

# Anomalous Andreev interferometer: Study of an anomalous Josephson junction coupled to a normal wire

Alberto Hijano<sup>1,2,\*</sup>, Stefan Ilić<sup>1,†</sup> and F. Sebastián Bergeret<sup>1,3,‡</sup>

<sup>1</sup>*Centro de Física de Materiales (CFM-MPC) Centro Mixto CSIC-UPV/EHU, E-20018 Donostia-San Sebastián, Spain*

<sup>2</sup>*Department of Condensed Matter Physics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain*

<sup>3</sup>*Donostia International Physics Center (DIPC), 20018 Donostia-San Sebastián, Spain*



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Josephson junctions (JJs), where both time-reversal and inversion symmetry are broken, exhibit a phase shift  $\varphi_0$  in their current-phase relation. This allows for an anomalous supercurrent to flow in the junction even in the absence of a phase bias between the superconductors. We show that a finite phase shift also manifests in the so-called Andreev interferometers—a device that consists of a mesoscopic conductor coupled to the  $\varphi_0$  junction. Due to the proximity effect, the resistance of this conductor is phase sensitive—it oscillates by varying the phase of the JJ. As a result, the quasiparticle current  $I_{qp}$  flowing through the conductor has an anomalous component, which exists only at finite  $\varphi_0$ . Thus, the Andreev interferometry could be used to probe the  $\varphi_0$  effect. We consider two realizations of the  $\varphi_0$  junction and calculate  $I_{qp}$  in the interferometer: a superconducting structure with spin-orbit coupling and a system of spin-split superconductors with spin-polarized tunneling barriers.

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## I. INTRODUCTION

The dc Josephson effect establishes that the current flowing between two superconductors with a phase difference  $\varphi$ , obtained for instance by applying a magnetic flux to the closed circuit, is given as  $I_J = I_c \sin \varphi$ . Here  $I_c$  is the critical current of the junction. In such junctions the phase difference of the ground state is  $\varphi = 0$ . In a system where (only) time-reversal symmetry is broken, such as superconductor/ferromagnet/superconductor (S/F/S) structures, it was shown that the current-phase relation can acquire a phase shift of  $\pi$ , and therefore such junctions are called  $\pi$  junctions [1–4].

In junctions where both time-reversal and inversion symmetries are broken the current-phase relation takes a more general form [5]

$$I_S = I_c \sin(\varphi + \varphi_0) = I_0^S \sin \varphi + I_{an}^S \cos \varphi. \quad (1)$$

Such JJs are known as  $\varphi_0$  junctions by analogy. This effect is referred to as the anomalous Josephson effect (AJE). In general, the current-phase relation of a JJ given by Eq. (1) can be decomposed into the usual current  $I_0^S$  and anomalous current  $I_{an}^S$ .  $I_{an}^S$  is nonzero only if the appropriate symmetries are broken, leading to a finite supercurrent even at zero phase difference between the superconductors.

AJE reflects the interplay between spin-dependent fields and superconductivity. This interaction is the basis of several effects and applications that are attracting the interest of a large community, such as topological [6–8] and unconventional [9,10] superconductivity, superconducting spintronics

[11], and novel superconducting electronic elements [12]. The most well-known proposals for AJE involve superconducting structures with spin-orbit interaction [5,13–20], some of which have been successfully tested in experiment [21–24]. Other theoretical studies have proposed numerous alternative realizations of AJE: in S/F/S junctions with a nonhomogeneous magnetization texture [25–32], junctions of unconventional superconductors [33–36], and between topologically nontrivial superconducting leads [37]. An anomalous current-phase relation can also be obtained under non-equilibrium situation in multiterminal structures [38–40].  $\varphi_0$  junctions could prove to be a key component for quantum electronics, as they can provide a stable phase bias to quantum circuits, and could therefore be particularly useful in phase-coherent superconducting electronics and spintronics [11,23].

In this paper, we consider a  $\varphi_0$  junction coupled to a mesoscopic conductor, in a device known as Andreev interferometer [41–44]. The basic physical idea behind such devices is the following: Superconducting correlations are induced in the conductor by the proximity effect, and as a consequence, its resistance becomes sensitive to the phase of the Josephson junction. This means that a simple resistance measurement performed on the conductor can reveal details about the phase dynamics of the adjacent superconducting structure. In the 90s this topic was particularly active, and several types of Andreev interferometers were theoretically proposed [45–48] and experimentally realized [49,50]. Andreev interferometers have been used to study the magnetoresistance oscillations [51], electric transport [43,52–57], and thermopower and thermal transport [58–61] in S/N structures.

Our goal is to establish how the anomalous phase shift  $\varphi_0$  manifests on the quasi-particle transport through the Andreev interferometer shown in Fig. 1(a). An important advantage of this geometry is that it allows for a decoupling of the superconducting loop with the  $\varphi_0$  junction, and the

\*alberto.hijano@ehu.eus

†stefan.ilic@csic.es

‡fs.bergeret@csic.es

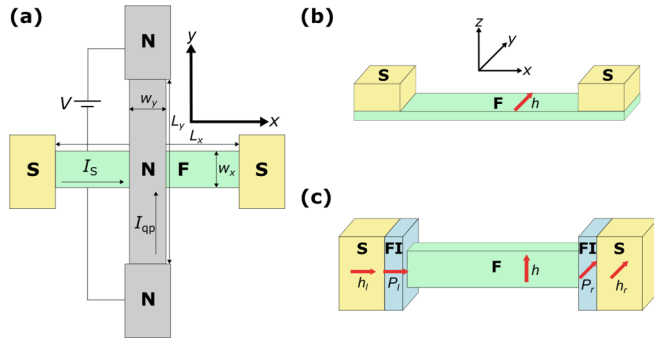


FIG. 1. (a) Schematic structure of the Andreev interferometer. (b) S/F/S structure. Here, F is a wire with Rashba spin-orbit coupling, and a spin-splitting field  $h$ . (c) S/FI/F/FI/S structure. FI layers act as spin-filtering barriers with polarizations  $P_{r/l}$ , and they induce spin-splitting fields  $h_{r/l}$  in adjacent S layers. F is a ferromagnet with an exchange field  $h$ .

normal wire where the conductance measurement is done, such that the noise associated with the measurement process does not perturb the  $\varphi_0$  junction. Our main result is that the phase-dependent contribution to the dissipative (quasiparticle) current through the vertical arm of the interferometer can be written as

$$\delta I_{qp}(\varphi) = I_c^{qp} \cos(\varphi + \varphi_0^{qp}) = I_0^{qp} \cos \varphi + I_{an}^{qp} \sin \varphi. \quad (2)$$

Therefore, this current also exhibits an anomalous phase shift  $\varphi_0^{qp}$ . Here  $I_0^{qp}$  is the usual component, which exists in Andreev interferometers with conventional junctions.  $I_{an}^{qp}$  is the anomalous component, which can only exist in the presence of a  $\varphi_0$  junction. Note that the phase shift in the Josephson current  $\varphi_0$  and in the quasiparticle current  $\varphi_0^{qp}$  are in general not equal, but they have similar magnitude and can be directly related to each other (see the Fig. 2). Our result suggests a way to experimentally obtain the value of  $\varphi_0$  from  $\varphi_0^{qp}$  by performing conductance measurements.

