Letter

Robustness of vortex-bound Majorana zero modes against correlated disorder

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We investigate the effect of correlated disorder on Majorana zero modes (MZMs) bound to magnetic vortices in two-dimensional topological superconductors. By starting from a lattice model of interacting fermions with a $p_x \pm ip_y$ superconducting ground state in the disorder-free limit, we use perturbation theory to describe the enhancement of the Majorana localization length at weak disorder and a self-consistent numerical solution to understand the breakdown of the MZMs at strong disorder. We find that correlated disorder has a much stronger effect on the MZMs than uncorrelated disorder and that it is most detrimental if the disorder correlation length ℓ is on the same order as the superconducting coherence length ξ. In contrast, MZMs can survive stronger disorder for $\ell \ll \xi$ as random variations cancel each other within the length scale of ξ, while an MZM may survive up to very strong disorder for $\ell \gg \xi$ if it is located in a favorable domain of the given disorder realization.

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Introduction. Topological phases of matter harbor exotic nonlocal quasiparticles and have been proposed as a promising platform for fault-tolerant quantum computation [1,2]. In particular, topological superconducting systems, including one-dimensional (1D) and two-dimensional (2D) heterostructures [3–8] as well as intrinsic 2D superconductors with p-wave pairing symmetry [9], are predicted to host Majorana zero modes (MZMs) [10–13], which can implement the Clifford gate set via braiding. While most proposals for MZM braiding have focused on 1D systems such as nanowire T-junctions [14], MZMs bound to superconducting vortices in 2D systems have distinct advantages as the 2D geometry allows a greater degree of freedom in the motion of the MZMs.

Due to their inherently nonlocal nature, MZMs are known to be protected against infinitesimal local perturbations, including random disorder. However, given that real-world materials contain disorder in varying forms and strength, it is also important to understand the robustness of MZMs against disorder beyond the infinitesimal limit. For example, weak disorder may make the MZMs less localized, leading to a smaller qubit density and/or more gate errors, whereas strong disorder may lead to a complete breakdown of the MZMs. While there have been numerous studies along these lines, most of them focus on 1D nanowires [15–25], while those studying 2D superconductors do not consider vortex-bound MZMs [26] or only concentrate on uncorrelated disorder [27,28].

In this Letter, we consider a simple microscopic model of interacting fermions with a $p_x \pm ip_y$ superconducting ground state [29,30] and study the effect of *correlated* disorder by combining analytical and numerical approaches. Specifically, we investigate vortex-bound MZMs in this model and understand how their robustness depends on the correlation length of the disorder. Our main result is that correlated disorder is significantly more detrimental to the MZMs than uncorrelated disorder. In particular, disorder has the most adverse effect if its correlation length ℓ is similar to the superconducting coherence length ξ, while disorders with $\ell \ll \xi$ and $\ell \gg \xi$ are both more benign, even though for completely different reasons. Since our results naturally extend to the continuum limit of the model and are expressed in terms of measurable length and energy scales, they should apply universally for $p_x \pm ip_y$ superconductors and provide useful guidelines for the realization of MZM braiding in realistic experimental systems.

Model. We consider a tight-binding Hamiltonian of interacting spinless fermions on the square lattice,

$$\hat{H} = -\sum_{\mathbf{r}} \mu_{\mathbf{r}} c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}} - \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} (t_{\mathbf{r},\mathbf{r}'} c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}'} + \text{H.c.}) - g \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}} c_{\mathbf{r}'}^{\dagger} c_{\mathbf{r}'}, \tag{1}$$

where the three terms describe a site-dependent chemical potential, a nearest-neighbor hopping amplitude, and a nearest-neighbor attractive interaction, respectively. In the presence of a magnetic field, the hopping amplitude is spatially modulated by the vector potential $\mathbf{A}(\mathbf{r})$ through the Peierls substitution, $t_{\mathbf{r},\mathbf{r}'} = t e^{iA_{\mathbf{r},\mathbf{r}'}}$, where $A_{\mathbf{r},\mathbf{r}'} = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}(\hat{\mathbf{r}}) \cdot d\hat{\mathbf{r}}$. We expand the chemical potential as $\mu_{\mathbf{r}} = \bar{\mu} + \delta\mu_{\mathbf{r}}$, where $\bar{\mu}$ is a constant background, while $\delta\mu_{\mathbf{r}}$ describes random disorder of strength $\delta\bar{\mu}$ that is correlated within a length scale ℓ. Mathematically, $\delta\mu_{\mathbf{r}}$ are real Gaussian random variables characterized by

