


Negative group velocity and Kelvin-like wake pattern

Eugene B. Kolomeisky¹, Jonathan Colen², and Joseph P. Straley³

¹*Department of Physics, University of Virginia, P. O. Box 400714, Charlottesville, Virginia 22904-4714, USA*

²*Department of Physics, University of Chicago, 5720 South Ellis Avenue, Chicago, Illinois 60637, USA*

³*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506-0055, USA*

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Wake patterns due to a uniformly traveling source are a result of the resonant emission of the medium's collective excitations. When there exists a frequency range where such excitations possess a negative group velocity, their interference leads to a wake pattern resembling the Kelvin ship wake: while there are “transverse” and “divergent” wave fronts trailing the source, they are oriented oppositely to Kelvin's. This is illustrated by an explicit calculation of “roton” wake patterns in superfluid ⁴He where a Kelvin-like wake emerges when the speed of the source marginally exceeds the Landau critical roton velocity.

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Physics experiments typically measure a response of a system to an external disturbance. A familiar type of disturbance is an object traveling uniformly relative to a medium. This can give rise to a range of effects including Mach waves behind a supersonic projectile [1], Cherenkov radiation emitted by a rapidly moving charge [2], and ship waves [3,4], all of which are examples of coherent generation of the medium's collective excitations [5].

If the disturbance is weak, the response is proportional to it and dictated by the properties of the medium. This is the case for far-field wake patterns that are governed by the frequency spectrum $\Omega(\mathbf{k})$ of the relevant collective excitations of the medium (here \mathbf{k} is the wave vector). For any physical system, a wake is present whenever there is a wave mode whose phase velocity $\mathbf{k}\Omega/k^2$ (here $k = |\mathbf{k}|$) matches the projection of the velocity of the source \mathbf{v} onto the direction of radiation \mathbf{k}/k [4,5]. This implies that the wave pattern is stationary relative to the source. For a source moving in the $+x$ direction with velocity \mathbf{v} , this requires the existence of a wave vector \mathbf{k} satisfying

$$\Omega(\mathbf{k}) = \mathbf{k} \cdot \mathbf{v} \equiv kv \cos \varphi \equiv k_x v, \quad (1)$$

where $v = |\mathbf{v}|$ and φ is the angle between the vectors \mathbf{k} and \mathbf{v} . Equation (1) is the Mach-Cherenkov-Landau (MCL) resonant radiation condition, which also describes the onset of Landau damping in a plasma [6] and the breakdown of superfluidity [7].

When the excitation spectrum is linear,

$$\Omega = uk, \quad (2)$$

where u is the speed of sound (or light), the MCL condition (1) becomes $\cos \varphi = u/v$. It can be satisfied only if $u \leq v$, i.e., if the source is supersonic (or superluminal).

Contrarily, the spectrum of gravity waves is [1,4]

$$\Omega^2 = gk, \quad (3)$$

where g is the free fall acceleration. Now the MCL condition (1) becomes $\cos \varphi = \sqrt{g/kv^2}$ and can always be satisfied for sufficiently large wave number k . As a result, the wake appears for any velocity.

The complexity of wake patterns is governed by the spatial dispersion, i.e., by the difference between the group and the phase velocities. Specifically, the simple cone geometry of Mach-Cherenkov wakes is due to the dispersionless spectrum (2) while the “feathered” appearance of ship wakes has its origin in the dispersive character of gravity waves (3).

Here we show that when the group velocity is negative (i.e., it is opposite in direction to the phase velocity) in a range of the wave vectors satisfying Eq. (1), the corresponding excitations interfere and create a wake pattern that resembles the Kelvin ship wake. This effect does not rely on the specific form of the dispersion law $\Omega(\mathbf{k})$ as long as the latter contains region(s) with negative group velocity. It is also different from a reversed Cherenkov effect in “left-handed” materials [8] that we do not consider.

Several physical systems possess excitations which in certain frequency ranges are characterized by negative group velocity. A classic example is the optical branch of vibrations in crystals [9]. We will illustrate the impact of negative group velocity on wake patterns both generally for an arbitrary isotropic medium and explicitly using the example of superfluid ⁴He. Past studies of wakes in a superfluid ⁴He [10,11] overlooked this effect.

