Analytic exposition of the graviton modes in fractional quantum Hall effects and its physical implications

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Neutral excitations in a fractional quantum Hall droplet define the incompressibility gap of the topological phase. In this paper, we derive a set of analytical results for the energy gap of the graviton modes with two-body and three-body Hamiltonians in both the long-wavelength and the thermodynamic limit. These allow us to construct model Hamiltonians for the graviton modes in different FQH phases, and to elucidate a hierarchical structure of conformal Hilbert spaces (null spaces of model Hamiltonians) with respect to the graviton modes and their corresponding ground states. Using the analytical tools developed, we perform numerical analysis with a particular focus on the Laughlin $\nu = 1/5$ and the Gaffnian $\nu = 2/5$ phases. Our calculation shows that for gapped phases, low-lying neutral excitations can undergo a "phase transition" even when the ground state is invariant. We discuss the compressibility of the Gaffnian phase, the possibility of multiple graviton modes, and the transition from the graviton modes to the "hollow-core" modes, as well as their experimental consequences.

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I. INTRODUCTION

The cryogenic two-dimensional electrons in a strong magnetic field can form an incompressible quantum fluid from strong interactions, leading to a large number of fractional quantum Hall (FQH) states describing a zoology of strongly correlated topological phases [1,2]. The characteristic fractional plateau of the Hall resistivity and the quantized thermal Hall conductance in these phases have been observed in experiments, revealing the topological nature of the quantum fluids [3–5]. The charged excitations in the FQH systems are predicted to carry fractional charge, with anyonic and even non-Abelian statistics [6–19]. Under the right conditions, these topological properties are expected to be invariant against local disturbance, making them desirable for the robust manipulation of quantum information [20–22].

Strictly speaking, topological systems are ideal systems where all energy scales in the system are sent to either zero or infinity. In realistic experiments or materials, all energy scales are finite, thus the dynamical aspects involving low-lying excitations cannot be ignored. In FQH systems, the low-lying excitations not only determine the quantization of certain transport properties in the thermodynamic limit, they can even form topological quantum fluids of their own [23–26]. The robustness of different topological indices (e.g., the Hall conductivity, the topological shift, and the central charge from the quasihole counting, etc.) can depend on different energy scales, and the interplay between the corresponding low-lying excitations leads to rich physics even within the same topological phase.

Understanding the low-lying excitations in strongly correlated systems is, however, a difficult task due to the lack of theoretical tools available. One can numerically study the energy spectrum of only small systems given the exponential increase of the Hilbert space with the system size, and the extrapolation to the thermodynamic limit is oftentimes unreliable. Analytically, perturbative calculations are not viable for FQH systems, because the kinetic energy is quenched and we are left with a purely interacting Hamiltonian. A popular approach is to construct model wave functions of these excitations either from the Jack polynomial formalism and its generalisation, or using the composite fermion theory [3,27–35]. These model states are indispensable for finite size numerical analysis, as well as offering insights into the universal nature of these excitations. The construction of these states also hints at the tantalising possibility of analytical treatment for very large system sizes, but so far the ability to rigorously compute physically relevant quantities in the thermodynamic limit has been mostly lacking.

In this paper, we focus on the neutral excitations of the FQH fluids, and derive a number of analytical results that are valid in the long-wavelength and the thermodynamic limit. Such excitations define the incompressibility gap of the topological phase, and they can also be responsible for quantum phase transitions within the same topological order [36–40]. In the long-wavelength limit, neutral excitations have a quadrupolar structure and can be understood as "spin-2 gravitons", which are closely related to the geometric deformation of the incompressible ground state [41–44]. We will thus denote excitations in this limit as quadrupole excitations or graviton modes. For short-range interactions, numerical analysis shows the energy of the graviton is large as compared to the charge gap given by the roton minimum [45,46]. However, it also seems possible to lower the graviton excitation

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energy by tuning the interaction between electrons, while maintaining the charge gap [47–50]. The analytical results we derive in this paper help us construct model Hamiltonians not only for the graviton modes, but also for capturing such transitions related to the graviton gap.

It is also important to note that the neutral excitations can be experimentally measured with inelastic photon scattering. This is especially true for graviton modes, since the momentum transfer of the photons is small [51–56]. Acoustic crystalline wave is also predicted to act like gravitational wave, which can interact with the graviton modes as a probe [57]. Furthermore, an optimal-control-based variational quantum algorithm has been designed for realizing the graviton mode in quantum computers [58]. There has also been much interest in the graviton mode recently due to its spin structure, allowing it to couple selectively to circularly polarized light, making them useful for experimentally distinguishing different topological phases [43]. In addition, it has been recently suggested that the coupling of the incompressible ground state to the graviton mode from geometric deformation can be responsible for the quench dynamics in FQH [59]. In the context of these experimental proposals and numerical results, Dirac composite fermion theory conjectures that certain FQH phases may have more than one graviton mode [60,61]. Thus both from the theoretical and the experimental perspectives, analytic results of long-wavelength neutral excitations can help us understand the fundamental nature of the geometric aspects of the FQH topological phases.

The organization of this paper will be as follows. The characteristic matrix formalism for calculating the energy of the graviton modes with two-body pseudopotentials is reviewed and a novel hierarchy structure of the density modes of the Laughlin states is rigorously proved with this method in Sec. II. This allows us to introduce the model Hamiltonians for the graviton modes in several FQH phases. Section III provides a more general formalism for the graviton modes with the three-body interactions, which can be used to study the graviton modes of non-Abelian FQH states such as the Moore-Read state and the Gaffnian state. In Sec. IV we perform numerical analysis of the low-lying excitations of the Laughlin $\nu = 1/5$ state (Laughlin-1/5 state for short) and the Gaffnian $\nu = 2/5$ state (Gaffnian state for short), with the theoretical tools we have developed. The experimental implications of the graviton modes in the Laughlin-1/5 phase are discussed when the model interaction is tuned, and we also discuss about the gaplessness of the Gaffnian phase. In Sec. V we summarize our results and discuss about the future outlooks.

II. TWO-BODY INTERACTIONS

Let us start with the dynamics of the long-wavelength neutral excitations for FQH states with two-body interactions. This formalism was first developed in Ref. [38], and we will summarize the results here and set the notations. This will be followed by a number of new results that are relevant to model two-body interactions, with rigorous statements on the nature of the neutral excitations that are valid in the thermodynamic limit (notations are listed in Table I).

TABLE I. Definition of various symbols used in the text.

$\hat{R}_i^a$	Guiding center operator
$\hat{\rho}_{\boldsymbol{q}}$	Guiding center density operator
$\delta\hat{\rho}_{\boldsymbol{q}}$	Regularized guiding center density operator
$V_{\boldsymbol{q}}$	Effective potential
$\tilde{S}_{\boldsymbol{q}}$	Static structure factor
$s_{\boldsymbol{q}}$	Reduced structure factor
$L_k(q^2)$	Laguerre polynomials
c^m	Expansion coefficients of effective potential
d^n	Expansion coefficients of reduced structure factor
$\vert\psi_{\boldsymbol{q}}\rangle$	SMA model wave function
$\Gamma_{mn}^{2\text{bdy}}$	Two-body characteristic matrix

The most general Hamiltonian for two-body interaction is

$$\hat{H}_{2\text{bdy}} = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} V_{\boldsymbol{q}} \sum_{i\neq j} e^{iq_a(\hat{R}_i^a - \hat{R}_j^a)}$$
$$= \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} V_{\boldsymbol{q}} \hat{\rho}_{\boldsymbol{q}} \hat{\rho}_{-\boldsymbol{q}} - N_e \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} V_{\boldsymbol{q}}. \quad (1)$$

Here N_e is the number of electrons, $\hat{H}_{2\text{bdy}}$ is a gapped two-body interaction with the guiding center density operator $\hat{\rho}_{\boldsymbol{q}} = \sum_i e^{iq_a\hat{R}_i^a}$ and $\boldsymbol{q} = (q_x, q_y)$, where i is the electron index, and $\hat{R}_i^a$ is the guiding center coordinates satisfying the commutation relation $[\hat{R}_i^a, \hat{R}_j^b] = -i\epsilon^{ab}\delta_{ij} l_B^2$. The length scale in the problem is the magnetic length $l_B = \sqrt{1/eB}$, with the perpendicular magnetic field B. Throughout this paper we assume translational invariance, so the ground state $|\psi_0\rangle$ of $\hat{H}_{2\text{bdy}}$ is both rotationally and translational invariant, with $\hat{H}_{2\text{bdy}}|\psi_0\rangle = E_0|\psi_0\rangle$.

The static structure factor for the unperturbed ground state is given by

$$\tilde{S}_{\boldsymbol{q}} = \frac{1}{N_e}\langle\psi_0|\hat{\rho}_{\boldsymbol{q}}\hat{\rho}_{-\boldsymbol{q}}|\psi_0\rangle = 1 + s_{\boldsymbol{q}} \quad (2)$$

where the reduced structure factor is its own Fourier transform, given by

$$s_{\boldsymbol{q}} = \frac{1}{N_e}\langle\psi_0|\sum_{i\neq j} e^{iq_a(\hat{R}_i^a - \hat{R}_j^a)}|\psi_0\rangle = -\int \frac{d^2q'}{2\pi} e^{i\boldsymbol{q}\times\boldsymbol{q}'} s_{\boldsymbol{q}'}. \quad (3)$$

Since the Laguerre polynomials $L_k(x)$ are eigenfunctions of the two-dimensional Fourier transform, we can thus expand $s_{\boldsymbol{q}}$ in the basis of the Laguerre-Gaussian series as follows:

$$s_{\boldsymbol{q}} = d^k L_k(q^2) e^{-\frac{1}{2}q^2} \quad (4)$$

where Einstein summation rule is assumed throughout this work for repeated indices. Note that d^k is only nonzero for odd k due to the fermion statistics, and Eq. (3) and Eq. (4) is true for generic rotationally invariant many-body states in a single Landau level. The regularized structure factor can be defined using the regularized guiding center density operator:

$$\delta\hat{\rho}_{\boldsymbol{q}} = \hat{\rho}_{\boldsymbol{q}} - \langle\psi_0|\hat{\rho}_{\boldsymbol{q}}|\psi_0\rangle = \hat{\rho}_{\boldsymbol{q}} - \frac{N_e}{2\pi q}\delta_{q,0} \quad (5)$$

where $q = |\boldsymbol{q}|$. so we have the following:

$$S_{\boldsymbol{q}} = \frac{1}{N_e}\langle\delta\hat{\rho}_{\boldsymbol{q}}\delta\hat{\rho}_{\boldsymbol{q}}\rangle_0 = \tilde{S}_{\boldsymbol{q}} - \frac{1}{N_e}\langle\hat{\rho}_{\boldsymbol{q}}\rangle_0\langle\hat{\rho}_{-\boldsymbol{q}}\rangle_0 \tag{6}$$

$$= 1 + \sum_{k=0}^{\infty} d_{2k+1}^{(0)} L_{2k+1}(q^2)e^{-\frac{1}{2}q^2} - \frac{1}{N_e}\langle\hat{\rho}_{\boldsymbol{q}}\rangle_0\langle\hat{\rho}_{-\boldsymbol{q}}\rangle_0 \tag{7}$$

and we define $\langle\hat{O}\rangle_0$ to be the expectation value of $\hat{O}$ with respect to the ground state.