$$\overline{\delta\mu_{\mathbf{r}}} = 0, \quad \overline{\delta\mu_{\mathbf{r}}\delta\mu_{\mathbf{r}'}} = \delta\bar{\mu}^2 e^{-|\mathbf{r}-\mathbf{r}'|^2/\ell^2}, \tag{2}$$

where the overline denotes averaging over many disorder realizations. In practice, these real-space random variables are generated through $\delta\mu_{\mathbf{r}} = \sum_{\mathbf{k}} \text{Re}[\delta\tilde{\mu}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}]$ from the independent momentum-space complex variables $\delta\tilde{\mu}_{\mathbf{k}}$ satisfying

$$\overline{\delta\tilde{\mu}_{\mathbf{k}}} = \overline{\delta\tilde{\mu}^*_{\mathbf{k}}} = 0, \quad \overline{\delta\tilde{\mu}_{\mathbf{k}}\delta\tilde{\mu}_{\mathbf{k}'}} = \overline{\delta\tilde{\mu}^*_{\mathbf{k}}\delta\tilde{\mu}^*_{\mathbf{k}'}} = 0,$$
$$\overline{\delta\tilde{\mu}_{\mathbf{k}}\delta\tilde{\mu}^*_{\mathbf{k}'}} = 2\mathcal{N}\delta\bar{\mu}^2\delta_{\mathbf{k},\mathbf{k}'}e^{-\ell^2|\mathbf{k}|^2/4}, \tag{3}$$

where the normalization constant is $\mathcal{N} = [\sum_{\mathbf{k}} e^{-\ell^2|\mathbf{k}|^2/4}]^{-1}$ for a large enough system size $L \gg \ell$.

In the absence of interactions ($g = 0$), disorder ($\mu_{\mathbf{r}} = \bar{\mu}$), and magnetic field ($t_{\mathbf{r},\mathbf{r}'} = t$), the tight-binding Hamiltonian in Eq. (1) is quadratic and translation invariant. By means of a Fourier transform, one then obtains a single fermion band with energy-momentum dispersion $\varepsilon_{\mathbf{k}} = -\bar{\mu} - 2t(\cos k_x + \cos k_y)$ for a normalized lattice constant $a = 1$. For $|\bar{\mu}| < 4t$, the low-energy physics is governed by a Fermi surface characterized by $\varepsilon_{\mathbf{k}} = 0$. In the following, we consider $\bar{\mu} = -4t + \varepsilon_F$ with $0 < \varepsilon_F < t$ to get an approximately circular Fermi surface around $\mathbf{k} = \mathbf{0}$. From an expansion to the lowest order in $\mathbf{k}$, the energy-momentum dispersion is then $\varepsilon_{\mathbf{k}} = -\varepsilon_F + |\mathbf{k}|^2/2m$, where $m = 1/(2t)$ is an effective mass. Thus, in this approximation, the Fermi surface is indeed circular with Fermi energy ε_F and Fermi wave vector $k_F = \sqrt{2m\varepsilon_F} = \sqrt{\varepsilon_F/t}$.

Bulk superconductivity. We first consider the Hamiltonian in Eq. (1) with attractive interactions ($g > 0$) but without disorder ($\mu_{\mathbf{r}} = \bar{\mu}$) or magnetic field ($t_{\mathbf{r},\mathbf{r}'} = t$). It has been shown numerically [29] and analytically [30] that the ground state is then a gapped $p_x \pm ip_y$ superconductor, which spontaneously breaks time-reversal symmetry. To describe this ground state on the mean-field (i.e., saddle-point) level, we employ a standard Hubbard-Stratonovich decoupling in Eq. (1) to obtain a quadratic Bogoliubov-de Gennes (BdG) Hamiltonian,

$$H = -\sum_{\mathbf{r}} \mu_{\mathbf{r}} c^\dagger_{\mathbf{r}} c_{\mathbf{r}} - \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} (t_{\mathbf{r},\mathbf{r}'} c^\dagger_{\mathbf{r}} c_{\mathbf{r}'} + t^*_{\mathbf{r},\mathbf{r}'} c^\dagger_{\mathbf{r}'} c_{\mathbf{r}})$$
$$- \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} (\Delta^*_{\mathbf{r},\mathbf{r}'} c_{\mathbf{r}} c_{\mathbf{r}'} + \Delta_{\mathbf{r},\mathbf{r}'} c^\dagger_{\mathbf{r}'} c^\dagger_{\mathbf{r}}), \tag{4}$$

