Letter

Phantom Bethe excitations and spin helix eigenstates in integrable periodic and open spin chains

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We demonstrate the existence of a special chiral "phantom" mode with some analogy to a Goldstone mode in the anisotropic quantum XXZ Heisenberg spin chain. The phantom excitations contribute zero energy to the eigenstate, but a finite fixed quantum of momentum k_0. The mode exists not due to symmetry principles, but results from nontrivial scattering properties of magnons with momentum k_0 given by the anisotropy via $\cos k_0 = \Delta$. Different occupations of the phantom mode lead to energetical degeneracies between different magnetization sectors in the periodic case. This mode originates from special string-type solutions of the Bethe ansatz equations with unbounded rapidities, the phantom Bethe roots (PBRs). We derive criteria under which the spectrum contains eigenstates with PBRs, both in open and periodically closed integrable systems, for spin-1/2 and higher spins, and discuss the respective chiral eigenstates. The simplest of such eigenstates, the spin helix state, which is a periodically modulated state of chiral nature, is built up from the phantom excitations exclusively. Implications of our results for experiments are discussed.

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Interacting quantum spin systems are a vibrant research field as fascinating kinds of order are realized with rather complex order parameters or of topological nature. Even the spin-1/2 XXZ chain, despite its long history and being one of the best studied paradigmatic models in quantum statistical mechanics [1], remains a source of inspiration and fascinating new progress. This model is integrable and in principle allows for the calculation of objects that in generic systems are usually not accessible in the thermodynamic limit. Among the relatively recent results the discovery of a set of quasilocal conserved quantities [2] with strong implications on the theory of finite-temperature quantum transport [3] and successes in the calculation of finite-temperature correlation functions [4,5] are exciting achievements.

In this Letter we are interested in the phenomena of anisotropic quantum spin chains requiring the understanding of energetical degeneracies in uncharted territory. A first example is the physics of so-called spin helix states (SHS) (4) which show sharp local polarization with respect to site-dependent axes. These states are routinely created, and widely used in coherent experimental protocols [6–8]. SHS can also be generated as nonequilibrium steady states via a dissipative quantum protocol [9–11] or via controlled local boundary dissipation. Remarkably, the needed boundary dissipation is of the type which allows the system to retain, partly, its integrability [12].

The eigenvalue degeneracies of isotropic quantum spin chains are well understood on the basis of the $su(2)$ symmetry algebra. Simple eigenstates that are fully polarized with respect to any axis form a multiplet of degeneracy $N+1$ for the spin-1/2 chain of length N.

The high degeneracy of this ferromagnetic multiplet can alternatively be explained by magnon excitations with a soft Goldstone mode at wave number $k=0$. Contrary to the usual situation when all magnons carry different momenta, this precise $k=0$ mode can be occupied up to N times.

A z anisotropy of the spin exchange interaction lifts the high degeneracy, leaving just two degenerate eigenstates with spins fully polarized in the $+z$ or $-z$ direction. The $su(2)$-type degeneracies can be restored by the so-called "quantum deformation" $U_q[su(2)]$ of the symmetry algebra [13–15], involving special possibly non-Hermitian boundary terms.

Remarkably, an analog of a Goldstone mode scenario can happen in periodic spin systems with z-exchange anisotropy, namely a multiple occupation of a single mode can occur, but now with a nonzero wave vector k_0 fine-tuned to the system's anisotropy $J_z/J_x = \Delta$ via $\cos k_0 = \Delta$. The corresponding excitation can be created at zero energetic cost. As in the isotropic case, the possibility of multiple occupations of the same zero-energy phantom mode leads to the high degeneracy. Unlike in the isotropic case, the eigenstates form a multiplet of degenerate chiral states carrying finite current.

Excitations with momentum mode $\pm k_0$ were discussed in Refs. [16–21] for accounting for the energetical degeneracies of the spin-1/2 XXZ chain and related systems. For certain systems with commensurable values of k_0, extended symmetry algebras are realized and the completeness of the Bethe ansatz has been investigated [17–21].

