Anomalous non-Abelian statistics for non-Hermitian generalization of Majorana zero modes

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In condensed matter physics, non-Abelian statistics for Majorana zero modes (or Majorana Fermions) is very important, really exotic, and completely robust. The race for searching Majorana zero modes and verifying the corresponding non-Abelian statistics becomes an important frontier in condensed matter physics. In this paper we generalize the Majorana zero modes to non-Hermitian (NH) topological systems that show universal but quite different properties from their Hermitian counterparts. Based on the NH Majorana zero modes, the orthogonal and nonlocal Majorana qubits are well defined. In particular, due to the particle-hole-symmetry breaking, NH Majorana zero modes have anomalous non-Abelian statistics with continuously tunable braiding Berry phase from $\pi/8$ to $3\pi/8$. This is quite different from the usual non-Abelian statistics with fixed braiding Berry phase $\pi/4$. The one-dimensional NH Kitaev model is taken as an example to numerically verify the anomalous non-Abelian statistics for two NH Majorana zero modes. The numerical results are exactly consistent with the theoretical prediction. With the help of braiding these two zero modes, the $\pi/8$ gate can be reached and thus universal topological quantum computation becomes possible.

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I. INTRODUCTION

Majorana zero modes (MZMs) have recently attracted much attention due to their potential application in topological quantum computations (TQCs) [1–18]. MZMs have been predicted to be induced by vortices in a two-dimensional (2D) spinless $p_x + ip_y$-wave superconductor (SC) [2] or localized at the ends of a one-dimensional (1D) p-wave SC [3]. For these topological superconductors (TSCs) with MZMs, topologically protected degenerate ground states (referred to as Majorana qubits) exist. Based on braiding these MZMs that obey non-Abelian statistics [1,11], a TQC is proposed [6,7]. Unfortunately, because the $\pi/8$ gate cannot be reached by braiding processes, a universal TQC based on MZMs has become unrealistic and still remains a challenge.

Alternatively, in recent years non-Hermitian (NH) physics has become an active research area that has attracted considerable research [19–73]. Researchers have investigated some NH effects on topological SCs and MZMs. Previous work has focus mainly on two types of NH terms on TSCs: gain/loss in SCs induced by imaginary chemical potentials [25–30] and imbalanced pairing [31], where the MZMs show similar properties as to those in a Hermitian system. However, many open questions still exist regarding the MZMs in NH TSCs:

(1) Can we generalize the MZMs to NH systems that show universal but quite different properties from their Hermitian counterparts?

(2) Can the NH effect change the non-Abelian statistics of MZMs?

(3) Do NH MZMs provide an alternative approach to universal TQC beyond their Hermitian counterparts?

In this paper we aim to answer the above questions and develop a theory for the NH generalized MZMs and the corresponding NH generalization for non-Abelian statistics (referred to as *anomalous non-Abelian statistics*). This paper is organized as follows. In Sec. II we generalize the MZMs and Majorana qubit to NH systems. In Sec. III we illustrate the continuously tunable Berry phase of the qubit states based on the braiding processes of non-Hermitian MZMs. The one-dimensional non-Hermitian Kitaev model is taken as an example to numerically verify the anomalous non-Abelian statistics for two NH MZMs in Sec. IV. In Sec. V we propose an alternative approach to universal topological quantum computations via NH MZMs in principle. Finally, we provide a summary and discussion in Sec. VI.

II. NON-HERMITIAN MAJORANA ZERO MODES

In certain TSCs, MZMs always emerge around defects, for example, the quantized vortex in 2D TSCs or the end in 1D TSCs. In general, a single MZM (sometimes referred to as Majorana fermion for Hermitian TSCs) can be described by a real fermionic field γ, i.e., $\gamma^\dagger = \gamma$. We can label two MZMs by complex fermions as $\gamma_1 = c_1 + c_1^\dagger$, $\gamma_2 = -i(c_2 - c_2^\dagger)$ and use them to represent the basis states of a nonlocal Majorana qubit:

$$|0\rangle_{\rm M} \equiv \frac{1}{\sqrt{2}}(|\overline{11}\rangle + |\overline{00}\rangle),\ |1\rangle_{\rm M} \equiv \frac{1}{\sqrt{2}}(|\overline{10}\rangle + |\overline{01}\rangle), \quad (1)$$

where $|\overline{mn}\rangle = |\overline{m}\rangle_1 \otimes |\overline{n}\rangle_2$ with $m, n = 0, 1$, $(|\overline{0}\rangle_i, |\overline{1}\rangle_i) = (|\overline{0}\rangle_i, c_i^\dagger|\overline{0}\rangle_i)$ are the eigenstates for complex fermions $c_i^\dagger$, $i = 1, 2$. In addition, $|0\rangle_{\rm M}$ is a fermion-empty state,

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$|1\rangle_{\rm M} = C_{\rm M}^\dagger|0\rangle_{\rm M}$ is a fermion-occupied state where $C_{\rm M}^\dagger$ is the composite fermionic operator,

$$C_{\rm M}^\dagger = (\gamma_1 - i\gamma_2)/2 = (c_1 + c_1^\dagger - c_2 + c_2^\dagger)/2, \quad (2)$$

and $C_{\rm M} = (\gamma_1 + i\gamma_2)/2$. The fermion parities for the two states of the Majorana qubit are different: the fermion parity of $|0\rangle_{\rm M}$ is even and the fermion parity of $|1\rangle_{\rm M}$ is odd. By introducing the fermionic parity operator $\hat{P}_F = (-1)^{\sum_j c_j^\dagger c_j}$, we have $\hat{P}_F|0\rangle_{\rm M} = |0\rangle_{\rm M}$ and $\hat{P}_F|1\rangle_{\rm M} = -|1\rangle_{\rm M}$.

