Letter

# Shear modulus anomaly of unconventional superconductors in a symmetry breaking field

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Using Ginzburg-Landau formalism, we theoretically study the isothermal shear modulus anomaly of an unconventional superconductor with a two-component order parameter, where the superconducting transition is split by a symmetry-breaking field. Experimental signatures are proposed for both chiral and nematic superconductors. Particularly striking is the vanishing of $C_{66}$ across the lower transition to a nematic superconducting state. Our findings can guide future experiments and shed new light on materials such as $Sr_2RuO_4$ and $M_xBi_2Se_3$.

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A superconductor described by a multicomponent order parameter falls under the category of unconventional superconductivity [1]. In the presence of a symmetry-breaking field (SBF) explicitly breaking the lattice symmetry, the superconducting transition is split into two branches [2,3]. $UPt_3$ is the earliest material that lead to the conception of this scenario, where the SBF is an SDW order above superconductivity [4–9].

Controversy abounds, however, around $Sr_2RuO_4$ and $M_xBi_2Se_3$, two other materials that may fit into this scenario. The superconductivity of $Sr_2RuO_4$ appears to break the time-reversal symmetry (TRS) [10]. Chiral superconductivity seems a natural interpretation [11], but the associated edge current has never been observed [12–15], casting doubt on the scenario. Experiments with uniaxial strain (as an SBF) turn out mixed results: no thermodynamic signature for split transitions is observed [16,17], yet in a $\mu$-SR experiment [18,19] the breaking of TRS occurs below the superconducting $T_c$, indicating two branches of transitions.

The nematic superconductivity model of $M_xBi_2Se_3$ [20] necessitates a yet-unidentified pre-existing "pinning field" [21] to explain the persistent nematic orientation of any given sample in repeated experiments [22,23]. A small distortion of crystal lattice has been observed [22–24], and it is suggested that this distortion accounts for the pinning effect [22,23]. The current body of experimental evidence supporting this so-called "nematic hypothesis" consists of large twofold anisotropic responses incompatible with lattice symmetry [25–33] emerging together with superconductivity. However, this is not logically inevitable: the pinning field alone already breaks the trigonal lattice symmetry and gives a twofold direction, regardless of the nature of the order parameter itself. The pair of split transitions remains the most definitive evidence for multicomponent superconductivity. To our knowledge, none of the hallmarks [2,3,21] pertaining to the split transitions have been tested for in existing experiments.

Recent ultrasound experiments reveal a shear modulus anomaly in $Sr_2RuO_4$ [34,35] across the superconducting transition in the absence of an SBF, supporting the hypothesis of a multicomponent order parameter; ultrasound measurements of the shear moduli can be a fruitful direction in the search of split transitions. Taking Ginzburg-Landau (GL) theory of a two-component order parameter as our starting point, we examine the discontinuity in the isothermal shear modulus across both branches of transitions. We find that, in particular, one of the shear moduli *vanishes* at the lower transition of $M_xBi_2Se_3$. Such a drastic signal should be readily observed in experiment. Prior results on $UPt_3$ exists [36] but, as we will presently explain, we disagree with their theoretical analysis. In contrast, the order parameter of an $s$-wave superconductor transforms trivially under any crystal symmetry, and shear strains cannot couple linearly to the superconductivity. Thus it would show *none* of the anomalies discussed in this paper.

We will start by introducing the GL theory of a two-component order parameter and coupling it to the planar shear strain and an SBF. Next, we evaluate the shear moduli, paying particular attention to their qualitative behavior across each of the split transitions. There are three separate cases: upper transition, lower transition into the chiral phase, and lower transition into another nematic phase. We then proceed to introduce the modified GL theory that describe nematic superconductivity in a trigonal crystal (e.g., $M_xBi_2Se_3$) and work out the behavior of the shear moduli for the case. Finally we discuss the implication of our result on the verification of split superconducting transition in such materials.

*Ginzburg-Landau theory.* For the materials highlighted in the introduction ($UPt_3$, $Sr_2RuO_4$, and $M_xBi_2Se_3$), the lattice point groups are respectively $D_{6h}$, $D_{4h}$, and $D_{3d}$. They all admit two-dimensional irreducible representations. We assume the (complex) superconducting order parameter $\vec{\eta} = (\eta_x, \eta_y)$ transforms under such representation. The most general homogeneous GL free energy, invariant under the lattice symmetry, is conventionally written as [37]

$$\mathcal{F}_0 = a|\eta|^2 + b_1|\eta|^4 + \frac{b_2}{2}[(\eta_x^*\eta_y)^2 + (\eta_y^*\eta_x)^2] + b_3|\eta_x|^2|\eta_y|^2. \quad (1)$$