It is natural to expand the effective potential $V_{\boldsymbol{q}}$ using the same orthogonal basis, which gives

$$V_{\boldsymbol{q}} = c^m L_m(q^2)e^{-\frac{q^2}{2}}. \tag{8}$$

Due to the orthogonality between the Laguerre polynomials, the ground-state energy has the following simple expression:

$$E_0 = c^m d^n \delta_{m,n} \tag{9}$$

where $\delta_{m,n}$ is the Kronecker delta. Once the expansion coefficients are given, one can easily calculate the ground-state energy. In general d^n still needs to be computed numerically as the variational energy of the ground state with respect to the two-body Haldane pseudopotential $\hat{V}_n^{2\text{bdy}} = \int d^2q L_n(q^2)e^{-\frac{q^2}{2}} \sum_{i\neq j} e^{iq_a(\hat{R}_i^a - \hat{R}_j^a)}$. However for model wave functions, certain coefficients of d^n are known exactly, since they are exact zero-energy states of their respective model Hamiltonians.

A. Energetics of the graviton modes

It is well known that the model wave functions for the graviton modes in the long-wavelength limit can be exactly constructed using the single mode approximation (SMA), which is defined as follows [36]:

$$|\psi_{\boldsymbol{q}}\rangle = \delta\hat{\rho}_{\boldsymbol{q}}|\psi_{\boldsymbol{0}}\rangle. \tag{10}$$

This family of wave functions is orthogonal to the ground state and at the same time retains some of the intrinsic correlation properties of the ground state. Our task is to calculate the variational energy of the graviton mode given by $\lim_{\boldsymbol{q}\to 0}|\psi_{\boldsymbol{q}}\rangle$:

$$\begin{aligned}\delta E_{\boldsymbol{q}\to 0} &= \lim_{\boldsymbol{q}\to 0}\frac{\langle\psi_{\boldsymbol{q}}|\hat{H}_{2\text{bdy}}|\psi_{\boldsymbol{q}}\rangle}{\langle\psi_{\boldsymbol{q}}|\psi_{\boldsymbol{q}}\rangle} - E_0\\ &= \lim_{\boldsymbol{q}\to 0}\frac{\langle\psi_0|[\delta\hat{\rho}_{-\boldsymbol{q}},[\hat{H}_{2\text{bdy}},\delta\hat{\rho}_{\boldsymbol{q}}]]|\psi_0\rangle}{2S_{\boldsymbol{q}}}\\ &= \lim_{\boldsymbol{q}\to 0}\sum_{i\neq j}\int\frac{d^2\boldsymbol{q}'}{(2\pi)^2}V_{\boldsymbol{q}'}\\ &\quad\times\frac{\langle\psi_0|[\delta\hat{\rho}_{-\boldsymbol{q}},[\hat{\rho}^i_{\boldsymbol{q}'}\hat{\rho}^j_{-\boldsymbol{q}'},\delta\hat{\rho}_{\boldsymbol{q}}]]|\psi_0\rangle}{2S_{\boldsymbol{q}}}.\end{aligned} \tag{11}$$

To calculate the double commutator, recall the GMP algebra for the density operators [36]:

$$[\delta\hat{\rho}_{\boldsymbol{q}_1},\delta\hat{\rho}_{\boldsymbol{q}_2}] = 2i\sin\frac{\boldsymbol{q}_1\times\boldsymbol{q}_2}{2}\delta\hat{\rho}_{\boldsymbol{q}_1+\boldsymbol{q}_2}. \tag{12}$$

This allows us to derive the final result, as shown in Ref. [38]:

$$\delta E_{\boldsymbol{q}\to 0} = \Gamma_{mn}^{2\text{bdy}} c^m d^n + O(q^2) \tag{13}$$

where m, n are both positive odd numbers because of the fermionic nature of electrons, and the characteristic matrix has an exact form:

$$\begin{aligned}\Gamma_{mn}^{2\text{bdy}} = \frac{(-1)^m}{2^8\eta\pi}\times[&2(m^2+m+1)\delta_{m,n}\\ &-(m+1)(m+2)\delta_{m,n-2} - m(m-1)\delta_{m,n+2}].\end{aligned} \tag{14}$$

It is worth noticing that in this characteristic matrix, the expansion coefficients of the interaction and the structure factor are symmetric. Furthermore, only the "nearest neighbors" ($n = m \pm 2$) are involved for any given m, which implies that the interplay between the components of different orders in the expansions of the interaction and the state is short ranged.

An immediate result is the variational energy of the graviton modes for the model Laughlin states at filling factor $\nu < 1/3$: for example, the variational energy of the graviton mode of the Laughlin-1/5 states (Laughlin-1/5 graviton mode for short) with respect to a generic two-body interaction can be written as [here $d^1 = d^3 = 0$ can be derived from Eq. (9) with the fact that the Laughlin-1/5 state lives within the null space of $\hat{V}_1^{2\text{bdy}} + \hat{V}_3^{2\text{bdy}}$]:

$$\delta E_{\boldsymbol{q}\to 0} = \sum_{m>3}\left(\Gamma_{mm}^{2\text{bdy}}c^m d^m + \Gamma_{m\pm 2,m}^{2\text{bdy}}c^{m\pm 2}d^m\right). \tag{15}$$

Thus with respect to the model Hamiltonian $\hat{V}_1^{2\text{bdy}}$ (only $c^1 > 0$) the variational energy is exactly zero: the Laughlin-1/5 graviton mode is in the null space of $\hat{V}_1^{2\text{bdy}}$.

B. Model Hamiltonians for the graviton modes

From both the theoretical and the experimental point of view, it is useful to find the model Hamiltonians, the exact eigenstates of which are the graviton modes. This will not only allow us to construct minimal models for phase transitions involving the softening of the graviton modes, but also help implement more systematic tuning of the realistic interactions in experiments for the observation of such transitions. For many topological phases, we can write down the root configurations of the graviton modes explicitly in the second quantized language. The model Hamiltonians can be identified accordingly using the numerically established local exclusion condition (LEC) formalism [62,63]. We illustrate this useful procedure for the graviton modes of the Laughlin phase at filling factor $\nu = 1/3$ and $\nu = 1/5$. This will be followed by a rigorous proof for all the Laughlin states with $\nu < 1/3$. The proof for the $\nu = 1/3$ case involves three-body interactions, and will be given in the next section.

Let us start with a brief introduction of the Jack polynomial formalism on the spherical geometry. The pseudospin structure in the quantized Landau levels enables us to define second-quantized wave functions in an occupation-number-like basis for the many-body wave functions. The magnetic field can be introduced by putting a Dirac magnetic monopole at the center of the sphere with a total of $2S$ magnetic fluxes, rendering a spinor structure of the single particle the wave function with total spin $S + N$, where N is the Laudau level (LL) index [64]. Without loss of generality we use the lowest LL (LLL) with $N = 0$, so the total number of single particle orbitals in the LLL is $N_o = 2S + 1$. We can thus express a many-body state with a string of N_o binary numbers, corresponding to the single particle orbitals sequentially from the north pole to the south pole. We use the integer 1 to denote an occupied orbital, and 0 for an empty orbital [3,30,31]. For example, if $S = 3$ and we have two electrons around the north pole of the sphere, this state can be denoted as $|1100000\rangle$, with a total number of seven orbitals.

On the disk geometry, the occupation basis also corresponds to the first-quantized wave functions with the symmetric gauge (we ignore the unimportant Gaussian factor). Each digit in the occupation basis from left to right corresponds to the orbital from the origin (z_k^0) to the edge ($z_k^{N_o-1}$), where $z_k = x_k + iy_k$ is the holomorphic coordinate of the kth electron. The many-body wave functions for the FQH states are linear combinations of such monomials. For example, the monomial $|1101\rangle$ in the first quantized form is given by

$$|1101\rangle \sim (z_1 - z_2)z_3^3 + (z_3 - z_1)z_2^3 + (z_2 - z_3)z_1^3. \tag{16}$$