For small wave numbers k the collective (or elementary) excitations in Bose liquids correspond to hydrodynamic sound waves with a linear spectrum (2) [7]. In liquid ⁴He the function $\Omega(\mathbf{k}) \equiv \Omega(k)$ reaches a maximum after an initial increase, followed by a “roton” minimum at some k_0 [7] (incidentally, this is also the case in a dipolar quantum gas [12]). The excitations with wave numbers sandwiched between these extrema are characterized by a negative group velocity $d\Omega/dk \equiv \Omega'(k) < 0$.

In the vicinity of $k = k_0$ it is customary to expand the function $\Omega(k)$ in powers of $k - k_0$:

$$\Omega = \frac{\Delta}{\hbar} + \frac{\hbar(k - k_0)^2}{2\mu}, \quad (4)$$

where Δ , μ , and k_0 are empirically known parameters which depend on the pressure [13]. Hereafter for illustration purposes and without the loss of generality we employ their values extrapolated to zero pressure [7]. The parameters of the roton part of the spectrum (4) can be conveniently assembled into a dimensionless combination,

$$a = \frac{\hbar^2 k_0^2}{\mu \Delta} \approx 31. \quad (5)$$

Additionally, it is useful to introduce a velocity scale

$$v_0 = \frac{\Delta}{\hbar k_0} \approx 60 \text{ m/s}, \quad (6)$$

which is the slope of the straight line connecting the origin $\Omega(0) = 0$ to the roton minimum $\Omega(k_0) = \Delta/\hbar$.

The excitation spectrum of superfluid ^4He ends at $\Omega = 2\Delta/\hbar$, $k = k_e \approx 3 \times 10^8 \text{ cm}^{-1}$ [7]. Below we largely focus on the source velocities $v \leq 2\Delta/\hbar k_e \approx 76 \text{ m/s}$ so that the end part of the spectrum does not contribute into the wake pattern.

Hereafter it is convenient to measure the wave number and the source velocity in units of k_0 and v_0 , respectively. The roton part of the spectrum (4) will then be given by

$$\Omega = 1 + \frac{1}{2}a(k - 1)^2. \quad (7)$$

The MCL requirement (1) is satisfied above a critical velocity v_c such as $\Omega = kv$ and $\Omega' = v$. Subjecting Eq. (7) to these conditions determines the threshold velocity to generate a wake which is also Landau's critical roton velocity to destroy superfluidity in ^4He :

$$v_c = a \left(\sqrt{1 + \frac{2}{a}} - 1 \right) \approx 0.98 \text{ (59 m/s, original units)}. \quad (8)$$

These conditions also define a critical wave number where the group and the phase velocities coincide, $\Omega' = \Omega/k$,

$$k_c = \sqrt{1 + \frac{2}{a}} \approx 1.03. \quad (9)$$

Thanks to the very large parameter a (5) (which further increases with the pressure [13]) the velocities v_0 (6) and v_c (8), and respective wave numbers k_0 and k_c , are extremely close to each other.

Superfluidity can also be destroyed by vortex loop excitations above a velocity that depends on the specific conditions of the flow. In the past this made it impossible to attain the Landau critical velocity (8) in practice. Subsequent experiments with isotopically pure ^4He [14] indicated that the critical velocity of vortex nucleation u_c is significantly larger than the Landau value (8) (at a pressure of 12 bars and larger). There also is numerical indication [11] relating u_c to the speed of sound $u > v_c$ (2). Hereafter vortex loop excitations are omitted since already at a moderate pressure their role is negligible.

When $v > v_c$, the MCL condition (1) holds within a range of the wave numbers $[k_-, k_+]$. In the roton approximation (7)

the bounds k_{\pm} are given by

$$k_{\pm} = 1 + \frac{1}{a}(v \pm \sqrt{v^2 + 2a(v - 1)}). \quad (10)$$

This shows the significance of the velocity v_0 (6): if $v_c < v < v_0 (=1)$, one has $k_- > k_0 (=1)$ and all the waves with wave numbers in the $[k_-, k_+]$ range have positive group velocity [inset to Fig. 1(a)]. As the source velocity increases, the $[k_-, k_+]$ interval widens and at $v = v_0 (=1)$ one finds that $k_- = k_0 (=1)$. For $v > v_0$ the wave numbers in the $[k_-, k_0]$ segment correspond to waves with negative group velocity [inset to Fig. 1(b)].