However, *can we generalize the MZMs and the corresponding Majorana qubit to NH systems*? The answer is *yes*! For a Hermitian system, a global phase transformation $\mathcal{S}$ for fermion operators is defined as

$$(c, c^\dagger) \mapsto \mathcal{S}(c, c^\dagger)\mathcal{S}^{-1} = (e^{-i\phi}c, e^{i\phi}c^\dagger), \quad (3)$$

with a real ϕ. Here, we generalize the phase ϕ from a real number to an imaginary number $\phi = -i\beta$, and the imaginary phase transformation becomes a NH particle-hole (PH) similarity transformation, i.e., $(c, c^\dagger) \mapsto (e^{-\beta}c, e^{\beta}c^\dagger)$ with real β. Therefore, with the help of the NH PH similarity transformation $\mathcal{S}$, we define the NH MZMs as $\gamma_i^\beta = \mathcal{S}\gamma_i\mathcal{S}^{-1}, i = 1, 2$ and we have

$$\gamma_1^\beta = e^{-\beta}c_1 + e^{\beta}c_1^\dagger, \quad \gamma_2^\beta = -i(e^{-\beta}c_2 - e^{\beta}c_2^\dagger), \quad (4)$$

where the NH strength β is a real number ($\beta = \beta^* \neq 0$). In particular, the NH MZMs satisfy $(\gamma_i^{-\beta})^\dagger = \gamma_i^\beta$, $(\gamma_i^\beta)^\dagger \neq \gamma_i^\beta$. Therefore the properties of NH MZMs are characterized by β. The NH PH similarity transformation $\mathcal{S}$ breaks intrinsic PH symmetry in a TSC, i.e., the symmetry between c and $c^\dagger$ is broken. The corresponding TSCs with NH MZMs are no longer Hermitian and the corresponding operators γ^β are no longer real.

We consider a TSC with two NH MZMs $\gamma_1^\beta,\ \gamma_2^\beta$ and the corresponding fermionic operators are defined as

$$\tilde{C}_{\rm M}^\dagger = \left(\gamma_1^\beta - i\gamma_2^\beta\right)/2, \quad \tilde{C}_{\rm M} = \left(\gamma_1^\beta + i\gamma_2^\beta\right)/2, \quad (5)$$

with $\{\tilde{C}_{\rm M}, \tilde{C}_{\rm M}^\dagger\} = 1$, $(\tilde{C}_{\rm M})^2 = (\tilde{C}_{\rm M}^\dagger)^2 = 0$. We therefore introduce a NH Majorana qubit $(|0\rangle_{\rm M}^\beta, |1\rangle_{\rm M}^\beta) = (|0\rangle_{\rm M}^\beta, \tilde{C}_{\rm M}^\dagger|0\rangle_{\rm M}^\beta)$ based on the NH MZMs, which can be derived from a Hermitian case under a global NH PH similarity transformation $\mathcal{S}$:

$$|0\rangle_{\rm M}^\beta = \mathcal{S}|0\rangle_{\rm M}, \quad |1\rangle_{\rm M}^\beta = \mathcal{S}|1\rangle_{\rm M}. \quad (6)$$

From the definition of the NH MZMs, the energy difference between $|0\rangle_{\rm M}^\beta$ and $|1\rangle_{\rm M}^\beta$ disappears. Thus there is almost no coupling between γ_1^β and γ_2^β.

For the NH Majorana qubits, according to $\mathcal{S}\hat{P}_F\mathcal{S}^{-1} = \hat{P}_F$, the fermion parity is also a good quantum number and the fermion parities for the NH qubits are the same as their Hermitian counterpart: $\hat{P}_F|0\rangle_{\rm M}^\beta = |0\rangle_{\rm M}^\beta,\ \hat{P}_F|1\rangle_{\rm M}^\beta = -|1\rangle_{\rm M}^\beta$. In addition, we emphasize that the PH symmetry for the "empty" state $|0\rangle_{\rm M}^\beta$ is broken, but the PH symmetry for the "occupied" state $|1\rangle_{\rm M}^\beta$ is unbroken. Under PH transformation we have

$$|0\rangle_{\rm M}^\beta \mapsto |0'\rangle_{\rm M}^\beta \neq |0\rangle_{\rm M}^\beta, \quad |1\rangle_{\rm M}^\beta \mapsto |1'\rangle_{\rm M}^\beta = |1\rangle_{\rm M}^\beta. \quad (7)$$

The PH symmetry breaking of the NH Majorana qubit plays an important role in changing the typical non-Abelian statistics to anomalous non-Abelian statistics.

III. ANOMALOUS NON-ABELIAN STATISTICS

First we summarize the quantum properties of MZMs in Hermitian cases. MZMs obey SU(2) level-2 non-Abelian statistics. On the one hand, the fusion rule of MZMs is given by

$$\sigma \times \sigma = \mathbf{1} + \psi, \quad \psi \times \psi = \mathbf{1}, \quad \psi \times \sigma = \sigma, \quad (8)$$

where $\mathbf{1}$ is a vacuum sector, ψ is the (complex) fermion sector, and σ is the MZM sector. Two σ particles (MZMs) may either annihilate to the vacuum or fuse into a ψ particle. On the other hand, if we exchange two MZMs $(\gamma_1,\ \gamma_2)$, the result of the braiding is

$$\gamma_1 \to -\gamma_2, \quad \gamma_2 \to \gamma_1 \quad (9)$$

and the exchange operator (the braiding operator) $\mathcal{R}_{\rm M}$ can be described by $\mathcal{R}_{\rm M} = e^{i\frac{\pi}{4}\gamma_1\gamma_2}$. We may call $\mathcal{R}_{\rm M}$ to be Ivanov's braiding operator [1]. During the braiding process, the Berry phases for $|0\rangle_{\rm M}$ and $|1\rangle_{\rm M}$ are 0 and $\pi/2$, respectively. So for the Majorana qubit $(|0\rangle_{\rm M}, |1\rangle_{\rm M})$, the braiding operator is obtained as

$$\mathcal{R}_{\rm M} = e^{-i\Delta\Phi\tau_z} = e^{-i\frac{\pi}{4}\tau_z}, \quad (10)$$

which is the *Ivanov's braiding operator*. Here, τ_z denotes a Pauli matrix on the Majorana qubit $(|0\rangle_{\rm M}, |1\rangle_{\rm M})$. According to the topological feature of SU(2) level-2 non-Abelian statistics, the Berry phase during braiding processes $\Delta\Phi$ is fixed to be $\pi/4$ and cannot be changed.