which must be solved self-consistently in terms of the superconducting pairing potentials,

$$\Delta_{\mathbf{r},\mathbf{r}'} = -\Delta_{\mathbf{r}',\mathbf{r}} = g\langle c_{\mathbf{r}} c_{\mathbf{r}'}\rangle, \tag{5}$$

where $\langle \mathcal{O}\rangle$ means the expectation value of the operator $\mathcal{O}$ with respect to the ground state of H. These pairing potentials can generally be parameterized as

$$\Delta^x_{\mathbf{r}} \equiv \Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{r}}_x} = \sum_{\mathbf{q}} (\Delta^+_{\mathbf{q}} + \Delta^-_{\mathbf{q}}) e^{i\mathbf{q}\cdot\mathbf{r}},$$
$$\Delta^y_{\mathbf{r}} \equiv \Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{r}}_y} = i\sum_{\mathbf{q}} (\Delta^+_{\mathbf{q}} - \Delta^-_{\mathbf{q}}) e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{6}$$

where $\hat{\mathbf{r}}_x = (1, 0)$ and $\hat{\mathbf{r}}_y = (0, 1)$ are the lattice vectors, and the component $\Delta^\pm_{\mathbf{q}}$ corresponds to $p_x \pm ip_y$ superconductivity with a spatial modulation of wave vector $\mathbf{q}$. In the absence of disorder ($\mu_{\mathbf{r}} = \bar{\mu}$) and magnetic field ($t_{\mathbf{r},\mathbf{r}'} = t$), the superconductivity is translation symmetric [29,30]. Assuming $p_x + ip_y$ pairing symmetry without loss of generality, the components in Eq. (6) then become

$$\Delta^+_{\mathbf{q}} = \bar{\Delta}\delta_{\mathbf{q},\mathbf{0}}, \qquad \Delta^-_{\mathbf{q}} = 0, \tag{7}$$

corresponding to $\Delta_{\mathbf{r}} \equiv \Delta^x_{\mathbf{r}} = -i\Delta^y_{\mathbf{r}} = \bar{\Delta}$. The constant $\bar{\Delta}$ can be determined from a self-consistent solution of Eqs. (4) and (5). In the universal continuum limit ($k_F \ll 1$), we show in the Supplemental Material (SM) [31] that $\bar{\Delta}$ satisfies

$$1 = \frac{g}{N}\sum_{\mathbf{k}} \frac{|\mathbf{k}|^2}{\sqrt{\varepsilon^2_{\mathbf{k}} + 4|\mathbf{k}|^2|\bar{\Delta}|^2}} \approx g\nu \int \frac{d\varepsilon\, k_F^2}{\sqrt{\varepsilon^2 + 4k_F^2|\bar{\Delta}|^2}}, \tag{8}$$

where N is the number of lattice sites, and ν is the density of states at the Fermi level. If we then choose $\bar{\Delta}$ to be real and positive without loss of generality, it is approximately given by the standard superconducting gap formula,

$$\bar{\Delta} \sim \frac{E}{2k_F} \exp\left(-\frac{1}{2gk_F^2\nu}\right), \tag{9}$$

where E is an energy scale governing the high-energy cutoff (whose precise value is irrelevant), while $2gk_F^2$ is an effective interaction strength reflecting the p-wave symmetry of the superconductivity. Importantly, because of the factor $k_F^2 \propto \varepsilon_F$ within the exponential, the pairing potential $\bar{\Delta}$ strongly depends on the Fermi energy ε_F [30].

Next, we include a weak disorder in the chemical potential ($\delta\bar{\mu} \ll \bar{\mu}$) and study its effect on the pairing potentials $\Delta_{\mathbf{r},\mathbf{r}'}$ via perturbation theory. Formally, we restore $\mu_{\mathbf{r}} = \bar{\mu} + \delta\mu_{\mathbf{r}}$ in Eq. (4) and modify Eq. (7) by writing $\Delta^+_{\mathbf{q}} = \bar{\Delta}\delta_{\mathbf{q},\mathbf{0}} + \delta\Delta^+_{\mathbf{q}}$ and $\Delta^-_{\mathbf{q}} = \delta\Delta^-_{\mathbf{q}}$. We can then employ $\delta\mu_{\mathbf{r}} = \sum_{\mathbf{q}} \delta\hat{\mu}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$, where $\delta\hat{\mu}_{\mathbf{q}} = \frac{1}{2}[\delta\tilde{\mu}_{\mathbf{q}} + \delta\tilde{\mu}^*_{-\mathbf{q}}]$, and obtain the self-consistent solution of Eqs. (4) and (5) up to linear order in $\delta\hat{\mu}_{\mathbf{q}}$ and $\delta\Delta^\pm_{\mathbf{q}}$. In the continuum limit ($|\mathbf{q}| \ll k_F \ll 1$) of weak superconductivity ($\xi^{-1} \ll k_F$), this approach gives (see the SM [31])