 

This can be cast into an elegant alternate form. Let $\{\sigma_i, i = 1, 2, 3\}$ be the usual Pauli matrices, and define $\langle\sigma\rangle_i \equiv (\vec{\eta}^\dagger\sigma_{i-1}\vec{\eta})$ (identifying $\sigma_{1-1} = \sigma_3$). Each $\langle\sigma\rangle_i$ forms a one-dimensional representation of $D_{4h}$. Under $D_{6h}$ and $D_{3d}$, $(\langle\sigma\rangle_1, \langle\sigma\rangle_2)$ transforms like a headless vector in the basal plane, while $\langle\sigma\rangle_3$ transforms trivially. The free energy becomes

$$\mathcal{F}_0 = a|\eta|^2 + \frac{1}{2}\sum_{i=1}^{3}\Lambda_i\langle\sigma\rangle_i^2;$$
$$\Lambda_1 = 2b_1, \quad \Lambda_2 = 2b_1 + \frac{1}{2}(b_3 + b_2), \tag{2}$$
$$\Lambda_3 = 2b_1 + \frac{1}{2}(b_3 - b_2).$$

Stability requires all $\Lambda_i > 0$. $D_{3d}$ and $D_{6h}$ crystals are additionally constrained to $\Lambda_1 = \Lambda_2$. Throughout this paper, there will be *no summation over indices* unless explicitly indicated.

The theory (2) has exactly one critical point at $a = 0$. For $a < 0$, the equilibrium state is characterized by $\langle\sigma\rangle_i \neq 0$ where $i$ corresponds to the smallest $\Lambda_i$; all other $\langle\sigma\rangle_j = 0$. The nematic state with a real $\vec{\eta}$ requires $i = 1$ or 2, while $i = 3$ yields the complex chiral state.

We introduce a background SBF $\Delta$ to the free energy:

$$\mathcal{F}_{\text{SBF}} = -\Delta\langle\sigma\rangle_1. \tag{3}$$

A positive (or negative) $\Delta$ explicitly favors $\eta \propto (1, 0)$ (or (0,1)). We assume $\Delta > 0$, and the opposite case is obtained through trivial sign changes. This breaks the crystal symmetry down to $D_{2h}$ ($C_{2h}$ for the $D_{3d}$ case). Reader's attention is drawn to the twofold axis parallel to (1,0): this twofold symmetry characterizes the normal phase and will be referred to as $\mathcal{C}_2'$.

This addition was first proposed to explain the split transitions of $UPt_3$ [5–7]. For $M_xBi_2Se_3$, the reported pinning field is along either the $a$ or $a^*$ lattice direction for each individual sample [23,38], and (3) is also appropriate. For $Sr_2RuO_4$, an applied uniaxial strain may act as an SBF.

The (upper) critical temperature is now raised to $a_u = |\Delta| > 0$, and the order parameter is *pinned* along (1,0): this orientation is symmetry-protected by $\mathcal{C}_2'$[1]. However, the "natural preference" of the material is encoded in the quartic order terms in (2), and this may not be compatible with (1,0). If that is indeed the case, the competition drives a second transition at a lower temperature [21].

Specifically, if $\Lambda_2$ is the smallest, $\vec{\eta}$ remains nematic but tilts away from (1,0) below this lower transition, breaking $\mathcal{C}_2'$. If $\Lambda_3$ is smallest, $\vec{\eta}$ becomes a combination of (1,0) and an isotropic chiral component $(1, \pm i)$, breaking the TRS but preserving the $D_{2h}$ (with the $\mathcal{C}_2'$ redefined to be followed by time reversal). If $\Lambda_i$ ($i \neq 1$) is the smallest, the lower critical temperature is

$$a_l = -\left(\frac{\Lambda_i}{\Lambda_1 - \Lambda_i}\right)\Delta < 0. \tag{4}$$

[1]For $\Delta < 0$, the (0,1) orientation is also protected by the same $\mathcal{C}_2'$. All subsequent symmetry arguments apply equally.

Up to an overall phase, $\vec{\eta}$ is parameterized as

$$\vec{\eta} = |\eta|\begin{pmatrix}\cos\theta \\ e^{i\chi}\sin\theta\end{pmatrix}. \tag{5}$$

In the upper phase $a_u > a > a_l$, we find the equilibrium solution:

$$|\bar{\eta}|^2 = \frac{\Delta - a}{\Lambda_1}, \quad \bar{\theta} = 0, \quad \chi \text{ drops out.} \tag{6}$$

The over-bar quantities such as $\bar{\theta}$ denote the equilibrium solution. The associated specific heat discontinuity is

$$C_{\text{upper}} - C_{\text{normal}} = (T_0 + \Delta)/\Lambda_1, \tag{7}$$

where $T_0$ is the critical temperature of the single transition in the absence of SBF. It is related to $a$ by $a = (T - T_0)$.

