

Nonlinear magnetoresistivity in two-dimensional systems induced by Berry curvatureShibalik Lahiri,¹ Tanmay Bhore,² Kamal Das^{1,*} and Amit Agarwal^{1,†}¹*Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India*²*Department of Physics, Indian Institute of Science Education and Research Bhopal, Bhopal 462066, India*

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We investigate the effect of band geometric quantities on nonlinear magnetoresistivity, which dictates the quadratic dependence of the nonlinear voltage generated by the applied current. We propose that the interplay of the Berry curvature, the orbital magnetic moment, and the Lorentz force can induce a finite nonlinear resistivity in two-dimensional systems in the presence of a perpendicular magnetic field. The induced nonlinear magnetoresistivity scales linearly with the magnetic field and is purely quantum mechanical in origin. Our proposed transport signature can be used as an additional experimental probe for the geometric quantities in intrinsically time reversal symmetric systems.

DOI: [10.1103/PhysRevB.105.045421](https://doi.org/10.1103/PhysRevB.105.045421)**I. INTRODUCTION**

The band geometric properties of quantum materials, such as the Berry curvature and the orbital magnetic moment (OMM), play a fundamental role in the linear and nonlinear (NL) transport and optical properties [1,2]. Some prominent examples within the linear response include phenomena like the anomalous Hall effect (AHE) [3–5], valley Hall effect [6], magnetic field induced AHE [7], intrinsic Hall effect [8–10], and magnetoresistance [11,12]. Several very exciting NL Hall effects and other NL transport phenomena are also being actively explored [13–22]. However, the exploration of NL transport induced by geometric quantities in the presence of a magnetic field is still at a nascent stage.

In the presence of a magnetic field, the spin-orbit coupling has been shown to induce a unidirectional magnetoresistance in two-dimensional (2D) magnetic systems [23–25]. More recently, bilinear magnetoresistance [26,27] and NL planar Hall effect [28] were demonstrated in nonmagnetic spin-orbit coupled 2D systems based on the conversion of spin current to charge current. This was facilitated by including the magnetic field via Zeeman coupling. In this paper, we propose an alternative origin for NL magnetoresistance in 2D systems, which is purely quantum mechanical in nature. We show that band geometric quantities, such as the Berry curvature and the OMM, induce second-order NL magnetoresistance in the presence of a *perpendicular* magnetic field in 2D systems. This is more significant in a time reversal symmetric system, where the entire contribution to the NL resistance arises from the band geometric properties. The predicted NL resistivity is a different transport signature of band geometric properties, and it can be used as an experimental tool to probe the Berry curvature and the OMM.

To obtain the NL resistivity, we use the semiclassical electron dynamics and the Boltzmann transport equation. We calculate all the NL conductivity contributions that arise from

the Berry curvature, the OMM, and the Lorentz force in 2D systems. We find that in addition to broken space inversion symmetry (SIS), the anisotropy of the band dispersion is also a necessary criterion for the NL conductivities to be finite. Intriguingly, in contrast to the quadratic magnetic field dependence of the magnetoresistivity in the linear response regime [11], the predicted NL magnetoresistivity varies linearly with the magnetic field. Furthermore, it also persists in systems with broken time reversal symmetry (TRS), although there are additional contributions arising from the Drude conductivity. We explicitly calculate all the NL conductivities and the NL resistivities for 2D systems which host tilted massive Dirac fermions.

This paper is organized as follows: In Sec. II we describe the experimentally relevant NL resistivity matrix in terms of the theoretically calculated NL conductivities. This is followed by a detailed calculation of all the different contributions to the NL magnetoconductivities in Sec. III. Section IV presents a specific example of the predicted NL magnetoconductivity and magnetoresistivity in a tilted massive Dirac Hamiltonian. This is followed by a discussion in Sec. V, and finally, we summarize our findings in Sec. VI.

II. NONLINEAR RESISTIVITY

In this section, we will define the second-order NL resistivity matrix and obtain its general expression in terms of the NL conductivities and the linear resistivities. Before we proceed, it is useful to understand the commonly followed experimental setup of the NL transport, shown in Fig. 1. In NL transport experiments [17,27,28], typically, an ac current (or current density) of frequency ω is sent through the device, and as a response, the induced longitudinal [27] (parallel to the current) and transverse [17,28] (perpendicular to the current) voltage drops (or electric fields) are measured. The induced NL voltages or electric fields $E^{(2)}$ are distinguished from their linear counterparts by specifically measuring them at 2ω or zero frequency, using a lock-in amplifier.

Specifically, for an input current density j^ω , the induced linear response electric field E^ω , the induced NL responses in

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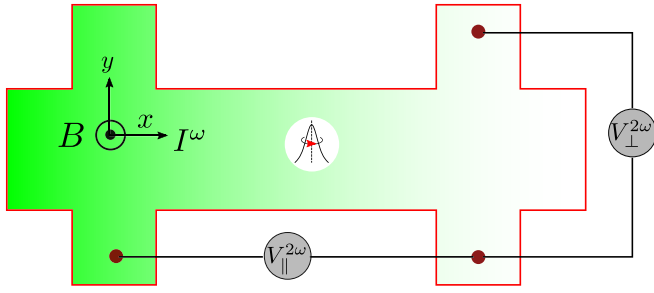


FIG. 1. Schematic of the experimental setup for the measurement of the NL magnetoresistivity. The magnetic field is applied perpendicular to the x - y plane. The current (at frequency ω) flows along the x axis, and the voltage drops along the direction of the current, $V_{\parallel}^{2\omega}$, and perpendicular to it, $V_{\perp}^{2\omega}$, are measured.

the form of the dc electric field E^{nldc} , and the second harmonic electric field $E^{2\omega}$ are usually measured. This measurement scheme introduces the general concept of second-order NL resistivity $\tilde{\rho}_{abc}^{(2)}$ via the relation $E_a^{(2)} = \tilde{\rho}_{abc}^{(2)} j_b j_c$, where j_b is the applied current density. Here, the subscripts a and b denote the coordinate axes, and summation over the repeated indices is implied. Particularly, we can define the NL dc (zero frequency) resistivity, $E_a^{\text{nldc}} = \tilde{\rho}_{abb}^{\text{nldc}} (j_b^\omega)^2$, and the second harmonic resistivity, $E_a^{(2\omega)} = \tilde{\rho}_{abb}^{(2\omega)} (j_b^\omega)^2$. In addition to these, there will also be an induced electric field in the linear response, which defines the linear resistivity ρ_{ab} , $E_a^\omega = \rho_{ab} j_b^\omega$.

To connect the experimentally measured NL resistivity to the theoretically calculated NL conductivity, it is essential to establish a well-defined relation between the two. Theoretically, we calculate the second harmonic current via $j_a^{(2\omega)} = \sigma_{abc}^{(2\omega)} E_b E_c$ and the NL dc current as $j_a^{\text{nldc}} = \sigma_{abc}^{\text{nldc}} E_b E_c^*$, where E_b and E_c denote the complex components of the applied electric field. One simple way to connect the NL conductivity with the NL resistivity is to calculate j_a^{nldc} or $j_a^{(2\omega)}$ in terms of the components of j_b^ω , making use of the linear response resistivity. This yields $j_a^{(2\omega)} = [\sigma_{ab'c'}^{2\omega} \times \rho_{bb'} \times \rho_{cc'}] j_b^\omega j_c^\omega$. Now, using this and defining the total resistivity ρ^{total} in terms of the measured electric field and current, we obtain the expression of NL resistivity. The details of the calculations are summarized in Appendix A.