$$\delta\Delta^+_{\mathbf{q}} = f\left(\frac{\xi|\mathbf{q}|}{2}\right)\frac{\partial\bar{\Delta}}{\partial\varepsilon_F}\,\delta\hat{\mu}_{\mathbf{q}},$$
$$\delta\Delta^-_{\mathbf{q}} = -h\left(\frac{\xi|\mathbf{q}|}{2}\right)e^{2i\vartheta_{\mathbf{q}}}\,\frac{\partial\bar{\Delta}}{\partial\varepsilon_F}\,\delta\hat{\mu}_{\mathbf{q}}, \tag{10}$$

where $\xi = v_F/(2k_F\bar{\Delta}) = 1/(2m\bar{\Delta})$ is the superconducting coherence length, $v_F = k_F/m$ is the Fermi velocity, $\vartheta_{\mathbf{q}}$ is the angle between $\mathbf{q}$ and $\hat{\mathbf{r}}_x$, while $f(x)$ and $h(x)$ are dimensionless functions with asymptotic forms

$$f(x) \approx \begin{cases} 1 - \frac{x^2}{6} & (x \ll 1),\\ \frac{1}{\ln x} & (x \gg 1),\end{cases}$$
$$h(x) \approx \begin{cases} \frac{x^2}{6} & (x \ll 1),\\ \frac{1}{2(\ln x)^2} & (x \gg 1).\end{cases} \tag{11}$$

For $\mathbf{q} = \mathbf{0}$, the disorder component $\delta\hat{\mu}_{\mathbf{0}}$ simply corresponds to a shift in the Fermi energy ε_F, and the pairing potential $\bar{\Delta}$ with $p_x + ip_y$ symmetry is renormalized accordingly. For finite $\mathbf{q}$, however, the disorder gives rise to reduced variations in the $p_x + ip_y$ pairing due to $f(\xi|\mathbf{q}|/2) < 1$ and also generates a finite $p_x - ip_y$ pairing due to $h(\xi|\mathbf{q}|/2) > 0$. Both of these effects are more pronounced if the disorder wave vector $\mathbf{q}$ exceeds the inverse coherence length ξ^{-1}. We note that, while the mean-field results in Eqs. (10) and (11) may not be quantitatively right for $|\mathbf{q}| \gg \xi^{-1}$, any corrections beyond the mean-field level are expected to strengthen our main conclusions by suppressing $f(\xi|\mathbf{q}|/2)$ and $h(\xi|\mathbf{q}|/2)$.

Finally, we describe the real-space correlations in the pairing potentials $\Delta_{\mathbf{r},\mathbf{r}'}$ as a result of disorder. Since $h(x) \ll f(x)$ for all x, we neglect the components $\delta\Delta^-_{\mathbf{q}}$ and use Eq. (6) to introduce $\delta\Delta_{\mathbf{r}} \equiv \delta\Delta^x_{\mathbf{r}} = -i\delta\Delta^y_{\mathbf{r}} = \sum_{\mathbf{q}} \delta\Delta^+_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$. From Eqs. (3) and (10), the disorder correlations in $\delta\Delta_{\mathbf{r}}$ are then

$$\overline{\delta\Delta_{\mathbf{r}}\delta\Delta_{\mathbf{r}'}} = \mathcal{N}\alpha^2\delta\bar{\mu}^2 \mathrm{Re}\sum_{\mathbf{q}} e^{-\frac{1}{4}\ell^2|\mathbf{q}|^2 + i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} f^2\left(\frac{\xi|\mathbf{q}|}{2}\right), \tag{12}$$

where $\alpha = \partial\bar{\Delta}/\partial\varepsilon_F$ and $f^2(x) \equiv [f(x)]^2$. Since $f(x)$ depends only logarithmically on its argument, it is a reasonable approximation to substitute $f(\xi|\mathbf{q}|/2)$ with $f(\xi/\ell)$ in Eq. (12) and work with the resulting simplified correlations,

$$\overline{\delta\Delta_{\mathbf{r}}\delta\Delta_{\mathbf{r}'}} = \alpha^2 f^2(\xi/\ell)\delta\bar{\mu}^2 e^{-|\mathbf{r}-\mathbf{r}'|^2/\ell^2}. \tag{13}$$