In our Letter we show why a macroscopic occupation of precisely $\pm k_0$ becomes possible, despite magnons of the wave number k_0 having nontrivial scattering. These states are realized by nonstandard string-type solutions of the Bethe ansatz equations with infinite rapidities. The Bethe ansatz equations for singular roots are satisfied with a universal choice for their arrangement (11), which makes them effectively "disappear" from the set of Bethe ansatz equations. For this reason we call the roots with infinite rapidities phantom Bethe roots and the respective excitations phantom excitations.

We find phantom Bethe roots in other integrable systems including open quantum systems and also for higher spins.

Finally, we find that the role of the fully polarized eigenstates in the isotropic case is taken, in anisotropic systems, by simple but rather nontrivial chiral states, the spin helix states. The SHS have ballistic current and a harmonic modulation (with period $2\pi/k_0$) of transversal magnetization. Remarkably, the SHS are created with exclusively phantom excitations, both in open and in periodic spin chains.

Factorized eigenstates at commensurate values of anisotropy. We consider the XXZ spin-1/2 Hamiltonian for periodic and open boundary conditions. For the periodically closed chain we have

$$H_{XXZ} = \sum_{n=1}^{N} h_{n,n+1}(\Delta),$$

$$h_{n,n+1}(\Delta) = J\big[\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\big(\sigma_n^z\sigma_{n+1}^z - I\big)\big], \quad (1)$$

with boundary conditions $\vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$. For convenience we put $J = 1$ throughout this Letter. For the open chain we have

$$H_{XXZ} = \sum_{n=1}^{N-1} h_{n,n+1}(\Delta) + \vec{h}_l\vec{\sigma}_1 + \vec{h}_r\vec{\sigma}_N, \quad (2)$$

with boundary fields $\vec{h}_l$ and $\vec{h}_r$ on the first and on the last sites. In both cases a shift $-J\Delta$ in the nearest-neighbor interaction (1) is added for convenience. Both models (1) and (2) are integrable and solvable via Bethe ansatz methods [22–24]. We parametrize the anisotropy Δ of the exchange interaction as $\Delta = \cos\gamma$ or $\Delta = \cosh\eta$ with $\eta = i\gamma$.

We want to construct factorized eigenstates of the Hamiltonians and introduce for each site the qubit state

$$|y\rangle = \begin{pmatrix} 1 \\ \mathrm{e}^{y} \end{pmatrix}. \quad (3)$$

The qubit state (3) with $y = f + iF$ corresponds to a fully polarized spin-1/2 pointing into the direction $\vec{n} = (\sin\theta\cos F, \sin\theta\sin F, \cos\theta)$ with $\tan\frac{\theta}{2} = \mathrm{e}^{f}$. A site-factorized state, the so-called spin helix state (SHS) [9,10]

$$|\mathrm{SHS}(y_0, \varphi)\rangle = |y_0\rangle_1|y_0 + i\varphi\rangle_2 \cdots |y_0 + i(N-1)\varphi\rangle_N, \quad (4)$$

with the subscripts indicating the site number, with uniformly increasing angles on some offset y_0, becomes an eigenstate of the XXZ Hamiltonian if (i) the increase φ of the angle is identical to $\pm\gamma$, the parameter of the anisotropy $\Delta = \cos\gamma$, and (ii) the boundary conditions can be accounted for. The parameter φ is real (imaginary) for easy plane (easy axis) anisotropy corresponding to a state with a uniformly increasing azimuthal (polar) angle.

The bulk interaction of the XXZ Hamiltonian applied to any SHS state (4) results in 0 due to the “divergence” relation

$$h(\Delta)|y\rangle \otimes |y + i\gamma\rangle = |y\rangle \otimes (\kappa\sigma^z|y + i\gamma\rangle) - (\kappa\sigma^z|y\rangle) \otimes |y + i\gamma\rangle, \quad (5)$$

where $\kappa = i\sin\gamma$. For the periodic model (1), the SHS will be an eigenstate if the periodic closure condition $\gamma N = 2\pi m$ with integer m is satisfied.