Assuming that the current is applied only along the x direction, for the case of 2D systems, the NL resistivity matrix elements are calculated to be

$$\begin{pmatrix} \tilde{\rho}_{xxx}^{(2)} \\ \tilde{\rho}_{yxx}^{(2)} \end{pmatrix} = -[\rho] \begin{pmatrix} \sigma_{xxx} & \sigma_{xxy} & \sigma_{xyx} & \sigma_{xyy} \\ \sigma_{yxx} & \sigma_{yyx} & \sigma_{yyx} & \sigma_{yyy} \end{pmatrix} \begin{pmatrix} \rho_{xx}^2 \\ \rho_{xx}\rho_{yx} \\ \rho_{xx}\rho_{yx} \\ \rho_{yx}^2 \end{pmatrix}. \quad (1)$$

Here, $\tilde{\rho}_{xxx}^{(2)}$ stands for the NL resistivity, and $\tilde{\rho}_{yxx}^{(2)}$ stands for NL Hall resistivity. In Eq. (1), the first term on the right-hand side denotes the 2×2 linear response resistivity matrix for 2D systems. Equation (1) calculates the NL resistivity differently than earlier studies [17,26]. Note that Eq. (1) holds for both the NL dc resistivity as well as the second harmonic resistivity, depending on which NL conductivity is used on the right-hand side. However, in the transport limit, $\omega\tau \ll 1$, both the NL conductivities are identical [$\sigma^{\text{nldc}}(\omega \rightarrow 0) = \sigma^{2\omega}(\omega \rightarrow 0)$],

and this is also reflected in the NL resistivity. For the rest of the paper, we work in this limit, and thus, we have $\tilde{\rho}_{abc}^{(2)} = \tilde{\rho}_{abc}^{(2\omega)} = \tilde{\rho}_{abc}^{\text{nldc}}$. Simplifying the notation further, we use $\tilde{\rho}_{xx}^{(2)}$ for $\tilde{\rho}_{xxx}^{(2)}$ and $\tilde{\rho}_{yx}^{(2)}$ for $\tilde{\rho}_{yxx}^{(2)}$ in the rest of the paper.

To explore the NL magnetoresistivity in 2D systems induced by the Berry curvature and the OMM, we first calculate the different contributions to the NL conductivity. We will specifically consider the case of a magnetic field perpendicular to the 2D plane, i.e., device geometry in the normal Hall configuration, as shown in Fig. 1.

III. NONLINEAR CONDUCTIVITIES

In this section, we calculate the general expressions of all the components of the NL conductivity tensor in the presence of a magnetic field. As discussed in the last section, the NL conductivities are related to the NL current via the relation $j_a^{(2\omega)} = \sigma_{abc}^{(2\omega)} E_b E_c$. In the semiclassical Boltzmann transport formalism, the charge current can be expressed as $\mathbf{j}(t) = -e \int [d\mathbf{k}] D^{-1} \hat{\mathbf{r}} g(t)$. Here, $g(t)$ denotes the nonequilibrium distribution function (NDF), and $[d\mathbf{k}]$ stands for $g_s d\mathbf{k}/(2\pi)^2$, with g_s denoting the spin degeneracy. In a 2D system, in the presence of perpendicular magnetic field, the band geometric quantities modify (i) the dynamics of the charge carriers in the phase space $(\hat{\mathbf{r}}, \hat{\mathbf{k}})$, (ii) the phase-space volume D^{-1} , and (iii) the band dispersion. These impact the NDF of the charge carriers in the presence of applied electric field, which in turn gives rise to additional band geometry induced contributions to the NL conductivities.

For the Hall configuration ($\mathbf{E} \perp \mathbf{B}$) that we consider for this paper, the equation of motion is given by [8,29,30]

$$\dot{\mathbf{r}} = D \left[\tilde{\mathbf{v}} + \frac{e}{\hbar} \mathbf{E}(t) \times \boldsymbol{\Omega} \right], \quad (2)$$

$$\hbar \dot{\mathbf{k}} = D [-e\mathbf{E}(t) - e(\tilde{\mathbf{v}} \times \mathbf{B})]. \quad (3)$$

Here, $-e$ (with $e > 0$) is the electronic charge, and the phase-space modifying factor is given by $1/D = [1 + \frac{e}{\hbar} (\mathbf{B} \cdot \boldsymbol{\Omega})]$, with $\boldsymbol{\Omega}$ being the Berry curvature. In the above equations, $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v}_m$ is the OMM modified velocity, where $\hbar \mathbf{v} = \partial \epsilon / \partial \mathbf{k}$ and $\hbar \mathbf{v}_m = \partial \epsilon_m / \partial \mathbf{k}$, with $\epsilon_m = \mathbf{m} \cdot \mathbf{B}$. This is due to the fact that in the presence of a magnetic field, the Zeeman-like coupling of the OMM with the magnetic field modifies the electronic band energy via the relation $\tilde{\epsilon} = \epsilon - \mathbf{m} \cdot \mathbf{B}$. The Berry curvature and the OMM for the n th band can be computed using the relation [1,6]

$$\Omega_a^n = -2\epsilon_{abc} \sum_{n \neq n'} \frac{\text{Im} \langle n | \partial \mathcal{H} / \partial k_b | n' \rangle \langle n' | \partial \mathcal{H} / \partial k_c | n \rangle}{(\epsilon^n - \epsilon^{n'})^2}, \quad (4)$$

$$m_a^n = -\frac{e}{\hbar} \epsilon_{abc} \sum_{n \neq n'} \frac{\text{Im} \langle n | \partial \mathcal{H} / \partial k_b | n' \rangle \langle n' | \partial \mathcal{H} / \partial k_c | n \rangle}{\epsilon^n - \epsilon^{n'}}. \quad (5)$$

Here, ϵ_{abc} is the Levi-Civita symbol, and the band energies and eigenstates are for an unperturbed system, $\mathcal{H} |n\rangle = \epsilon^n |n\rangle$. For 2D systems, both the Berry curvature and the OMM have only one finite component: the component pointing out of the plane. In our case, we consider the 2D system to be in the x - y plane; thus, only the z components of these quantities are defined.

In the presence of an ac electric field, of the form $\mathbf{E}(t) = \mathbf{E}e^{i\omega t} + \mathbf{E}^*e^{-i\omega t}$, and a static magnetic field [31,32] the NDF can be calculated using the Boltzmann kinetic equation with the relaxation time approximation [33], which reads

$$\frac{\partial g(t)}{\partial t} + \dot{\mathbf{k}} \cdot \nabla_{\mathbf{k}} g(t) = -\frac{g(t) - \tilde{f}}{\tau}. \quad (6)$$

Here, \tilde{f} is the Fermi-Dirac distribution function given by $\tilde{f} = 1/[1 + e^{\beta(\epsilon - \mu)}]$ at chemical potential μ and inverse temperature $\beta = 1/(k_B T)$, with k_B being the Boltzmann constant and T being the temperature. In Eq. (6), τ is the relaxation time, and for simplicity we ignore its energy dependence. The NDF can be expressed as a sum of the equilibrium and nonequilibrium parts, $g(t) = \tilde{f} + \delta g(t)$. Furthermore, the nonequilibrium part $\delta g(t)$ can be expressed as a power series of the applied electric field as $\delta g = \sum_{\nu} \delta g_{\nu}$, with $\delta g_{\nu} \propto |\mathbf{E}|^{\nu}$. In this paper we are interested in current $\propto |\mathbf{E}|^2$, and thus, we calculate the NDF up to quadratic order in electric field. To this end, we use the ansatz

$$\delta g_2(t) = f_2^0 + f_2^{0*} + f_2^{2\omega} e^{i2\omega t} + f_2^{2\omega*} e^{-i2\omega t}. \quad (7)$$