From a direct comparison with Eq. (2), this result has a simple physical interpretation. For $\ell \gg \xi$, the local pairing potential is determined by the local chemical potential via Eq. (9). For $\ell \ll \xi$, the variations in the pairing potential still follow those in the chemical potential, but the constant of proportionality is reduced by a factor $f^2(\xi/\ell) \ll 1$.

Majorana localization length. We now consider a superconducting vortex hosting a MZM and understand the effect of weak disorder on the localization length of the MZM. Taking the continuum limit, $c_{\mathbf{r}} \to \psi(\mathbf{r})$, assuming a pure $p_x + ip_y$ pairing symmetry, $\Delta_{\mathbf{r}} \equiv \Delta^x_{\mathbf{r}} = -i\Delta^y_{\mathbf{r}} \to \Delta(\mathbf{r})$, and including a magnetic field, the BdG Hamiltonian in Eq. (4) takes the form $H = \int d^2\mathbf{r}\, \boldsymbol{\psi}^\dagger \cdot \mathcal{H} \cdot \boldsymbol{\psi}$, where $\boldsymbol{\psi} = (\psi, \psi^\dagger)^T$ and

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix} \frac{1}{2m}(-i\nabla + \mathbf{A})^2 - \varepsilon_F & 2\Delta(\partial_x + i\partial_y) \\ -2\Delta^*(\partial_x - i\partial_y) & -\frac{1}{2m}(-i\nabla - \mathbf{A})^2 + \varepsilon_F \end{bmatrix}. \tag{14}$$

Focusing on a single vortex at the origin and using polar coordinates, $\mathbf{r} = (r, \vartheta)$, the π magnetic flux of the vortex can be represented by a vector potential with components

$$A_r(\mathbf{r}) = 0, \qquad A_\vartheta(\mathbf{r}) = \frac{2\pi\delta(\vartheta) - a(r)}{2r}, \tag{15}$$

where the term $\propto \delta(\vartheta)$ corresponds to a $\mathbb{Z}_2$ flux string [1,32], while $a(r) \approx 1$ for $r \ll \lambda$ and $a(r) \sim e^{-r/\lambda}$ for $r \gg \lambda$ in terms of the London penetration depth λ. In this gauge, the pairing potential $\Delta(\mathbf{r})$ does not have any angular winding and simply takes its bulk value for $r \gg \xi$. The Hamiltonian matrix in Eq. (14) can then be written in polar coordinates as

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix} -\frac{1}{2m}\left(D_r^2 + D_{\vartheta,-}^2\right) - \varepsilon_F & 2\Delta e^{i\vartheta}\left(\partial_r + \frac{i}{r}\partial_\vartheta\right) \\ -2\Delta^* e^{-i\vartheta}\left(\partial_r - \frac{i}{r}\partial_\vartheta\right) & \frac{1}{2m}\left(D_r^2 + D_{\vartheta,+}^2\right) + \varepsilon_F \end{bmatrix}, \tag{16}$$

where $D_r^2 \equiv \partial_r^2 + (1/r)\partial_r$ and $D_{\vartheta,\pm} \equiv (\partial_\vartheta \pm ia/2)/r$, while the $\mathbb{Z}_2$ flux string induces antiperiodic boundary conditions, $\psi(r, 2\pi) = -\psi(r, 0)$, in the polar angle ϑ. If we take $\Delta \in \mathbb{R}$ without loss of generality, search for the MZM in the form

$$\gamma = \int d^2\mathbf{r}\, \phi(r)\left[ie^{-i\vartheta/2}\,\psi(\mathbf{r}) - ie^{i\vartheta/2}\,\psi^\dagger(\mathbf{r})\right], \tag{17}$$

which naturally satisfies the antiperiodic boundary conditions, and demand $\gamma = \gamma^\dagger$ as well as $[H, \gamma] = 0$, the radial MZM wave function $\phi(r)$ must be a real solution of

$$\frac{1}{2m}\left[\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{(1-a)^2\phi}{4r^2}\right] + \varepsilon_F\phi + 2\Delta\left[\frac{d\phi}{dr} + \frac{\phi}{2r}\right] = 0. \tag{18}$$