The model wave functions for the FQH states on the sphere in many cases are Jack polynomials (or Jacks), which are a family of symmetric polynomials characterised by the so-called generalized Pauli principle. One important characteristic of the Jack polynomial states is the existence of a root configuration, with all of the occupation bases of the state "squeezed" from the root configuration [3,30,31]. For instance, if one considers the Laughlin-1/3 state with 3 electrons, the wave function $(z_1 - z_2)^3(z_1 - z_3)^3(z_2 - z_3)^3$ is a Jack polynomial denoted by $J^{\alpha=-2}_{|1001001\rangle}$. Here 1001001 is the root configuration, and $\alpha = -2$ in the superscript is derived from the admission rule of the root configuration. All coefficients of the monomials in the Jack polynomial are determined by α, and these monomials are "squeezed" from the root configuration. We denote two monomials m_1, m_2 and that m_2 is squeezed from m_1 by $m_1 \succ m_2$. That implies m_2 is obtained from m_1 by repeatedly moving two electrons in the binary string towards each other, without changing the total angular momentum of the monomial. Explicitly for the Laughlin-1/3 state with three electrons we have the following:

$$\begin{aligned} J^{-2}_{|1001001\rangle} &\sim (z_1 - z_2)^3(z_1 - z_3)^3(z_2 - z_3)^3 \\ &= |1001001\rangle - 3|0110001\rangle - 3|1000110\rangle \\ &\quad + 6|0101010\rangle - 15|0011100\rangle \end{aligned} \tag{17}$$

which consists of only the monomials squeezed from the root configuration $|1001001\rangle$, and the monomials are:

$$\begin{aligned} |1001001\rangle &= (z_1^3 - z_2^3)z_3^6 + (z_3^3 - z_1^3)z_2^6 + (z_2^3 - z_3^3)z_1^6 \\ |0110001\rangle &= (z_2^1 z_1^2 - z_1^1 z_2^2)z_3^6 + (z_1^1 z_3^2 - z_3^1 z_1^2)z_2^6 \\ &\quad + (z_3^1 z_2^2 - z_2^1 z_3^2)z_1^6 \\ |1000110\rangle &= (z_1^4 - z_2^4)z_3^5 + (z_3^4 - z_1^4)z_2^5 + (z_2^4 - z_3^4)z_1^5 \\ |0101010\rangle &= (z_2^1 z_1^3 - z_1^1 z_2^3)z_3^5 + (z_1^1 z_3^3 - z_3^1 z_1^3)z_2^5 \\ &\quad + (z_3^1 z_2^3 - z_2^1 z_3^3)z_1^5 \\ |0011100\rangle &= (z_2^2 z_1^3 - z_1^2 z_2^3)z_3^4 + (z_1^2 z_3^3 - z_3^2 z_1^3)z_2^4 \\ &\quad + (z_3^2 z_2^3 - z_2^2 z_3^3)z_1^4 \end{aligned} \tag{18}$$

Let us now move on to the model wave functions of the graviton modes, as constructed in Ref. [45]. In contrast to the FQH ground states, these wave functions are not Jack polynomials. Nevertheless, they have rich algebraic structure with a root configuration. Similar to the Jack polynomials, only monomials squeezed from the root configuration have nonzero coefficients in the basis expansion. We will also show later on that they can be linear combinations of Jack polynomials of different admission rules in some cases. The root configurations of the graviton modes for the Laughlin-1/3 state and 1/5 state for comparison are shown as follows:

$$1100001001001001001\cdots \quad \text{(Laughlin-1/3)} \tag{19}$$

$$1001000000010000100001\cdots \quad \text{(Laughlin-1/5).} \tag{20}$$

Since all the occupation bases are squeezed from the root configurations above, we can immediately apply the local exclusion condition (LEC) [62,63]. An LEC is defined by a triple, denoted by $\hat{n} = \{n, n_e, n_h\}$, giving the constraint that there can be no more than n_e electrons or n_h holes in a circular droplet containing n fluxes anywhere in the quantum fluid. For a spherical geometry, one can simply look at the droplet at the north pole for the *highest weight states*. Applying LEC to the graviton mode of the Laughlin-1/3 state immediately leads to the conclusion that it is a "Haffnian quasihole state" (thus a zero-energy state of the Haffnian model Hamiltonian; more details will be given in the next section). This is because the LEC of the Haffnian state is given by $\hat{n} = \{4, 2, 4\}$ and the droplet at the north pole of the Laughlin-1/3 graviton mode does obey this condition as Eq. (19) shows. We would like to emphasize that the admissible rules for Jack polynomials cannot be used here, since the graviton modes are not Jack polynomials (note that the Haffnian states are also not Jack polynomials). Using the same reasoning, it is easy to see that the graviton modes of the Laughlin-1/5 state is the zero-energy state of the Laughlin-1/3 model Hamiltonian, given that the LEC of the Laughlin-1/3 state is $\hat{n} = \{2, 1, 2\}$.

It is worth noting that although using LEC offers a very simple way of determining if the graviton mode belongs to the null space of some model Hamiltonian, it only applies to the cases where the root configurations are easy to find, and fundamentally the LEC scheme is only "proven" numerically. We will now proceed to analytically prove the more general cases for the Laughlin phases at $\nu_k = \frac{1}{2k+1}$, $k \in \mathrm{N}^+$. The $\nu = 1/3$ case will be left to the next section focusing on the three-body model Hamiltonians.

In the following discussion, we will denote the Laughlin state with the filling factor $\nu_k = \frac{1}{2k+1}$ by $|\psi_{\nu_k}\rangle$, the null space of the corresponding model Hamiltonian $\hat{H}_{\nu_k}$ by $\mathcal{H}_{\text{Laughlin}-\nu_k}$ ($\mathcal{H}_{\text{L}-\nu_k}$ for short in the figures) and the corresponding graviton mode $|\psi_{\nu_k,\boldsymbol{q}}\rangle = \delta\hat{\rho}_{\boldsymbol{q}}|\psi_{\nu_k}\rangle$ with $q \to 0$. The variational energy of $|\psi_{\nu_{k+1},\boldsymbol{q}}\rangle$ with respect to the model Hamiltonian $\hat{H}_{\nu_k}$ is given by

$$\begin{aligned} \delta E_{\boldsymbol{q}\to 0} &= \Gamma^{\text{2bdy}}_{mn} c^m d^n \\ &= \sum_m^{2k+1} \frac{(-1)^m c^m}{2^8 \eta\pi} \times [2(m^2 + m + 1)d^m \\ &\quad - (m+1)(m+2)d^{m+2} - m(m-1)d^{m-2}] \end{aligned} \tag{21}$$

where $\hat{H}_{\nu_k} = \sum_{m=1}^{2k-1} c^m L_m(q^2) e^{-\frac{q^2}{2}}$, $c^m > 0$ and m can only be odd. From Eq. (9) we know that

$$E_0 = c^m d^n \delta_{m,n} = \sum_m^{2k+3} c^m d^m = 0. \tag{22}$$

Thus as long as $c^m > 0$, the ground-state energy should always be zero. This is only possible when all the $d^m = 0$ with $m < 2k + 3$. We thus obtain

$$\delta E_{q\to 0} = 0. \tag{23}$$

Hence we proved that the graviton mode of the Laughlin state at $\nu = 1/(2k+3)$ [Laughlin-$1/(2k+3)$ graviton mode for short] should be contained in the null space of $\hat{H}_{\nu_k}$. In particular, the graviton mode at $\nu = 1/5$ is the exact zero-energy state of the $\hat{V}_1^{2\text{bdy}}$ pseudopotential. The latter can serve as the model Hamiltonian for the graviton mode. This is true for all the Laughlin states with $\nu < 1/3$.

III. THREE-BODY INTERACTIONS

To study the low-lying excitations and especially the graviton modes of the non-abelian FQH phases, it is important to extend the characteristic matrix formalism to the model Hamiltonians with three-body interactions. This is because the model Hamiltonians for the non-Abelian phases consist of few-body pseudopotentials involving clusters of more than two electrons [65,66]. Such interactions can also physically arise from LL mixing [67–70]. Analogous to the two-body case, for three-body interactions we can define the following reduced structure factor for the unperturbed ground state:

$$\begin{aligned}\bar{S}_{q_1,q_2} &= \sum_{i\neq j\neq k} \langle\psi_0|\hat{\rho}^i_{q_1}\hat{\rho}^j_{q_2}\hat{\rho}^k_{-q_1-q_2}|\psi_0\rangle \\ &= \sum_{i\neq j\neq k} \langle\psi_0|e^{iq_{1a}\hat{R}^a_i} e^{iq_{2a}\hat{R}^a_j} e^{-i(q_{1a}+q_{2a})\hat{R}^a_k}|\psi_0\rangle\end{aligned} \tag{24}$$

where the indices i, j, and k denote different electrons. Additional notations for three-body calculations can be found in Table II. A generic three-body Hamiltonian is given by

$$\begin{aligned}\hat{H}_{3\text{bdy}} &= \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2}\hat{\rho}_{q_1}\hat{\rho}_{q_2}\hat{\rho}_{-q_1-q_2} \\ &\quad - \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2}(\hat{\rho}_{q_1+q_2}\hat{\rho}_{-q_1-q_2} \\ &\quad + \hat{\rho}_{q_1}\hat{\rho}_{-q_1} + \hat{\rho}_{q_2}\hat{\rho}_{-q_2}) \\ &\quad - N_e \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2} \\ &= \sum_{i\neq j\neq k} \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2}\hat{\rho}^i_{q_1}\hat{\rho}^j_{q_2}\hat{\rho}^k_{-q_1-q_2}.\end{aligned} \tag{25}$$

Here V_{q_1,q_2} denotes the effective three-body interaction. The ground-state energy is thus given by the expectation value of

TABLE II. Definition of various symbols used in the three-body calculations.

$\hat{\rho}^i_{q_1}$	Guiding center density operator of the ith electron $\hat{\rho}^i_{q_1} = e^{iq_{1a}\hat{R}^a_i}$
$\bar{S}_{q_1,q_2}$	Reduced three-body structure factor
$V_{q_1 q_2}$	Three-body effective potential
$\tilde{q}_i$	Momentum components in Jacobi coordinates
$\tilde{Q}_i$	Square of $\tilde{q}_i$
$L_k^{(\alpha)}(q^2)$	Generalized Laguerre polynomials
$c^{m_1 m_2}$	Expansion coefficients of three-body effective potential effective potential
$\bar{d}^{n_1 n_2}$	Expansion coefficients of reduced three-body structure factor
$\tilde{\Gamma}^{3\text{bdy}}_{m_1 m_2 n_1 n_2}$	Three-body characteristic tensor
$\tilde{\Gamma}^{0/+/-}_{m_1 m_2 n_1 n_2}$	Diagonal/off-diagonal parts of the three-body characteristic tensor

the Hamiltonian with respect to the ground state:

$$\begin{aligned}E_0 &= \sum_{i\neq j\neq k} \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2}\langle\psi_0|\hat{\rho}^i_{q_1}\hat{\rho}^j_{q_2}\hat{\rho}^k_{-q_1-q_2}|\psi_0\rangle \\ &= \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2}\bar{S}_{q_1,q_2},\end{aligned} \tag{26}$$

which is analogous to the result in the two-body case. By constructing the SMA state as Eq. (10) shows, the energy of the graviton mode can be written as

$$\begin{aligned}\delta E_q &= \frac{\langle\psi_q|\hat{H}_{3\text{bdy}}|\psi_q\rangle}{\langle\psi_q|\psi_q\rangle} - E_0 \\ &= \frac{\langle\psi_0|[\delta\hat{\rho}_{-q},[\hat{H}_{3\text{bdy}},\delta\hat{\rho}_q]]|\psi_0\rangle}{2S_q} \\ &= \sum_{i\neq j\neq k} \int \frac{d^2q_1 d^2q_2}{(2\pi)^4} V_{q_1,q_2} \\ &\quad \times \frac{\langle\psi_0|[\delta\hat{\rho}_{-q},[\hat{\rho}^i_{q_1}\hat{\rho}^j_{q_2}\hat{\rho}^k_{-q_1-q_2},\delta\hat{\rho}_q]]|\psi_0\rangle}{2S_q}.\end{aligned} \tag{27}$$