Here, f_2^0 or f_2^{0*} is the rectification (or dc) part, and $f_2^{2\omega}$ or $f_2^{2\omega*}$ is the second harmonic (2ω) part of the NDF. Using this ansatz in Eq. (6), we calculate the second harmonic part to be [34,35],

$$f_2^{2\omega} = \sum_{\nu=0}^{\infty} (D\tau_{2\omega}\hat{L}_B)^{\nu} D \frac{e^{\tau_{2\omega}}}{\hbar} \mathbf{E} \cdot \nabla_{\mathbf{k}} f_1^{\omega}. \quad (8)$$

Here, f_1^{ω} is the linear order correction to the distribution function [see Appendix B for details], and we have defined

$$\hat{L}_B = \frac{e}{\hbar} (\tilde{\mathbf{v}} \times \mathbf{B}) \cdot \nabla_{\mathbf{k}}, \quad \tau_{2\omega} = \frac{\tau}{1 + i2\omega\tau}. \quad (9)$$

The rectification counterpart can be obtained from Eq. (8) simply by replacing $\tau_{2\omega} \rightarrow \tau$ and $\mathbf{E} \rightarrow \mathbf{E}^*$. Using Eq. (8), it is straightforward to calculate the NDF as a power series of the magnetic field [2,9,36–38]. The explicit form of the distribution function is presented in Appendix C.

Using the obtained NL distribution function, one can calculate the rectification current $\mathbf{j}^0(t)$, the second harmonic current $\mathbf{j}^{2\omega}(t)$, and the corresponding conductivities. In this paper, we restrict ourselves to the lowest-order magnetic field corrections to the NL conductivities. The details of the calculations and the general expressions for all the different conductivity terms are presented in Appendix C. Here, we list the second harmonic contributions and focus only on those contributions, which are nonzero in systems which intrinsically preserve TRS (nonmagnetic systems).

Before doing explicit calculations, we note that the scattering time dependence of the NL conductivities can be inferred from very general symmetry arguments. For example, if a current component $j \propto (B)^a (\tau)^b$, then under time reversal we have $-j \propto (-B)^a (-\tau)^b \propto j \times (-1)^{a+b}$, and consequently, $a + b$ should be an odd integer. Thus, terms with odd (even) powers of B in the magnetoconductivity will always have even (odd) powers of τ . This symmetry argument is also applicable for magnetic field independent NL conductivities, and in that case we consider $a = 0$.

In intrinsically time reversal symmetric systems in two dimensions, we find that only the NL Hall conductivity [15,16,39–41] survives in the absence of a magnetic field. Since this contribution originates from the anomalous velocity, it is called NL anomalous Hall (NAH) conductivity and it is given by

$$\sigma_{abc}^{\text{NAH}} = -\frac{e^3 \tau_{\omega}}{2\hbar} \varepsilon_{abd} \int [d\mathbf{k}] \Omega_d v_c f' + (b \leftrightarrow c). \quad (10)$$

Here, $f' \equiv \partial_{\epsilon} f$ is the derivative of the Fermi function with respect to energy, and we have defined $\tau_{\omega} \equiv \tau/(1 + i\omega\tau)$. As the name suggests, this term contributes only to the Hall current, and the diagonal components $\sigma_{aaa}^{\text{NAH}}$ vanish. In the presence of a magnetic field, we find that in addition to Eq. (10) there are three other contributions to the NL conductivities: (i) a contribution arising solely from the OMM $\sigma_{abc}^{\text{OMM}}$, (ii) a contribution arising from the interplay of the anomalous velocity and the Lorentz force σ_{abc}^{AL} , and (iii) a Berry curvature dependent contribution arising from the phase-space factor σ_{abc}^{B} .

The OMM induced NL conductivity is given by

$$\sigma_{abc}^{\text{OMM}} = \frac{e^3 \tau_{\omega} \tau_{2\omega}}{2\hbar} \int [d\mathbf{k}] [v_{ma} \partial_{k_b} v_c f' + v_a \partial_{k_b} (v_{mc} f' + \epsilon_m v_c f'')] + (b \leftrightarrow c). \quad (11)$$

Here, we have defined $f'' \equiv \partial_{\epsilon}^2 f$ and note that the derivative operator acts on all the terms appearing on its right side. The anomalous velocity and the Lorentz force combine to give

$$\sigma_{abc}^{\text{AL}} = -\frac{e^3 \tau_{\omega}^2}{2\hbar} \varepsilon_{abd} \int [d\mathbf{k}] \Omega_d \frac{e\mathbf{B}}{\hbar} (v_y \partial_{k_x} v_c - v_x \partial_{k_y} v_c) f' + (b \leftrightarrow c). \quad (12)$$

This conductivity also contributes only to the Hall current. The NL conductivity contribution induced by the phase-space factor is given by

$$\sigma_{abc}^{\text{B}} = \frac{e^3 \tau_{\omega} \tau_{2\omega}}{2\hbar} \int [d\mathbf{k}] v_a (\Omega_B \partial_{k_b} + \partial_{k_b} \Omega_B) v_c f' + (b \leftrightarrow c). \quad (13)$$

Here, we have defined $\Omega_B \equiv \frac{e}{\hbar} \boldsymbol{\Omega} \cdot \mathbf{B}$. It is clear from Eqs. (10)–(13) that all these NL conductivities depend on either the Berry curvature or the OMM. Based on this, we conclude that in intrinsically TRS preserving systems, the second-order NL responses are induced only by the geometric properties of the electron wave function. In other words, in intrinsically TRS preserving systems, the Lorentz force by itself, without the Berry curvature or the OMM, cannot induce the second-order NL response in 2D systems.

To calculate the NL resistivity, we also need the linear response conductivity matrix. The general expression of the linear response current $\mathbf{j}^{\omega}(t)$ is calculated in Appendix B. The Drude conductivity is given by $\sigma_{ij}^{\text{D}} = -e^2 \tau_{\omega} \int [d\mathbf{k}] v_i v_j f'$. In an intrinsically TRS preserving system, the nonzero conductivity up to linear order in the magnetic field is given

by [9,42,43]

$$\sigma_{ab}^L = -e^2 \tau_\omega^2 \frac{eB}{\hbar} \int [dk] v_a (v_y \partial_{k_x} v_b - v_x \partial_{k_y} v_b) f', \quad (14)$$

$$\sigma_{ab}^O = \frac{e^2}{\hbar} \varepsilon_{abd} \int [dk] \epsilon_m \Omega_d f'. \quad (15)$$

The superscript L implies the Lorentz force contribution (the normal Hall effect), and the superscript O denotes the OMM contribution (OMM induced Hall effect).

This summarizes the general framework for calculating the second-order responses. We now explicitly calculate the NL magnetoconductivity in quantum systems which can be described via pair of valleys hosting tilted massive Dirac cones. This is one of the simplest systems which has an anisotropic band dispersion along with finite band geometric quantities and can be treated analytically. We show explicitly in the next section that the anisotropy of the band dispersion is a necessary condition to obtain nonzero NL conductivities.

IV. TILTED MASSIVE DIRAC SYSTEMS

In this section, we calculate the NL resistivity for 2D systems with a pair of tilted massive Dirac fermions. Each one of the ‘‘Dirac valleys’’ is specified by the Hamiltonian [15,44]

$$\mathcal{H}_s = v_F (k_x \sigma_y - s k_y \sigma_x) + s v_t k_x + \Delta \sigma_z. \quad (16)$$

Here, $s = \pm$ is the valley index, σ_i 's are the Pauli matrices representing the sublattice degree of freedom, Δ is the band gap, v_F denotes the Fermi velocity, and the v_t term introduces tilt in the band dispersion along the k_x axis. The model Hamiltonian in Eq. (16) lacks SIS, and the two valleys are related by the TRS. Furthermore, the mirror symmetry is broken along the k_x line and preserved along the k_y line [39]. This model acts as a building block of realistic band structures in systems like the surface states of topological crystalline insulators such as SnTe and transition metal dichalcogenides such as WTe₂ [15,44].