For large distances, $r \gg \lambda$, in the disorder-free limit, we can set $\Delta = \bar{\Delta}$ and neglect $a(r) \sim e^{-r/\lambda} \ll 1$. The *exact* general solution of Eq. (18) then takes the form

$$\phi(r) = \frac{C}{\sqrt{r}} \exp(-2m\bar{\Delta}r)\cos\left[\sqrt{2m\varepsilon_F - (2m\bar{\Delta})^2}\, r + \varphi\right] = \frac{C}{\sqrt{r}} \exp\left(-\frac{r}{\xi}\right)\cos\left[q_F r + \varphi\right], \tag{19}$$

where C and φ are arbitrary constants, while ξ is the coherence length and $q_F = \sqrt{k_F^2 - \xi^{-2}} \approx k_F$ is the Fermi wave vector for weak superconductivity. Importantly, the solution in Eq. (19) is approximately valid even for $\xi \ll r \ll \lambda$ as the correction to Eq. (18) from a finite $a(r)$ is subdominant due to $|\phi/r^2| \ll |d^2\phi/dr^2|$ for any $r \gg \xi$. As expected, the Majorana localization length is thus simply the coherence length ξ in the disorder-free limit.

If we include a weak disorder in the chemical potential μ (i.e., the Fermi energy ε_F), it affects the decay of the MZM wave function $\phi(r)$ and, hence, the localization length via the pairing potential Δ. Ignoring the power-law prefactor, the approximate disorder average of $|\phi(r)|$ from Eq. (19) is

$$\overline{|\phi(r)|} \sim \overline{\exp\left(-2m\int_0^r d\hat{r}[\bar{\Delta} + \delta\Delta(\hat{r})]\right)} = \exp[-2m\bar{\Delta}r]\,\overline{\exp\left[-2m\int_0^r d\hat{r}\,\delta\Delta(\hat{r})\right]}. \tag{20}$$

Since $O = 2m\int_0^r d\hat{r}\,\delta\Delta(\hat{r})$ is a Gaussian random variable with vanishing mean, $\overline{O} = 0$, it satisfies the standard identity $\overline{\exp(-O)} = \exp(\overline{O^2}/2)$. Using the correlations of $\delta\Delta(\hat{r})$ in Eq. (13), the disorder average for large distances r satisfying both $r \gg \ell \gg k_F^{-1}$ and $r \gg \xi \gg k_F^{-1}$ then becomes

$$\overline{|\phi(r)|} \sim \exp\left[-2m\bar{\Delta}r + 2m^2\int_0^r d\hat{r}\int_0^r d\hat{r}'\,\overline{\delta\Delta(\hat{r})\delta\Delta(\hat{r}')}\right] \approx \exp\left[-\frac{r}{\xi} + \frac{\sqrt{\pi}}{2}\kappa^2\frac{\ell r}{\xi^2} f^2\left(\frac{\xi}{\ell}\right)\right], \tag{21}$$

and corresponds to an enhanced Majorana localization length

$$\xi' = \xi\left[1 - \frac{\sqrt{\pi}}{2}\kappa^2\frac{\ell}{\xi} f^2\left(\frac{\xi}{\ell}\right)\right]^{-1}, \tag{22}$$

where $\kappa = \delta\bar{\mu}(\partial\bar{\Delta}/\partial\varepsilon_F)/\bar{\Delta}$ is the relative change in the pairing potential due to a shift $\delta\bar{\mu}$ in the Fermi energy.

According to Eq. (22), the localization length is more sensitive to disorder with larger correlation length ℓ. Indeed, for $\ell \ll \xi$, the correction to the localization length is suppressed as a result of $\ell/\xi \ll 1$ and $f(\xi/\ell) \ll 1$. It should be emphasized, however, that Eq. (22) is no longer relevant for $\ell \gg \xi$ as the corresponding disorder average is only appropriate for $r \gg \ell$. Instead, for $\xi \ll r \ll \ell$, the behavior is entirely determined by the specific disorder realization. Since the chemical