However, for the NH MZMs γ^β, their non-Abelian statistics are different from the Hermitian case and become a new type of non-Abelian statistics, namely, *anomalous non-Abelian statistics*.

On the one hand, there exists a typical fusion rule for the NH MZMs:

$$\begin{aligned} \sigma^\beta \times \sigma^\beta &= \mathbf{1}^\beta + \psi^\beta, \\ \psi^\beta \times \psi^\beta &= \mathbf{1}^\beta, \\ \psi^\beta \times \sigma^\beta &= \sigma^\beta, \end{aligned} \quad (11)$$

where $\mathbf{1}^\beta$ is the NH vacuum sector, ψ^β is the NH (complex) fermion sector, and σ^β is the NH MZM sector. Two NH σ^β particles may either annihilate to the NH vacuum $\mathbf{1}^\beta$ or fuse into a NH ψ^β particle.

On the other hand, anomalous braiding processes exist for the NH MZMs γ^β. According to the case with two NH MZMs γ_1^β and γ_2^β, two degenerate quantum states always exist. Consequently, the braiding process for the NH MZMs is also defined by $\gamma_1^\beta \to -\gamma_2^\beta,\ \gamma_2^\beta \to \gamma_1^\beta$. Then, a question is, *can the braiding operator for NH MZMs $\mathcal{R}_{\rm M}^\beta$ be derived by performing a similarity transformation on the braiding operator for the Hermitian MZMs $\mathcal{R}_{\rm M}$*? The answer is *no*, i.e.,

$$\mathcal{R}_{\rm M}^\beta \neq \mathcal{S}\mathcal{R}_{\rm M}\mathcal{S}^{-1} = e^{-i\frac{\pi}{4}\tau_z}. \quad (12)$$

To show why, let us derive the braiding matrix $\mathcal{R}_{\rm M}^\beta$ on the Majorana qubit during the braiding processes. The Berry phases for the quantum states of the Majorana qubits $|0\rangle_{\rm M}^\beta$ and $|1\rangle_{\rm M}^\beta$ from the braiding operation are calculated by the Wilson

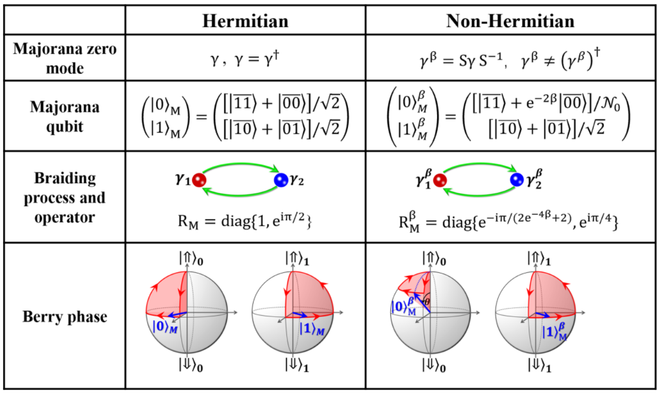

FIG. 1. An illustration to show the comparison between the typical Majorana qubits and NH Majorana qubits. We list the MZMs, orthogonality of the qubit states, braiding operator, and Berry phase in rows 2–5, respectively.

loop method,

$$|A_i|e^{\Delta\phi_i^{\beta}}=\prod_{n=0}^{n=N_s}{}_{\mathrm{M}}^{\ \beta}\langle i(\theta,\varphi_n)|i(\theta,\varphi_{n+1})\rangle_{\mathrm{M}}^{\beta},\qquad(13)$$

where $i=0,1$, the amplitude $|A_i|=1$ when the evolution step number N_s is sufficiently large, and $|i(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta}$ is the ith state of the Majorana qubit at the n step during the braiding process, which is labeled by the two parameters $\theta=2\arctan(e^{-2\beta})$ and φ_n. In particular, we have

$$|0(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta}=\frac{1}{\mathcal{N}_0}[e^{i\varphi_n}|\overline{11}\rangle+e^{-2\beta}|\overline{00}\rangle],$$
$$|1(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta}=\frac{1}{\sqrt{2}}[|\overline{10}\rangle+e^{-i\varphi_n}|\overline{01}\rangle],\qquad(14)$$

where $\mathcal{N}_0=\sqrt{1+e^{-4\beta}}$ is the self-normalization coefficient of the state $|0(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta}$.

First, we derive the effects of the braiding operator $\mathcal{R}_{\mathrm{M}}^{\beta}$ on the quantum state $|0\rangle_{\mathrm{M}}^{\beta}$. We map the states $(|\overline{00}\rangle,|\overline{11}\rangle)$ onto a pseudospin $(|\Uparrow\rangle_0,|\Downarrow\rangle_0)$ and use the Bloch sphere to label the quantum states. In the Hermitian case $\beta=0$ (the left side in the last row of Fig. 1), the initial state is $|0\rangle_{\mathrm{M}}^{\beta}=|\Uparrow\rangle_0+|\Downarrow\rangle_0$, which is denoted by a spot at the equator on the Bloch sphere $[\theta,\varphi]=[\pi/2,0]$. During the braiding process, $|0\rangle_{\mathrm{M}}^{\beta}$ adiabatically deforms into $(e^{i\varphi_n}|\Uparrow\rangle_0+|\Downarrow\rangle_0)$ and finally changes into $(e^{i\pi/2}|\Uparrow\rangle_0+|\Downarrow\rangle_0)$ denoted by another spot $[\theta,\varphi]=[\pi/2,\pi/2]$. So the geometry phase (Berry phase) is $\Delta\varphi(1-\cos\theta)/2$, where $\theta=\pi/2$ and $\Delta\varphi=\pi/2$, while in the NH case $\beta\neq0$ (the right side in the last row of Fig. 1), the initial state becomes $(|\Uparrow\rangle_0+e^{-2\beta}|\Downarrow\rangle_0)$, which is denoted by a spot away from the equator of the Bloch sphere, $[\theta,\varphi]=[2\arctan(e^{-2\beta}),0]$. During the braiding processes, it adiabatically deforms into $(e^{i\varphi_n}|\Uparrow\rangle_0+e^{-2\beta}|\Downarrow\rangle_0$ and finally changes into $(e^{i\pi/2}|\Uparrow\rangle_0+e^{-2\beta}|\Downarrow\rangle_0)$, denoted by $[\theta,\varphi]=[2\arctan(e^{-2\beta}),\pi/2]$. After the braiding processes we obtain the geometry phase as $\Delta\varphi(1-\cos\theta)/2$, where $\tan(\theta/2)=e^{-2\beta}$ and $\Delta\varphi=\pi/2$.