The energy dispersion for this two-band model is given by $\epsilon^\pm = s v_t k_x \pm \epsilon_0$, where $\epsilon_0 = (v_F^2 k^2 + \Delta^2)^{1/2}$, with $k = (k_x^2 + k_y^2)^{1/2}$. Here, the + (−) sign stands for the conduction (valence) band. The band dispersions for both valleys are shown in Figs. 2(a) and 2(b). The tilt-modified band velocity along the x direction is given by $v_x^\pm = s v_t \pm v_F^2 k_x / \epsilon_0$, and $v_y^\pm = \pm v_F^2 k_y / \epsilon_0$. For the model Hamiltonian in Eq. (16), the Berry curvature and OMM have been calculated from Eqs. (4) and (5), respectively, and are given by [15,44]

$$\Omega_z = \mp s \frac{v_F^2 \Delta}{2(v_F^2 k^2 + \Delta^2)^{3/2}}, \quad (17)$$

$$m_z = -s \frac{e v_F^2 \Delta}{2\hbar(v_F^2 k^2 + \Delta^2)}. \quad (18)$$

As expected, the band edges are rich in both the Berry curvature and the OMM and act as hot spots. The distributions of the Berry curvature with constant-energy contours (in the conduction band) are shown for both valleys in Figs. 2(c) and 2(d). We note that both the OMM and the Berry curvature are independent of the tilt.

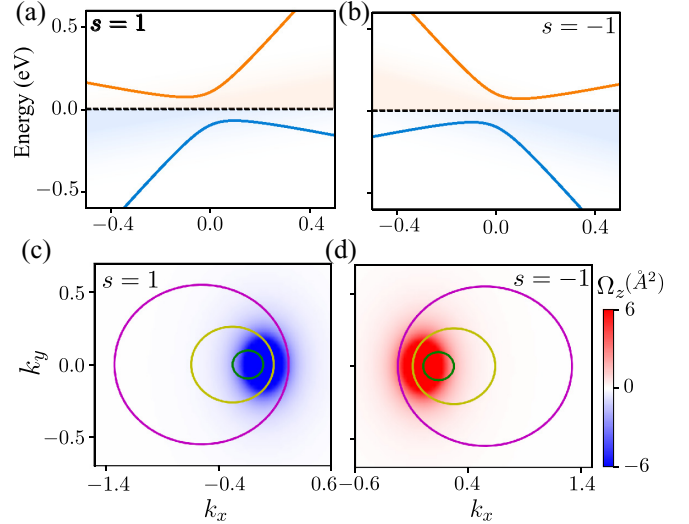


FIG. 2. The band dispersion of the tilted massive Dirac Hamiltonian along the k_x axis for nodes with (a) $s = 1$ and (b) $s = -1$. (c) and (d) show the Berry curvature distribution in the momentum space for the conduction band for nodes $s = 1$ and $s = -1$, respectively. The different energy contours are indicated by the green ($\mu = 0.1$ eV), yellow ($\mu = 0.2$ eV), and magenta ($\mu = 0.4$ eV) lines. The various parameters associated with the Hamiltonian are $v_F = 1$ eV Å, $v_t = 0.1 v_F$, and $\Delta = 0.1$ eV.

A. Nonlinear conductivities

Next, we calculate the NL conductivities for the tilted massive Dirac Hamiltonian in Eq. (16), using the general equations (10)–(13) for the NL conductivities. For analytical insights, we calculate the NL conductivities up to leading order (linear order) in the tilt, and the results are summarized in Table I in terms of the parameter $r \equiv \Delta/\mu$ for $\mu > \Delta$. We find that the NL conductivities are valley degenerate; thus, we simply multiply the results for one valley with a valley degeneracy factor g_v . As a double check of our calculations, we have also done numerical computations of the NL conductivities, including the tilt to all orders, and we find reasonable agreement between the analytical and numerical results (see Fig. 3).

From the NL conductivities summarized in Table I, it is evident that the tilt in the dispersion plays an important role. More specifically, the tilt manifests in the anisotropy of the Fermi surface and the x component of the band velocity. Among the different components of the NL Hall conductivity, we find $\sigma_{yx}^{\text{NAH}} (= \sigma_{xy}^{\text{NAH}})$ and $\sigma_{yxx}^{\text{NAH}}$ to be nonzero. This can be attributed to the nonzero Berry curvature dipole [15] of the system which arises due to the broken mirror symmetry along the x axis. The chemical potential dependence of $\sigma_{yx}^{\text{NAH}} (= \sigma_{xy}^{\text{NAH}})$ and $\sigma_{yxx}^{\text{NAH}}$ is shown in Fig. 3(a). Similarly, for the other Hall component which originates from the combined effect of anomalous velocity and Lorentz force, we find $\sigma_{yyx}^{\text{AL}} (= \sigma_{xyx}^{\text{AL}})$ and σ_{yyy}^{AL} to be nonzero. The chemical potential dependence of these terms is shown in Fig. 3(b). Unlike the above-mentioned conductivity components, the NL conductivities induced by the phase-space contribution, shown in Fig. 3(c), and the OMM contribution, shown in Fig. 3(d), contribute to both the

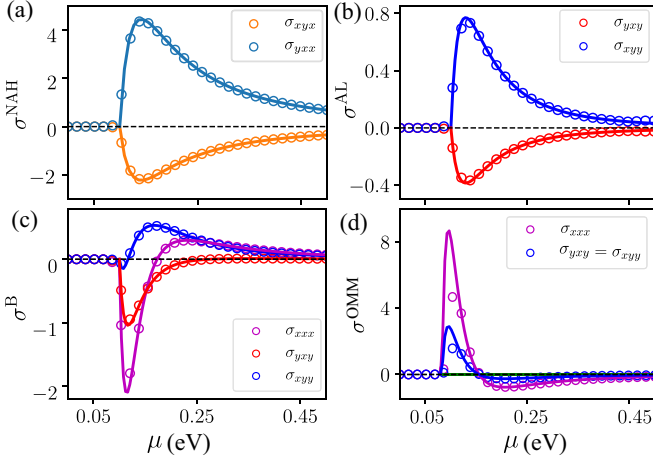


FIG. 3. Variation of the four different contributions to the NL conductivity with the chemical potential μ . The open circles are the results of numerical calculations at temperature $T = 50$ K, while the solid lines are the results of analytical calculations at zero temperature, up to linear order in v_t . (a) The nonlinear anomalous Hall conductivity. (b) The anomalous velocity and Lorentz force induced conductivity. (c) The Berry curvature of the phase-space factor induced conductivity. (d) Orbital magnetic moment induced nonlinear conductivity. The conductivities are plotted in units of 10^{-3} nA m/V². The Hamiltonian parameters are the same as in Fig. 2. Additionally, we have considered $\tau = 1$ ps.

diagonal and Hall components. Both of these contributions induce a nonzero σ_{xxx} and the Hall conductivities $\sigma_{yyx} = \sigma_{yx}$ and σ_{xyy} . We emphasize that the magnetic field dependence of the NL resistivity (discussed below) originates primarily from the diagonal component of the NL conductivity σ_{xxx} . This particular NL conductivity component is generally finite in a system where mirror symmetry is broken along the x axis.