The resulting wake patterns can be understood through linear response theory [15–18]. We suppose that every particle of the medium is perturbed by a traveling external field of the potential energy $U(\mathbf{r} - \mathbf{v}t)$. Then the average value of the Fourier transform of induced density due to the perturbation is given by $\delta\bar{n}(\omega, \mathbf{k}) = -2\pi\alpha(\mathbf{k} \cdot \mathbf{v}, \mathbf{k})U(\mathbf{k})\delta(\omega - \mathbf{k} \cdot \mathbf{v})$ where $\alpha(\omega, \mathbf{k})$ is a generalized susceptibility [7,15,16] and $U(\mathbf{k})$ is the Fourier transform of $U(\mathbf{r})$. Inverting the Fourier transform and changing the frame of reference to that of the source, $\mathbf{r} - \mathbf{v}t \rightarrow \mathbf{r}$, the induced density is given by

$$\delta\bar{n}(\mathbf{r}) = - \int \frac{d^d k}{(2\pi)^d} \alpha(\mathbf{k} \cdot \mathbf{v}, \mathbf{k}) U(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (11)$$

where d is the space dimensionality. The susceptibility $\alpha(\omega, \mathbf{k})$ has a pole at $\omega \equiv \mathbf{k} \cdot \mathbf{v} = \Omega(\mathbf{k})$ [7,16] which is the MCL condition (1).

The wake pattern due to a point source [$U(\mathbf{k}) = \text{const}$] will be determined by a combination of Kelvin's stationary phase argument [3–5] and numerical evaluation of the integral (11) [18], with generalized susceptibility

$$\alpha(\omega, \mathbf{k}) \propto \frac{1}{(\omega + i0)^2 - \Omega^2(\mathbf{k})}. \quad (12)$$

Far from the source, the phase $f = \mathbf{k} \cdot \mathbf{r}$ is large and the exponential in (11) is highly oscillatory. This is where contributions of elementary plane waves interfere destructively leaving almost no net result, unless their wave vectors satisfy the MCL condition (1) and have a phase which is stationary with respect to \mathbf{k} . This is the condition of constructive interference leading to a wake.

We start by discussing two-dimensional wake patterns. The wake is formed by interference of plane waves with positive x components of the wave vector, $k_x > 0$. We also take $k_y > 0$ ($k_y < 0$ contributions can be found via reflection $y \rightarrow -y$). Then the wake is found at $y > 0$ if the group velocity is positive or at $y < 0$ if the group velocity is negative. The phase is given by

$$f = \mathbf{k} \cdot \mathbf{r} = k_x x + k_y y = \frac{\Omega(k)}{v} x + \sqrt{k^2 - \frac{\Omega^2(k)}{v^2}} y, \quad (13)$$

where we used the MCL condition (1) and $k^2 = k_x^2 + k_y^2$ to re-express $k_{x,y}$ in terms of k . Therefore the stationary phase condition $df/dk = 0$ becomes

$$\frac{y}{x} = \frac{\Omega' \sqrt{v^2 k^2 - \Omega^2}}{\Omega \Omega' - v^2 k}. \quad (14)$$

Since the phase f is constant along the wave front, Eqs. (13) and (14) can be solved to give equations of the wave