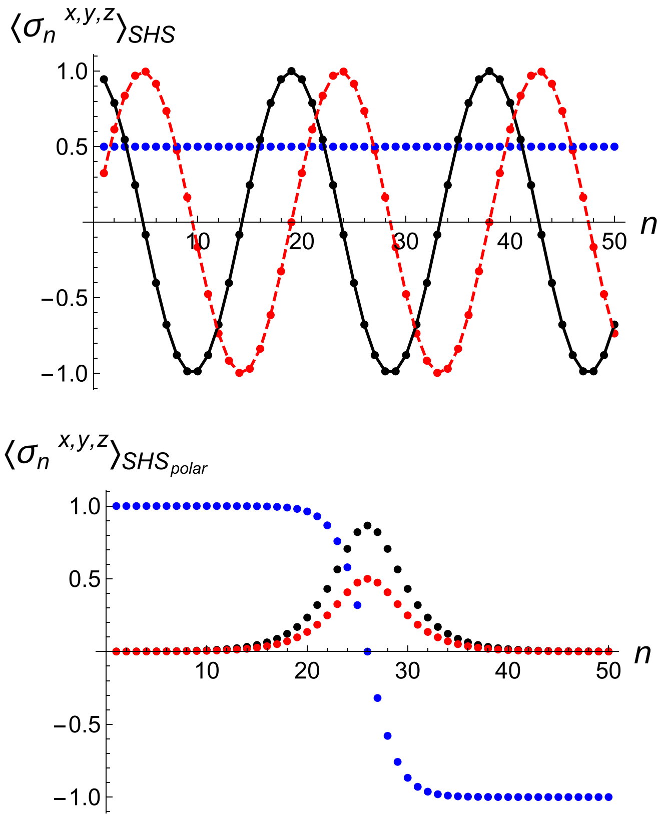

FIG. 1. Components of local magnetization $\langle\sigma_n^x\rangle$, $\langle\sigma_n^y\rangle$, $\langle\sigma_n^z\rangle$ for SHS/phantom Bethe states vs site number n, for the easy plane (upper panel) and the easy axis case (lower panel), indicated with black, red, and blue points, respectively. Upper panel: SHS (4) with increasing azimuthal angle, the phantom Bethe eigenstate of (2) or (1) for $|\Delta| < 1$. Parameters: $\varphi = \gamma = 2\pi/19$, $y_0 = i\gamma + 1/\sqrt{3}$. Curves connecting points serve as a guide for the eye. Lower panel: SHS (4) with increasing polar angle, the phantom Bethe eigenstate of (2) for $\Delta > 1$. Parameters: $i\varphi = \eta = 2\pi/19$, $y_0 = i\pi/6 + N\eta/2$.

This can only happen for anisotropy $|\Delta| \leqslant 1$.

For the open chain condition (ii) on the boundary can be satisfied not only in the case $|\Delta| \leqslant 1$, but also for $|\Delta| > 1$. For $|\Delta| > 1$ we may use expression (4) with the replacement $\varphi = i\eta$ which results in a spin helix state with fixed azimuthal angle and uniformly increasing polar angles. The eigenstate condition is fulfilled, if the boundary interactions $h_l = \vec{h}_l\vec{\sigma}_1$ and $h_r = \vec{h}_r\vec{\sigma}_N$ satisfy

$$h_l|y_0\rangle = \kappa\,\sigma^z|y_0\rangle + \lambda_-|y_0\rangle, \quad (6)$$

$$h_r|y_{N-1}\rangle = -\kappa\,\sigma^z|y_{N-1}\rangle + \lambda_+|y_{N-1}\rangle, \quad (7)$$

where $y_{N-1} = y_0 + i(N-1)\gamma$, and $\lambda_\pm$ are some boundary-dependent constants. The energy eigenvalue is $E = \lambda_- + \lambda_+$. Note that in the open chain case a condition on the anisotropy Δ as in the periodic case is absent and φ in (4) can be real or imaginary.