For three-body interactions, it will be more convenient to use the Jacobi coordinates, which can separate the degree of freedom of the center-of-mass from other ones while maintaining the commutation relations between coordinates:

$$\begin{aligned}\hat{R}^a_{ij} &= \frac{1}{\sqrt{2}}(\hat{R}^a_i - \hat{R}^a_j) \\ \hat{R}^a_{ij,k} &= \frac{1}{\sqrt{6}}(\hat{R}^a_i + \hat{R}^a_j - 2\hat{R}^a_k) \\ \hat{R}^a_{ijk} &= \frac{1}{\sqrt{3}}(\hat{R}^a_i + \hat{R}^a_j + \hat{R}^a_k).\end{aligned} \tag{28}$$

As expected, after doing the coordinate transformations, the Hamiltonian in Eq. (25) contains no center-of-mass term and can be written as

$$\hat{H}_{3\text{bdy}} = \sum_{i\neq j\neq k} \int \frac{d^2\tilde{q}_1 d^2\tilde{q}_2}{3\times(2\pi)^4} V_{\tilde{q}_1\tilde{q}_2} e^{i\tilde{q}_{1a}\hat{R}^a_{ij}} e^{i\tilde{q}_{2a}\hat{R}^a_{ij,k}}, \tag{29}$$

which leads to a global linear transformation in the k space:

$$\tilde{\boldsymbol{q}}_1 = \frac{1}{\sqrt{2}}(\boldsymbol{q}_1 - \boldsymbol{q}_2),\quad \tilde{\boldsymbol{q}}_2 = \sqrt{\frac{3}{2}}(\boldsymbol{q}_1 + \boldsymbol{q}_2). \tag{30}$$

The transformation to the Jacobi coordinates also enables us to properly decompose the three-body calculations into the product of two symmetric two-body ones, with the Laguerre-Gaussian expansions of the effective three-body potential given by

$$V_{\tilde{\boldsymbol{q}}_1,\tilde{\boldsymbol{q}}_2} = \sum_{m_1m_2} c^{m_1m_2} L_{m_1}^{(0)}\left(\frac{\tilde{Q}_1}{2}\right) L_{m_2}^{(0)}\left(\frac{\tilde{Q}_2}{2}\right) e^{-\frac{1}{4}(\tilde{Q}_1+\tilde{Q}_2)} \tag{31}$$

where $L_m^{(i)}(x)$ is the generalised Laguerre polynomials, and $i=0$ gives the usual Laguerre polynomials. Note that the steps above are standard to deal with the three-body interactions in FQH and a detailed derivation can also be found in Ref. [47]. Without loss of generality only model Hamiltonians are considered here, so for the three-body interactions we take $i=0$. The reduced three-body structure factor can be expanded as

$$\bar{S}_{\tilde{\boldsymbol{q}}_1,\tilde{\boldsymbol{q}}_2} = \sum_{i;n_1n_2} d_i^{n_1n_2} \sqrt{\frac{n_1!n_2!}{(n_1+i)!(n_2-i)!}} \left(\frac{\tilde{\boldsymbol{q}}_1}{\tilde{\boldsymbol{q}}_2}\right)^i \times L_{n_1}^{(i)}\left(\frac{\tilde{Q}_1}{2}\right) L_{n_2}^{(-i)}\left(\frac{\tilde{Q}_2}{2}\right) e^{-\frac{1}{4}(\tilde{Q}_1+\tilde{Q}_2)} \tag{32}$$

where i must be even integers because of the fermionic statistics, $|i| \leqslant n_1+n_2$ due to rotational invariance and $\tilde{Q}_j \equiv |\tilde{\boldsymbol{q}}_j|^2$. Note that unlike the generic two-body case, the expansion of three-body interaction also contains the generalized Laguerre polynomials (when $i \neq 0$). Because the generalised Laguerre polynomials form a complete basis, any function of $\tilde{\boldsymbol{q}}_1$ and $\tilde{\boldsymbol{q}}_2$ can be expanded as in Eq. (32). Moreover, there seems to exist a singularity in the expansion when $|\tilde{\boldsymbol{q}}_j| \to 0$, but in fact this is not the case given that $\forall i \in \mathbb{Z}\backslash\mathbb{N}$, $|i| \leqslant n$:

$$\lim_{|\tilde{\boldsymbol{q}}_j|\to 0} \tilde{\boldsymbol{q}}_j^i \times L_n^{(i)}\left(\frac{\tilde{Q}_j}{2}\right) = 0. \tag{33}$$

It is useful to consider the expansion of a three-electron rotationally invariant state with zero center-of-mass angular momentum in the magnetic field $|\psi_3\rangle = \sum_{n_1,n_2} \alpha^{n_1,n_2}|n_1,n_2\rangle$, where the quantum number n_1 denotes the relative momentum between the first and the second electron, and n_2 represents the relative momentum between the center-of-mass of two electrons and the third electron [28]. Due to the fermionic statistics, the coefficients of expansion α^{n_1,n_2} are fixed with an overall factor and their specific values can be found in Ref. [71] and are also attached in the Supplemental Materials [72]. Thus for any generic rotationally invariant many-body wave function with zero center-of-mass angular momentum, $d_i^{n_1n_2}$ should be proportional to the product of two expansion coefficients of $|n_1,n_2\rangle$ based on Eq. (24):

$$d_i^{n_1n_2} \propto \alpha^{*n_1,n_2}\alpha^{n_1+i,n_2-i}. \tag{34}$$

This immediately leads to the conclusion that if $d_0^{n_1n_2}=0$, all the other $d_i^{n_1n_2}$ with the same indices n_1 and n_2 vanish as well.

By substituting Eq. (32) to the ground-state energy Eq. (26), a set of equations on the expansion coefficients can be derived from the orthogonality condition of the generalized Laguerre polynomials, which gives the ground-state energy:

$$E_0 = \delta_{m_1,n_1}\delta_{m_2,n_2} c^{m_1m_2} d_0^{n_1n_2}. \tag{35}$$

For model wave functions and the corresponding model Hamiltonians, the ground-state energy is zero. Using the same technique as the two-body case, we can write down the zeroth-order term of the energy gap $\delta\tilde{E}_{\boldsymbol{q}}$ in the long-wavelength limit, given as follows:

$$\begin{aligned}\delta\tilde{E}_{\boldsymbol{q}\to 0} =& \Gamma^i_{m_1m_2n_1n_2} c^{m_1m_2} d_i^{n_1n_2}\\ &\propto (\Gamma^i_{m_1m_2n_1n_2}\alpha_{n_1+i,n_2-i}) c^{m_1m_2}\alpha^{*n_1,n_2},\end{aligned} \tag{36}$$

which means that the i-dependence in $d_i^{n_1n_2}$ can be absorbed into the characteristic tensor. The final result is given by

$$\begin{aligned}\delta\tilde{E}_{\boldsymbol{q}\to 0} &= \tilde{\Gamma}^{\text{3bdy}}_{m_1m_2n_1n_2} c^{m_1m_2}\bar{d}^{n_1n_2}\\ &= (\Gamma^0_{m_1m_2n_1n_2} + \Gamma^+_{m_1m_2n_1n_2} + \Gamma^-_{m_1m_2n_1n_2})\\ &\quad\times c^{m_1m_2}\bar{d}^{n_1n_2} \times \mathcal{C}\end{aligned} \tag{37}$$

where the constant $\mathcal{C} = -1/(2^6 \times 3\eta\pi^2)$ and we have defined the factor:

$$\begin{aligned}\bar{d}^{n_1n_2} &\equiv \int \frac{d^2\tilde{\boldsymbol{q}}_1 d^2\tilde{\boldsymbol{q}}_2}{(2\pi)^4} \bar{S}_{\tilde{\boldsymbol{q}}_1,\tilde{\boldsymbol{q}}_2} L_{n_1}^{(0)}\left(\frac{\tilde{Q}_1}{2}\right) L_{n_2}^{(0)}\left(\frac{\tilde{Q}_2}{2}\right) e^{-\frac{1}{4}(\tilde{Q}_1+\tilde{Q}_2)}\\ &\propto \alpha^{n_1,n_2} = \sum_{n_1,n_2}\langle n_1,n_2|\psi_3\rangle\end{aligned} \tag{38}$$

The explicit expressions of the characteristic tensor components are given by

$$\begin{aligned}\Gamma^0_{m_1m_2n_1n_2} &= 2(n_1^2+n_2^2+n_1+n_2+2)\delta_{n_1,m_1}\delta_{n_2,m_2}\\ &\quad - n_1(n_1-1)\delta_{n_1,m_1+2}\delta_{n_2,m_2}\\ &\quad - (n_1+1)(n_1+2)\delta_{n_1,m_1-2}\delta_{n_2,m_2}\\ &\quad - n_2(n_2-1)\delta_{n_1,m_1}\delta_{n_2,m_2+2}\\ &\quad - (n_2+1)(n_2+2)\delta_{n_1,m_1}\delta_{n_2,m_2-2},\end{aligned} \tag{39}$$

which corresponds to the diagonal terms ($i=0$) in the Laguerre-Gaussian expansion of the structure factor. The expression is also reminiscent of two copies of two-body characteristic matrices given in Eq. (14). Meanwhile the contributions from the off-diagonal terms in the reduced ground-state structure factor are given by

$$\begin{aligned}\Gamma^+_{m_1m_2n_1n_2} &= \frac{\alpha_{n_1+2,n_2-2}}{\alpha_{n_1,n_2}} \times \sqrt{\frac{n_1!n_2!}{(n_1+2)!(n_2-2)!}}\\ &\quad\times (n_1+1)(n_1+2)(\delta_{m_1,n_1}-\delta_{m_1,n_1+2})\\ &\quad\times (\delta_{m_2,n_2}-\delta_{m_2,n_2-2})\end{aligned} \tag{40}$$

and

$$\begin{aligned}\Gamma^-_{m_1m_2n_1n_2} &= \frac{\alpha_{n_1-2,n_2+2}}{\alpha_{n_1,n_2}} \times \sqrt{\frac{n_1!n_2!}{(n_1-2)!(n_2+2)!}}\\ &\quad\times (n_2+1)(n_2+2)(\delta_{m_2,n_2}-\delta_{m_2,n_2+2})\\ &\quad\times (\delta_{m_1,n_1}-\delta_{m_1,n_1-2}),\end{aligned} \tag{41}$$

which correspond to $i = +2$ and -2 in Eq. (36). These terms physically captures the contributions to the energy of the graviton modes from different $|n_1, n_2\rangle$ components. Based on the antisymmetric property of fermionic wave functions, n_1 must be odd. Rigorous and detailed derivations on all the results we have got so far can be found in the Supplemental Material [72].