For $a < a_l$, the lower phase solution is

$$|\bar{\eta}|^2 = -\frac{a}{\Lambda_i}, \quad |\bar{\eta}|^2\cos 2\bar{\theta} = \left(\frac{\Delta}{\Lambda_1 - \Lambda_i}\right),$$
$$\bar{\chi} = \begin{cases}0 & (i = 3, \text{nematic}) \\ \pm\pi/2 & (i = 2, \text{chiral})\end{cases}. \tag{8}$$

For this lower transition, $\theta$ can serve as an order parameter. It exhibits the usual meanfield critical behavior $\bar{\theta} \propto \pm|a - a_l|^{1/2}$ for $a < a_l$. The associated specific heat jump is

$$C_{\text{lower}} - C_{\text{upper}} = (T_0 + a_l)\left(\frac{1}{\Lambda_i} - \frac{1}{\Lambda_1}\right). \tag{9}$$

We next consider the coupling exclusively to shear strains in the basal plane:

$$\epsilon_1 = \epsilon_{xx} - \epsilon_{yy}, \quad \epsilon_2 = 2\epsilon_{xy}. \tag{10}$$

These form a 2D representation under $D_{3d}$ or $D_{6h}$; under $D_{4h}$ they are separate one-dimensional representations. One can write down invariant combinations of these with $(\langle\sigma\rangle_1, \langle\sigma\rangle_2)$. The strain part of the free energy is then

$$\mathcal{F}_\epsilon = -g_1\epsilon_1\langle\sigma\rangle_1 - g_2\epsilon_2\langle\sigma\rangle_2 + \frac{c_1}{2}\epsilon_1^2 + \frac{c_2}{2}\epsilon_2^2. \tag{11}$$

Here, $c_1$ and $c_2$ are the shear moduli in the normal phase; $c_1 = c_2$ and $g_1 = g_2$ for $D_{3d}$ and $D_{6h}$.

The coupling to the strain renormalizes the quartic coefficients [34] when the strain is eliminated using GL equations. The quartic coefficients in (1) must be replaced with the "bare" ones: define

$$\lambda_3 = \Lambda_3; \qquad \lambda_i = \Lambda_i + \frac{g_i^2}{c_i}, \quad i = 1, 2. \tag{12}$$

We expect this correction to be quite small.[2] However, it will prove to be qualitatively important in the case of trigonal crystal.

[2]The jumps of elastic moduli across the unsplit (or upper) transition indicates the size of the renormalization effect (12) (cf. Eqs. (18), and also Refs. [34–36]), and is usually found in the range of $10^{-4}$ [34–36]. In addition, the Testardi relation [44] misses this renormalization [39], but still works fairly well: see Supplemental material of Ref. [34].

At the end, the full GL free energy is

$$\mathcal{F} = a|\eta|^2 - \Delta\langle\sigma\rangle_1 + \frac{1}{2}\sum_{i=1}^{3}\lambda_i\langle\sigma\rangle_i^2 + \sum_{i=1,2}\left(-g_i\epsilon_i\langle\sigma\rangle_i + \frac{1}{2}c_i\epsilon_i^2\right). \quad (13)$$

One imposes $\partial\mathcal{F}/\partial\epsilon_i = 0$ for strains at equilibrium:

$$\bar{\epsilon}_i = \frac{g_i}{c_i}\langle\sigma\rangle_i, \quad i = 1, 2. \quad (14)$$

Equations (4)–(9) applies equally to the free energy (13).

For the trigonal and hexagonal cases ($D_{3d}$ and $D_{6h}$), the symmetry forces $\Lambda_1 = \Lambda_2$, $g_1 = g_2$ and $c_1 = c_2$. The free energy (1) becomes invariant under arbitrary rotation about the principal axis. To describe the lower transition into nematic phase for these crystals, the emergent $O(2)$ symmetry needs to be broken down to a hexagonal one by the addition of sixth order terms in the free energy; this will be discussed later.

*Upper transition.* Let us first analyze the normal-to-superconducting transition at $a = \Delta$. To be precise, let $F(\epsilon_1, \epsilon_2) = \mathcal{F}$ with the condition $\partial\mathcal{F}/\partial\vec{\eta} = 0$ enforced, and define $c_{i,j} = \partial^2 F/\partial\epsilon_i\partial\epsilon_j$, to be evaluated at equilibrium.[3] The modulus $c_{i,j}$ can be expressed in Voigt notation as

$$c_{1,1} = \tfrac{1}{4}(C_{11} + C_{22} - 2C_{12}),$$
$$c_{2,2} = C_{66},$$
$$c_{1,2} = \tfrac{1}{2}(C_{16} - C_{26}). \quad (15)$$

In the normal phase, $c_{1,1} = c_1$, $c_{2,2} = c_2$, and $c_{1,2} = 0$.