From Fig. 3 it is evident that all of the NL conductivities broadly follow two features: (i) all the NL conductivity contributions are peaked near the band edge, and they decrease as we move away from the band edge after the initial rise, and (ii) the peaks are not located exactly at the band edge. The appearance of the peaks near the band edge can be understood from the fact that all these conductivity contributions originate from the Berry curvature or the OMM, which are primarily concentrated near the band edge. As we move away from these hot spots, the strength of the Berry curvature and the OMM

decreases. The fact that the peaks are not exactly at the band edges is related to the reason that NL conductivities originate from the combined effect of tilt and geometric quantities. The position of the peaks of the conductivities in the μ axis can be explicitly calculated from the analytical expressions provided in Table I. If we consider $\sigma_{abc} \propto (a + br^2 + cr^4)/\mu^v$, then the peak position is found to be $\mu_0 = \Delta[\{ -(v+2)b \pm \sqrt{[(v+2)^2b^2 - 4v(v+4)ac]} / (2va) \}]^{1/2}$. Another interesting finding from the analytical results for σ^{OMM} (see the last column of Table I) is that unlike the other three contributions (NAH, AL, and B), the OMM contribution does not vanish exactly at $\Delta = \mu$. We note that unlike σ^{NAH} and σ^{AL} , the NL conductivities σ^{B} and σ^{OMM} change sign as a function of μ . This can be anticipated from analytical results which have two roots in the μ axis.

B. Nonlinear resistivity

Having calculated the NL conductivities, we now turn our focus to the NL magnetoresistivity. For the tilted massive Dirac model in Eq. (16), the Drude conductivity is calculated to be $\sigma_{xx} = g_s g_v \frac{e^2 \tau \omega}{\hbar^2} \frac{\mu}{4\pi} (1 - r^2)$. The linear classical Hall conductivity is calculated to be $\sigma_{xy} = -\sigma_{yx}^{\text{L}} = -g_s g_v \frac{e^3 \tau \omega^2 B}{\hbar^4} \frac{v_F^2}{4\pi} (1 - r^2)$, and the OMM induced intrinsic Hall conductivity is found to be $\sigma_{xy}^{\text{O}} = -\sigma_{yx}^{\text{O}} = -g_s g_v \frac{e^3 B}{\hbar^2} \frac{v_F^2}{8\pi \mu^2} r^2$. Note that the linear conductivities presented above are calculated up to zeroth order in the tilt for simplicity, and this does not change our results qualitatively.

To understand the dependence of the NL resistivity on the magnetic field to the lowest order, we use the magnetic field dependence of the different linear and NL conductivities in Eq. (1) to obtain

$$\begin{pmatrix} \tilde{\rho}_{xx}^{(2)} \\ \tilde{\rho}_{yx}^{(2)} \end{pmatrix} \sim \begin{pmatrix} B^0 & B^1 \\ B^1 & B^0 \end{pmatrix} \begin{pmatrix} B^1 & B^0 & B^0 & B^1 \\ B^0 & B^1 & B^1 & 0 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^1 \\ 0 \end{pmatrix}. \quad (19)$$

Since we are interested in only the lowest-order magnetic field dependence of the NL resistivity, we have neglected the quadratic B dependence and have thus put $\rho_{xy}^2 = 0$. Focusing on the lowest-order magnetic field dependence, we find $\tilde{\rho}_{xx}^{(2)} \propto B + O(B^3) + \dots$, and $\tilde{\rho}_{yx}^{(2)} \propto B^0 + O(B^2) + \dots$. Interestingly, this magnetic field dependence of the second-order

TABLE I. Analytical results of the different contributions to the NL conductivities. For conciseness we have defined $r = \Delta/\mu$. The Berry curvature dipole, Lorentz force and anomalous velocity, phase-space factor, and OMM induced NL conductivities are written in units of $\{\tilde{\sigma}^{\text{NAH}}, \tilde{\sigma}^{\text{AL}}, \tilde{\sigma}^{\text{B}} = \tilde{\sigma}^{\text{OMM}}\} = g_s g_v \frac{e^3 \tau \omega}{\hbar^2} \frac{\Delta v_t}{4\pi \mu^2} \{1, \frac{e v_F^2 \tau \omega}{\hbar^2 \mu} B, \frac{e v_F^2 \tau \omega}{\hbar^2 \mu} B\}$.

NL conductivities	Anomalous Hall NAH [Eq. (10)]	Lorentz force and Berry curvature AL [Eq. (12)]	Phase-space factor B [Eq. (13)]	Orbital magnetic moment OMM [Eq. (11)]
σ_{xxx}	0	0	$\tilde{\sigma}^{\text{B}} (\frac{15}{4} - 15r^2 + \frac{45}{4}r^4)$	$-\tilde{\sigma}^{\text{OMM}} (\frac{27}{4} - 21r^2 + \frac{45}{4}r^4)$
$\sigma_{yyx} = \sigma_{yx}$	0	$-\tilde{\sigma}^{\text{AL}} \frac{3}{4} (1 - r^2)$	$\tilde{\sigma}^{\text{B}} (\frac{1}{2} - \frac{17}{4}r^2 + \frac{15}{4}r^4)$	$-\tilde{\sigma}^{\text{OMM}} (\frac{9}{4} - 7r^2 + \frac{15}{4}r^4)$
σ_{xyy}	0	$\tilde{\sigma}^{\text{AL}} \frac{3}{2} (1 - r^2)$	$\tilde{\sigma}^{\text{B}} (\frac{11}{4} - \frac{13}{2}r^2 + \frac{15}{4}r^4)$	$-\tilde{\sigma}^{\text{OMM}} (\frac{9}{4} - 7r^2 + \frac{15}{4}r^4)$
$\sigma_{yx} = \sigma_{xy}$	$-\tilde{\sigma}^{\text{NAH}} \frac{3}{4} (1 - r^2)$	0	0	0
σ_{yxx}	$\tilde{\sigma}^{\text{NAH}} \frac{3}{2} (1 - r^2)$	0	0	0

resistivity is in contrast to the linear order resistivity, for which we have $\rho_{xx}^{(1)} \propto B^0$, while $\rho_{xy}^{(1)} \propto B$.

In a more general form valid for all 2D materials, we obtain

$$\tilde{\rho}_{xx}^{(2)} = -\rho_{xx}^2 [\rho_{xx}\sigma_{xx} + 2\rho_{yx}\sigma_{xy} + \rho_{xy}\sigma_{yx}], \quad (20)$$

$$\tilde{\rho}_{yx}^{(2)} = -\rho_{xx}^2 \rho_{yy}\sigma_{yxx}. \quad (21)$$

Equation (20) for the NL resistivity is main result of this paper, while Eq. (21) is the NL (anomalous) Hall resistivity, an experimental manifestation of the NL Hall conductivity predicted by Sodemann and Fu [15]. From Eq. (20) it can be clearly seen that the magnetic field dependence of the NL resistivity comes from both the linear and NL conductivities. However, all the NL conductivities arise from the presence of a finite OMM and the Berry curvature. Thus, we conclude that the NL resistivities are induced by the quantum geometric properties of the electron wave function, and they are of purely quantum mechanical origin.

In the limit $\omega\tau \ll 1$ (generally valid for transport experiments), we can express the magnetoresistivity and Hall resistivity for the massive tilted Dirac Hamiltonian in a simple form as

$$\tilde{\rho}_{xx}^{(2)} = -\frac{3\pi^2\hbar^2\Delta v_t v_F^2 r^2}{e^2\tau\mu^6(1-r^2)^3}B, \quad (22)$$

$$\tilde{\rho}_{yx}^{(2)} = -\frac{3\pi^2\hbar^2\Delta v_t}{2e^3\tau^2\mu^5(1-r^2)^2}. \quad (23)$$

Here, we have neglected the contribution of the OMM induced intrinsic Hall conductivity in the linear response, which is relatively smaller than the classical Hall conductivity (see Appendix D for the exact form). Clearly, the magnetoresistivity and Hall resistivity have different scattering times as well as chemical potential dependences. The Δ factor signifies the broken inversion symmetry, which enables finite values of the Berry curvature and the OMM. The factor v_t highlights the effect of tilt, or the anisotropy of the Fermi surface or band velocity, which is pivotal to obtain NL conductivities, as discussed earlier.