We study the two main realizations of  $\varphi_0$  junctions. Namely, Josephson junctions with Rashba spin-orbit coupling

(SOC) and multilayer ferromagnetic structures [Figs. 1(b) and 1(c)]. In both cases the anomalous phase is related to the existence of a Lifshitz invariant in the free energy [62–64]. In the first example such invariant stems from an interplay between a Zeeman field and the SOC, whereas in the second example it stems from non-coplanar magnetizations of magnetic layers.

## II. THE SETUP

We consider the geometry shown in Fig. 1(a). The  $\varphi_0$  junction lies along the  $x$  direction, and consists of a ferromagnetic wire placed between two superconducting reservoirs. These superconductors are connected in a loop (not shown), so that when a magnetic field is applied through the loop, the resulting flux creates a phase difference between them and leads to a Josephson current flowing along the  $x$  wire. An additional normal wire (N) is placed perpendicularly to the F wire (on the  $y$  direction). N is connected to two normal reservoirs. We assume that the F and N wires intersect at their midpoints. A voltage difference between the normal electrodes leads to a quasiparticle current  $I_{qp}$  in the  $y$  wire, which can be decomposed in two contributions:  $I_{qp} = I_\Omega + \delta I_{qp}(\varphi)$ , where  $I_\Omega$  is the usual Ohmic contribution, whereas  $\delta I_{qp}(\varphi)$  is the phase-dependent part given in Eq. (2). The latter is affected by the proximity effect with the  $x$  wire.

In the rest of this paper we determine the usual and anomalous components of  $\delta I_{qp}(\varphi)$  for two different realizations of a  $\varphi_0$  junction: a S/F/S junction with Rashba SOC [Fig. 1(b), Sec. III] and a S/FI/F/FI/S junction, where FI stands for a ferromagnetic insulator [Fig. 1(c), Sec. IV]. In the first example, a magnetic field and a spin-orbit coupling provide time-reversal and inversion symmetry breaking, respectively, which leads to the anomalous phase shift [5,17]. Here the anomalous Josephson current is  $I_{an}^S \propto h\kappa_\alpha$ , with  $\kappa_\alpha$  being a parameter associated with singlet-triplet conversion due to the SOC, and  $h$  is a weak exchange or Zeeman field [see Eq. (3)]. In the second example, the  $\varphi_0$  effect occurs if the FI tunneling barriers are spin-polarized, so that the barrier polarizations  $P_{r/l}$  and the magnetization direction in the F layer  $h$  are

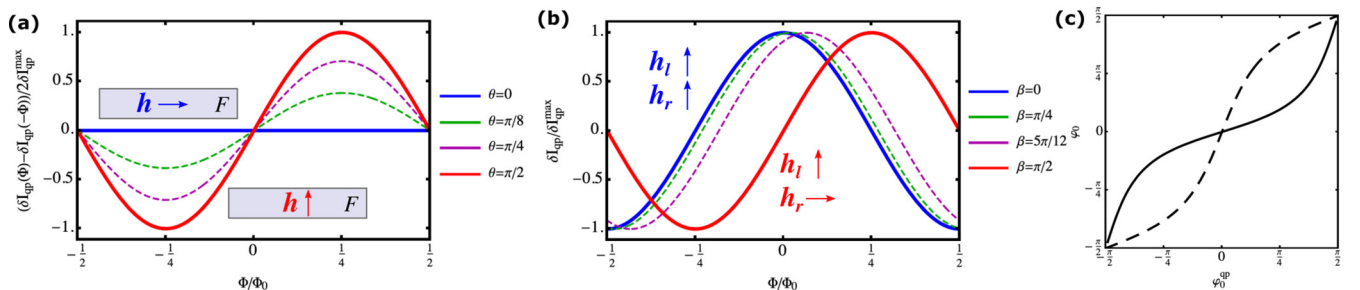


FIG. 2. (a) Component of the quasiparticle current odd in flux, which corresponds to  $I_{an}^{qp}$ , for the configuration shown in Fig 1(b).  $\theta$  is an angle between the in-plane exchange field in the F layer and the  $x$  axis. The Josephson phase is given by  $\varphi = 2\pi\Phi/\Phi_0$ , where  $\Phi$  is the applied flux and  $\Phi_0$  is the flux quantum. (b) Quasiparticle current along the  $y$  wire for different magnetization directions for the configuration shown in Fig 1(c). The exchange fields on the S electrodes  $h_{r/l}$  are taken to be perpendicular to the exchange field on the F wire in order to maximize the current, while they form an angle  $\beta$ . (c) Relation between the anomalous phase shifts in the Josephson current  $\varphi_0$  and in the quasiparticle current  $\varphi_0^{qp}$  for the junction with Rashba SOC (solid) and S/FI/F/FI/S junction (dashed), calculated from the expressions provided in Appendices A and B, respectively. In the first case, the value of the exchange field is  $h = 0.1\Delta$  and the Rashba coupling constant ranges from  $\kappa_\alpha \xi_0 \in [-0.5, 0.5]$ . In the second case, the exchange field is  $h = 0.1\Delta$  and the splitting of the superconducting electrodes are  $h_{r/l} = 0.1\Delta$ , where they form an angle  $\beta \in [-\pi/2, \pi/2]$ . The temperature is  $T = 0.01T_{c0}$  in both cases.

noncoplanar, and therefore the magnetization inversion symmetry is broken [25–32]. The anomalous Josephson current is  $I_{\text{an}}^S \propto \chi$ , where  $\chi = \mathbf{n}_h \cdot (\mathbf{n}_l \times \mathbf{n}_r)$  is the so-called chirality of the junction (see Fig. 1). Here,  $\mathbf{n}_h$  and  $\mathbf{n}_{r/l}$  are the unit vectors along the exchange field and polarization directions. We will show that in both examples, the interferometer quasiparticle anomalous current  $I_{\text{an}}^{\text{qp}}$  has the same dependence on the spin-dependent fields as the anomalous Josephson current  $I_{\text{an}}^S$ ; namely,  $I_{\text{an}}^{\text{qp}} \propto h\kappa_\alpha$  in the first example, and  $I_{\text{an}}^{\text{qp}} \propto \chi$  in the second example.

### III. JOSEPHSON JUNCTION WITH RASHBA SOC

We first study a S/F/S structure, shown in Fig. 1(b). Here, F is a wire with Rashba spin-orbit coupling (SOC), and an exchange field  $\mathbf{h}$ , which comes either from intrinsic magnetization (F is a ferromagnet) or from an externally applied magnetic field. For this configuration, the anomalous Josephson current is only affected by the component of the exchange field perpendicular to the current direction  $x$ , so we consider a field oriented along the  $y$  direction  $\mathbf{h} = h\mathbf{y}$  in order to maximize the  $\varphi_0$  effect [17].