Second, using a similar operation on $|1\rangle_{\mathrm{M}}^{\beta}$, we map the qubit $(|\overline{01}\rangle,|\overline{10}\rangle)$ onto a pseudospin $(|\Uparrow\rangle_1,|\Downarrow\rangle_1)$, as shown in the last row of Fig. 1. As a result, the braiding operators for the Majorana qubit $(|0\rangle_{\mathrm{M}}^{\beta},|1\rangle_{\mathrm{M}}^{\beta})$ are obtained as

$$\mathcal{R}_{\mathrm{M}}^{\beta}\begin{pmatrix}|0\rangle_{\mathrm{M}}^{\beta}\\|1\rangle_{\mathrm{M}}^{\beta}\end{pmatrix}=\begin{pmatrix}e^{i\Delta\phi_0^{\beta}}&0\\0&e^{i\Delta\phi_1^{\beta}}\end{pmatrix}\begin{pmatrix}|0\rangle_{\mathrm{M}}^{\beta}\\|1\rangle_{\mathrm{M}}^{\beta}\end{pmatrix},\qquad(15)$$

where the Berry phases are obtained as $\Delta\phi_0^{\beta}=-\frac{\pi}{2(e^{-4\beta}+1)}$ and $\Delta\phi_1^{\beta}=\pi/4$. It is obvious that the Berry phase for $|0\rangle_{\mathrm{M}}^{\beta}$ is different from $|0\rangle_{\mathrm{M}}$.

The braiding operator is obtained as

$$\mathcal{R}_{\mathrm{M}}^{\beta}=e^{-i\Delta\Phi'\tau_z},\qquad(16)$$

which is the NH generalization of the Ivanov's braiding operator and is different from the Hermitian case shown in Eq. (10). Here, τ_z denotes a Pauli matrix on the Majorana qubit $(|0\rangle_{\mathrm{M}}^{\beta},|1\rangle_{\mathrm{M}}^{\beta})$, and $\Delta\Phi'=\frac{1}{2}(\Delta\phi_1^{\beta}-\Delta\phi_0^{\beta})=\frac{\pi}{8}+\frac{\pi}{4(e^{-4\beta}+1)}$ denotes a Berry phase during braiding processes that can continuously tuned from $\pi/8$ to $3\pi/8$. Thus $\Delta\Phi'$ can be an arbitrary value in the region of $(\pi/8,3\pi/8)$, including a rational number or irrational number. As a result, we call it *anomalous non-Abelian statistics*. By contrast, the Berry phase from braiding processes for usual non-Abelian statistics is fixed to $\Delta\Phi=\pi/4$. Besides, when we fix β, for two non-Hermitian Majorana zero modes far away, the braiding rule and the corresponding Berry phase $\Delta\Phi'$ will never change, no matter how you change the braiding path. In this sense, this can be considered as non-Abelian generalization of Abelian statistics for U(1) Abelian anyons with arbitrary Berry phase according to the Wilczek flux-binding picture.

IV. NUMERICAL SIMULATIONS ON VERIFYING THE ANOMALOUS NON-ABELIAN STATISTICS

A. Non-Hermitian MZMs in the 1D Kitaev chain

A 1D NH Kitaev model [3] with imbalanced p-wave SC paring is taken as an example to illustrate the anomalous non-Abelian statistics of NH MZMs, and the numerical simulations are performed during the braiding processes. The Hamiltonian for the NH Kitaev model is written as

$$\hat{\mathrm{H}}_{\mathrm{NHK}}(\beta)=-\sum_{j=1}^{N}[t(c_j^{\dagger}c_{j+1}+c_{j+1}^{\dagger}c_j)+\Delta^{+}c_j^{\dagger}c_{j+1}^{\dagger}$$
$$+\Delta^{-}c_{j+1}c_j+\mu(1-2n_j)],\qquad(17)$$

where c_j ($c_j^{\dagger}$) annihilates (creates) a fermion on site j. t, $\Delta^{\pm}$, μ, and N denote the hopping amplitude, the strength of p-wave pairing, the chemical potential, and the lattice number, respectively. We set $\Delta^{\pm}=\Delta_0e^{\pm2\beta}$, where $\beta\in\mathbb{R}$ represents the NH strength and $\Delta_0>0$. When $\beta\neq0$, we have $\hat{\mathrm{H}}_{\mathrm{NHK}}\neq\hat{\mathrm{H}}_{\mathrm{NHK}}^{\dagger}$, which can be achieved by the NH similarity transformation from its Hermitian counterpart. In this paper we focus on the case of $t=\Delta_0$.