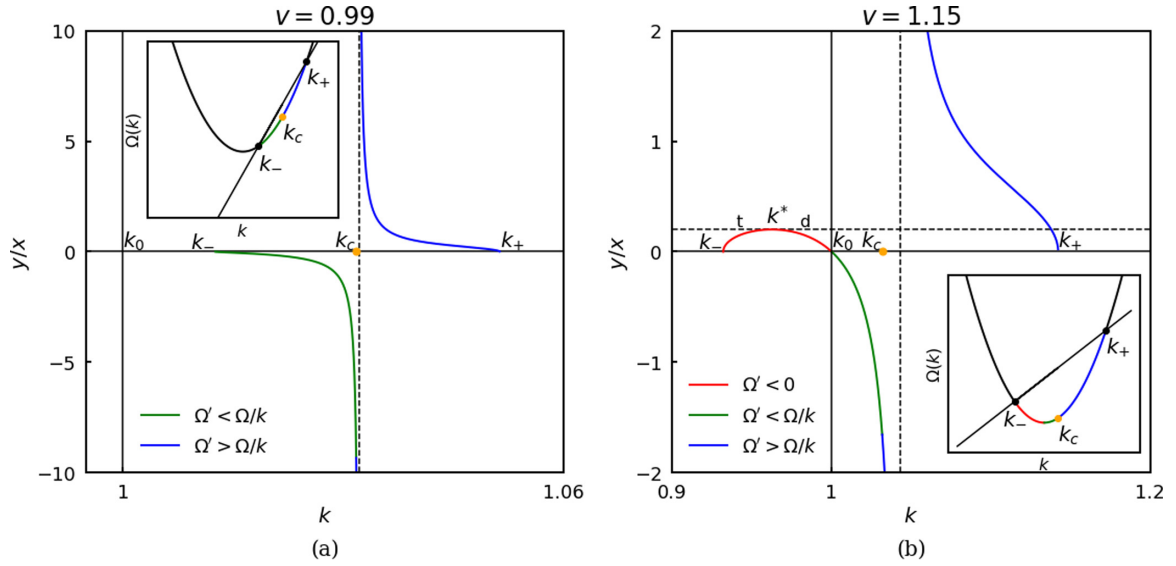


FIG. 1. The right-hand side of the equation of the stationary phase (14) in the roton approximation (7) with v and k measured in units of v_0 (6) and k_0 . (a) Source velocity $v = 0.99$ ($v_c < v < 1$) versus (b) source velocity $v = 1.15$ ($v > 1$). Insets illustrate the spectrum $\Omega(k)$ and the origin of the critical wave number k_c (orange) (9) and boundary values k_{\pm} (10) as solutions to $\Omega(k) = kv$. (b) includes waves with negative group velocity whose interference is responsible for transverse (t) and divergent (d) parts of the Kelvin-like wake in Fig. 2(b).

fronts in the parametric form:

$$x(k) = \frac{\Omega\Omega' - v^2k}{vk(\Omega'k - \Omega)}f, \quad y(k) = \frac{\Omega'\sqrt{v^2k^2 - \Omega^2}}{vk(\Omega'k - \Omega)}f. \quad (15)$$

Not only do Eqs. (13)–(15) describe wake patterns in two dimensions ([3–5,17,18]), they also apply to the three-dimensional case. The main difference is that the xy plane becomes a plane intersecting the three-dimensional wake pattern along the path of the source. Indeed, the different dimensionality does not change the phase function. The way that the amplitude of the $d = 3$ wake pattern falls off with distance will be different but the wave pattern itself will be the same, according to the stationary phase argument.

The right-hand side (rhs) of Eq. (14) diverges at a wave number k_v that can be bounded by evaluating $\Omega(k_c)\Omega'(k_c) - v^2k_c = k_c(v_c^2 - v^2) \leq 0$ and $\Omega(k_+)\Omega'(k_+) - v^2k_+ = k_+v[\Omega'(k_+) - v] \geq 0$, meaning that $k_c \leq k_v \leq k_+$.

If the source velocity belongs to the $[v_c, v_0]$ interval, the group velocity Ω' is positive in the $[k_-, k_+]$ range. In the roton approximation (7) the behavior of the rhs of Eq. (14) is shown in Fig. 1(a). Here y/x is a monotonically decreasing function of k both in the $[k_-, k_v]$ range where $y/x < 0$ and in the $[k_v, k_+]$ interval where $y/x > 0$. Thus the wake is present both ahead ($x > 0$) and behind ($x < 0$) the source.

Moreover, since for $k > k_c$ the group velocity Ω' is larger than the phase velocity Ω/k , while for $k < k_c$, the opposite is true, $\Omega' < \Omega/k$, and signs of y and Ω' coincide, the second of Eqs. (15) implies that there are two families of wave fronts distinguished by the choice of the phase:

$$f = 2\pi n, \quad \text{if } k \in [k_c, k_+] (\Omega' > \Omega/k), \quad (16)$$

$$f = -2\pi n, \quad \text{if } k \in [k_-, k_c] (\Omega' < \Omega/k), \quad (17)$$

where n takes on positive integers. Since for the first of these the rhs of the stationary phase condition (14) can be both

positive and negative [blue-colored parts of the curves in Fig. 1(a)], the wave fronts (15) and (16) are found both at $x < 0$ and $x > 0$. Specifically, $k = k_v$, the point of divergence of y/x , corresponds to $x = 0$. On the other hand, the wave fronts described by Eqs. (15) and (17) are only found at $x < 0$ because in the range of wave numbers (17) the rhs of Eq. (14) is negative [green-colored part of the curves in Fig. 1(a)].