We describe the system using the quasiclassical Green's function (GF) formalism [65]. In the diffusive limit, GFs are obtained as a solution of the Usadel equation [66]. SOC can be included as a background SU(2) field [67–69]. Superconducting correlations are described by the condensate GF,  $\hat{f}$ , which is a  $2 \times 2$  matrix in spin space that consists of a singlet component,  $f_0$  and, in general, three triplet components,  $f_j \sigma_j$ , where  $j = 1, 2, 3$  and  $\sigma_j$  are the three Pauli matrices. We assume that the proximity effect in the F wire is weak due, for example, to low S/F transmission coefficient. In this case, the Usadel equation can be linearized [17]. For the situation under consideration, transport in  $x$  direction and  $h$  field in  $y$  direction, only the condensate components  $f_0$  and  $f_2$  are finite and satisfy:

$$\pm \partial_{xx}^2 f_0^{R/A} + i\kappa_\varepsilon^2 f_0^{R/A} - i\kappa_F^2 f_2^{R/A} - \kappa_\alpha \partial_x f_2^{R/A} = 0 \quad (3a)$$

$$\pm \partial_{xx}^2 f_2^{R/A} + i\kappa_\varepsilon^2 f_2^{R/A} - i\kappa_F^2 f_0^{R/A} - \kappa_\alpha \partial_x f_0^{R/A} = 0 \quad (3b)$$

where  $\kappa_\varepsilon^2 = 2\varepsilon/D$ ,  $\kappa_F^2 = 2h/D$  and  $\kappa_\alpha = 4\alpha^3\tau/m$ . Here,  $\varepsilon$  is the energy,  $D$  is the diffusion constant,  $h$  is the exchange field, and  $\alpha$  is the Rashba coupling constant. The upper and lower sign correspond to the retarded and advanced condensate GFs  $\hat{f}^{R/A}$  respectively. In the following we omit the superscript to simplify the notation. Moreover, to simplify the calculation, in Eq. (3) we have neglected the renormalization of the exchange field by the SOC, and the relaxation of the triplet component due to SOC [23].

The Usadel equation (3) is supplemented by boundary conditions describing the interfaces between different materials. The S/F junctions are described by the generalized Kuprianov-Lukichev conditions [70]

$$\pm \partial_n f_{0,r/l} + \eta_{r/l} \kappa_\alpha f_{2,r/l} = \mp \frac{1}{\gamma} \mathcal{F}_0 e^{i\eta_{r/l}\varphi/2} \quad (4a)$$

$$\partial_n f_{2,r/l} = 0. \quad (4b)$$

Here,  $\mathcal{F}_0 = \Delta/\sqrt{\Delta^2 - \varepsilon^2}$  is the anomalous GF of the superconducting electrode,  $\partial_n$  is the normal derivative at the surface and  $\gamma = \sigma_F R_b$  is the parameter describing the barrier

strength, where  $R_b$  is the normal-state tunneling resistance per unit area and  $\sigma_F$  is the conductivity of the ferromagnet.  $\eta_{r/l} = \pm 1$  for the right ( $x = L_x/2$ ) and left boundaries ( $x = -L_x/2$ ).

The condensate function in the  $y$  wire  $\hat{f}_y$  is induced by the proximity effect with the  $x$  wire. To find  $\hat{f}_y$ , we start from the Kuprianov-Lukichev condition describing the interface between the two wires, and the Usadel equation in the  $y$  wire. Provided that the widths of the wires  $w_{x,y}$  are much smaller than the superconducting coherence length, we can integrate the Usadel equation over the cross-section of the wire. If the interface resistance is much larger than the resistance of the wires,  $R_b \gg L_{x,y}/\sigma_{F,N}$ , we find the equation determining  $\hat{f}_y$ :

$$\pm \partial_{yy}^2 \hat{f}_y + i\kappa_\varepsilon'^2 \hat{f}_y = -\frac{w_x}{\gamma_0^2} \hat{f}(0) \delta(y). \quad (5)$$

Here,  $\gamma_0^2 = R_b \sigma_N w_y$  and  $\kappa_\varepsilon'^2 = 2\varepsilon/D_y$ , with  $R_b$  being the resistance per unit area of the interface of the  $x$  and  $y$  wires and  $\sigma_N$  is the normal-state conductance of the  $y$  wire. The Dirac delta term describes the proximity effect, and is a source term. The contact of the  $y$  wire with the normal reservoirs is assumed to be ideal so that the condensate functions vanish at the ends of the wire is  $\hat{f}_y(\pm L_y/2) = 0$ .

A voltage bias  $V$  is applied between the normal electrodes. Due to our assumption of large  $R_b$  we can neglect the inverse proximity effect. Thus, in leading order the phase-dependent correction to the quasiparticle current is given by [71–73]

$$\delta I_{\text{qp}} = \frac{-\sigma_N}{16eL_y} \int d\varepsilon F_T(\varepsilon, V/2) \langle \text{Tr}(\hat{f}_y^R - \hat{f}_y^A)^2 \rangle. \quad (6)$$

Here  $\langle \dots \rangle = 1/L_y \int_{-L_y/2}^{L_y/2} dy \dots$  denotes average over the length,  $\hat{f}^{R/A}$  is the  $4 \times 4$  matrix GF in Nambu-spin space [see Eq. (A1) in the Appendix] and  $F_T$  is defined as  $F_T(\varepsilon, V) = \frac{1}{2} [\tanh \frac{\varepsilon+eV}{2T} - \tanh \frac{\varepsilon-eV}{2T}]$ . Solving the boundary value problem, Eqs. (3) and (4), we first calculate the  $\hat{f}$  for the  $x$ -wire, and then  $\hat{f}_y$  for the  $y$ -wire from Eq. (5). Using Eq. (6) we then obtain the usual and anomalous quasiparticle currents entering Eq. (2).

Up to the leading order terms in exchange field and Rashba SOC, the quasiparticle current takes the following form:

$$I_0^{\text{qp}} = c_1, \quad (7)$$

$$I_{\text{an}}^{\text{qp}} = c_2 h \kappa_\alpha. \quad (8)$$

The factors  $c_1$  and  $c_2$  depend on  $T$  and  $L_{x,y}$ , and their exact form is given by Eqs. (A14) and (A15) in the Appendix. Both components of the quasiparticle current depend on the spin-dependent fields in the same way as the components of Josephson current [17]:  $I_0^S$  and  $I_0^{\text{qp}}$  are independent of these fields, whereas  $I_{\text{an}}^S, I_{\text{an}}^{\text{qp}} \sim h\kappa_\alpha$ . Note that the result for the anomalous current, Eq. (8), also holds in the case when exchange field in the F layer is not fully aligned with the  $y$ -direction, by taking  $h = h_{\text{tot}} \sin \theta$ . Here  $h_{\text{tot}}$  is an arbitrarily oriented in-plane field, and  $\theta$  is an angle between the field and the  $x$  direction. To illustrate this, in Fig. 2(a) we plot the odd component of the anomalous quasiparticle current,  $I_{\text{an}}^{\text{qp}} \sin \varphi = \frac{1}{2} [I_{\text{qp}}(\varphi) - I_{\text{qp}}(-\varphi)]$ , for different values of  $\theta$ . The current is normalized with respect to its maximum value for clarity. For  $\theta = 0$ , the exchange field is parallel to the wire,

so there is no anomalous phase shift and  $I_{\text{an}}^{\text{qp}}$  vanishes. The  $\varphi_0$  effect is maximized for  $\theta = \pi/2$ , where the exchange field is perpendicular to the wire. In Fig. 2(c) we plot the relation between the  $\varphi_0$  phase-shift in the Josephson current, and the phase-shift measured in the quasiparticle current  $\varphi_0^{\text{qp}}$ . All expressions presented in this paper are valid for arbitrary temperature. For the numerical computations however, we only focus on low temperatures,  $T = 0.01T_{c0}$ , where the magnitude of the quasiparticle current is maximized.