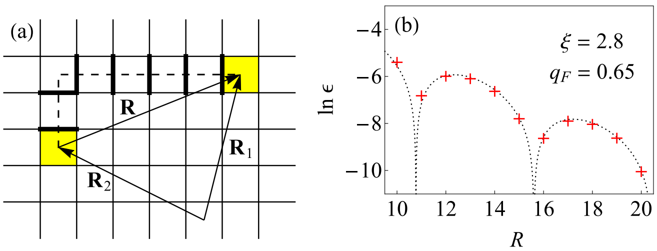

FIG. 1. (a) Two vortices centered at the yellow plaquettes with separation $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2 = (5, 2)$. The $\mathbb{Z}_2$ flux string (dashed line) intersects several links denoted by thick lines. (b) MZM hybridization energy ϵ as a function of the separation $\mathbf{R} = (R, 0)$ for a 50×30 system. The dotted line is a fit of Eq. (19) with $\xi = 2.8$ and $q_F = 0.65$.

potential only fluctuates on length scales larger than ℓ, the MZM is located within a region of size ℓ with approximately constant chemical potential. Therefore, the relevant localization length, corresponding to $\xi \ll r \ll \ell$, is the same as in a *disorder-free* system with the given chemical potential μ. Interestingly, the localization length may then even *decrease* if the MZM is located in a region with $\mu > \bar{\mu}$.

Numerical solution. To qualitatively check the validity of our results, we numerically obtain self-consistent solutions of Eqs. (4) and (5) through an iterative procedure. Since MZMs must appear in pairs for any closed system, we consider two superconducting vortices centered at two square plaquettes with positions $\mathbf{R}_{1,2}$ [see Fig. 1(a)]. In this case, the $\mathbb{Z}_2$ flux string connects the two vortices, and the hopping amplitudes in Eq. (4) become $t_{\mathbf{r},\mathbf{r}'} = t u_{\mathbf{r},\mathbf{r}'} e^{iA'_{\mathbf{r},\mathbf{r}'}}$, where $u_{\mathbf{r},\mathbf{r}'}$ is -1 $(+1)$ if the $\mathbb{Z}_2$ flux string intersects (does not intersect) the link $\langle \mathbf{r}, \mathbf{r}' \rangle$, while $A'_{\mathbf{r},\mathbf{r}'}$ is only nonzero within a radius λ of each vortex. The precise form of $A'_{\mathbf{r},\mathbf{r}'}$ and the details of the iterative procedure are described in the SM [31,33].

We choose the parameters of Eq. (1) to be $t = 1$, $\bar{\mu} = -3.5$, and $g = 5.0$, which correspond to $m = 0.5$, $\varepsilon_F = 0.5$, and $k_F \approx 0.7$. In the absence of disorder, the self-consistent solution for a vortex-free system gives a bulk pairing potential $\bar{\Delta} \approx 0.33$ and a bulk fermion gap $E_0 \approx 0.41$. If we then include two vortices with separation $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$, we find a low-energy fermion in the bulk gap whose energy exhibits an oscillating exponential decay [34,35] as a function of the distance $R \equiv |\mathbf{R}|$ [see Fig. 1(b)]. Since this fermion consists of the two MZMs bound to the vortices, and its finite energy results from a hybridization between the MZM wave functions, we fit its energy ϵ with the functional form of Eq. (19) to extract $\xi \approx 2.8$ and $q_F \approx 0.65$. We note that these values agree with $1/(2m\bar{\Delta}) \approx 3.0$ and $k_F \approx 0.7$ even though the system is not in the continuum limit.

Finally, we include two vortices with $R \gg 1$ and investigate how the energy ϵ of the lowest-energy fermion behaves as the disorder strength $\delta\bar{\mu}$ is gradually increased. The disorder-averaged results are shown in Fig. 2 for different disorder correlation lengths, corresponding to (a) $\ell = 0$, (b) $\ell < \xi$, (c) $\ell \sim \xi$, and (d) $\ell > \xi$, respectively. In all cases, we find that the energy ϵ increases by two orders of magnitude from the MZM expectation, $e^{-R/\xi} \sim 10^{-5}$, to the generic disordered expectation, $1/N \sim 10^{-3}$, which indicates a disorder-induced breakdown of the MZMs. This breakdown occurs at $\delta\bar{\mu} \sim \varepsilon_F$ for $\ell \gtrsim \xi$ and at $\delta\bar{\mu} > \varepsilon_F$ for $\ell < \xi$. In the former case, we hypothesize that it results from a hybridization between the MZMs and the gapless edge modes that are expected to separate topological superconducting regions with local $\varepsilon_F > 0$ and trivial insulating regions with local $\varepsilon_F < 0$.