The spatial periodicities along the central line $y = 0$ can be found by setting $k = k_{\pm}$ in the first of Eqs. (15):

$$\Delta x_{\pm} = \frac{2\pi}{k_{\pm}}, \quad (18)$$

The two wave-front families are separated by a marginal wave front that can be obtained by evaluating Eq. (14) at $k = k_c$ [orange dot in Fig. 1(a)]:

$$\frac{y}{x} = -\frac{v_c}{\sqrt{v^2 - v_c^2}}. \quad (19)$$

This is a Mach-Cherenkov cone with the critical velocity v_c playing a role of the limiting velocity. The same equation describes the asymptotic behavior of the wave fronts (15)–(17) far away from the source.

In the roton approximation (7) the resulting wake pattern is shown in Fig. 2(a) where predictions of the stationary phase analysis, Eqs. (15)–(17) and (19) [color matching Fig. 1(a)] are overlaid (for $y > 0$) with results of numerical evaluation of the integral (11) for $d = 2$. The latter additionally supplies the information about the size of the waves (in arbitrary units) varying from crests (deep red) to troughs (deep blue).

A wake pattern of this kind will be found in any system that has an analog of the Landau velocity if the source velocity only slightly exceeds v_c . The pattern can be deduced via an expansion of a dispersion law $\Omega(k)$ about $k = k_c$ up to the second order in $(k - k_c)$. Such a wake is indeed observed in water where due to the effects of capillarity omitted in Eq. (3), one finds $v_c \approx 23$ cm/s [4].

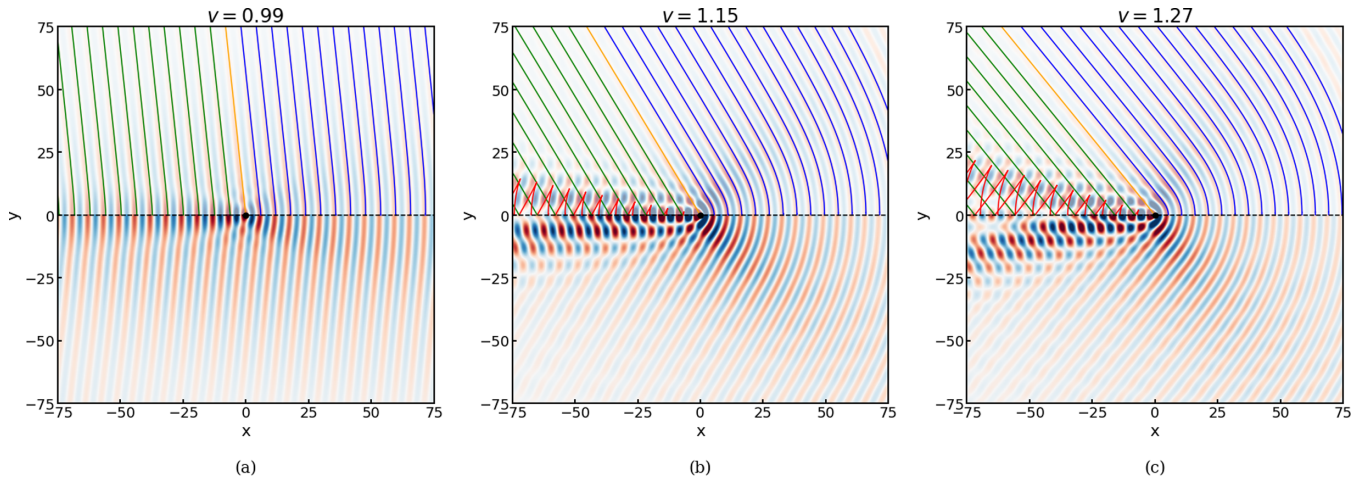


FIG. 2. Numerically evaluated two-dimensional roton wake patterns (11) and (12) due to a point source traveling to the right for different velocities measured in units of v_0 (6). The unit of length is $1/k_0$, and the color field indicates the size of the waves (arbitrary units). The predictions of the stationary phase argument, Eqs. (15)–(17) and (19), color matching Fig. 1, are overlaid for $y > 0$ where the wake pattern is made more transparent for visibility. The source velocities in (a) and (b) match Figs. 1(a) and 1(b). Kelvin-like wakes, highlighted by a series of red triangles tracing wave crests, are visible in (b) and (c) for $x < 0$.