#### IV. S/FI/F/FI/S JUNCTION

Another configuration to obtain a  $\varphi_0$  junction is a S/FI/F/FI/S junction with noncoplanar magnetizations [Fig. 1(c)]. This configuration has not yet been realized in experiment, but has been theoretically predicted to show AJE [25–32]. In these structures, the role of the FI layers is two-fold: firstly, they induce an exchange field  $\mathbf{h}_{r/l}$  in the adjacent S layer, and secondly, they act as spin-polarized tunneling barriers with a polarization  $\mathbf{P}_{r/l}$ . The linearized Usadel equation in the F layer reads

$$\pm \partial_{xx}^2 \hat{f} + i\kappa_F^2 \hat{f} - i\frac{\kappa_F^2}{2} \{\sigma_3, \hat{f}\} = 0, \quad (9)$$

where  $\{.,.\}$  is an anticommutator. We have assumed, without loss of generality, that the exchange field in the F-wire points on the  $z$  direction.

The S/F junctions with spin-filtering barriers are described by the generalized Kuprianov-Lukichev boundary condition [74,75]. The exchange fields  $\mathbf{h}_{r/l}$  induced via the magnetic proximity effect in the S electrodes point in the same direction as the polarization vectors  $\mathbf{P}_{r/l}$ . The linearized boundary conditions read

$$\begin{aligned} \pm \gamma \partial_n \hat{f}_{r/l} &= \frac{1}{2} [\hat{\mathcal{G}}_{r/l} \mathbf{P}_{r/l} \cdot \boldsymbol{\sigma}, \hat{f}_{r/l}] \\ &+ \frac{1}{2} \{\hat{\mathcal{G}}_{r/l}, \hat{f}_{r/l}\} \mp \sqrt{1 - P_{r/l}^2} \hat{\mathcal{F}}_{r/l} e^{i\eta_{r/l}\varphi/2}. \end{aligned} \quad (10)$$

Here,  $\hat{\mathcal{G}}_{r/l}$  and  $\hat{\mathcal{F}}_{r/l}$  are the normal and anomalous GFs of the spin-split superconducting electrode, respectively. In the weak exchange field limit, they are given by

$$\hat{\mathcal{G}}_{r/l} = \mathcal{G}_0 - \mathbf{h}_{r/l} \cdot \boldsymbol{\sigma} \frac{d\mathcal{G}_0}{d\varepsilon} \quad (11a)$$

$$\hat{\mathcal{F}}_{r/l} = \mathcal{F}_0 - \mathbf{h}_{r/l} \cdot \boldsymbol{\sigma} \frac{d\mathcal{F}_0}{d\varepsilon} \quad (11b)$$

with  $\mathcal{G}_0 = -i\varepsilon/\sqrt{\Delta^2 - \varepsilon^2}$ .

Generally, the coherence length in the ferromagnetic layer is much shorter than in a normal metal  $\kappa_F^{-1} \ll \kappa_N^{-1}$ , where  $\kappa_N = \sqrt{2T/D}$ . We assume the long-junction regime, so that  $\kappa_F^{-1} \ll L_x \ll \kappa_N^{-1}$ . In this regime the length of the  $x$  wire  $L_x$  is much longer than the penetration length of the Cooper pairs  $\kappa_F^{-1}$  in the F layer, so that the condensate functions  $\hat{f}$  and  $\hat{f}_y$  are mediated primarily by the long-range triplet superconducting correlations [9,76,77], whereas the singlet and short-range triplet correlations decay over the length  $\kappa_F^{-1}$ .

To calculate the interferometer current, we proceed similarly as in the previous example. First, from Eqs. (9) and (10), we find  $\hat{f}$  in the  $x$  wire, and then calculate the current

in the  $y$  wire from Eqs. (5) and (6). The usual and anomalous quasiparticle currents are given by [32]

$$I_0^{\text{qp}} = c_3 \sqrt{1 - P_r^2} \sqrt{1 - P_l^2} \gamma^{-2} \mathbf{h}_{l\perp} \cdot \mathbf{h}_{r\perp} \quad (12)$$

$$I_{\text{an}}^{\text{qp}} = c_4 \sqrt{1 - P_r^2} \sqrt{1 - P_l^2} \gamma^{-3} (P_l h_r + P_r h_l) \mathbf{z} \cdot (\mathbf{n}_l \times \mathbf{n}_r), \quad (13)$$

where  $\mathbf{h}_{r/l\perp} = \mathbf{h}_{r/l} - (\mathbf{h}_{r/l} \cdot \mathbf{h})\mathbf{h}/h^2$  are the components of  $\mathbf{h}_{r/l}$  perpendicular to  $\mathbf{h}$  [see Eqs. (B12) and (B13) in the Appendix for the exact form of the coefficients  $c_3$  and  $c_4$ ].

From Eq. (13), we see that the anomalous quasiparticle current is proportional to the scalar triple product of the magnetizations  $I_{\text{an}}^{\text{qp}} \propto \chi = \mathbf{z} \cdot (\mathbf{n}_l \times \mathbf{n}_r)$  [32]. Here,  $\chi$  is the junction chirality, and it is nonzero only if the barrier polarizations and the magnetization direction are noncoplanar. As in the previous example, the quasiparticle current and the Josephson current have the same dependence on the spin-dependent fields: for the usual components  $I_0^S, I_0^{\text{qp}} \propto \mathbf{h}_{l\perp} \cdot \mathbf{h}_{r\perp}$ , and for the anomalous components  $I_{\text{an}}^S, I_{\text{an}}^{\text{qp}} \propto \chi$ .

Usually,  $I_0^{\text{qp}}$  is the dominant contribution to the interferometer current, as it is of the lower order in the small barrier parameter  $\gamma^{-1}$ , namely  $I_{\text{an}}^{\text{qp}}/I_0^{\text{qp}}, I_{\text{an}}^S/I_0^S \sim \gamma^{-1} \ll 1$ . However, if  $\mathbf{h}_{l\perp} \cdot \mathbf{h}_{r\perp} = 0$  the usual component vanishes and only the anomalous current contributes. In other words, the measured quasiparticle current is directly linked to the  $\varphi_0$  effect. To illustrate this point further, in Fig. 2(b) we use the analytical formulas (B12) and (B13) to plot the normalized quasiparticle current for different magnetic configurations. Here,  $\beta$  is the angle formed by the exchange fields on the S electrodes  $\mathbf{h}_{r/l}$ , which are taken to be perpendicular to  $\mathbf{h}$ . Unlike in panel (a), here we plot the total anomalous quasiparticle current  $I_{\text{an}}^{\text{qp}}$  to stress the shift from an even-in-phase to odd-in-phase behavior when the angle between the electrode magnetizations increases. For  $\beta = 0$ , there is no  $\varphi_0$  effect, so the current is even in the phase.  $I_0^{\text{qp}}$  decreases with increasing  $\beta$  and vanishes for  $\beta = \pi/2$ . In this case, the oscillation of  $\delta I_{\text{qp}}$  are given by the anomalous quasiparticle current, so that  $\delta I_{\text{qp}}$  becomes odd in  $\varphi$ . In Fig. 2(c) we plot the relation between the  $\varphi_0$  and  $\varphi_0^{\text{qp}}$ .