Although having the same algebraic form, the SHS for the easy plane and easy axis cases have rather different physical properties as visualized in Fig. 1. Note that we call by the $\mathrm{SHS}_{\mathrm{polar}}$ the SHS (4) for the easy axis case.

The factorized SHS state is after the ferromagnetic state the simplest eigenstate of XXZ spin chains. Yet the SHS (4) is quite nontrivial, and describes a "frozen" spin precession around the z axis with period $2\pi/\varphi$ (see Fig. 1). Due to the chiral nature, the SHS carries a remarkably high magnetization current, finite in the thermodynamic limit,

$$\langle j^z\rangle_{\mathrm{SHS}} = \langle 4i(\sigma_n^+\sigma_{n+1}^- - \mathrm{H.c.})\rangle_{\mathrm{SHS}} = \pm 2\frac{\sin\gamma}{\cosh^2(\mathrm{Re}[y_0])},$$

where the sign $\pm$ corresponds to the choice $\varphi = \pm\gamma$ in (4). Remarkably, the SHS (4) with an adjustable wavelength can be realized in cold atom experiments [6,7].

The very existence of an eigenstate (4) for the periodic spin chain, characterized by periodic modulations in the magnetization profile, seems to contradict the $U(1)$ symmetry: XXZ eigenvectors split in blocks with well-defined values of the global magnetization $S^z = \sum_n \sigma_n^z$ and expectation values $\langle\sigma_n^+\rangle = \langle\sigma_n^-\rangle = 0$ vanish, and so do $\langle\sigma_n^x\rangle = \langle\sigma_n^y\rangle = 0$.

This paradox is resolved by the energetical degeneracy of eigenstates with different values of the total magnetization S^z. We will show that a superposition of states from different blocks yields the state (4) which is not an eigenstate of the operator S^z.

Phantom Bethe roots at commensurate anisotropies in periodic XXZ chains. The eigenstates and eigenvalues are given in terms of rapidities μ_j ($j = 1, 2, \dots, n$) whose total number n may take any value out of $0, 1, \dots, N$. For any solution of the Bethe ansatz equations (BAEs)

$$\frac{\sinh^N(\mu_j - i\gamma/2)}{\sinh^N(\mu_j + i\gamma/2)} = \prod_{l\neq j}^{n}\frac{\sinh(\mu_j - \mu_l - i\gamma)}{\sinh(\mu_j - \mu_l + i\gamma)}, \tag{8}$$

there is an eigenstate with energy and total momentum

$$E = -\sum_{j=1}^{n} e(\mu_j), \quad K = \sum_{j=1}^{n} k(\mu_j), \tag{9}$$

with single particle energy and momentum defined by

$$e(\mu_j) = \frac{4\sin^2\gamma}{\cosh(2\mu_j) - \cos\gamma}, \quad \mathrm{e}^{ik(\mu)} = \frac{\sinh\left(\mu + i\frac{\gamma}{2}\right)}{\sinh\left(\mu - i\frac{\gamma}{2}\right)}. \tag{10}$$

The Bethe eigenvector $\Psi_{\mu_1,\dots,\mu_n} = B(\mu_1)\cdots B(\mu_n)|0\rangle$ is obtained by the application of magnon creation operators $B(\mu_j)$ to the reference state $|0\rangle = |\uparrow\uparrow\dots\uparrow\rangle$ of fully polarized spins [23,25].