The 1D NH SC chain may have nontrivial topological properties. For the translation variables ansatz, we transform the fermion Hamiltonian into momentum space, $\hat{\mathrm{H}}_{\mathrm{NHK}}(k)=\sum_k\psi_k^{\dagger}h(k,\beta)\psi_k$, with

$$h(k,\beta)=(t\cos k+\mu)\cdot\sigma^z+\Delta_0\sin k\cdot\sigma^{y,\beta},\qquad(18)$$

by introducing $\psi_k = (c_k, c_{-k}^\dagger)^T$, where

$$\sigma_j^{y,\beta} = \mathcal{S}\sigma_j^y\mathcal{S}^{-1} = \cosh(\beta)\sigma_j^y - i\sinh(\beta)\sigma_j^x \tag{19}$$

is a 2×2 matrix and $\mathcal{S}(\beta) = \mathrm{diag}\{e^{\frac{\beta}{2}}, e^{-\frac{\beta}{2}}\}$. With the help of the biorthogonal set, we define right/left eigenstates for the NH systems as $\hat{\mathrm{H}}_{\mathrm{NHK}}|\Psi\rangle_{\mathrm{R}} = E|\Psi\rangle_{\mathrm{R}}$, and $\hat{\mathrm{H}}_{\mathrm{NHK}}^\dagger|\Psi\rangle_{\mathrm{L}} = E^*|\Psi\rangle_{\mathrm{L}}$. The dispersion of the quasiparticle is given by

$$E(\beta, k) = \sqrt{(\mu - t\cos k)^2 + (\Delta_0 \sin k)^2}, \tag{20}$$

which is independent with the NH parameter β. To describe the topological structure of $\hat{\mathrm{H}}_{\mathrm{NHK}}$, we define *biorthogonal* Z_2 *topological invariant* [74],

$$\omega = \mathrm{sgn}(\eta_{k=0}\cdot\eta_{k=\pi}), \tag{21}$$

where $\eta_{k=0/\pi} = {}_{\mathrm{L}}\langle\Psi|c_{k=0/\pi}^\dagger c_{k=0/\pi}|\Psi\rangle_{\mathrm{R}}$ and $\eta_{k=0/\pi}(\beta) = \eta_{k=0/\pi}(\beta=0)$. Therefore we have $\eta_{k=0} = \mathrm{sgn}(t+\mu)$, $\eta_{k=\pi} = \mathrm{sgn}(-t+\mu)$. For the case of $\omega = 1$ ($|t| < |\mu|$), the SC is trivial, but for $\omega = -1$ ($|t| > |\mu|$), the SC becomes topological.

Then we analytically solve the NH model with open boundary condition. When $\omega = -1$, there exist two edge states with (nearly) zero energy, i.e., the NH MZMs $\gamma_{L/R}^\beta$ located at the left/right end of the 1D chain. Using the transformation shown in Eq. (4), the operator of NH MZMs is written as

$$\gamma_L^\beta = \frac{1}{\sqrt{\mathcal{N}}}[(e^\beta + e^{-\beta})a_1 + i(e^\beta - e^{-\beta})b_1],$$
$$\gamma_R^\beta = \frac{1}{\sqrt{\mathcal{N}}}[-i(e^\beta - e^{-\beta})a_N + (e^\beta + e^{-\beta})b_N], \tag{22}$$

where $a_i = c_i + c_i^\dagger$, $b_i = -i(c_i - c_i^\dagger)$ denote the Majorana fermions at site i, and $\mathcal{N}$ is the biorthogonal normalized coefficient. We can construct a pair of canonical operators $(\widetilde{C}_{\mathrm{M}}, \widetilde{C}_{\mathrm{M}}^\dagger)$ as

$$\widetilde{C}_{\mathrm{M}} = \tfrac{1}{2}(\gamma_L^\beta + i\gamma_R^\beta), \quad \widetilde{C}_{\mathrm{M}}^\dagger = \tfrac{1}{2}(\gamma_L^\beta - i\gamma_R^\beta), \tag{23}$$

with $\{\widetilde{C}_{\mathrm{M}}, \widetilde{C}_{\mathrm{M}}^\dagger\} = 1$, $(\widetilde{C}_{\mathrm{M}})^2 = (\widetilde{C}_{\mathrm{M}}^\dagger)^2 = 0$. In general, the corresponding Majorana model can be written with the eigenenergy as

$$H_{\mathrm{NHK}} = \sum_{j=1}^{N-1}\varepsilon_j\left(\widetilde{C}_j^\dagger\widetilde{C}_j - \frac{1}{2}\right) + 0\times\widetilde{C}_{\mathrm{M}}^\dagger\widetilde{C}_{\mathrm{M}}, \tag{24}$$

where the explicit expression for $\widetilde{C}_j$ is not necessary since we are only concerned with zero-energy mode. Therefore the wave function of the quasidegenerate ground states for the 1D SC model can be defined as

$$|\Psi_0(\beta)\rangle = \widetilde{C}_{\mathrm{M}}|F\rangle, \quad |\Psi_1(\beta)\rangle = \widetilde{C}_{\mathrm{M}}^\dagger|0\rangle_{\mathrm{M}}^\beta, \tag{25}$$

where $|F\rangle$ is the many-body quantum state with occupied single-particle states for $E < 0$ and empty single-particle states for $E \geqslant 0$, and $|\Psi_{0/1}(\beta)\rangle$ satisfies the self-normalization condition $|\langle\Psi_{0/1}(\beta)|\Psi_{0/1}(\beta)\rangle| = 1$. The similarity of these two MZMs is defined as

$$\chi(\beta) = |\langle\Psi_0(\beta)|\Psi_1(\beta)\rangle|. \tag{26}$$

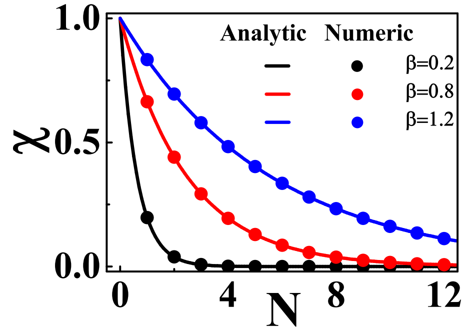

FIG. 2. The numerical results (dots) and the analytical results (lines) for the similarity between two degenerate ground states in NH Kitaev model with $t = \Delta_0$, $\mu = 0$, and $\beta = 0.2,\ 0.8$, and 1.2. These results indicate the orthogonality of two degenerate ground states in thermodynamic limit ($N \mapsto \infty$).