Note that compared to the existing methods of studying the graviton modes, the characteristic tensor formalism can determine the neutral gap with just the information of the ground state, making the exact diagonalization to obtain excited energy spectra unnecessary. Since $\Gamma^{3\text{bdy}}_{m_1 m_2 n_1 n_2}$ is universal and independent of the microscopic details, for a given FQH state with a model Hamiltonian or even realistic interaction, once the numerical properties of the expansion coefficients are determined, the behavior of the graviton mode can be fully depicted by Eq. (37).

A. Model Hamiltonian of Laughlin-1/3 graviton mode

With the help of the three-body characteristic tensor formalism, now the behavior of the Laughlin-1/3 graviton mode can be analytically discussed. From Eq. (38) and Eq. (34), we can decompose $\bar{d}^{n_1,n_2}$ as follows:

$$\bar{d}^{n_1,n_2} = \sum_{i=1}^{k_{n_1+n_2}} \lambda_{n_1+n_2,i} A_i^{n_1,n_2} \tag{42}$$

where $k_{n_1+n_2}$ denotes the degeneracy of three-body pseudopotentials (or the highest-weight three-body eigenstates of the total angular momentum operator) with total relative angular momentum $n_1 + n_2$; $\lambda_{n_1+n_2,i}$ depends on the ground-state wave function, but $A_i^{n_1,n_2}$ are well defined as shown in Ref. [71]. When there is no degeneracy in the highest weight state wave function (i.e., $k_{n_1+n_2} = 1$, which is true for $n_1 + n_2 < 9$), $\bar{d}^{n_1,n_2}$ can be considered to be proportional to $A_1^{n_1,n_2} = |\alpha^{n_1,n_2}|^2$. Thus from the ground-state energy in Eq. (26), considering $d^1 = 0$ which gives $d^{1,n_2} = 0$ for this state, we have

$$\begin{aligned} &\lambda_{1+n_2,1} = 0, \quad \text{for} \quad n_2 = 2, 4, 5, 6, 7 \\ &\quad \Rightarrow \lambda_{a,1} = 0, \text{for} \quad a = 3, 5, 6, 7, 8 \\ &\quad \Rightarrow \bar{d}^{n_1 n_2} = 0, \text{for} \quad n_1 + n_2 = 3, 5, 6, 7, 8. \end{aligned} \tag{43}$$

We have thus proved that the graviton mode of the Laughlin-1/3 state lives in the null space of the following three-body Hamiltonian:

$$\hat{H}^{3\text{bdy}} = \sum_{i=3}^{6} \tilde{c}_i \hat{V}_i^{3\text{bdy}} \tag{44}$$

where $\tilde{c}_4 = 0$ and $\tilde{c}_i > 0$ otherwise. In another word, the graviton mode is in the null space of the Haffnian model Hamiltonian $\hat{H}_{\text{Haffnian}} = \hat{V}_3^{3bdy} + \hat{V}_5^{3bdy} + \hat{V}_6^{3bdy}$ as Fig. 1 shows. This Hamiltonian provides us with nonvanishing coefficients of c_{m_1,m_2} with $m_1 + m_2 = 3, 5$, and 6. Thus based on Eq. (39), the energy of the graviton mode of the Laughlin-1/3 state is given by

$$\delta \tilde{E}_{\boldsymbol{q}} = \tilde{\Gamma}^{3\text{bdy}}_{m_1 m_2 n_1 n_2} c^{m_1 m_2} \bar{d}^{n_1 n_2} = 0 \tag{45}$$

where the algebraic properties of $\tilde{\Gamma}^{3\text{bdy}}_{m_1 m_2 n_1 n_2}$ make sure that only the coefficients $\bar{d}^{n_1 n_2}$ with $n_1 + n_2 \leqslant 8$ are involved.

We also illustrate the behaviours of $\bar{d}^{n_1,n_2}$ in Fig. 2, all the bold coefficients are zero given by the ground-state energy in Eq. (35) of the corresponding model Hamiltonian, denoted by different colors. Meanwhile the fermionic statistics ensures that all the gray coefficients have to vanish. Although the black coefficients are generally not zero, none of them are involved in the result. Thus one can prove that the graviton mode energy of the Laughlin-1/3 state with the Haffnian Hamiltonian is zero.

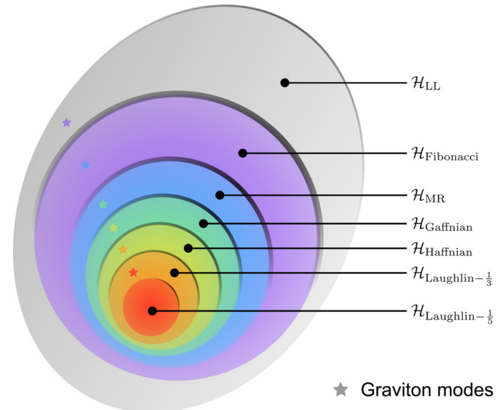

FIG. 1. Hierarchy of the null spaces of different FQH states. The null space of different model Hamiltonians can be organized as a hierarchical structure in the full Hilbert space. Meanwhile the density modes or more precisely, the graviton modes (denoted by the stars with the corresponding colors) constructed from model ground states live in the next larger null space. For example, the graviton modes of Laughlin-1/3 phase (orange star) live in the Haffnian null space (green circle), which contains the ground state and all the quasihole states of the Haffnian model Hamiltonian. It is efficient to verify this structure with the characteristic tensor formalism proposed in this paper.

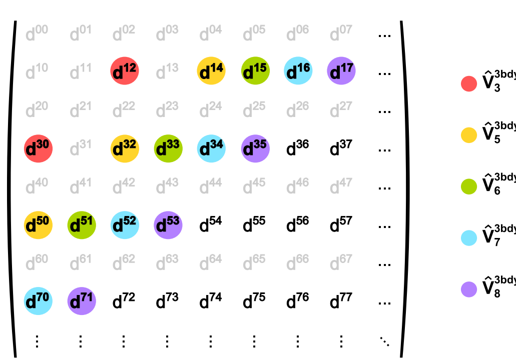

FIG. 2. Structure of the $\bar{d}_{n_1 n_2}$ coefficients of the Laughlin-1/3 state. Here gray coefficients are zero because of the fermionic statistics. Bold coefficients are zero because of the vanishing ground-state energy (contributions from different three-body model Hamiltonians are denoted by different colors). Black coefficients are unknown and not necessarily zero. Note that here we omit the bars over the $\bar{d}_{n_1 n_2}$ coefficients for simplicity.

B. Moore-Read graviton mode

Based on the same idea, only two coefficients $\bar{d}^{30}$ and $\bar{d}^{12}$ of the Moore-Read state vanish from the ground-state energy, so the energy gap of the corresponding graviton mode with the Hamiltonian $\hat{H}_{\mathrm{MR}} = \hat{V}_3^{3bdy}$ is given by

$$\begin{aligned}\delta\tilde{E}_{\boldsymbol{q}} &= \tilde{\Gamma}^{3\mathrm{bdy}}_{m_1m_2n_1n_2} c^{m_1m_2}\bar{d}^{n_1n_2}\\ &= c^{30}\big(\tilde{\Gamma}^{3\mathrm{bdy}}_{3032}\bar{d}^{32} + \tilde{\Gamma}^{3\mathrm{bdy}}_{3050}\bar{d}^{50}\big) + c^{12}\big(\tilde{\Gamma}^{3\mathrm{bdy}}_{1232}\bar{d}^{32} + \tilde{\Gamma}^{3\mathrm{bdy}}_{1214}\bar{d}^{14}\big)\\ &= -\frac{1}{2^8\times 3\eta\pi^2}\times[c^{30}(8\bar{d}^{32} - 16\bar{d}^{50})\\ &\quad + c^{12}(-24\bar{d}^{32} - 16\bar{d}^{14})]\\ &\propto \lambda_{5,1}\times(c^{30} + 3c^{12}) \end{aligned} \tag{46}$$

where $\lambda_{5,1}$ is the coefficient of the structure factor of the states with the total angular momentum quantum number $n_1 + n_2 = 5$ as in Eq. (42), which is proportional to the expectation value of $\hat{V}_5^{3\mathrm{bdy}}$ with respect to the Moore-Read state and we have used the ratio of the coefficients given in Ref. [71]:

$$\alpha^{50} : \alpha^{32} : \alpha^{14} = -\frac{\sqrt{5}}{4} : \frac{1}{2\sqrt{2}} : \frac{3}{4}. \tag{47}$$

Thus the graviton mode energy of the Moore-Read state can be determined by tuning c^{30} and c^{12}. Furthermore, similar to the Laughlin-1/3 state, the graviton mode of the Moore-Read state should live in the null space of four-body interactions, which will not be discussed in detail here.