Definition. We shall call a Bethe root μ_p satisfying (8), a *phantom* Bethe root, if it does not give a contribution to the respective energy eigenvalue (9), i.e., if $\mathrm{Re}[\mu_p] = \pm\infty$. The next Lemma affirms that such phantom Bethe roots do exist:

Lemma 1. For anisotropy $\gamma = 2\pi m/N$ with integer m there exist the following "phantom" solutions of the BAE (8) for any given $n = 1, 2, \dots, N$,

$$\mu_p = \pm\infty + i\pi\frac{p}{n}, \quad p = 1, 2\dots, n. \tag{11}$$

These distributions remind us of the string solutions to the Bethe ansatz equations. Note, however, that (11) holds for any finite system size N with a total number n of roots equidistantly distributed with separation π/n. Note that our Lemma describes the precise arrangement of the infinite roots appearing in Refs. [1–6]. Upon introducing a finite magnetic flux respectively twisted boundary conditions, the roots become finite while the imaginary parts stay close to the values of Lemma 1. This is relevant for the dependence of the energy as function of the twist and has important consequences for the transport properties [25].

Proof. Assume $\mu_j = \pm\mu_\infty + i\pi j/n$, where μ_∞ has a large real part which we let to ∞ when evaluating the left-hand side (LHS) of the Bethe ansatz equations. As $\gamma = 2\pi m/N$ the LHS of (8) becomes LHS $\to \mathrm{e}^{\mp i\gamma N} = 1$. On the right-hand side (RHS) the term μ_∞ drops out, leaving finite differences $\mu_j - \mu_l = i\pi(j-l)/n$. Denoting $\omega = \mathrm{e}^{i\pi/n}$, we have

$$\begin{aligned}\mathrm{RHS}_j &= \prod_{\substack{l=1\\ l\neq j}}^{n}\frac{\omega^{j-l}\mathrm{e}^{-i\gamma} - \omega^{-(j-l)}\mathrm{e}^{i\gamma}}{\omega^{j-l}\mathrm{e}^{i\gamma} - \omega^{-(j-l)}\mathrm{e}^{-i\gamma}}\\ &= \prod_{l=1}^{n-1}\frac{\omega^{l}\mathrm{e}^{-i\gamma} - \omega^{-l}\mathrm{e}^{i\gamma}}{\omega^{l}\mathrm{e}^{i\gamma} - \omega^{-l}\mathrm{e}^{-i\gamma}}\\ &= \prod_{l=1}^{n-1}\frac{\omega^{l}\mathrm{e}^{-i\gamma} - \omega^{-l}\mathrm{e}^{i\gamma}}{-\omega^{-l}\mathrm{e}^{i\gamma} + \omega^{l}\mathrm{e}^{-i\gamma}} = 1.\end{aligned}$$

Here, we used that the set of ω^{j-l} with $l = 1, \dots, n$ (and $\neq j$) is identical to the set of ω^l with $l = 1, \dots, n-1$ as we have $\omega^n = -1$. ■

Phantom Bethe vectors for periodic chains. The Bethe vectors corresponding to the phantom Bethe root (PBR) solution (11), under the conditions of Lemma 1, can be constructed as described below (10). The two signs $\pm$ in (11) correspond to different Bethe vectors which upon normalization reads

$$|\pm, n\rangle = \frac{1}{n!\sqrt{\binom{N}{n}}}\sum_{l_1,\dots,l_n=0}^{N-1}\mathrm{e}^{\pm i\gamma(l_1+\dots+l_n)}\sigma_{l_1}^-\cdots\sigma_{l_n}^-|0\rangle,$$