In the thermodynamic limit ($N \mapsto \infty$), the two MZMs are orthogonal ($\chi(\beta) = 0$) and the energy splitting of them is zero. For example, when $t = \Delta_0$ and $\mu = 0$ we have $\chi(\beta) = (\tanh\beta)^N$, and it is obvious that $\chi(\beta) \mapsto 0$ with $N \mapsto \infty$ for $|\tanh\beta| < 1$, i.e., the two degenerate ground states are orthogonal. The proof of the orthogonal properties is shown in Fig. 2, where the numeric results (the dots) are consisted with the analytic results (the lines). Due to the orthogonality and the parities of the twofold degenerate ground states, they can be used to construct the two basis states of the NH Majorana qubit:

$$|0\rangle_{\mathrm{M}}^\beta \equiv |\Psi_0(\beta)\rangle, \quad |1\rangle_{\mathrm{M}}^\beta \equiv |\Psi_1(\beta)\rangle. \tag{27}$$

B. T-type braiding of the non-Hermitian MZMs

The non-Abelian statistics of two NH MZMs can be verified in the T-junction Majorana chain systems [11,75], which contain at least four lattice sites or eight Majorana sites, as shown in Fig. 3. Here, the braiding processes of the two NH MZMs are denoted by blue dotted arrows. We denote the quantum states of Majorana modes by $|\gamma\rangle_{L/R}^\beta(T_n)$ with $n \in (0, 7)$, where $T_0 = 0$ represents the initial time and T_n for the nth step of the braiding process. At the beginning of the braiding processes, two unpaired NH MZMs are located at the left end $\gamma_L^\beta(T_0)$ (green ball) and right end $\gamma_R^\beta(T_0)$ (black ball) of the 1D Majorana chain. Then we can adiabatically tune the chemical potential and hopping parameters to exchange the two NH MZMs with seven steps from T_0 to T_7. Firstly, during the period from T_0 to T_2, the leftmost fermion is driven to the bottom site of site 4. Then, during the period from T_2 to T_5, the rightmost fermion is driven to the leftmost site. At last, during the period from T_5 to T_7, the bottom fermion is driven to the rightmost site. For example, during the period $t \in (T_0, T_1)$, we adiabatically turn on intrasite potential μ_1 to a sufficiently large value and turn off the intersite link t to zero, simultaneously. As a result, the left mode is moved to site 2. The rest of the process can be done in the similar way. Finally, the result of the NH Ivanov's braiding operator $\mathcal{R}_{\mathrm{M}}^\beta$ is described by $\gamma_L^\beta(T_7) = -\gamma_R^\beta(T_0)$ and $\gamma_R^\beta(T_7) = \gamma_L^\beta(T_0)$.

We perform the numerical simulations to verify the non-Hermitian Ivanov's braiding operator $\mathcal{R}_{\mathrm{M}}^\beta$ for two NH MZMs

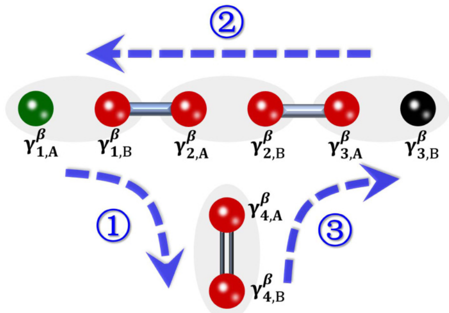

FIG. 3. Schematic diagram for the T-type braiding process to exchange the two NH MZMs. We take a system with eight Majorana fermions as an example. Here, the braiding processes of the two NH MZMs are denoted by blue dotted arrows. At the beginning of the braiding processes, two unpaired NH MZMs are almost located at the left end $\gamma_L^{\beta}(T_0)$ (green ball) and right end $\gamma_R^{\beta}(T_0)$ (black ball) of the 1D Majorana chain. Then we can adiabatically tune the chemical potential and hopping parameters to exchange the two NH MZMs with seven steps from T_0 to T_7. Firstly, during the period from T_0 to T_2, the leftmost fermion is driven to the bottom site of site 4. Then, during the period from T_2 to T_5, the rightmost fermion is driven to the leftmost site. Finally, during the period from T_5 to T_7, the bottom fermion is driven to the rightmost site.

by mapping the original fermionic model $\hat{\mathrm{H}}_{\mathrm{NHK}}(\beta)$ onto a NH transverse Ising model [76] via the Jordan-Wigner transformation, which is defined by the stringlike annihilation and creation operators

$$c_j = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^-, \quad c_j^{\dagger} = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+. \tag{28}$$

As a result, the Hamiltonian $\hat{\mathrm{H}}_{\mathrm{NHK}}$ in Eq. (17) becomes

$$\hat{\mathrm{H}}_{\mathrm{NHS}}(\beta) = -\frac{1}{4}\sum_j \left(J\sigma_j^{x,\beta}\sigma_{j+1}^{x,\beta} - 4h\sigma_j^z\right), \tag{29}$$

where $\sigma_j^{x,\beta} = \cosh(\beta)\sigma_j^x + i\sinh(\beta)\sigma_j^y$ and $J = t = \Delta_0$, $h = \mu$. Meanwhile, the braiding process for the Majorana qubit $(|0\rangle_{\mathrm{M}}^{\beta}, |1\rangle_{\mathrm{M}}^{\beta})$ is mapped onto that for the two degenerate ground states in the spin representation. The NH Ivanov's braiding operator $\mathcal{R}_{\mathrm{M}}^{\beta}$ for two MZMs is mapped onto the corresponding operator $R^z(\varphi)$, which rotates the spin φ angle around the z axis in spin representation, i.e.,

$$\mathcal{R}_{\mathrm{M}}^{\beta} \leftrightarrow R^z(\varphi). \tag{30}$$

This is the same as the case in the Hermitian system. Here, we also take the four-sites system as an example. What needs to be emphasized is that $T_3 - T_4$ plays a key role during the braiding process, i.e., the MZM in site 2B is driven to site 2A (as shown in Fig. 3). In spin representation, this operation can be mapped to rotate the spin in site 2 $\pi/4$ around the z axis.

