$ , as in Eq. (B2). The calculation of these terms is therefore identical, and their coefficients are also  $p+1$ . However, one should be careful in the interpretation of this

result, since  $\partial_0 A'_i$  is not the exactly the physical electric field, as discussed in Sec. IV A. Indeed,  $A'$  does not even transform covariantly under left-acting U(1) gauge transformations. Instead, it satisfies the modified infinitesimal transformation law,

$$A'_i \rightarrow A'_i + i \partial_i \lambda + i[\lambda, \delta A_i + A'_i]_*, \quad (\text{B22})$$

This means that the Seiberg-Witten map cannot be applied directly to the probe,  $A'_i$ , and it must be modified accordingly. Rather than doing so here, particularly given the difficulties with constructing a gauge invariant mutual Chern-Simons term, we leave this to future work. Instead, we emphasize the physically transparent derivation of the Hall conductivity in Sec. IV A, which should be consistent with such an analysis.

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