One needs to solve $\partial\mathcal{F}/\partial\vec{\eta}$ for arbitrary $\epsilon_1$ and $\epsilon_2$ near equilibrium. Given the equilibrium $\bar{\epsilon}_2 \propto \langle\sigma\rangle_2 = 0$ in the upper phase, a small-$\epsilon_2$ expansion to leading order suffices. We use tilde to denote solution in the presence of a fixed, externally applied $\vec{\epsilon}$:

$$|\tilde{\eta}|^2 \approx \frac{g_1\epsilon_1 + \Delta - a}{\lambda_1}; \quad \tilde{\theta} \approx \frac{g_2\epsilon_2/2}{g_1\epsilon_1 + \Delta - (\lambda_1 - \lambda_2)|\tilde{\eta}|^2}, \quad (16)$$

and $\tilde{\chi} = 0$. [While $\bar{\eta}^2$ admits correction at $O(\epsilon_2^2)$, the condition $\partial\mathcal{F}/\partial|\eta| = 0$ ensures that $c_{22}$ is independent of this correction term.] The shear moduli are found to be

$$c_{1,1}^{(u)} = c_1 - \frac{g_1^2}{\lambda_1},$$
$$c_{2,2}^{(u)} = c_2 - \frac{g_2^2(\Delta - a)}{(\lambda_2 - \Lambda_1)(\Delta - a) + \Lambda_1\Delta}. \quad (17)$$

The off-diagonal $c_{1,2}^{(u)} = 0$ since it is odd under $\mathcal{C}_2'$ symmetry.

The modulus $c_{1,1}$ is discontinuous across the upper transition:

$$\delta_u c_{1,1} \equiv c_{1,1}^{(u)} - c_1 = -\frac{g_1^2}{\lambda_1}, \quad (18)$$

[3] See Ref. [39].

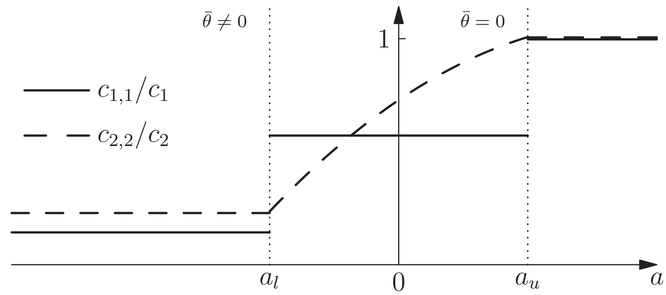


FIG. 1. Qualitative behaviors of the dimensionless $c_{i,i}/c_i$ across the pair of split transitions, assuming the lower phase is chiral. If the crystal is trigonal or hexagonal (e.g., $UPt_3$), one has $c_{1,1}/c_1 = c_{2,2}/c_2$ in the lower phase. Otherwise the relative heights of the low-temperature plateaus are dependent on GL parameters.

but $c_{2,2}$ remains continuous. However, in the limit of $\Delta \to 0$, $c_{2,2}$ develops a step, too: the presence of $\Lambda_1\Delta$ in the denominator smooths out the would-be singular behavior.

*Lower transition into the chiral phase.* Assuming that $\Lambda_3$ is the smallest, the order parameter acquires a chiral component when $a < a_l$. The equilibrium solution is (8). We carry out the same procedure to extract the elastic moduli in the lower phase. Given $\bar{\epsilon}_2 = 0$, we again expand the solution only to linear order in $\epsilon_2$ (picking the positive branch for $\chi$).

$$|\tilde{\eta}|^2 = -\frac{a}{\Lambda_3}, \quad |\tilde{\eta}|^2\cos 2\tilde{\theta} = \left(\frac{g_1\epsilon_1 + \Delta}{\lambda_1 - \Lambda_3}\right),$$
$$\tilde{\chi} = \frac{\pi}{2} + \frac{g_2\epsilon_2}{(\Lambda_3 - \lambda_2)|\tilde{\eta}|^2\sin 2\tilde{\theta}}. \quad (19)$$

The lower chiral phase moduli are

$$c_{1,1}^{(c)} = c_1 - \frac{g_1^2}{\lambda_1 - \Lambda_3}, \quad c_{2,2}^{(c)} = c_2 - \frac{g_2^2}{\lambda_2 - \Lambda_3}, \quad (20)$$

and $c_{1,2}^{(c)} = 0$ again, as dictated by the $D_{2h}$ symmetry. Compared with the upper phase result (17), at the lower transition $c_{1,1}$ is discontinuous and $c_{2,2}$ is continuous with a kink:

$$\delta_c c_{1,1} \equiv c_{1,1}^{(c)} - c_{1,1}^{(u)} = -\frac{g_1^2\Lambda_3}{\lambda_1(\lambda_1 - \Lambda_3)}. \quad (21)$$

The behavior is plotted in Fig 1.