During T_3 to T_4, the system contains two spins in site 2 and site 4. For certain times in this period $t \in (T_3, T_4)$, the spin in site 2 is rotated with angle φ_n. Then, in σ^z representation, the eigen wave function of the two lowest eigenenergies for $\hat{\mathrm{H}}_{\mathrm{NHS}}(\beta)$ in Eq. (29) are read as

$$|0(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta} = \frac{1}{\mathcal{N}_0}\begin{bmatrix}\cos\varphi_n + i\sin\varphi_n\\ 0\\ 0\\ e^{-2\beta}\end{bmatrix}\begin{bmatrix}\uparrow\uparrow\\ \uparrow\downarrow\\ \downarrow\uparrow\\ \downarrow\downarrow\end{bmatrix},$$

$$|1(\theta,\varphi_n)\rangle_{\mathrm{M}}^{\beta} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ \cos\varphi_n - i\sin\varphi_n\\ 0\end{bmatrix}\begin{bmatrix}\uparrow\uparrow\\ \uparrow\downarrow\\ \downarrow\uparrow\\ \downarrow\downarrow\end{bmatrix}, \tag{31}$$

where $\theta = 2\arctan(e^{-2\beta})$ and $\mathcal{N}_0 = \sqrt{1+e^{-4\beta}}$ is the self-normalization coefficient. The Berry phases of the qubit states $|0\rangle_{\mathrm{M}}^{\beta}$, $|1\rangle_{\mathrm{M}}^{\beta}$ obtained from rotation (or Majorana braiding) are defined by the Wilson loop in Eq. (13). For a sufficiently large N_s, the Berry phase for the qubit state $|0\rangle_{\mathrm{M}}^{\beta}$ is

$$\begin{aligned}\Delta\phi_0^{\beta} &= \sum_{n=0}^{N_s}\arctan\left[-\frac{\sin(\pi/2N_s)}{e^{-4\beta}+\cos(\pi/2N_s)}\right]\\ &\simeq \sum_{n=0}^{N_s}\arctan\left(-\frac{\pi}{2N_s(e^{-4\beta}+1)}\right)\\ &\simeq -\frac{\pi}{2(e^{-4\beta}+1)}.\end{aligned} \tag{32}$$

Similarly, the Berry phases for the qubit $|1\rangle_{\mathrm{M}}^{\beta}$ is

$$\Delta\phi_1^{\beta} = \sum_{n=0}^{N_s}\arctan\left[\frac{\sin(\pi/2N_s)}{1+\cos(\pi/2N_s)}\right] \simeq \frac{\pi}{4}. \tag{33}$$

Therefore the result of braiding two NH MZMs is same as Eq. (15). We can see that the NH factor β has the effect of adjusting the braiding phase, and the phase region for $|0\rangle_{\mathrm{M}}^{\beta}$ is $\Delta\phi_0^{\beta} \in [-\pi/2, 0]$, as shown in Fig. 4(a), where the numerical results (dots) are exactly consistent with the theoretical prediction (lines). The variation of the cumulative phases during the braiding processes for the Hermitian case ($\beta = 0$) are shown in Fig. 4(c) and for the non-Hermitian cases ($\beta = \mp 2$) in Figs. 4(b) and 4(d). In the limit of $\beta \to -\infty$, we have

$$\mathcal{R}_{\mathrm{M}}^{\infty} = \mathrm{diag}\{1, e^{i\pi/4}\}. \tag{34}$$

This is just the $\pi/8$ gate needed in the universal TQC, $\mathbf{T} = \mathcal{R}_{\mathrm{M}}^{\infty}$.

In general, to realize a $\pi/8$ gate through single braiding operation, the Majorana qubit must be realized in the limit of negative infinite β. However, by using the Solovay-Kitaev construction [4], to realize universal quantum computation we just need to realize an arbitrary phase gate with phase changing $\Delta\Phi' \neq 0$, $\pm\pi/4$, $\pm\pi/2$, and π, where the arbitrary phase gate is defined as $\mathcal{R}_{\mathrm{M}}^{\beta} = e^{-i\Delta\Phi'\tau_z}$. Therefore the gate is also topological with the help of an arbitrary phase changing with $\Delta\Phi'$ but not must be fixed to $\pi/8$. As a result, the gate with an arbitrary phase changing can be realized by finite β but do not have to be fixed to infinitely small, i.e $\beta \to -\infty$.

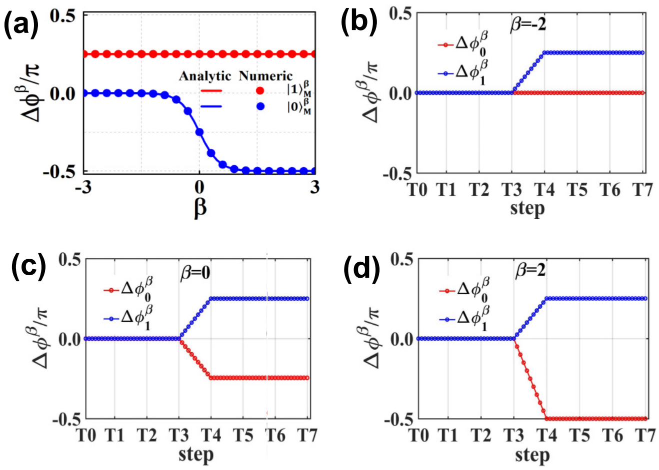

FIG. 4. Non-Abelian statistics of MZMs realized in the T junction. The Berry phase for the quantum state $|0\rangle_{\rm M}^{\beta}$ ($|1\rangle_{\rm M}^{\beta}$) is $\Delta\phi_0^{\beta}$ ($\Delta\phi_1^{\beta}$), and the numerical results (dots) are exactly consistent with the theoretical prediction (lines). (a) The Berry phase for $|0\rangle_{\rm M}^{\beta}$ is always $\pi/4$ for different β, and the Berry phase for $|1\rangle_{\rm M}^{\beta}$ can vary from $-\pi/2$ to 0 with β. The variation of the cumulative phases during the braiding processes for the Hermitian case ($\beta = 0$) is shown in (c), and for the non-Hermitian cases ($\beta = \mp 2$) are shown in (b) and (d).