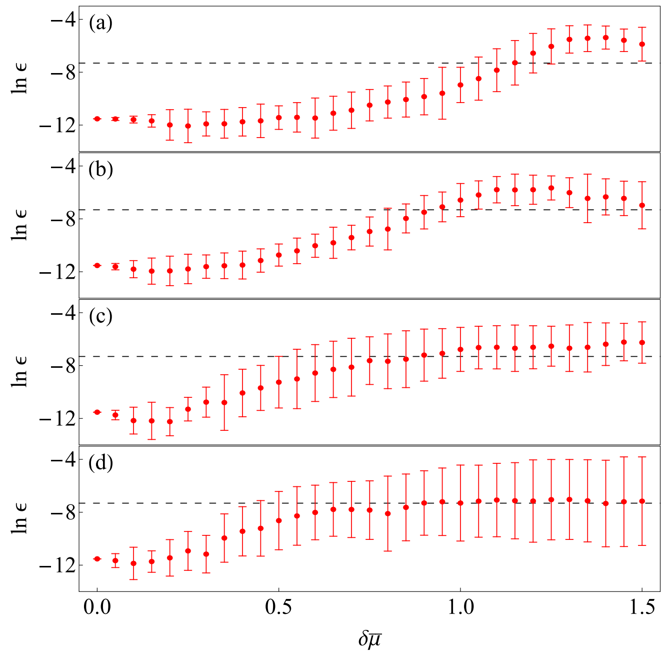

FIG. 2. Disorder-averaged MZM hybridization energy ϵ against disorder strength $\delta\bar{\mu}$ for (a) $\ell = 0$, (b) $\ell = 1$, (c) $\ell = 3$, and (d) $\ell = 10$ if the two vortices hosting the MZMs are separated by $\mathbf{R} = (25, 15)$ inside a 50×30 system. Each data point is averaged over 25 disorder realizations, and its error bar shows the variation among the individual realizations. The dashed line marks the expectation for a generic disordered system, $\epsilon \sim 1/N = 1/1500$.

Remarkably, the breakdown of the MZMs in Fig. 2 is consistent with our weak-disorder results in at least three different ways. First, in the case of $\ell \sim \xi$, the breakdown at $\delta\bar{\mu} \sim \varepsilon_F$ roughly corresponds to $\kappa \sim 1$ at which Eq. (22) predicts a divergent localization length for $\ell/\xi \sim 1$ and, correspondingly, $f(\xi/\ell) \sim 1$. Second, the MZMs can *generally* survive stronger disorder for $\ell < \xi$. Third, the energy ϵ has larger variations for $\ell > \xi$ as the MZMs can survive up to very strong disorder for *certain* disorder realizations.

Discussion. We have studied the effect of correlated disorder on vortex-bound MZMs in $p_x \pm ip_y$ superconductors and demonstrated that it is much more detrimental than uncorrelated disorder. The general picture is that disorder gradually increases the MZM localization length until the MZMs eventually break down due to a divergent localization length. However, according to Eq. (22), the correction to the localization length strongly depends on the disorder correlation length ℓ and is suppressed for short-range-correlated disorder ($\ell \ll \xi$) because random variations cancel each other within the superconducting coherence length ξ. We note that, while Eq. (22) is only valid for $\ell \gg k_F^{-1}$, our numerical results confirm this effect even in the uncorrelated limit ($\ell \to 0$).

For long-range-correlated disorder ($\ell \gg \xi$), the MZM localization length from Eq. (22), which characterizes the decay of the wave function $\phi(r)$ at large distances, $r \gg \ell$, is strongly renormalized and rapidly diverges. However, the MZM is located inside a large "disorder domain" of size ℓ within which the chemical potential μ is approximately constant due to the correlated nature of the disorder. Thus, within this domain, the wave function $\phi(r)$ decays as in a disorder-free system with the same chemical potential μ. For $\mu > \bar{\mu}$, the MZM then survives even in the presence of strong disorder because its wave function is already exponentially small, $\phi(\ell) \sim e^{-\ell/\xi}$, at the boundary of the domain, $r \sim \ell$. In this case, any actual braiding of the MZMs is restricted to such favorable disorder domains, but effective braiding may still be achievable through a measurement-only protocol [36,37].

Therefore, we conclude that disorder has the most adverse effect on the MZMs if its correlation length is similar to the superconducting coherence length. In this regime, all MZMs are expected to break down as soon as disorder is strong enough to induce topologically distinct regions surrounded by gapless edge modes. We emphasize that, even though we concentrate on a specific lattice model and only include disorder in the chemical potential, our results naturally extend to the continuum limit and should be universally applicable to disordered $p_x \pm ip_y$ superconductors.

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