It is worth mentioning, that to simplify equations and obtain analytical solutions, we have assumed a low barrier transmission at the S/F interfaces and between the  $x$  and  $y$  wires. Consequently, the obtained quasiparticle current, being proportional to powers of a small interface parameter, is also small. However, the findings of our work should still hold qualitatively in setups with smaller interface resistances, where the quasiparticle currents should be significantly larger. In other words, our results give a lower bound of the current amplitude. Moreover, the expressions for the quasiparticle current, Eqs. (7), (8), (12), and (13) are valid for all temperatures. Indeed, the temperature dependence enters the coefficients  $c_i$ , ( $i = 1, 2, 3, 4$ ), given in Eqs. (A14), (A15), (B12), and (B13). From these expressions one can show that the quasiparticle current amplitudes are maximized at low temperatures, and they decrease monotonically towards zero at the superconducting critical temperature. This behavior of  $\delta I^{\text{qp}}$  is in contrast with the temperature behavior of the critical Josephson current, whose sign may be reversed by changing the temperature [2] (i.e., when a  $0-\pi$  transition occurs).



## V. CONCLUSION

In summary, we have studied the current-phase relation of an Andreev interferometer with an anomalous Josephson junction. We have shown how the quasiparticle current through the normal arm of the interferometer is affected by the appearance of an anomalous phase  $\varphi_0$ . Specifically, we have studied the AJE in S/F/S structures with spin-splitting field and Rashba SOC or spin-filtering barriers. Our results show that there is also an anomalous contribution to the phase-dependent part of the quasiparticle current proportional to  $\sin \varphi$  when  $\varphi_0$  is different from 0 and  $\pi$ . Moreover, the usual and anomalous quasiparticle currents have the same dependence on the spin-dependent fields as the anomalous Josephson current. Suitable materials for the realization of the anomalous Andreev interferometer are InSb [21], Bi wires [78], Bi<sub>2</sub>Se<sub>3</sub> [22], and InAs [24,79] due to the large spin-orbit coupling, in combination with conventional superconductors and normal metals. In particular, in systems with InAs a large

phase shift  $\varphi_0 \approx \pi/2$  was experimentally observed [23] in the Josephson current, and based on our findings, we expect equally strong effect in the Andreev interferometer geometry. For the ferromagnetic interferometers we propose EuS/Al structures to engineer a  $\varphi_0$  junction in a S/FI/FI/S junction due to the well-defined splitting and strong barrier polarization [80], while the F layer can consist of a Co wire [81,82].

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## APPENDIX A: JOSEPHSON JUNCTION WITH RASHBA SOC

In this Appendix we present a detailed derivation of the expressions used in the main text for the currents in the presence of Rashba SOC. In Sec. A 1 we first present the derivation of the Josephson current in the  $x$  wire, followed by the derivation of the quasiparticle current in the  $y$  wire in Sec. A 2.

### 1. Current along the $x$ wire

We start by solving the linearized Usadel equation, Eq. (3), with the appropriate boundary conditions—Eq. (4). First, let us note that the retarded and advanced anomalous Green's functions (GFs) in Nambu-spin space have the following structure

$$\check{f} = \begin{pmatrix} 0 & \hat{f}(\varepsilon) \\ \hat{f}(-\varepsilon) & 0 \end{pmatrix}, \quad (\text{A1})$$

where  $\hat{X} = \mathcal{T}\hat{X}\mathcal{T}^{-1}$ , and  $\mathcal{T} = i\sigma_2\mathcal{K}$  is the time-reversal transformation, with  $\mathcal{K}$  being the complex conjugate operation. Moreover, we can relate  $\check{f}^A$  to  $\check{f}^R$  as

$$\check{f}^A(\varepsilon, \mathbf{h}, \alpha) = \check{f}^R(-\varepsilon, -\mathbf{h}, -\alpha). \quad (\text{A2})$$

In the following we only write the retarded GF and omit the superscript to simplify the notation.

We find the condensate function  $\hat{f}$  perturbatively in  $\kappa_\alpha$ , keeping terms up to the first order in this parameter. The solution is

$$\begin{aligned} f_0 = & \left[ \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{1,+} + B_{1,+} \right) e^{i\varphi/2} + \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{2,+} + B_{2,+} \right) e^{-i\varphi/2} \right] e^{\kappa_+ x} \\ & + \left[ \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{2,+} - B_{2,+} \right) e^{i\varphi/2} + \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{1,+} - B_{1,+} \right) e^{-i\varphi/2} \right] e^{-\kappa_+ x} \\ & + \left[ \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{1,-} - B_{1,-} \right) e^{i\varphi/2} + \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{2,-} - B_{2,-} \right) e^{-i\varphi/2} \right] e^{\kappa_- x} \\ & + \left[ \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{2,-} + B_{2,-} \right) e^{i\varphi/2} + \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{1,-} + B_{1,-} \right) e^{-i\varphi/2} \right] e^{-\kappa_- x}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} f_2 = & \left[ \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{1,+} + B_{1,+} \right) e^{i\varphi/2} + \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{2,+} + B_{2,+} \right) e^{-i\varphi/2} \right] e^{\kappa_+ x} \\ & + \left[ \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{2,+} - B_{2,+} \right) e^{i\varphi/2} + \left( \left( 1 + \frac{\kappa_\alpha x}{2} \right) A_{1,+} - B_{1,+} \right) e^{-i\varphi/2} \right] e^{-\kappa_+ x} \\ & - \left[ \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{1,-} - B_{1,-} \right) e^{i\varphi/2} + \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{2,-} - B_{2,-} \right) e^{-i\varphi/2} \right] e^{\kappa_- x} \\ & - \left[ \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{2,-} + B_{2,-} \right) e^{i\varphi/2} + \left( \left( 1 - \frac{\kappa_\alpha x}{2} \right) A_{1,-} + B_{1,-} \right) e^{-i\varphi/2} \right] e^{-\kappa_- x}, \end{aligned} \quad (\text{A4})$$

where  $\kappa_\pm = \sqrt{-i\kappa_\varepsilon^2 \pm i\kappa_F^2}$ , and the coefficients are given by

$$A_{1,\pm} = \frac{\mathcal{F}_0}{4\gamma\kappa_\pm} \frac{e^{\kappa_\pm L_x/2}}{\sinh \kappa_\pm L_x}, \quad (\text{A5a})$$

$$A_{2,\pm} = \frac{\mathcal{F}_0}{4\gamma\kappa_{\pm}} \frac{e^{-\kappa_{\pm}L_x/2}}{\sinh \kappa_{\pm}L_x}, \quad (\text{A5b})$$