From Eqs. (22) and (23), we define two experimentally relevant quantities which are independent of the scattering timescale. For the resistivity we define $\tilde{\rho}_{xx}^{(2)}/\rho_{xx}$, which is equivalent to the ratio of the NL longitudinal voltage to the linear voltage (multiplied by current) $V_{\parallel}^{(2)}/(V_{\parallel}^{(1)}I_x)$. The variation of this quantity with the chemical potential is shown in Fig. 4(a). For the NL Hall resistivity, we define $\tilde{\rho}_{yx}^{(2)}/\rho_{xx}^2$, which is equivalent to the ratio of the NL Hall voltage to the square of the linear voltage $V_{\perp}^{(2)}/(V_{\parallel}^{(1)})^2$, and the corresponding chemical potential dependence is highlighted in Fig. 4(b). We find that the ratio of $\tilde{\rho}_{xx}^{(2)}/\rho_{xx}$ is finite in a small region in the vicinity of the band edge, while the ratio $\tilde{\rho}_{yx}^{(2)}/\rho_{xx}^2$ is finite in the vicinity of the band edge over a relatively larger region of μ .

V. DISCUSSION

In this paper, we have considered time reversal symmetric systems for two reasons. First, for TRS-preserving systems the NL Drude conductivity vanishes; hence, the predicted NL magnetoconductivities make the total NL resistivity finite.

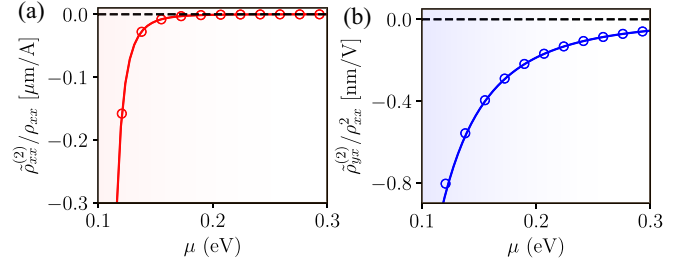


FIG. 4. The variation of the scattering time independent (a) NL resistivity ratio $\tilde{\rho}_{xx}^{(2)}/\rho_{xx}$ and (b) NL Hall resistivity ratio $\tilde{\rho}_{yx}^{(2)}/\rho_{xx}^2$ with the chemical potential μ . The open circles correspond to the numerical result ($T = 50$ K), and the solid line represents our analytical result at zero temperature. The Hamiltonian parameters are the same as those in Fig. 2, and we have considered $\tau = 1$ ps.

Second, the NL resistivity is caused solely by the geometric quantities, namely, the Berry curvature and the OMM. These two observations make the experimental realization of the predicted NL resistivity and its physical origin very clear. In contrast, in systems where both SIS and the TRS are broken, like doped magnetic semiconductors [45], the NL magnetoresistivity will be accompanied by an additional contribution from Drude resistivity and the classical Lorentz force contribution, as shown explicitly in Appendix C. Interestingly, this novel NL magnetoresistivity of quantum mechanical origin is not restricted to 2D systems, and it can also be finite for three-dimensional systems [46–48].

We emphasize that the relaxation time approximation, which we have used in our paper, while being very insightful, is a simplified approach. More rigorous approaches, developed in Refs. [11,12,49,50], among others, can be used to include the field dependence in the scattering time. This remains a future direction for us to explore NL magnetoresistivity after including the electric field dependence of the scattering timescale, along with skew scattering. One can also extend our semiclassical approach to the quantum kinetic framework [51–53] with a magnetic field, although we believe that this is likely to produce results similar to what we have in this paper. Furthermore, going beyond the semiclassical regime which is valid for small magnetic fields, it will be interesting to see how our results change for large magnetic fields which give rise to Landau levels.

VI. CONCLUSION

To summarize, we explored the impact of band geometric quantities on second-order NL magnetotransport in 2D anisotropic systems. Specifically, we studied second-order NL magnetoresistivity, which relates the quadratic NL voltage generated in response to an applied current in crystalline materials. We showed that the interplay of the Berry curvature, the OMM, and the Lorentz force can induce NL resistivity in 2D systems which is purely quantum mechanical in origin. We found that in the presence of a *perpendicular* magnetic field the NL magnetoresistivity has nontrivial linear B dependence. In intrinsically time reversal symmetric systems, where the Drude contribution to the longitudinal NL conductivity is identically zero, the predicted NL magnetoresistivity is the

only finite NL resistivity contribution. Our findings pave the way for further understanding of the nontrivial transport signature of band geometry in quantum materials.

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APPENDIX A: DEFINING THE NONLINEAR RESISTIVITY MATRIX

In this Appendix, we present a general approach of defining the NL resistivity matrix from the linear and NL conductivities. The definition of resistance follows from the familiar Ohm's law. If the voltage drop V is measured in the presence of constant current I , then the resistance R is defined as the ratio of these two quantities, $R = \frac{V(I)}{I}$. Within the linear response theory, voltage is considered to be linearly proportional to current, and hence, the resistance is current independent. However, this dependence of voltage on current can be nonlinear in general, which makes the resistance current dependent. Up to second order in current, the voltage can be expressed as [28]

$$V_a = V_a^{(1)} + V_a^{(2)} = R_{ab}^{(1)} I_b + \tilde{R}_{abc}^{(2)} I_b I_c. \quad (\text{A1})$$

Here, the voltage $V^{(1)}$ ($V^{(2)}$) is linear (quadratic) in current, $R_{ab}^{(1)}$ is the linear resistance, and $\tilde{R}_{abc}^{(2)}$ represents NL resistance. Strictly speaking, $\tilde{R}_{abc}^{(2)}$ does not have the dimension of resistance (hence we use the tilde), but we will still call it NL resistance in spirit. In the linear response regime, one measures the voltage $V = V^{(1)}$ in response to an applied current I , and resistance can be obtained using $R_{ab}^{(1)} = V_a/I_b$. From a theoretical point of view, instead of resistance we calculate the conductivity σ from the relation $j_a = \sigma_{ab} E_b$, where E and j are the applied electric field and generated current density, respectively. If we identify the measured voltage V with E and the current I with j , then the resistance can be easily connected to the conductivity as $R^{(1)} \sim \rho = [\sigma]^{-1}$, with ρ being the linear resistivity. However, the scenario for $\tilde{R}_{abc}^{(2)}$ is not as simple as for the linear resistivity, and to remedy this we define the quantity $\tilde{R}_{abc}^{(2)}$ in terms of the NL conductivities below.