As $v \rightarrow v_c + 0$, the wake in Fig. 2(a) transforms into a one-dimensional periodic pattern of induced density $\delta\bar{n}(x)$ with the period $\Delta x = 2\pi/k_c$, which was also predicted to occur when the velocity of a bulk uniform flow slightly exceeds the Landau critical roton velocity [19].

If $v > v_0$, the conclusions found for the $v_c < v < v_0$ regime carry over for waves in the range $[k_0, k_+]$ where the group velocity Ω' remains positive. This is illustrated in Fig. 1(b) where we plotted Eq. (14) for $v > v_0$ in the roton approximation (7). The behavior of y/x in the $[k_0, k_+]$ range is qualitatively the same as that in Fig. 1(a). This explains a part of the wake pattern in Fig. 2(b) that is qualitatively similar to that in Fig. 2(a).

For waves in the range $[k_-, k_0]$, the group velocity Ω' is negative, and the rhs of Eq. (14) is positive. Since the signs of Ω' and y coincide, part of the wake described by Eqs. (15) and (17) due to excitations of the $[k_-, k_0]$ range will still be found behind the source, $x < 0$, but at $y < 0$. Moreover, a function y/x positive in the $[k_-, k_0]$ interval and vanishing at its ends must have at least one maximum within it. This general reasoning is illustrated in Fig. 1(b) which shows that in the roton approximation the rhs of Eq. (14) is positive in the $[k_-, k_0]$ range (red curve) and has a maximum at a wave number k^* .

If $0 \leq y/x < (y/x)^* \equiv (y/x)(k^*)$ [shown in Fig. 1(b) as a dashed horizontal line] Eq. (14) has three solutions. One of them immediately to the left of $k = k_+$ corresponds to already discussed wave fronts (15) and (16). The remaining two solutions within the $[k_-, k_0]$ interval, one to the left and one to the right of $k = k^*$, are new. As y/x increases approaching $(y/x)^*$ the two solutions tend to each other, join at $y/x = (y/x)^*$, and none are found if $y/x > (y/x)^*$. This part of the wake pattern confined within an angle $2 \arctan(y/x)^*$ is represented in Figs. 2(b) and 2(c) by a series of curved red triangles.

There is a substantial similarity between this pattern and the classic Kelvin ship wake formed behind a point pressure source uniformly traveling a calm water surface [3,4].

Specifically, Kelvin's terminology of "transverse" and "divergent" [3] wave fronts applies: interfering excitations with wave numbers in the $[k_-, k^*]$ ($[k^*, k_0]$) range produce the transverse (divergent) wave fronts which smoothly (discontinuously) connect the edges of the pattern $y/x = \pm(y/x)^*$ across the central line $y = 0$. The notable difference from Kelvin's is that the pattern is reversed (i.e., it faces away from the source) as the wake is formed by elementary waves with negative group velocity.

Like the Landau critical roton velocity $v = v_c$, the instant $v = v_0$ where elementary waves having negative group velocity start participating in forming wake pattern, represents a critical phenomenon. It is expected that it will be accompanied by an increase in the wave resistance which we are planning to study in the future.

As the source velocity increases beyond $v = v_0$, the size of the waves increases, the opening angle of the Kelvin-like wake grows, and the roton approximation (4) eventually breaks down. However, qualitatively the wake pattern will not change as long as $v \lesssim 76$ m/s; the pattern in Fig. 2(c) corresponds to $v = 76$ m/s. At larger velocities the wake pattern will acquire new elements due to contributions coming from the end part of the spectrum. At the same time the Kelvin-like feature will persist and widen with v until the source velocity exceeds about 167 m/s. This is when the lower bound of the MCL interval k_- coincides with location of the maximum of $\Omega(k)$ and the range of wave numbers with $\Omega' < 0$ is largest.

To summarize, we demonstrated that if in response to a small heavy uniformly traveling source a medium radiates excitations possessing negative group velocity, their interference leads to a reversed Kelvin-like wake. In superfluid ^4He this effect should be commonplace because the Landau critical roton velocity of $v_c = 59$ m/s is extremely close to the Kelvin threshold of $v_0 = 60$ m/s. At the same time the details of the Kelvin feature may be somewhat obscured due to interference with overlapping "green" family of wave fronts, Figs. 2(b) and 2(c). A Kelvin-like wedge should be observable by light

scattering techniques while the fine structure of all the discussed wake patterns can be only resolved with the help of x-ray or neutron scattering.

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