$$n = 0, 1, \dots, N. \tag{12}$$

Each multiplication by a $B(\mu_j)$ operator adds a quasiparticle with momentum $k(\mu_j)$ and zero energy. Within the standard picture [23,24] quasiparticles obey a "Fermi rule": All $k(\mu_j)$ are usually different. This property is violated for phantom Bethe roots μ_p for which all $k(\mu_p)$ are exactly the same: Either $k(\mu_p) = +\gamma \equiv k_0$ or $k(\mu_p) = -\gamma \equiv -k_0$ depending on the sign of the singular part in (11). The repeated action of B generates "phantom" Bethe states (12) with "quantized" momenta $\pm n\gamma$ and zero energy for all magnetization sectors n, yielding the degeneracy of the eigenvalue $E = 0$ between different sectors. Note that the $E = 0$ state is not a ground state of (1), which is obtained by filling the Fermi sea with quasiparticles giving negative energy contributions to (9). The dimension of the degenerate subspace is deg $= 2(N-1) + 2 = 2N$ since the states $|+, n\rangle, |-, n\rangle$ for $n = 1, 2, \dots, N-1$ are linearly independent and for $n = 0, N$ the states $|+, n\rangle, |-, n\rangle$ coincide. The degeneracy between sectors with different magnetization leads to eigenstates with periodic modulations in the density profile. Indeed, the SHS (4) with positive chirality and $\varphi = +\gamma = 2\pi m/N \neq \pi$ is a linear combination of phantom Bethe states $|+, n\rangle$, and SHS

(4) with opposite chirality $\varphi = -\gamma$ is a linear combination of $|-, n\rangle$,

$$|\mathrm{SHS}(y_0, \pm 2\pi m/N)\rangle = \binom{N}{n}^{1/2} \sum_{n=0}^{N} e^{y_0 n} |\pm, n\rangle \tag{13}$$

(see Supplemental Material [25] for the proof). Finally, note that the states (12) are chiral, which is evidenced by the nonzero expectation values of the magnetization current [25],

$$\langle \pm, n | j^z | \pm, n \rangle = \pm \frac{8n(N-n)}{N(N-1)} \sin\gamma, \tag{14}$$

reaching its maximum of order $|j^z| \to 2\sin\gamma$ for $n = N/2$.

Mixtures of regular and phantom excitations for the periodic XXZ model. Here, we show that phantom Bethe roots can appear alongside the usual finite Bethe roots, for other special values of the anisotropy.

Let us assume that within a sector of n_0 flipped spins, there exists a BAE solution with n phantom Bethe roots $\mu_1, \dots, \mu_n$ and the remaining $r = n_0 - n$ Bethe roots are regular. We denote the regular roots as $x_1, \dots, x_r$ where $x_j = \mu_{n+j}$. Let us consider separately the BAE (8) subsets for phantom μ_p and for regular x_j. Substituting (11) in (8) we obtain

$$e^{i\gamma(N-2r)} = 1, \tag{15}$$

since each factor of the RHS containing a mixed pair μ_p, x_j contributes a term $\exp(2i\gamma)$. The product over factors of the RHS involving two phantom roots results in $+1$ precisely as in Lemma 1. The criterion (15) fixes the anisotropy parameter while the BAE subset for regular roots simplifies to

$$\frac{\sinh^N(x_j - i\gamma/2)}{\sinh^N(x_j + i\gamma/2)} = e^{\pm 2i\gamma n} \prod_{\substack{l=1 \\ l\neq j}}^{r} \frac{\sinh(x_j - x_l - i\gamma)}{\sinh(x_j - x_l + i\gamma)},$$

for all $j = 1, \dots, r$ (see also Refs. [19,20]). This has the structure of the BAE of a twisted XXZ chain, because of the presence of a constant phase factor. The signs $\pm$ match those in (11).

Phantom excitations in the open XXZ chain. The energy of Hamiltonian (2) is given by (9) with an additional offset, $E = \sum_{j=1}^{N} e(\mu_j) + E_0$, where

$$E_0 = -\sinh\eta(\coth\alpha_- + \coth\alpha_+ + \tanh\beta_- + \tanh\beta_+), \tag{16}$$

where the boundary fields $h_{l,r}$ are parametrized as

$$\vec{h} = \frac{\sinh\eta}{\sinh\alpha_\pm \cosh\beta_\pm}(\cosh\theta_\pm, i\sinh\theta_\pm, \mp\cosh\alpha_\pm \sinh\beta_\pm),$$