C. Gaffnian graviton mode

For the Gaffnian state, the coefficients $\bar{d}^{30}$, $\bar{d}^{12}$, $\bar{d}^{50}$, $\bar{d}^{32}$, and $\bar{d}^{14}$ vanish because of the model Hamiltonian $\hat{H}_{\mathrm{Gaffnian}} = \hat{V}_3^{3\mathrm{bdy}} + \hat{V}_5^{3\mathrm{bdy}}$ [only c_{m_1,m_2} with $m_1 + m_2 = 3$ and 5 can be nonzero, so the coefficients d_{m_1,m_2} with the same indices have to vanish to make the ground-state energy zero as Eq. (35) shows]. Thus the energy gap of the corresponding graviton mode can be written as

$$\begin{aligned}\delta\tilde{E}_{\boldsymbol{q}} &= c_{50}\big(\tilde{\Gamma}^{3\mathrm{bdy}}_{5052}\bar{d}^{52} + \tilde{\Gamma}^{3\mathrm{bdy}}_{5070}\bar{d}^{70}\big)\\ &\quad + c^{32}\big(\tilde{\Gamma}^{3\mathrm{bdy}}_{3234}\bar{d}^{34} + \tilde{\Gamma}^{3\mathrm{bdy}}_{3252}\bar{d}^{52}\big)\\ &\quad + c^{14}\big(\tilde{\Gamma}^{3\mathrm{bdy}}_{1416}\bar{d}^{16} + \tilde{\Gamma}^{3\mathrm{bdy}}_{1434}\bar{d}^{34}\big)\\ &= -\frac{1}{2^8\times 3\eta\pi^2}\times[c^{50}(40\bar{d}^{52} - 40\bar{d}^{70})\\ &\quad + c^{32}(-16\bar{d}^{34} - 80\bar{d}^{52})\\ &\quad + c^{14}(-40\bar{d}^{16} - 24\bar{d}^{34})]\\ &\propto \lambda_{7,1}\times(5c^{50} + 2c^{32} + 9c^{14}) \end{aligned} \tag{48}$$

where $\lambda_{7,1}$ is the coefficient of the structure factor of the states with the total angular momentum quantum number $n_1 + n_2 = 7$, which is proportional to the expectation value of $\hat{V}_7^{3\mathrm{bdy}}$ with respect to the Gaffnian state, and the ratio of the coefficients given in Ref. [71] has been used:

$$\alpha^{70} : \alpha^{52} : \alpha^{34} : \alpha^{16} = -\frac{\sqrt{21}}{8} : \frac{1}{8} : \frac{\sqrt{15}}{8} : \frac{3\sqrt{3}}{8}. \tag{49}$$

Thus the graviton mode energy of the Gaffnian state can be determined by tuning c^{50}, c^{32} and c^{14}. Furthermore, there exist no c^{30} or c^{12} terms in Eq. (48), which indicates that the graviton mode of the Gaffnian state should live in the null space of the Moore-Read model Hamiltonian $\hat{V}_3^{3bdy}$. In particular, the variational energy of the Gaffnian graviton mode is independent of the strength of $\hat{V}_3^{3\mathrm{bdy}}$ in the Hamiltonian.

IV. NUMERICAL STUDY

As shown in Fig. 1, the theoretical derivations have revealed the well-organized hierarchical structure of the null space of multiple model Hamiltonians, in which the graviton modes of different FQH states reside. For the graviton modes to be experimentally relevant, they have to be low-lying states in the excitation spectrum. For fully-gapped FQH phases, the graviton modes are also gapped. Moreover, given they are neutral excitations, for realistic interactions they may also become gapless without affecting the robustness of the Hall conductivity plateau [47,48,50]. We now proceed to perform numerical calculations for the dynamical properties of the graviton modes, using the theoretical tools developed in the previous sections. We focus in particular on the Laughlin-1/5 state and the Gaffnian state, and discuss possible theoretical and experimental consequences.

A. Laughlin-1/5 graviton mode

We have shown that the graviton mode of Laughlin-1/5 state lives within the null space of $\hat{V}_1^{2\mathrm{bdy}}$, and it is a quantum fluid of Laughlin-1/3 quasiholes. If we look at a short-range interaction with model Hamiltonians involving only $\hat{V}_1^{2\mathrm{bdy}}$ and $\hat{V}_3^{2\mathrm{bdy}}$, the dynamics of the graviton modes is completely controlled by $\hat{V}_3^{2\mathrm{bdy}}$. It is thus useful to understand the low-lying excitations of the following toy model:

$$\hat{H}_L = (1-\lambda)\hat{V}_1^{2\mathrm{bdy}} + \lambda\hat{V}_3^{2\mathrm{bdy}} \tag{50}$$

The ground state is invariant when tuning the value of λ. In contrast, the low-lying excitations can be qualitatively different. In particular, when λ is close to zero, the graviton mode and the magnetoroton modes should be the low-lying excitations. On the contrary, if there exist states that are punished by $\hat{V}_1^{2\mathrm{bdy}}$ but not $\hat{V}_3^{2\mathrm{bdy}}$, then they will become the low-lying states when λ is close to unity. Thus one can expect to see the transition of the low-lying states when λ is continuously increased from 0 to 1.

The results of the Laughlin states with 6 electrons are shown in Fig. 3. While the ground state is invariant (Laughlin-1/5 state, denoted by the dark red color in Fig. 3), the low-lying excitations show a clear cross-over behavior. When $\lambda \to 1$, the density modes including the graviton modes and the multi-magnetoroton modes, shown by red spectrum in Fig. 3(a), are no longer low-lying excitations. The structure of the Hilbert space in the LLL is illustrated in Fig. 3(b). The null space of the Laughlin-1/5 model Hamiltonian (Laughlin-1/5 null space for short) denoted by the red circle is a proper subspace of the Laughlin-1/3 null space(light-red part), the complement space of which contains the states either only punished by $\hat{V}_1^{2\mathrm{bdy}}$ (blue circle), or punished by both $\hat{V}_1^{2\mathrm{bdy}}$

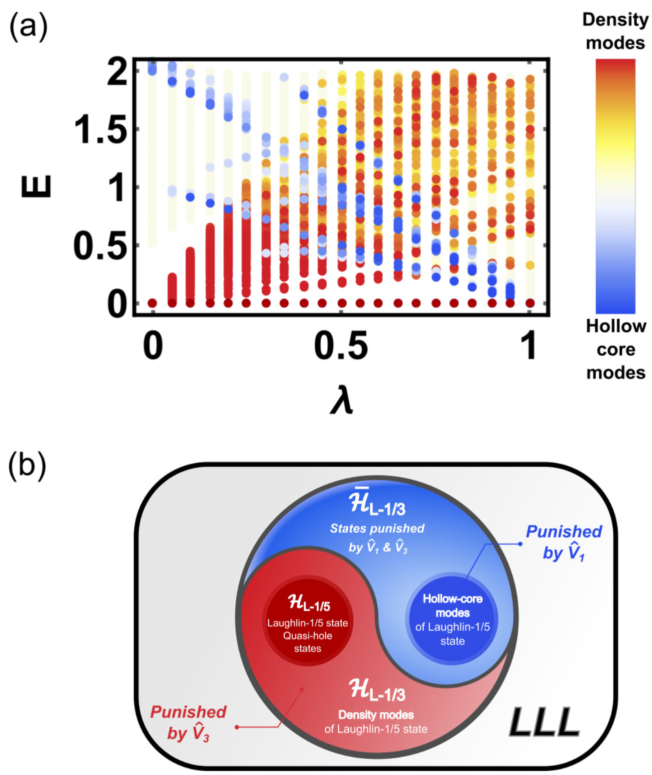

FIG. 3. Spectrum of the toy Hamiltonian $\hat{H}_L$ with respect to 6 electrons and 26 orbitals. (a) As the color bar shows, the red dots depict the spectrum of the density modes and the hollow-core modes are denoted by blue dots. The color of each dot is determined by calculating their collective overlap with all the density modes in the Laughlin-1/3 null space $\mathcal{H}_{\text{L-1/3}}$, or all the hollow-core modes in the complementary space $\bar{\mathcal{H}}_{\text{L-1/3}}$. As λ increases, though the ground state (denoted by dark red) stays invariant, the low-lying states show a clear cross over behavior and transform from density modes to hollow-core modes. (b) Illustrates the structure of the Hilbert space and the relationship between the states and the model Hamiltonians. The Laughlin-1/5 null space $\mathcal{H}_{\text{L-1/5}}$ (dark red circle) punished by neither $\hat{V}_1^{\text{2bdy}}$ nor $\hat{V}_3^{\text{2bdy}}$ is the sub-space of $\mathcal{H}_{\text{L-1/3}}$ (only punished by $\hat{V}_3^{\text{2bdy}}$). Meanwhile there exist the hollow-core modes (blue circle) only punished by $\hat{V}_1^{\text{2bdy}}$ in $\bar{\mathcal{H}}_{\text{L-1/3}}$. All of the other states are punished by both $\hat{V}_1^{\text{2bdy}}$ and $\hat{V}_3^{\text{2bdy}}$, living in the remaining part of $\bar{\mathcal{H}}_{\text{L-1/3}}$.

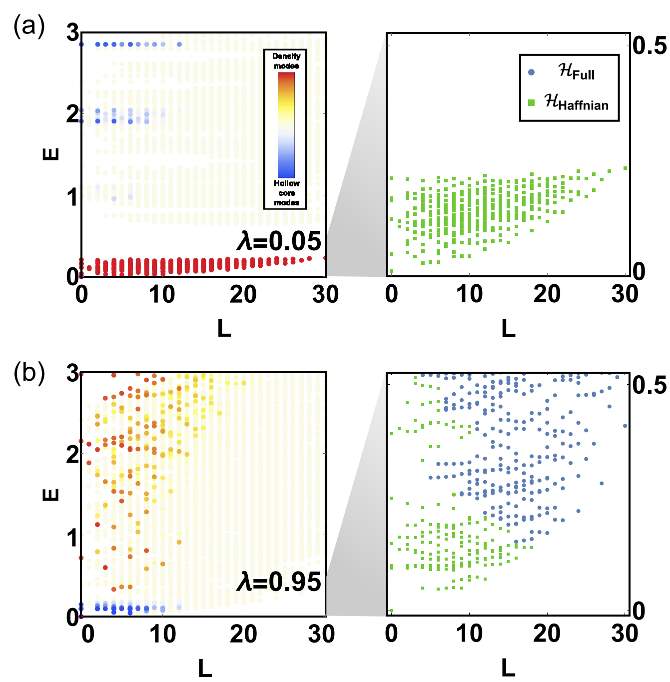

FIG. 4. Spectra of the toy Hamiltonian $\hat{H}_L$ diagonalized in different Hilbert spaces. The case with $\lambda = 0.05$ is shown in (a) and $\lambda = 0.95$ in (b). In the left panel, one can clear observe the transition of the low-lying states when λ increases from 0.05 to 0.95. These states have different nature as explained in Fig. 5. In the right panel, we zoom in on the spectra to the range $E \in [0, 0.5]$ and mark the states living in the Haffnian null space with green squares and other states with blue dots. As expected, by using the CF picture, both the density modes and the hollow-core modes live in the Haffnian null space. Note that the lowest angular momentum of the hollow-core modes is $L_{\min} = 4$ as can be seen in (b). Furthermore, the quantized energy of the hollow-core modes (especially when $\lambda \to 0$) can be understood by using the root configurations in Eq. (51).

and $\hat{V}_3^{\text{2bdy}}$ (light-blue part). We can refer to the blue states as the "hollow-core" modes, since they live in the null space of $\hat{V}_3^{\text{2bdy}}$ but out of the null space of $\hat{V}_1^{\text{2bdy}}$ [73–75].