From an experimental point of view, we have the relation between the electric field and the current,

$$E_a = \rho_{ab}^{\text{total}} j_b. \quad (\text{A2})$$

Here, ρ_{ab}^{total} is the general NL resistivity, which may vary with the current. Below, we will retain only the first-order NL term in the resistivity [27],

$$\rho_{ab}^{\text{total}} = \rho_{ab}^{(1)} + \rho_{abc}^{(2)} j_c + O(j^2), \quad (\text{A3})$$

and obtain an expression for $\rho_{abc}^{(2)}$ in terms of the linear and NL conductivities. To this end, we rewrite the left-hand side

of Eq. (A2) invoking the property of the Kronecker delta function as

$$\delta_{ac} E_c = \rho_{ab}^{\text{total}} j_b. \quad (\text{A4})$$

Now, we decompose the Kronecker delta function into a product of σ^{-1} and σ matrices and add and subtract $\chi_{lmn} E_m E_n$ on the left-hand side. This yields

$$\sigma_{al}^{-1} (\sigma_{lc} E_c + \chi_{lmn} E_m E_n) - \sigma_{al}^{-1} \chi_{lmn} E_m E_n = \rho_{ab}^{\text{total}} j_b. \quad (\text{A5})$$

Substituting Eq. (A3) in Eq. (A5) and identifying the total current as $j_l = \sigma_{lc} E_c + \chi_{lmn} E_m E_n$, we get

$$\sigma_{al}^{-1} j_l - \sigma_{al}^{-1} \chi_{lmn} E_m E_n = \rho_{ab}^{(1)} j_b + \rho_{abc}^{(2)} j_b j_c. \quad (\text{A6})$$

From this, it is straightforward to identify $\rho_{ab}^{(1)} = \sigma_{ab}^{-1} = \rho_{ab}$. Furthermore, converting the electric field to current via the relation $E_m = \rho_{mo} j_o$ (within the first-order approximation), we obtain

$$\rho_{abc}^{(2)} j_b j_c = -\rho_{al} \chi_{lmn} \rho_{mo} \rho_{np} j_o j_p. \quad (\text{A7})$$

Since the repeated indices are summed over in Eq. (A7), we can substitute $p \rightarrow c$ and $o \rightarrow b$ in the right-hand side to facilitate a direct comparison with the left-hand side. This yields the following relation for the second-order resistivity:

$$\rho_{abc}^{(2)} = -\rho_{al} \chi_{lmn} \rho_{mb} \rho_{nc}. \quad (\text{A8})$$

We have used this equation to write Eq. (1).

APPENDIX B: LINEAR CONDUCTIVITIES

In this Appendix, we provide the detailed calculation of the NDF up to linear order in E field and linear order in B field. For this we consider the ansatz $\delta g_1(t) = f_1^\omega e^{i\omega t} + f_1^{\omega*} e^{-i\omega t}$. Putting this in Eq. (6), we get

$$f_1^\omega = \sum_{\nu} (D\tau_\omega \hat{L}_B)^\nu \left(D \frac{e\tau_\omega}{\hbar} \mathbf{E} \cdot \nabla_{\mathbf{k}} \tilde{f} \right). \quad (\text{B1})$$

Considering a low magnetic field strength, we expand the series in Eq. (B1) to get various orders of magnetic field dependence. Separating the magnetic field dependences, we write the NDF as $f_1^\omega = f_{10}^\omega + f_{11}^\omega$, where the first subscript denotes the order of the electric field and the second subscript denotes the order of the magnetic field. This approach of expansion of the L_B operator is very common in textbooks and is known as the Zener-Jones method [2,42]. The scattering time independent [2] equilibrium part of the NDF, after the Taylor expansion, is given by

$$\tilde{f} = f - \epsilon_m f'. \quad (\text{B2})$$

The NDF proportional to the linear order of scattering time is given by

$$f_{10}^\omega = e\tau_\omega \mathbf{E} \cdot \mathbf{v} f', \quad (\text{B3})$$

$$f_{11}^\omega(\gamma, \xi) = -e\tau_\omega [\gamma \Omega_B \mathbf{v} f' + \xi (\mathbf{v}_m f' + \epsilon_m \mathbf{v} f'')] \cdot \mathbf{E}. \quad (\text{B4})$$

Equation (B3) is the magnetic field independent part from which the linear Drude conductivity originates. To point out the origin of magnetic field dependences in NDF, we use the coefficient γ for the phase-space factor and ξ for the OMM.

The quadratic scattering time dependent part of the NDF is given by

$$f_{11}^\omega(\alpha) = \alpha e \tau_\omega^2 \hat{L} \mathbf{v} \cdot \mathbf{E} f'. \quad (\text{B5})$$

Here, we have defined $\hat{L} \equiv \frac{e}{\hbar} (\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{k}}$, and α denotes the origin of the magnetic field dependence from the Lorentz force.

The magnetic field independent current of fundamental frequency can be written as $\mathbf{j}_{10}(t) = \mathbf{j}_{10}^\omega e^{i\omega t} + \mathbf{j}_{10}^{\omega*} e^{-i\omega t}$. From this we define the magnetic field independent conductivities as

$$\sigma_{ab}(\alpha \tau^0) = -\frac{e^2}{\hbar} \varepsilon_{abd} \int [d\mathbf{k}] \Omega_d f, \quad (\text{B6})$$

$$\sigma_{ab}(\alpha \tau) = -e^2 \tau_\omega \int [d\mathbf{k}] v_a v_b f'. \quad (\text{B7})$$

Equation (B6) is the anomalous Hall conductivity, which vanishes in the presence of TRS. Equation (B7) is the ordinary Drude conductivity.

Similarly, the magnetic field dependent current in fundamental frequency can be written as $\mathbf{j}_{11}(t) = \mathbf{j}_{11}^\omega e^{i\omega t} + \mathbf{j}_{11}^{\omega*} e^{-i\omega t}$. The corresponding conductivities, after separating various scattering time dependences, can be written as

$$\begin{aligned} \sigma_{ab}(\alpha \tau^0) &= \xi \frac{e^2}{\hbar} \varepsilon_{abd} \int [d\mathbf{k}] \Omega_d \epsilon_m f', \\ \sigma_{ab}(\alpha \tau^2) &= -\alpha e^2 \tau_\omega^2 \frac{eB}{\hbar} \int [d\mathbf{k}] v_a (v_y \partial_{k_x} v_b - v_x \partial_{k_y} v_b) f', \end{aligned} \quad (\text{B8})$$

and

$$\begin{aligned} \sigma_{ab}(\alpha \tau) &= e^2 \tau_\omega \int [d\mathbf{k}] [\xi v_{ma} v_b f' + \gamma v_a \Omega_B v_b f' \\ &\quad + \xi v_a (v_{mb} f' + \epsilon_m v_b f'')]. \end{aligned} \quad (\text{B10})$$

In the presence of TRS (broken SIS), $(\epsilon_m, \boldsymbol{\Omega})(-\mathbf{k}) = -(\epsilon_m, \boldsymbol{\Omega})(\mathbf{k})$, $\mathbf{v}(-\mathbf{k}) = -\mathbf{v}(\mathbf{k})$, and $\mathbf{v}_m(-\mathbf{k}) = \mathbf{v}_m(\mathbf{k})$; hence, all the conductivities $\propto \tau$ vanish. The only terms that survive are (i) the Lorentz force contribution $\propto \tau^2$, which gives rise to the classical Hall effect, and (ii) the anomalous velocity and OMM contribution that is $\propto \tau^0$. We have written these nonzero contributions in Eqs. (14) and (15) of the main text.