and $+$ $(-)$ corresponds to the right (left) field. The Bethe roots μ_j satisfy BAEs of a somewhat bulky form [25–28]. After some algebra [28] we find that if

$$\pm(\theta_+ - \theta_-) = (2M - N + 1)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ \quad \mathrm{mod}\ 2\pi i, \tag{17}$$

is satisfied with some integer $M = 0, 1, \dots, N-1$, each set of N Bethe roots contains n phantom Bethe roots of type (11), where n takes one of two values $n_+ = N - M$ and $n_- = M + 1$ [28,29]. The remaining $N - n$ Bethe roots x_j $(= \mu_{n+j})$ are regular and satisfy the reduced BAEs,

$$\frac{G_\pm\left(x_j - \frac{\eta}{2}\right) \sinh^{2N}\left(x_j + \frac{\eta}{2}\right)}{G_\pm\left(-x_j - \frac{\eta}{2}\right) \sinh^{2N}\left(x_j - \frac{\eta}{2}\right)} = \prod_{\substack{l=1 \\ l\neq j}}^{N-n_\pm} \frac{\sinh(x_j - x_l + \eta)}{\sinh(x_j - x_l - \eta)} \frac{\sinh(x_j + x_l + \eta)}{\sinh(x_j + x_l - \eta)}, \quad j = 1, \dots, N - n_\pm, \tag{18}$$

$$G_\pm(u) = \prod_{\sigma=\pm} \sinh(u \mp \alpha_\sigma)\cosh(u \mp \beta_\sigma),$$

while the total eigenvalue has contributions from the regular Bethe roots only. We would like to note that (16) holds literally for case $n = n_+$. For $n = n_-$ the $+E_0$ contribution in (16) is to be replaced by $-E_0$ (see Ref. [28]). We find that the BAE (18) for $n = N - M$ describes $\dim G_M^+ = \sum_{m=0}^{M} \binom{N}{m}$ Bethe states, while the remaining $2^N - \dim G_M^+$ eigenstates are contained in the other, complementary BAE set for $n = M + 1$ [28,29]. Unlike in the periodic setup, where some Bethe eigenstates contain PBR modes, and other eigenstates are fully regular, in open systems, satisfying criterion (17), *all* 2^N eigenstates include phantom Bethe roots. Remarkably, the condition (17) appears in Refs. [30–33] as a condition for the application of the algebraic Bethe ansatz. The BAE set (18) coincides with that found by an alternative method [30,31,33].

Now we focus on the simplest Bethe states, corresponding to all Bethe roots being phantom, $n_+ = N$, the respective energy given by E_0 (16). We demonstrate that such "phantom" Bethe states are spin helix states (4) with appropriately chosen parameters. The phantom Bethe states for mixtures of phantom and regular Bethe roots can be also obtained explicitly and show chiral features [28,29].

Phantom Bethe states: Open XXZ chain. Easy plane regime $|\Delta| < 1$. It is straightforward to verify that the SHS (4), with $\varphi = \gamma$, $\mathrm{Re}[y_0] = \beta_-$, and phase $\mathrm{Im}[y_0] = \pi + i\alpha_- + i\theta_-$ (note that α_-, θ_- are imaginary and $\beta_- = -\beta_+$ are real to ensure the Hermiticity of H), is an eigenstate of H. Indeed, one can check that (6) and (7) are satisfied with $\lambda_\pm = -\sinh\eta(\coth\alpha_\pm - \tanh\beta_\pm)$, so that this SHS is an eigenvector of (2) with eigenvalue $\lambda_- + \lambda_+$, which coincides with the phantom Bethe vector eigenvalue E_0 (16). For the magnetization profile of this SHS, see Fig. 1, top panel. Unlike for the periodic chain, here the eigenvalue E_0 is generically nondegenerate.