It is useful to look more closely at the spectra of $\mathcal{H}_L$ with $\lambda = 0.05$ and $\lambda = 0.95$ as shown in the left panel of Fig. 4, where the density modes including the graviton modes make up the low-lying states when λ is close to 0. In contrast when λ is close to 1, the energy of these states significantly increases as expected so the low-lying excitations are replaced by the hollow-core modes. To understand better the nature of the low-lying states, one can also diagonalize $\hat{H}_L$ in different sub-Hilbert spaces, instead of the full Hilbert space of a single LL, and to check if the truncation of the Hilbert space affects the low-lying excitations. In the right panel of Fig. 4, the spectra of the Hamiltonian diagonalized in the full Hilbert space are shown, where the states that live almost entirely within the Haffnian null space are denoted by green squares. For both cases, $\hat{H}_L$ with $\lambda = 0.05$ (low-lying excitations consisted of density modes) and $\lambda = 0.95$ (low-lying excitations consisted of hollow-core modes), numerical studies show strong evidence that all the low-lying states live in the Haffnian null space. On the other hand, the graviton and the magnetoroton modes live within the null space of $\hat{V}_1^{\text{2bdy}}$ (which itself is a subspace of Haffnian null space), while the hollow-core modes live outside of the $\hat{V}_1^{\text{2bdy}}$ null space.

We can also understand the differences between these two types of low-lying states, by appealing to the intuitive picture from the composite fermion (CF) theory [29,76]. Figure 5 illustrates the physical distinctions between the graviton modes (low-lying states when $\lambda = 0.05$) and the hollow-core modes (low-lying states when $\lambda = 0.95$). According to the CF theory, the Laughlin-1/5 state of electrons can be reinterpreted as the Laughlin-1/3 state of CFs as Fig. 5(a) shows, because each CF contains one electron and two fluxes so the filling factor becomes $\nu^* = 1/(5-2) = 1/3$. Similar to the Landau levels of electrons, the discrete levels of CFs are sometimes named as "Λ levels" [76]. The Laughlin-1/3 null space only contains the states in the first CF level. The graviton and the magnetoroton modes are thus excitations

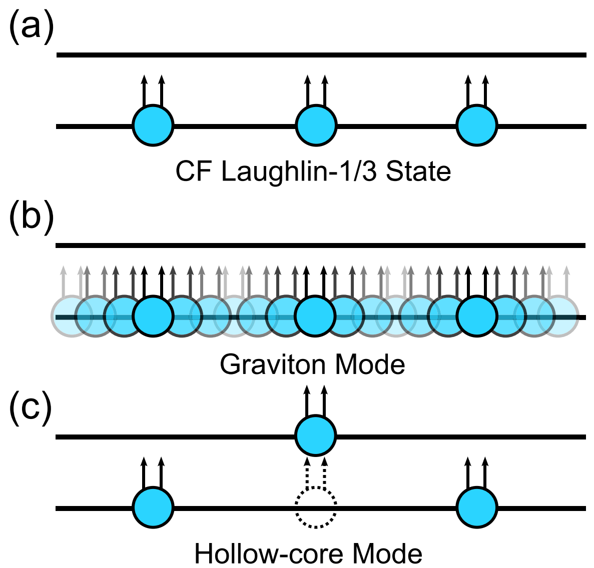

FIG. 5. Nature of the low-lying states with different model Hamiltonian in the composite fermion picture. (a) The Laughlin-$1/5$ state of the electrons can be reinterpreted as the Laughlin-$1/3$ state of CFs consisted of one electron and two fluxes. (b) The graviton modes can be understood as the excitations of CFs in the lowest CF level. (c) The hollow-core modes are created by exciting CFs to the second CF level, which still live in the Gaffnian null space.

within the partially filled first CF level, which are low-lying excitations for small λ. In contrast, when λ approaches one, the hollow-core modes come from the excitations of the CFs into the second CF level, in some sense similar to the graviton modes of the Laughlin-$1/3$ state. Some of the root configurations containing one or more of the "hollow-core" modes can be written as

$$\begin{aligned} &110000000010000100001000001\cdots && L=4\\ &110000000011000000000100001\cdots && L=8 \qquad (51)\\ &110000000011000000000110000\cdots && L=12 \end{aligned}$$

with the lowest angular momentum $L = 4$, agreeing well with Fig. 4(b). From these root configurations one can also understand the quantized energy of the hollow-core modes. Indeed, each pair of electrons in the root configuration (corresponding to a CF in the second CF level) contributes a unit of energy. Thus for six electrons, the highest energy should be ~ 3 as shown in Fig. 4(a).

B. Experimental significance

While we analyze the graviton modes above with only toy models, they can give insights on the experimental measurements of low-lying neutral excitations in FQH phases, using for example Raman scattering or inelastic photon scattering [52–54,77]. For the Laughlin phase at $\nu = 1/5$, a short-range realistic interaction (e.g., in the LLL, or with the Coulomb interaction renormalized by sample thickness or screening [78–80]), the graviton mode as well as the magnetoroton modes will be more prominent. However, since the realistic interaction cannot completely project out the complement of the null space of $\hat{V}_1^{\text{2bdy}}$, the graviton modes will always mix with the hollow-core modes, so their experimental signals will not be as clean as those from, for example, the Laughlin-$1/3$ phase.

With longer-range interactions (e.g., in higher LLs), it is still possible for the Laughlin-$1/5$ state to be robust in the sense that the plateau of the Hall conductivity can be observed [81]. However, for such interactions, we do not expect clear experimental signals of the graviton modes due to the strong mixing with the hollow-core modes. If the realistic interaction is short-ranged, but dominated by $\hat{V}_3^{\text{3bdy}}$, there will be no graviton modes (or quadruple excitations) at low energy. Instead, the low-lying excitations are in the complement of the null space of $\hat{V}_1^{\text{2bdy}}$, and in particular the quasihole excitations can be fractionalised and carry the charge of $e/10$. This is analogous to the nematic FQH phase at $\nu = 1/3$ observed in the experiments, and the fractionalization of the Laughlin-$1/3$ quasiholes near the phase transition [82]. It would thus be very interesting if the hollow-core modes, characterised by fractionalised Laughlin-$1/5$ quasiholes, can be observed in experiments.

There was also recent interest in the possibility of the multiple graviton modes in FQH states [83–85]. Here we show microscopically that at $\nu = 1/5$, the Laughlin phase has only a single graviton mode living in the null space of $\hat{V}_1^{\text{2bdy}}$. In particular, all the density modes are excitations in the lowest CF level, and their coupling to higher CF levels are suppressed by the short-range interaction. It is important to note from our analytical proof that this is the direct consequence of the fact that the Laughlin model wave function has exact zero energy with respect to $\hat{V}_1^{\text{2bdy}}$ and $\hat{V}_3^{\text{2bdy}}$. On the other hand, the CF state at $\nu = 2/7$, which can be understood as the particle-hole conjugate of the Laughlin-$1/5$ state within the lowest CF level, is no longer the exact zero-energy state with respect to $\hat{V}_1^{\text{2bdy}}$ and $\hat{V}_3^{\text{2bdy}}$. Thus the graviton mode of the $\nu = 2/7$ state will have components both within the null space of $\hat{V}_1^{\text{2bdy}}$ and the complement of it. One can reinterpret this as multiple graviton modes [55,59–61]: since the null space of $\hat{V}_1^{\text{2bdy}}$ corresponds to the lowest CF level, the two graviton modes indeed can be understood as geometric fluctuation within the lowest CF level, as well as the geometric fluctuation associated with the mixing between different CF levels. This will lead to two resonance peaks of opposite chirality with the Raman measurement, while the relative strength of the two peaks depends on the microscopic details of the electron-electron interaction.

C. Gaffnian graviton mode

The behaviours of the Gaffnian graviton modes at $\nu = 2/5$ are not entirely the same as the Laughlin-$1/5$ case. Based on the same idea, one can study these modes by diagonalizing the following Hamiltonian:

$$\hat{H}_G = (1-\lambda)\hat{V}_3^{\text{3bdy}} + \lambda\hat{V}_5^{\text{3bdy}} \qquad (52)$$

The spectrum of the droplet with 10 electrons is shown in Fig. 6. The density modes are still behaving as predicted by the theoretical derivations, i.e., occupying the low-lying states

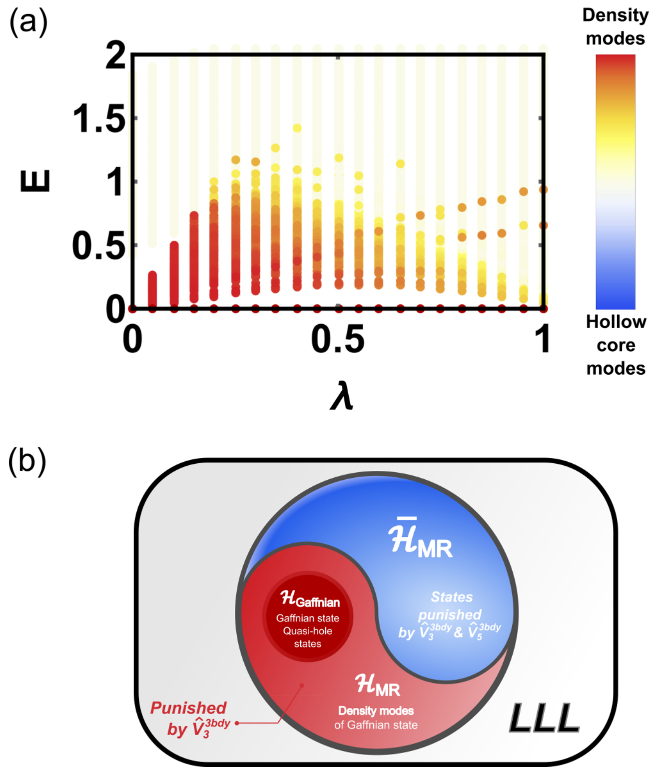

FIG. 6. Spectrum of the toy Hamiltonian $\hat{H}_G$ with respect to 10 electrons and 22 orbitals. (a) Shows the spectra of the model Hamiltonian $\hat{H}_G$ in Eq. (52). The low-lying states are density modes even when λ is quite large, and the absence of hollow core modes is different from the Laughlin-1/5 state in Fig. 3. (b) Illustrates the structure of the Hilbert space and the relationship between the states and the model Hamiltonians. All the states in the complementary space of the Moore-Read null space are punished by both $\hat{V}_3^{3\text{body}}$ and $\hat{V}_5^{3\text{body}}$.