The above result is applicable to hexagonal crystal by setting $\lambda_1 = \lambda_2$, $g_1 = g_2$ and $c_1 = c_2$ to meet the symmetry constraint. Thalmeier *et al.* [36] attempted a similar analysis for the hexagonal $UPt_3$, but we disagree with their result regarding $C_{66}$ ($c_{2,2}$ in our notation.) They theoretically found a $C_{66}$ discontinuity across the upper transition, while ours is continuous. Directly below the lower transition, their result for $C_{66}$ translates to $c_{2,2} = c_2 - g_2^2/\lambda_3$ in our notation, disagreeing with our (20). We point out that they themselves observed no $C_{66}$ discontinuity in the experiments [36].

*Lower transition into the nematic phase: tetragonal case.* Now we assume that $\Lambda_2$ is the smallest, and the lower phase remain nematic. Note that $\Lambda_2 \neq \Lambda_1$ is allowed only by a tetragonal crystal. While this scenario is not immediately relevant to $Sr_2RuO_4$, we still work out the detail, as the lesson learned here is applicable to the $D_{3d}$ version relevant to $M_xBi_2Se_3$.

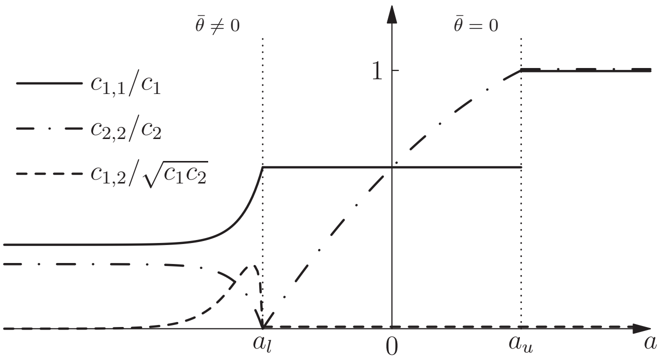


FIG. 2. Qualitative behaviors of the dimensionless $c_{i,j}/\sqrt{c_i c_j}$ across the pair of split transitions, assuming the lower phase is nematic. The qualitative features at each critical point (kinks, discontinuity, zero, and the $|\delta a|^{1/2}$ behavior of $c_{1,2}$) are dictated by symmetry, and shared by both tetragonal and trigonal/hexagonal cases.

Before jumping into the calculation, we note that $c_{2,2}^{(u)}$ from (17) vanishes at $a = a_l$ when $i = 2$ in (4). Specifically, let $\alpha = a/a_l$, and one has

$$c_{2,2}^{(u)} = c_2 \frac{\Lambda_2}{\Lambda_1}(\Lambda_1 - \Lambda_2)\frac{c_2}{g_2^2}|\alpha - 1| + O((\alpha - 1)^2) \tag{22}$$

directly above the lower transition. Shear strain $\epsilon_2$ becomes a soft mode: we will comment on this shortly.

We follow the same procedure as above to calculate the elastic moduli. The equilibrium solution is (8). It suffices to compute the perturbed solution to first order in $\delta\epsilon_i \equiv (\epsilon_i - \bar{\epsilon}_i)$ for both $i = 1, 2$. We will skip the intermediate expressions, and directly quote the end results:

$$\begin{aligned} c_{1,1}^{(n)} &= c_1 - \frac{g_1^2}{\lambda_1}\left[1 + \frac{(\alpha^2 - 1)\lambda_2\Lambda_2}{(\alpha^2 - 1)\lambda_2(\lambda_1 - \Lambda_2) + \lambda_1 g_2^2/c_2}\right], \\ c_{2,2}^{(n)} &= c_2(\alpha^2 - 1)\,\frac{\Lambda_2(\lambda_1 - \Lambda_2)}{\Lambda_2(\lambda_2 - \lambda_1) + \alpha^2\lambda_2(\lambda_1 - \Lambda_2)}, \\ c_{1,2}^{(n)} &= \sqrt{\alpha^2 - 1}\,\frac{\Lambda_2 g_1 g_2}{\Lambda_2(\lambda_2 - \lambda_1) + \alpha^2\lambda_2(\lambda_1 - \Lambda_2)}, \end{aligned} \tag{23}$$

valid for $\alpha > 1$. Compared with (17), it is seen that all three components are *continuous* at $\alpha = 1$ (i.e., $a = a_l$): $c_{1,1}$ shows a kink, $c_{2,2}$ *vanishes* linearly with $|\alpha - 1|$ on either side of the transition, and $c_{1,2}$ grows as $|\alpha - 1|^{1/2}$ below the transition. The general behavior of $c_{i,j}$ around $\alpha = 1$ is sketched in Fig. 2.