$$B_{1,\pm} = \frac{\kappa_{\alpha}\mathcal{F}_0}{8\gamma\kappa_{\pm}} \frac{1}{\sinh \kappa_{\pm}L_x} \left( \frac{1}{\kappa_{\mp} \sinh \kappa_{\mp}L_x} (e^{-\kappa_{\pm}L_x/2} - e^{\kappa_{\pm}L_x/2} \cosh \kappa_{\mp}L_x) - \frac{L_x}{2} e^{\kappa_{\pm}L_x/2} \right), \quad (\text{A5c})$$

$$B_{2,\pm} = \frac{\kappa_{\alpha}\mathcal{F}_0}{8\gamma\kappa_{\pm}} \frac{1}{\sinh \kappa_{\pm}L_x} \left( \frac{1}{\kappa_{\mp} \sinh \kappa_{\mp}L_x} (-e^{\kappa_{\pm}L_x/2} + e^{-\kappa_{\pm}L_x/2} \cosh \kappa_{\mp}L_x) + \frac{L_x}{2} e^{-\kappa_{\pm}L_x/2} \right). \quad (\text{A5d})$$

Having found the condensate function, we proceed to calculate the Josephson current in the F wire, which is given as

$$I_S = \frac{\pi\sigma_F T}{e} \sum_{\omega} \text{Im}(f_0^*(\partial_x f_0 - \kappa_{\alpha} f_2) - f_2^* \partial_x f_0). \quad (\text{A6})$$

Note that in Eq. (A6) we introduce the Matsubara frequencies  $\omega = 2\pi(n + 1/2)T$ . The Matsubara GF is obtained by analytic continuation of  $\check{f}$  to the complex plane  $\varepsilon \rightarrow i\omega$ . We can use the boundary conditions (4) to simplify the previous equation:

$$I_S = \frac{2\pi T}{eR_b} \sum_{\omega>0} \text{Im} f_0^*(L_x/2) \mathcal{F}_0 e^{i\varphi/2}. \quad (\text{A7})$$

After substitution the Eqs. (A3) and (A4) in Eq. (A7), we find the Josephson current:

$$I_S = \frac{2\pi\sigma_S T}{e\gamma^2} \sum_{\omega>0} \mathcal{F}_0^2 \left[ \text{Re} \frac{1}{\kappa_+ \sinh(\kappa_+ L_x)} \sin \varphi + \kappa_{\alpha} \text{Im} \left( \frac{-L_x/2}{\kappa_+ \sinh(\kappa_+ L_x)} + \frac{\cosh(\kappa_+ L_x)}{|\kappa_+|^2 \sinh(\kappa_+ L_x)^2} \right) \cos \varphi \right]. \quad (\text{A8})$$

Here we identify the usual  $I_0$  and anomalous  $I_{\text{an}}$  components of the Josephson current as the terms proportional to  $\sin \varphi$  and  $\cos \varphi$ , respectively. The usual component is independent of the SOC strength, while the anomalous current is proportional to  $\kappa_{\alpha}$  and is nonvanishing if the exchange field  $h$  is finite.

## 2. Current along the y wire

Starting from Eq. (5), and imposing  $\check{f}_y(\pm L_y/2) = 0$ , we find the condensate function in the y wire

$$\check{f}_y = \frac{w_x}{2\gamma_0^2 \sqrt{-i\kappa'_x} \cosh(\sqrt{-i\kappa'_x} L_y/2)} \check{f}(0) \sinh(\sqrt{-i\kappa'_x} (L_y/2 - |y|)), \quad (\text{A9})$$

where  $\check{f}(0)$  is the condensate function for the x wire (found in Sec. A 1), evaluated at the intersection of x and y wires. Next, we use  $\check{f}_y$  to calculate the quasiparticle current  $\delta I_{\text{qp}}$  from Eq. (6). We decompose this current as  $\delta I_{\text{qp}} = \delta I_1 + \delta I_2$ , where

$$\delta I_1 = -\frac{\sigma_N}{eL_y} \int d\varepsilon F_T(\varepsilon, V/2) \frac{1}{16} \langle \text{Tr}((\check{f}^R)^2 + (\check{f}^A)^2) \rangle = \frac{\pi T \sigma_N}{2eL_y} \text{Im} \sum_{\omega>0} \langle \text{Tr} \check{f}(\omega + ieV/2)^2 \rangle, \quad (\text{A10})$$

$$\delta I_2 = \frac{4\sigma_N}{eL_y} \int_0^{\infty} d\varepsilon F_T(\varepsilon, V/2) \frac{1}{16} \langle \text{Tr}(\check{f}^R \check{f}^A) \rangle. \quad (\text{A11})$$

Using solution (A9), the summands in Eq. (A10) can be written as

$$\langle \text{Tr} \check{f}(\omega + ieV/2)^2 \rangle = \frac{w_x^2}{\gamma_0^4 \kappa'_{\omega}{}^2 \cosh^2(\kappa'_{\omega} L_y/2)} \left( \frac{\sinh(\kappa'_{\omega} L_y)}{2\kappa'_{\omega} L_y} - \frac{1}{2} \right) \left( |f_0(0)|^2 - |f(0)|^2 \right) \Big|_{\omega=2\pi(n+1/2)T+ieV/2}, \quad (\text{A12})$$

where  $f_0(0)$  and  $f(0)$  are the singlet and triplet component of the GF of the x wire at the crossing point. Similarly, the integrand in Eq. (A11) can be written as

$$\begin{aligned} \langle \text{Tr}(\check{f}^R \check{f}^A) \rangle &= \frac{w_x^2 \left( \frac{\sinh(\sqrt{2}\kappa'_x L_y/2)}{\sqrt{2}\kappa'_x L_y} - \frac{\sinh(\sqrt{2}i\kappa'_x L_y/2)}{\sqrt{2}i\kappa'_x L_y} \right)}{2\gamma_0^4 |\kappa'_x|^2 \cosh(\sqrt{-i\kappa'_x} L_y/2)^2} (f_0(\varepsilon, \mathbf{h}, \alpha) f_0(\varepsilon, -\mathbf{h}, -\alpha)^* \\ &\quad + f_0(-\varepsilon, \mathbf{h}, \alpha)^* f_0(-\varepsilon, -\mathbf{h}, -\alpha) - f(\varepsilon, \mathbf{h}, \alpha) \cdot f(\varepsilon, -\mathbf{h}, -\alpha)^* - f(-\varepsilon, \mathbf{h}, \alpha)^* \cdot f(-\varepsilon, -\mathbf{h}, -\alpha)). \end{aligned} \quad (\text{A13})$$