Simplest experimental setup. Using our results, long-lived SHS can be obtained in experiments where effectively one-dimensional spin-1/2 XXZ chains with tunable anisotropy are realized [6,7]. A spin helix of the form (4) with an adjustable wavelength is created within cold atom setups by applying a magnetic field gradient in the z direction on an array of initially noninteracting qubits polarized along the x axis (see Methods of Ref. [6] for details). To make the SHS an eigenstate of the XXZ Hamiltonian, the wavelength Q of the spin helix and the z-anisotropy Δ must be related as $\Delta = \cos Qa$, where a is the lattice constant. Indeed, under this choice an SHS of type (4) $|\mathrm{SHS}_{\pm}\rangle := |\mathrm{SHS}(iF_0, \pm Qa)\rangle$ will remain invariant in the bulk and change initially only at the boundaries, since

$$\sum_{n=1}^{N-1} h_{n,n+1}(\Delta)|\mathrm{SHS}_{\pm}\rangle = \mp i \sin Qa\big(\sigma_1^z - \sigma_N^z\big)|\mathrm{SHS}_{\pm}\rangle,$$

as follows from (5). The ends of the spin chain will thus play the role of defects, and the state in the bulk will be altered only by propagation of the information from the boundaries. Thus the state can be destroyed only after times of order $t = Na/v_{\mathrm{char}}$, where v_{char} is the sound velocity, N is the number of spins, and a is the lattice constant. For example, in Refs. [6,7], the process of the expansion of the defect in the bulk can be monitored. On the other hand, if the SHS period does not match the anisotropy $\Delta \neq \cos Qa$, then the initial SHS will be destroyed after times of order $t = a/v_{\mathrm{char}}$. On one hand, the effect is robust (with respect to the phase of the helix and chain length N), and on the other hand, it is sensitive with respect to the matching condition for the anisotropy Δ. This sensitivity can be used as a benchmark for calibrating the anisotropy or the wavelength of the produced SHS, or both.

Phantom Bethe states: Easy axis $\Delta = \cosh\eta > 1$. The SHS of the form (4) with $y_0 = i\pi - \theta_- + \alpha_- + \beta_-$ satisfies (6) and (7) with $\kappa \to -\sinh\eta$ and $\lambda_{\pm} = -\sinh\eta(\coth\alpha_{\pm} + \tanh\beta_{\pm})$. Consequently, state (4) is an eigenstate of H with eigenvalue $\lambda_+ + \lambda_- = E_0$. Thus, state (4) is a phantom Bethe vector. It describes spins on the lattice with fixed azimuthal angle and changing polar angle along the chain (see Fig. 1, lower panel). Unlike the "azimuthal" spin helix state (4), the "polar" SHS carries no spin current, $\langle j^z\rangle_{\mathrm{SHS_{polar}}} = 0$.

Discussion. We have described a scenario of excitations in integrable systems, namely phantom excitations with phantom Bethe roots corresponding to unbounded rapidities. The existence criterion for these states is formulated and depends on the boundary conditions of the system. Under this criterion a certain subset of Bethe roots is located at infinity with relative positions at equidistant points. This resembles a perfect thermodynamic Bethe ansatz (TBA) string, but is of entirely different nature.

For models with periodic boundaries the PBRs are responsible for degeneracies between sectors with different total magnetization, and lead to factorized spin helix eigenstates at anisotropies given by (15). Also for the open XXZ model the PBR related eigenstates are spin helix states with a winding polarization vector, in the easy plane regime, and the "polar angle" version of the latter, in the easy axis regime. Our results can be used for the generation of stable spin helix states in experimental setups realizing XXZ chains [6,7].

While our discussion was restricted to the XXZ model, the presence of phantom Bethe roots, due to their simple analytic form (11), can be easily established in other integrable models, e.g., in the periodic spin-1 Fateev-Zamolodchikov model [34–36], and arbitrary spin s generalizations [37–40] (see Supplemental Material [25]). It would be interesting to search for PBR analogs in intrinsically non-Hermitian integrable models, e.g., Refs. [41,42].

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