of $\mathcal{H}_G$ with $\lambda \to 0$. However as shown in Fig. 6, when $\lambda \to 1$ there exists no state in the Hilbert space that is only punished by $\hat{V}_3^{3\text{bdy}}$, so the null space of $\hat{V}_5^{3\text{bdy}}$ lies entirely within the Gaffnian null space (also see Fig. 7). Thus there are no hollow-

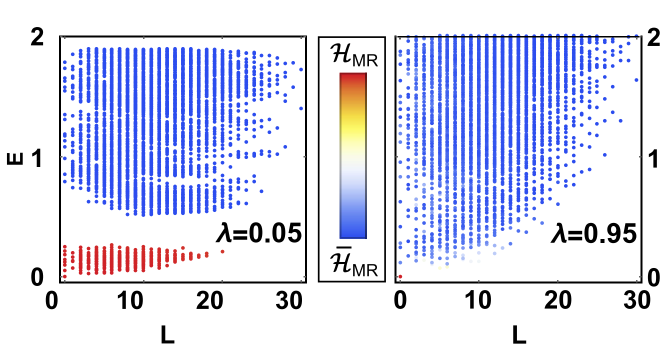

FIG. 7. Spectra of the toy Hamiltonian $\hat{H}_G$ diagonalized in the full Hilbert spaces. The spectrum with $\lambda = 0.05$ is shown in the left panel and $\lambda = 0.95$ in the right panel. As expected, when $\lambda = 0.05$ all the low-lying states (density modes) live within the Moore-Read null space $\mathcal{H}_{\text{MR}}$. Meanwhile when $\lambda = 0.95$, all the states except the ground state are in the complement of $\mathcal{H}_{\text{MR}}$ within the LLL, denoted by $\bar{\mathcal{H}}_{\text{MR}}$.

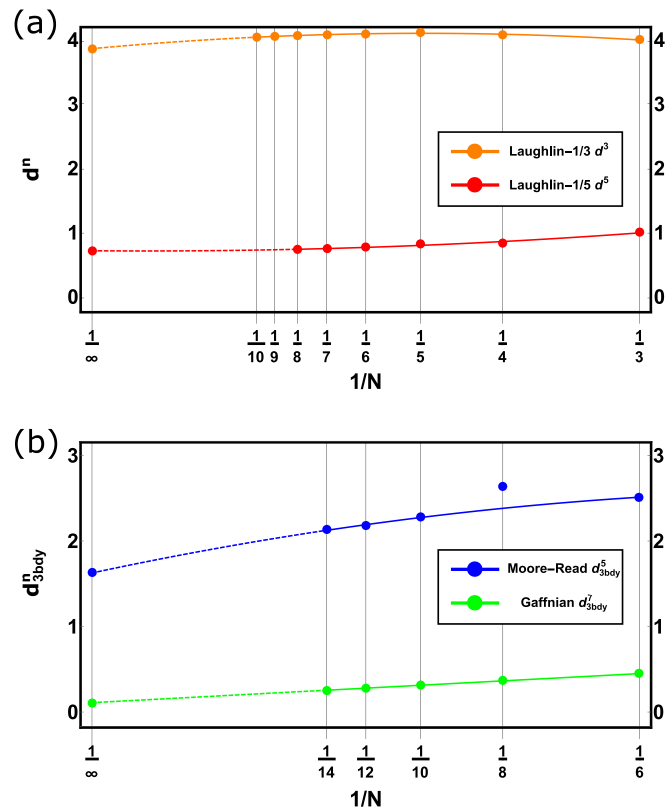

FIG. 8. Finite size scaling of the structure factor coefficients of different states. (a) The structure factor expansion coefficients of the Laughlin-1/3 and the Laughlin-1/5 state. According to the orthogonality of Laguerre polynomials, d^m of the Laughlin-$1/m$ state is equal to the expectation value of the model Hamiltonian $\hat{V}_m^{2\text{bdy}}$ (thus $c^m = 1$) acting on this state. Thus the numerical results shown here provide the value of the corresponding dimensionless coefficients despite with the dimension of energy, which is true for (b) as well. (b) The expectation value of $\hat{V}_7^{3\text{bdy}}$ with respect to the Gaffnian state (≈ 0.04 in the thermodynamic limit), denoted by $d_{3\text{bdy}}^7$, is significantly smaller than other coefficients in the plot, where the expectation value of $\hat{V}_5^{3\text{bdy}}$ with respect to the Moore-Read state is denoted by $d_{3\text{bdy}}^5$ (≈ 1.6 in the thermodynamic limit).

core modes here in contrast to the case for the Laughlin-1/5 phase. It would be interesting to see if this is related to the conjecture that the model Hamiltonian of Eq. (52) is gapless in the thermodynamic limit at $\nu = 2/5$, while the Laughlin-1/5 phase is gapped.

From Eq. (48), we know the graviton mode gap of the Gaffnian state at $\nu = 2/5$ is determined by the expectation value of just $\hat{V}_7^{3\text{bdy}}$ with respect to the ground state, denoted by $d_{3\text{bdy}}^7$ to be consistent with the two-body case in Fig. 8, where the finite-size scaling of the structure factor coefficients of different states is shown. Previous numerical calculations show evidence that in the thermodynamic limit, the gap of Eq. (52) at $\lambda = 0.5$ closes in the $L = 2$ sector [48]. This is indeed the sector of the graviton mode, and we have shown it is in the null space of $\hat{V}_3^{3\text{bdy}}$ and its energy is entirely determined by $\hat{V}_5^{3\text{bdy}}$. Our numerical calculation is thus valid for a family of model Hamiltonian of Eq. (52) parametrized by λ. It shows that the graviton mode of the Gaffnian phase will likely go soft in the thermodynamic limit, as its variational energy is an order of magnitude smaller than the graviton modes in the Moore-Read phase. It is however also important to note that the graviton mode energy gap of the Laughlin-$1/5$ phase is also an order of magnitude lower than that of the Laughlin-$1/3$ phase, as shown in Fig. 8(a).

While the Gaffnian model Hamiltonian is a theoretical model that is conjectured to be gapless and thus describing a possibly critical point, it is also closely related to the gapped Jain $\nu = 2/5$ phase from short-range two-body interactions [86–88]. It would be interesting to see how the graviton modes at $\nu = 2/5$ behave when we approach the critical point from the gapped Jain phase at $\nu = 2/5$. The finite numerical analysis seems to suggest that both the charge gap and the neutral gap will close, but with realistic interactions, we can also entertain the possibility that the graviton modes of the Jain $\nu = 2/5$ phase can close first while the charge gap remains open, in analogy to the nematic FQH phase that has been observed in experiments at $\nu = 2 + 1/3$ [89–91].

V. SUMMARY AND OUTLOOKS

In summary, we have presented a number of analytical results for the variational energies of the graviton modes in FQH phases. These results are rigorous in the thermodynamic limit, for FQH states with any arbitrary two-body or three-body interactions. In particular, we show that the variational energies of the graviton modes are fully determined by the ground-state wave function. In addition for short-range interactions, only the leading terms of the ground-state structure factor, when expanded in the proper Laguerre polynomial basis, are involved in the computation of the graviton mode energy. These analytical results allow us to construct model Hamiltonians for these graviton modes, which are exact zero-energy states of these Hamiltonians. We can thus determine analytically if the graviton mode lives entirely within a certain conformal Hilbert space, or if they have finite overlaps in different conformal Hilbert spaces. The latter gives microscopic understanding of the multiple graviton modes proposed in the effective field theory descriptions.

There are a number of proposals for the graviton modes to be detected in experiments, but in general it is a difficult task because of the high energy of such excitations. For many FQH phases with Coulomb-based interactions in simple experimental settings, the long-wavelength excitations are not the lowest energy ones. The graviton modes thus have to compete with multiroton and other neutral excitations. The analytical results we have derived can be useful in understanding how the graviton energy can be affected by realistic interactions, and how we can tune such interactions to lower their variational energies. The generalisation to three-body interactions in this work also allows us to treat Landau level mixing in realistic systems [67–69,92,93], which can be significant in higher LLs. Detailed studies of the graviton modes in the context of real experimental parameters will be carried out elsewhere. The softening of the graviton modes can also allow us to understand potential "phase transitions" in topological systems even when the ground-state topological properties are invariant, as we explored numerically with the Laughlin-$1/5$ state and the Gaffnian state with toy Hamiltonians in this paper.

Even with the methodology and the analytical tools developed in this paper, the quantitative values of the graviton energy in the thermodynamic limit cannot be determined without numerical computations and finite size scaling. It is, however, a much simpler procedure requiring the computation of only the ground state and partial information about its static structure factor, in contrast to the conventional ways requiring the computation of many low-lying states. The universal characteristic tensors derived in this paper show that the Hilbert space of the FQH states are highly structured, and this formalism can in principle be generalised to interactions involving more than three particles. The dispersion of the graviton mode can also be computed analytically by expanding the single mode approximation to higher orders in momentum. At this stage, both cases are algebraically very involved. It would be useful in the future to carry out a more general and systematic calculation of the graviton energy and its dispersion for any arbitrary Hamiltonians. This, combined with a numerically more efficient way to obtain information from the ground-state static structure factor (or the density correlation functions), can lead to much better understandings of the collective neutral excitations in non-Abelian FQH phases.

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