APPENDIX C: NONLINEAR CONDUCTIVITIES

In this Appendix, we calculate the NDF quadratic in the \mathbf{E} field and linear in the \mathbf{B} field. The corresponding master equation for the rectification part of the NDF is given by

$$f_2^0 = \sum_v (\alpha D \tau \hat{L}_B)^v D \frac{e\tau}{\hbar} \mathbf{E}^* \cdot \nabla_{\mathbf{k}} f_1^\omega. \quad (\text{C1})$$

It is straightforward to calculate the rectification part of the NDF from Eq. (C1). However, below we provide the second harmonic NDF obtained from Eq. (8). We expand Eq. (8) in orders of magnetic field and separate the NL NDF as $f_2^{2\omega} = f_{20}^{2\omega} + f_{21}^{2\omega}$. The magnetic field independent NL NDF is given by

$$f_{20}^{2\omega} = \frac{e^2 \tau_{2\omega} \tau_\omega}{\hbar} \mathbf{E} \cdot \nabla_{\mathbf{k}} (\mathbf{E} \cdot \mathbf{v} f'). \quad (\text{C2})$$

This coupled with band gradient velocity generates the NL Drude conductivity. The magnetic field dependent NL NDF proportional to the square of scattering time is given by

$$\begin{aligned} f_{21}^{2\omega}(\gamma, \xi) &= -\frac{e^2 \tau_{2\omega} \tau_\omega}{\hbar} \mathbf{E} \cdot [\gamma \Omega_B \nabla_{\mathbf{k}} (\mathbf{E} \cdot \mathbf{v} f') \\ &\quad \times \gamma \nabla_{\mathbf{k}} (\Omega_B \mathbf{E} \cdot \mathbf{v} f') \\ &\quad + \xi \nabla_{\mathbf{k}} \{\mathbf{E} \cdot (v_m f' + \epsilon_m \mathbf{v} f'')\}], \end{aligned} \quad (\text{C3})$$

The NL NDF proportional to the cubic order of scattering time is calculated to be

$$\begin{aligned} f_{21}^{2\omega} &= \frac{e^2 \tau_{2\omega} \tau_\omega}{\hbar} [\tau_{2\omega} \hat{L} (\mathbf{E} \cdot \nabla_{\mathbf{k}} (\mathbf{E} \cdot \mathbf{v} f')) \\ &\quad + \tau_\omega \mathbf{E} \cdot \nabla_{\mathbf{k}} (\hat{L} \mathbf{E} \cdot \mathbf{v} f')]. \end{aligned} \quad (\text{C4})$$

Using these expressions for NDF, we now calculate the general expressions for NL conductivities. Note that the NL responses can be generated either via coupling of anomalous velocity with linear NDF or through coupling of the band velocity of the Bloch electrons with the NL NDF.

The magnetic field independent second harmonic NL current can be written as $\mathbf{j}_{20}(t) = \mathbf{j}_{20}^{2\omega} e^{i2\omega t} + \mathbf{j}_{20}^{2\omega*} e^{-i2\omega t}$, and the corresponding NL conductivities are

$$\sigma_{abc}(\alpha \tau) = -\frac{e^3 \tau_\omega}{\hbar} \varepsilon_{abd} \int [d\mathbf{k}] \Omega_d v_c f', \quad (\text{C5})$$

$$\sigma_{abc}(\alpha \tau^2) = -\frac{e^3 \tau_\omega \tau_{2\omega}}{\hbar} \int [d\mathbf{k}] v_a \partial_{k_b} v_c f'. \quad (\text{C6})$$

In the presence of TRS, the NL conductivity $\propto \tau$ [Eq. (C5)] survives, while in the presence of SIS both Eqs. (C5) and (C6) vanish identically. So unlike the linear Drude conductivity, which is always nonzero, the NL Drude conductivity, Eq. (C6), vanishes in the presence of any of the symmetries (among TRS and SIS). In systems where both the symmetries are absent, the NL Drude conductivity may give rise to bilinear magnetoresistance if one considers the Zeeman coupling [26,54]. This theory was used to explain some recent experimental observations in the topological insulator surface states [27,28].

The magnetic field dependent second harmonic NL current can be expressed as $\mathbf{j}_{21}(t) = \mathbf{j}_{21}^{2\omega} e^{i2\omega t} + \mathbf{j}_{21}^{2\omega*} e^{-i2\omega t}$. The NL responses are induced by the OMM velocity, anomalous velocity, and the band gradient velocity. Below, we will express the NL conductivities as $\sigma_{abc} = \frac{e^3}{\hbar} \int [d\mathbf{k}] \tilde{\sigma}_{abc}$ for compactness. The anomalous velocity induced NL conductivity $\propto \tau$ is given by

$$\tilde{\sigma}_{abc}(\alpha \tau) = \tau_\omega \varepsilon_{abd} \Omega_d [\gamma \Omega_B v_c f' + \xi (v_{mc} f' + \epsilon_m v_c f'')], \quad (\text{C7})$$

and that $\propto \tau^2$ is given by

$$\tilde{\sigma}_{abc}(\alpha \tau^2) = -\tau_\omega^2 \varepsilon_{abd} \Omega_d \frac{eB}{\hbar} (v_y \partial_{k_x} v_c - v_x \partial_{k_y} v_c) f'. \quad (\text{C8})$$

The OMM velocity induced NL conductivity is given by

$$\tilde{\sigma}_{abc}(\alpha \tau^2) = \tau_\omega \tau_{2\omega} v_{ma} \partial_{k_b} v_c f'. \quad (\text{C9})$$

Finally, the band gradient velocity induced conductivity $\propto \tau^2$ is given by

$$\tilde{\sigma}_{abc}(\propto \tau^2) = \tau_\omega \tau_{2\omega} v_a [\Omega_B \partial_{k_b} v_c f' + \partial_{k_b} \{\Omega_B v_c f' + (v_{mc} f' + \epsilon_m v_c f'')\}], \quad (\text{C10})$$

and that $\propto \tau^3$ is given by

$$\tilde{\sigma}_{abc}(\propto \tau^3) = -\tau_{2\omega} \tau_\omega v_a \frac{eB}{\hbar} [\tau_{2\omega} (v_y \partial_{k_x} - v_x \partial_{k_y}) \partial_{k_b} v_c + \tau_\omega \partial_{k_b} (v_y \partial_{k_x} - v_x \partial_{k_y}) v_c] f'. \quad (\text{C11})$$

We find that in the presence of TRS, NL conductivities $\propto \tau$ and $\propto \tau^3$ vanish identically. Therefore, NL conductivities that are $\propto \tau^2$ survive and are highlighted in the main text in Eqs. (10)–(13). We emphasize that since the indices b and c are dummy indices, the expressions for NL conductivities have to be symmetrized. To facilitate this symmetry we have written the NL conductivities as $\sigma_{abc} = \sigma_{acb} = \frac{e^3}{\hbar} \int [d\mathbf{k}] [\tilde{\sigma}_{abc} + \tilde{\sigma}_{acb}] / 2$ in the main text. It is important to note that in the absence of both the SIS and TRS the NL conductivities $\propto \tau$ and $\propto \tau^3$ are expected to be nonzero, and

in that case NL resistivity can originate from the classical Drude and Lorentz force effects.

APPENDIX D: EXACT ANALYTICAL EXPRESSION OF NONLINEAR RESISTIVITY

In this Appendix, we provide the exact analytical expression for the NL resistivity including the effect of OMM induced intrinsic Hall conductivity. We calculate the linear resistivities using the expressions $\rho_{xx} = 1/\sigma_{xx}$ and $\rho_{xy} = -\rho_{yx} = -\sigma_{xy}/\sigma_{xx}^2$. Using these expressions in Eq. (20) and considering $\omega\tau \ll 1$, we calculate

$$\tilde{\rho}_{xx}^{(2)} = -\frac{3\pi^2 \hbar^2 \Delta v_t v_F^2 r^2}{2e^2 \tau \mu^6 (1-r^2)^3} \left[2 + \frac{\hbar^2}{\tau^2 \mu^2} \right] B. \quad (\text{D1})$$

Here, the second term in the parentheses originates from the OMM induced intrinsic Hall effect. It is evident from this expression that the OMM induced intrinsic Hall effect has a distinct signature in the NL resistivity. However, since in this paper we have considered $\hbar/\tau \ll \mu$, we neglect it in the main text. Another interesting feature of Eq. (D1) is that the NL resistivity diverges as we move towards $r \rightarrow 1$. This arises due to the fact that both the linear and NL conductivities vanish in this regime.

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