This asymptotic behavior is almost obvious in hindsight: suppose one is oblivious to the superconducting aspect of the problem, the strain part of the problem resembles a structural phase transition where the $C_2'$ twofold symmetry is spontaneously broken, and similar problems have been mapped to the transverse-field Ising model [40,41]. The asymptotic behavior of $c_{i,j}$ immediately follows from the well-known mean field results [42], with $\epsilon_2$ being a soft mode at the critical point [43]. We expect the vanishing of $c_{2,2}$ to have a particularly dramatic impact on the ultrasonic dispersion relation.

Also immediately recognizable from the Ising model meanfield result is that the (1,1)-component of *susceptibility* $\chi_{1,1} = (c^{-1})_{1,1} = 1/(c_{1,1} - c_{1,2}^2/c_{2,2})$ has a discontinuous step across the lower transition. This is easily confirmed with (23).

This is a good place to comment on the thermodynamic relation first due to Testardi [44], relating the discontinuities in specific heat $C$ and elastic modulus $c_{i,j}$ across a second-order transition, if $\partial T_c/\partial\epsilon_i$ and $\partial T_c/\partial\epsilon_j$ are also known:

$$\delta c_{i,j} = -\frac{\delta C}{T_c}\frac{\partial T_c}{\partial\epsilon_i}\frac{\partial T_c}{\partial\epsilon_j}. \tag{24}$$

This relation holds approximately for both the upper transition and the lower transition into chiral phase, albeit missing the renormalization effect (12).[4] Given the specific heat jump (9), one is tempted to invoke the (24) and concludes that $c_{1,1}$ is also discontinuous for the present case. This contradicts (23).

The Testardi relation assumes the existence of a critical surface $T_c(\vec{\epsilon})$: all strain components must be invariant under the broken symmetry. This is true for the two previous cases (the gauge U(1) and TRS) but not true for the present case: $\epsilon_2 \neq 0$ in the upper phase explicitly breaks the twofold rotational symmetry and destroys the transition. The relation is therefore not applicable. However, the critical line $T_c(\epsilon_1)$ is well-defined if one allows $\epsilon_2$ to reach equilibrium: $\epsilon_2 = \bar{\epsilon}_2(\epsilon_1)$. The Testardi relation would instead give the discontinuity in $1/\chi_{1,1}$, though it still misses the renormalization (12).

One interesting limit is $g_2 \to 0$. Since $\epsilon_2$ is decoupled from the problem, $c_{1,1} \to 1/\chi_{1,1}$ and become discontinuous at the transition. This is indeed readily seen in (23). Mathematically, $\lambda_1 g_2^2/c_2$ plays the some role as $\Lambda_1\Delta$ did for $c_{2,2}^{(u)}$ in (17), smoothing out the singularity. Since the renormalization $g_2^2/c_2$ is likely to be small compared to $\lambda_2$, in practice $c_{1,1}$ would still exhibit a "near-step":

$$\delta_n c_{1,1} \approx -\frac{g_1^2}{\lambda_1}\frac{\Lambda_2}{\lambda_1 - \Lambda_2}, \tag{25}$$

and the width of this broadened step is of the order of

$$\Delta\alpha = \frac{1}{2}\frac{g_2^2}{c_2\lambda_2}\frac{\lambda_1}{\lambda_1 - \Lambda_2}. \tag{26}$$

*Trigonal nematic superconductor.* The $M_x Bi_2 Se_3$ crystal has $D_{3d}$ symmetry that forces $\Lambda_1 = \Lambda_2 \equiv \Lambda$, $c_1 = c_2 \equiv c$ and $g_1 = g_2 \equiv g$. Without the SBF $\Delta$, the free energy (13) appears in-plane isotropic: it enjoys an emergent $O(2)$ symmetry. We assume $\Lambda < \Lambda_3$ and the material prefers the nematic state.