Finally, the usual ( $I_0^{\text{qp}}$ ) and anomalous ( $I_{\text{an}}^{\text{qp}}$ ) quasiparticle currents are

$$\begin{aligned} I_0^{\text{qp}} &= \frac{\pi\sigma_N T}{2eL_y} \text{Im} \sum_{n \geq 0} \frac{w_x^2 \left( \frac{\sinh(\kappa'_{\omega} L_y)}{2\kappa'_{\omega} L_y} - \frac{1}{2} \right)}{\gamma_0^4 \kappa'_{\omega}{}^2 \cosh^2(\kappa'_{\omega} L_y/2)} \frac{\mathcal{F}_0^2}{4\gamma^2} \left( \frac{1}{\kappa_+^2 \sinh^2(\kappa_+ L_x/2)} + \frac{1}{\kappa_-^2 \sinh^2(\kappa_- L_x/2)} \right) \Big|_{\omega=2\pi(n+1/2)T+ieV/2} \\ &\quad + \frac{\sigma_N}{4eL_y} \int_0^{\infty} d\varepsilon F_T(\varepsilon, V/2) \frac{w_x^2 \left( \frac{\sinh(\sqrt{2}\kappa'_x L_y/2)}{\sqrt{2}\kappa'_x L_y} - \frac{\sinh(\sqrt{2}i\kappa'_x L_y/2)}{\sqrt{2}i\kappa'_x L_y} \right)}{\gamma_0^4 |\kappa'_x|^2 \cosh(\sqrt{-i\kappa'_x} L_y/2)^2} \frac{|\mathcal{F}_0|^2}{4\gamma^2} \end{aligned}$$

$$\times \left( \frac{1}{|\kappa_+|^2 |\sinh(\kappa_+ L_x/2)|^2} + \frac{1}{|\kappa_-|^2 |\sinh(\kappa_- L_x/2)|^2} \right), \quad (\text{A14})$$

$$\begin{aligned} I_{\text{an}}^{\text{qp}} = & \kappa_\alpha \frac{\pi \sigma_N T}{2eL_y} \text{Im} \sum_{n \geq 0} \frac{w_x^2 \left( \frac{\sinh(\kappa'_\omega L_y)}{2\kappa'_\omega L_y} - \frac{1}{2} \right)}{\gamma_0^4 \kappa'_\omega{}^2 \cosh^2(\kappa'_\omega L_y/2)} \frac{\mathcal{F}_0^2}{4\gamma^2} \left( \frac{L_x}{2\kappa_+^2 \sinh^2(\kappa_+ L_x/2)} - \frac{L_x}{2\kappa_-^2 \sinh^2(\kappa_- L_x/2)} \right. \\ & \left. + \frac{\tanh(\kappa_- L_x/2)}{\kappa_+^2 \kappa_- \sinh^2(\kappa_+ L_x/2)} - \frac{\tanh(\kappa_+ L_x/2)}{\kappa_-^2 \kappa_+ \sinh^2(\kappa_- L_x/2)} \right) \Big|_{\omega=2\pi(n+1/2)T+ieV/2} \\ & + \kappa_\alpha \frac{\sigma_N}{4eL_y} \int_0^\infty d\varepsilon F_T(\varepsilon, V/2) \frac{w_x^2 \left( \frac{\sinh(\sqrt{2}\kappa'_\varepsilon L_y/2)}{\sqrt{2}\kappa'_\varepsilon L_y} - \frac{\sinh(\sqrt{2}i\kappa'_\varepsilon L_y/2)}{\sqrt{2}i\kappa'_\varepsilon L_y} \right)}{\gamma_0^4 |\kappa'_\varepsilon|^2 \cosh(\sqrt{-i}\kappa'_\varepsilon L_y/2)} \frac{|\mathcal{F}_0|^2}{4\gamma^2} \\ & \times \text{Im} \left\{ \frac{\tanh(\kappa_- L_x/2)}{|\kappa_+|^2 \kappa_- |\sinh(\kappa_+ L_x/2)|^2} - \frac{\tanh(\kappa_+ L_x/2)}{|\kappa_-|^2 \kappa_+ |\sinh(\kappa_- L_x/2)|^2} \right\}. \quad (\text{A15}) \end{aligned}$$

The factors  $c_1$  and  $c_2$  defined in Eqs. (7) and (8) of the main text can be extracted from Eqs. (A14) and (A15) by assuming the limit of weak exchange field  $h$ .

## APPENDIX B: S/FI/F/FI/S JUNCTION

In this Appendix we present a detailed derivation of the expressions used in the main text for the currents in the S/FI/F/FI/S geometry. In Sec. B 1 we first present the derivation of the Josephson current in the  $x$  wire, followed by the derivation of the quasiparticle current in the  $y$ -wire in Sec. B 2.

### 1. Current along the $x$ wire

The general solution of Eq. (9) for the condensate function in the  $x$  wire reads

$$\hat{f} = (A + A\sigma_3)e^{\kappa_+ x} + (B + B\sigma_3)e^{-\kappa_+ x} + (C - C\sigma_3)e^{\kappa_- x} + (D - D\sigma_3)e^{-\kappa_- x} + E\sigma_1 e^{i\kappa_\varepsilon x} + F\sigma_1 e^{-i\kappa_\varepsilon x} + G\sigma_2 e^{i\kappa_\varepsilon x} + H\sigma_2 e^{-i\kappa_\varepsilon x}. \quad (\text{B1})$$

Coefficients in Eq. (B1) can be found by applying the boundary conditions [Eq. (10)]. In the ferromagnetic wire, the supercurrent is given as

$$I_S = \pi \sigma_F \frac{T}{e} \sum_\omega \text{Im} \{ f_0^* \partial_x f_0 - \mathbf{f}^* \cdot \partial_x \mathbf{f} \}, \quad (\text{B2})$$

where  $\hat{f}_\omega = f_0 + \mathbf{f} \cdot \boldsymbol{\sigma}$  is decomposed into the scalar singlet amplitude  $f_0$  and the vector of triplet states  $\mathbf{f}$ . Assuming  $\kappa_F^{-1} \ll L_x \ll \kappa_\omega^{-1}$ , we can substitute the long range components  $f_1$  and  $f_2$  by their average values, given by

$$\langle f_{1/2} \rangle = \frac{\partial_x f_{1/2}|_{x=-L_x/2} - \partial_x f_{1/2}|_{x=L_x/2}}{\kappa_\omega^2 L_x}. \quad (\text{B3})$$

Then, using the boundary conditions [Eq. (10)], the long range triplet components are given by

$$\begin{aligned} \langle f_{1/2} \rangle = & \frac{1}{\kappa_\omega^2 L_x \gamma} (i\mathcal{G}_{l,0}(P_{l,2/3}f_{3/1}(-L_x/2) - P_{l,3/1}f_{2/3}(-L_x/2)) - \sqrt{1 - P_l^2} \mathcal{F}_{l,1/2} e^{-i\varphi/2} \\ & + i\mathcal{G}_{r,0}(P_{r,2/3}f_{3/1}(L_x/2) - P_{r,3/1}f_{2/3}(L_x/2)) - \sqrt{1 - P_r^2} \mathcal{F}_{r,1/2} e^{i\varphi/2}), \quad (\text{B4}) \end{aligned}$$

where  $\hat{\mathcal{G}}_{r/l}$  and  $\hat{\mathcal{F}}_{r/l}$  are given by Eq. (11). Using solution (B1), we can calculate the  $f_0$  and  $f_3$  near each boundary independently without overlapping. To first order in  $\gamma^{-1}$ , we obtain:

$$f_{0,r/l}^{(1)} = \frac{\sqrt{1 - P_{r/l}^2}}{2\gamma} \left( \frac{\mathcal{F}_{r/l,0} + \mathcal{F}_{r/l,3}}{\kappa_+} e^{-\kappa_+(L_x/2 \mp x)} + \frac{\mathcal{F}_{r/l,0} - \mathcal{F}_{r/l,3}}{\kappa_-} e^{-\kappa_-(L_x/2 \mp x)} \right) e^{i\eta_{r/l}\varphi/2} \quad (\text{B5})$$

$$f_{3,r/l}^{(1)} = \frac{\sqrt{1 - P_{r/l}^2}}{2\gamma} \left( \frac{\mathcal{F}_{r/l,0} + \mathcal{F}_{r/l,3}}{\kappa_+} e^{-\kappa_+(L_x/2 \mp x)} - \frac{\mathcal{F}_{r/l,0} - \mathcal{F}_{r/l,3}}{\kappa_-} e^{-\kappa_-(L_x/2 \mp x)} \right) e^{i\eta_{r/l}\varphi/2}. \quad (\text{B6})$$

The second-order terms in  $\gamma^{-1}$  are also important to obtain the anomalous Josephson effect:

$$f_{0,r/l}^{(2)} = \frac{-i\mathcal{G}_{r/l,0}}{2\gamma} (P_{r/l,1}\langle f_2 \rangle - P_{r/l,2}\langle f_1 \rangle) \left( \frac{e^{-\kappa_+(L_x/2 \mp x)}}{\kappa_+} - \frac{e^{-\kappa_-(L_x/2 \mp x)}}{\kappa_-} \right) \quad (\text{B7})$$

$$f_{3,r/l}^{(2)} = \frac{-i\mathcal{G}_{r/l,0}}{2\gamma} (P_{r/l,1}\langle f_2 \rangle - P_{r/l,2}\langle f_1 \rangle) \left( \frac{e^{-\kappa_+(L_x/2 \mp x)}}{\kappa_+} + \frac{e^{-\kappa_-(L_x/2 \mp x)}}{\kappa_-} \right). \quad (\text{B8})$$

To lowest order in  $\gamma^{-1}$ , the current (B2) can be written as

$$I_S = \frac{\pi\sigma_F T}{e\gamma} \sum_{\omega} \text{Im} \pm \sqrt{1 - P_{r/l}^2} (f_0^* \mathcal{F}_{r/l,0} - f^* \cdot \mathcal{F}_{r/l}) e^{i\eta_{r/l}\varphi/2} |_{x=\pm L_x/2}. \quad (\text{B9})$$

Finally, we find the usual and anomalous Josephson currents:

$$I_0^S = -\frac{2\pi}{eR_b} (\mathbf{h}_{l\perp} \cdot \mathbf{h}_{r\perp}) \frac{\sqrt{1 - P_r^2} \sqrt{1 - P_l^2}}{\sqrt{2}L_x\gamma^2} \sum_{\omega>0} \frac{T\mathcal{F}_0'^2}{\kappa_N^2} \quad (\text{B10})$$

$$I_{\text{an}}^S = -\frac{2\pi}{eR_b} (\chi_l - \chi_r) \frac{\sqrt{1 - P_r^2} \sqrt{1 - P_l^2}}{\kappa_F h L_x \gamma^3} \sum_{\omega>0} \frac{T\mathcal{G}_0 \mathcal{F}_0 \mathcal{F}_0'}{\kappa_{\omega}^2}. \quad (\text{B11})$$

For simplicity we have assumed the same amplitude of the order parameter in the two electrodes. The chiralities are defined as  $\chi_{r/l} = \mathbf{h} \cdot (\mathbf{P}_{r/l} \times \mathbf{h}_{l/r})$ , and  $\mathbf{h}_{r/l\perp} = \mathbf{h}_{r/l} - (\mathbf{h}_{r/l} \cdot \mathbf{h})\mathbf{h}/h^2$  are the components of  $\mathbf{h}_{r/l}$  perpendicular to  $\mathbf{h}$ .

## 2. Current along the y wire

We obtain the quasiparticle current in the y-wire by following the same procedure as in Sec. A 2. The coefficients  $c_3$  and  $c_4$  in Eqs. (12) and (13) are

$$c_3 = -\frac{\pi T \sigma_N}{2eL_y} \text{Im} \sum_{n \geq 0} \frac{w_x^2 \left( \frac{\sinh(\kappa'_\omega L_y)}{2\kappa'_\omega L_y} - \frac{1}{2} \right)}{\gamma_0^4 \kappa'_\omega{}^2 \cosh^2(\kappa'_\omega L_y/2)} \frac{2(\mathcal{F}'_0)^2}{\kappa_\omega^4 L_x^2} \Big|_{\omega=2\pi(n+1/2)T+ieV/2} \\ + \frac{\sigma_N}{4eL_y} \int_0^\infty d\varepsilon F_T(\varepsilon, V/2) \frac{w_x^2 \left( \frac{\sinh(\sqrt{2}\kappa'_\varepsilon L_y/2)}{\sqrt{2}\kappa'_\varepsilon L_y} - \frac{\sinh(\sqrt{2}i\kappa'_\varepsilon L_y/2)}{\sqrt{2}i\kappa'_\varepsilon L_y} \right)}{\gamma_0^4 |\kappa'_\varepsilon|^2 \cosh(\sqrt{-i}\kappa'_\varepsilon L_y/2)} \frac{2|\mathcal{F}'_0|^2}{|\kappa_\varepsilon|^4 L_x^2}, \quad (\text{B12})$$

$$c_4 = \frac{\pi T \sigma_N}{2eL_y} \text{Im} \sum_{n \geq 0} \frac{w_x^2 \left( \frac{\sinh(\kappa'_\omega L_y)}{2\kappa'_\omega L_y} - \frac{1}{2} \right)}{\gamma_0^4 \kappa'_\omega{}^2 \cosh^2(\kappa'_\omega L_y/2)} \frac{\sqrt{2}\mathcal{G}_0 \mathcal{F}_0 \mathcal{F}'_0}{\kappa_F \kappa_\omega^4 L_x^2} \Big|_{\omega=2\pi(n+1/2)T+ieV/2} \\ + \frac{\sigma_N}{4eL_y} \int_0^\infty d\varepsilon F_T(\varepsilon, V/2) \frac{w_x^2 \left( \frac{\sinh(\sqrt{2}\kappa'_\varepsilon L_y/2)}{\sqrt{2}\kappa'_\varepsilon L_y} - \frac{\sinh(\sqrt{2}i\kappa'_\varepsilon L_y/2)}{\sqrt{2}i\kappa'_\varepsilon L_y} \right)}{\gamma_0^4 |\kappa'_\varepsilon|^2 \cosh(\sqrt{-i}\kappa'_\varepsilon L_y/2)} \frac{-\sqrt{2}\text{Re}\{\mathcal{G}_0^* \mathcal{F}_0^* \mathcal{F}'_0\}}{\kappa_F |\kappa_\varepsilon|^4 L_x^2}. \quad (\text{B13})$$

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