If one goes beyond quartic order and writes down the most general $O(|\eta|^6)$ free energy allowed by symmetry, the new terms turn out to explicitly break the $O(2)$ symmetry down to sixfold [1,21]:

$$\mathcal{F}_6 = \Gamma_1|\eta|^6 + \Gamma_2\big(\langle\sigma\rangle_1^3 - 3\langle\sigma\rangle_2^2\langle\sigma\rangle_1\big). \tag{27}$$

The coefficient $\Gamma_2$ now dictates the "natural preference": $\Gamma_2 < 0$ favors $\vec{\eta} \propto (1, 0)$ and two other equivalent directions, while $\Gamma_2 > 0$ favors $\vec{\eta} \propto (0, 1)$ and the equivalent. The sign of $\Gamma_2$ is yet undetermined.

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[4] See Ref. [39] for the modified formula that correctly accounts for the renormalization. The original derivation [44] assumes a somewhat unphysical condition.

The original GL theory (1) is self-consistent in that it completely accounts for the leading $O(a^2)$ behavior of the free energy. We demand that $\mathcal{F}_6 \sim O(a^3)$ remains a small perturbation, so the modified GL theory with (27) added remains applicable. Specifically, let $D_c = |\bar{\eta}|^2$ at lower transition, we demand $\Gamma_i D_c \ll \lambda$ for $i = 1, 2$. Since the possible lower transition is caused by competition between $\mathcal{F}_6$ and $\Delta$, it may remain within the range of validity of our theory yet.

The SBF $\Delta$ breaks the crystal symmetry down to $C_{2h}$, and the aforementioned $\mathcal{C}_2'$ twofold axis is the only surviving spatial symmetry. The sign of $\Delta$ determines the direction of $\vec{\eta}$ in the upper phase. If it does not match the preference dictated by $\Gamma_2$, a lower transition similar to the tetragonal case takes place. The phase diagram has been thoroughly discussed by the present authors [21]. Experimentally, the sign of $\Delta$ has been reported to be sample-dependent [23,28,38]: *some* samples must exhibit this lower transition automatically. It what follows, we assume both $\Gamma_2$ and $\Delta$ to be *positive*; other cases are obtained by trivial sign flips.

The upper critical temperature is again $a_u = \Delta$. The upper phase equilibrium solution and moduli for trigonal crystal are

$$
\begin{aligned}
|\bar{\eta}|^2 &= \frac{-\Lambda + \sqrt{\Lambda^2 - 12(\Gamma_1 + \Gamma_2)(a - \Delta)}}{6(\Gamma_1 + \Gamma_2)},\\
c_{1,1}^{(ut)} &= c - \frac{g^2}{\lambda + 6(\Gamma_1 + \Gamma_2)|\bar{\eta}|^2},\\
c_{2,2}^{(ut)} &= c - \frac{g^2|\bar{\eta}|^2}{\Delta + (g^2/c)|\bar{\eta}|^2 - 9\Gamma_2|\bar{\eta}|^4},
\end{aligned}
\tag{28}
$$

and $c_{1,2}^{(ut)} = \bar{\theta} = \bar{\chi} = 0$. One sees that (28) and (17) coincides when $a \to \Delta^-$: the additional $\mathcal{F}_6$ is of no importance when $|\eta|$ is vanishingly small. In particular, the discontinuity in $c_{1,1}$ is still given by (18). We also note that the renormalization (12) must be kept in order to obtain a sensible answer for $c_{2,2}$ in the $\Delta \to 0$ limit: without the renormalization $c_{2,2}$ goes to negative infinity below the transition (utterly unphysical), but in (28), it merely becomes soft. The distinction between $\lambda$ and $\Lambda$ cannot be ignored for the trigonal problem.

Unfortunately, analytic solution cannot be obtained for the lower phase, and we resort to expanding in small $\delta t \equiv (a - a_l)/a_l$. Full expressions at leading order can be obtained.[5] But we will further invoke the condition $\Gamma_i D_c \ll \lambda$, and also assume that the renormalized $\Lambda$ is of the same order as the bare $\lambda$. One then goes through similar calculation and obtains in the lower nematic phase for trigonal crystal:

$$
\begin{aligned}
D_c &= \frac{1}{3}\sqrt{\frac{\Delta}{\Gamma_2}},\\
a_l &= -\frac{1}{3}\left[\Lambda\sqrt{\frac{\Delta}{\Gamma_2}} + (\Gamma_1 - 2\Gamma_2)\frac{\Delta}{\Gamma_2}\right] \approx -\Lambda D_c,\\
c_{1,1}^{(nt)} &\approx c\left[1 - \left(\frac{g^2}{c\Gamma_2 D_c}\right)\left(\frac{\Lambda}{\lambda}\right)\left(\frac{g^2/(c\Lambda) + 3\delta t/2}{g^2/(c\Gamma_2 D_c) + 36\delta t}\right)\right],\\
c_{2,2}^{(nt)} &\approx c\left[1 - \left(\frac{g^2}{c\Gamma_2 D_c}\right)\left(\frac{\Lambda}{\lambda}\right)\left(\frac{\lambda/\Lambda + 3\delta t/2}{g^2/(c\Gamma_2 D_c) + 36\delta t}\right)\right],\\
c_{1,2}^{(nt)} &\approx \left(\frac{g^2}{c\Gamma_2 D_c}\right)\left(\frac{\Lambda}{\lambda}\right)\left(\frac{\sqrt{3\delta t/2}}{g^2/(c\Gamma_2 D_c) + 36\delta t}\right).
\end{aligned}
\tag{29}
$$

The quantity $g^2/(c\Gamma_2 D_c)$ is the ratio between two characteristic scales: $g^2/c$ for the strain renormalization, and $\Gamma_2 D_c$ characterizing the lower transition. We have required $\Gamma_2 D_c \ll \lambda$, but typically the renormalization is also small: $g^2/c \ll \lambda$. Their ratio is undetermined and does not affect the validity of the GL theory. Thus we are justified in retaining the ratio in (29). However, if the renormalization effect is summarily ignored, both $c_{2,2}^{(nt)}$ and $c_{1,2}^{(nt)}$ will diverge at the lower critical point: again unphysical. We stress again that the renormalization (12) must be kept.

Although the detailed expressions become quite complicated, the qualitative physics of this lower transition is still governed by the spontaneous breaking of the $\mathcal{C}_2'$ symmetry: $c_{1,1}$ exhibits a kink, $c_{1,2}$ grows as $\sqrt{\delta t}$ in the lower phase, and $c_{2,2}$ *vanishes* at the transition, as sketched in Fig. 2. The (inverse) susceptibility $1/\chi_{1,1}$ is discontinuous, and the step height can be related to the specific heat discontinuity using the Testardi thermodynamic (24), up to necessary corrections (see footnote 2), but now there is no clear limit where it can be identified with $c_{1,1}$.

Our result has important implication to the supposed nematic superconductivity in $M_x Bi_2 Se_3$. The "nematic hypothesis" was first proposed as the origin for the twofold anisotropy responses that are incompatible with the lattice symmetry and observed only in the superconducting phase [20,25,26]. The persistence of the twofold direction, however, requires a pinning SBF that explicitly breaks the lattice symmetry in the normal state [21,22]. If one accepts the necessity of an explicit SBF, *the nematic superconductivity is no longer mandated by symmetry of the problem.* In addition, ARPES result [45] confirms that the conduction band dispersion remains very similar to the undoped crystal [46,47], and indicates that the normal state is a weakly interacting Fermi liquid. One is left with a dilemma: either there is some exotic phonon pairing mechanism that explicitly disfavors the $s$-wave pairing [48–52] (cf. Ref. [53]), or the superconductivity is single-component (i.e., *not nematic*) yet unusually susceptible to directional perturbations. To our knowledge, no existing experiments is able to differentiate the two possibilities, but the two scenarios suggest vastly different microscopic models.

Two key signatures for the nematic scenario are proposed here. First, the discontinuity in $c_{11}$ across the upper transition is unique to a multi-component order parameter. Moreover, the *existence* of a lower transition constitutes a proof beyond all doubt. The present authors suggested that the lower transition may be identified with calorimetric measurements [21], but we concede here that the specific heat discontinuity is proportional to $\sqrt{\Delta}$, and may well be too small to be observed. On the other hand, the *vanishing* of $c_{2,2}$ at the lower transition is a very dramatic signal, one that we hope would be easily picked up by ultrasound experiments. As discussed above, the sign of the SBF $\Delta$ varies sample to sample: some samples must exhibit a lower transition *without external manipulation.*

[5]See Ref. [39] for full expressions.

*Conclusion.* In this paper, we examine the shear modulus anomaly of an unconventional superconductor in the presence of an SBF. Such systems exhibits a pair of split superconducting transitions, and we work out the anomaly for both the upper and lower transitions within the GL theory.

For both $Sr_2RuO_4$ and $M_xBi_2Se_3$, two materials that are proposed to host unconventional superconductivity, the experimental evidence of split transitions has been inconclusive, to say the least. In light of recent ultrasound experiments, we hope that the measurement of shear moduli will prove fruitful in the experimental search of the lower transitions. This is especially the case for $M_xBi_2Se_3$, where we predict that one of the shear modes must become soft (its modulus vanishes) at the lower transition. We believe such a dramatic signature cannot go unnoticed.

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