V. NON-HERMITIAN ASSISTED TOPOLOGICAL QUANTUM COMPUTATION

Due to nonlocality and orthogonality, the NH MZMs may be utilized as a decoherence-free qubit, which play an important role in the realization of fault-tolerant universal TQC. We propose an alternative approach to universal TQC via NH MZMs in principle—*non-Hermitian assisted TQC*.

To perform universal TQC, four gates need to be carried out topologically: the Hadamard gate H, the phase gate S, the $\pi/8$ gate T, and the controlled NOT (CNOT) gate. By braiding the MZMs in Hermitian TSC, the Hadamard gate, phase gate, and CNOT gate can be attained. However, the T gate cannot be realized by the braiding process. Instead, a possible method is to use "magic states distillation," which is a combination of topological and nontopological approaches [77,78]. In contrast to the magic states method, the realization of the T gate in non-Hermitian assisted TQC is purely topological.

If one can realize $\hat{\rm H}_{\rm NHK}^{\beta}$ with the freely adjustable NH strength β, we can adiabatically tune β to construct a universal TQC. For the Hadamard gate, phase gate, and controlled NOT gate, we set the NH strength β to zero. For the $\pi/8$ gate, we set the NH strength β to be certain value and braid NH MZM for $\mathcal{N}$ times. For example, $\mathcal{N} = 4$ for $\beta = -(\ln 0.6)/4 \approx 0.128$. For this case, during the braiding processes, the $\pi/8$ gate can be reached $\mathbf{T} = [\mathcal{R}_{\rm M}^{\beta}]^{\mathcal{N}}$. In the end, to perform measurement, the NH strength β returns to zero again. What should be mentioned is that the $\mathbf{T}$ gate from the braiding process is based on "the anomalous Berry phase," which has huge advantages over other nontopological methods, such as the method of "magic state distillation" [77,78].

In Fig. 5, an illustration of two phase gates S and a $\pi/8$ gate for NH assisted TQC is shown. Here, we take a braiding process with three steps as an example: a phase gate S by exchanging two Hermitian MZMs with $\beta = 0$, a $\pi/8$ gate by exchanging two NH MZMs with $\beta \neq 0$, and a phase gate S by exchanging two MZMs with $\beta = 0$.

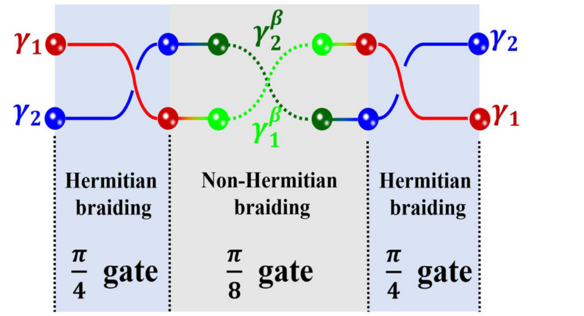

FIG. 5. An illustration of NH assisted topological quantum computation. In step 2, a $\pi/8$ gate is realized by tuning the NH strength β. Dotted lines indicate that multiple braiding operations can be performed.

Furthermore, the non-Hermitian model in Eq. (29) can be simulated using three-level atoms in a variety of setups, including trapped ions, cavity QED, and atoms in optical lattices. The dynamics by $H(\beta)$ can be decomposed as

$$e^{-iH(\beta)t} = e^{-i\mu\sigma^z t} Q(\beta)\left(\prod_i e^{i\frac{J}{4}\sigma_j^x\sigma_{j+1}^x t}\right)Q^{-1}(\beta), \qquad (35)$$

where the nonunitary dynamics $Q(\beta)$ and $Q^{-1}(\beta)$ are from measuring whether a spontaneous decay has occurred [79–81]. This process can be measured with a high degree of accuracy [82–84].

In a real experiment, β cannot be "fixed," which will introduce errors and become a big problem for topological quantum computation. Therefore noise in the control parameter β will easily change the Berry phase. For example, $\mathcal{R}_{\rm M}^{\beta} = e^{-i\Delta\Phi'\tau_z}$ is the ideal operation, and due to errors, the actual operation becomes $e^{-i\Delta\Phi'(1+\xi)\tau_z}$, where the angle of rotation differs from the desired Φ' by a factor $1+\xi$. So to perform accurate operations, one may consider replacing single operations by composite sequences of pulsed operations [85]. By this approach the errors and even those from the inability to control β can be corrected to $O(\xi^n)$ for arbitrary n.

VI. CONCLUSION AND DISCUSSION

In this paper we developed a theory for NH generalization of MZMs, i.e., $\gamma^{\beta} = \mathcal{S}\gamma\mathcal{S}^{-1}$, where $\mathcal{S}$ is the NH PH similarity transformation and β is the NH strength. The key point of NH generalization of MZMs is anomalous non-Abelian statistics. Due to the particle-hole symmetry breaking, the Berry phase from braiding processes become an arbitrary number in a region, i.e., $\Delta\Phi \in (\pi/8, 3\pi/8)$. The anomalous non-Abelian statistics for the NH MZMs indicates that in NH topological systems the theory for the usual unitary modular tensor category would be generalized to a theory for a certain nonunitary modular tensor category, and the theory for the usual topological field theories would be generalized to a theory for certain non-Hermitian topological field theories. In the future, we will study these issues. In addition, we plan to apply the theory to other TSCs, such as the 2D NH $p_x + ip_y$ TSC and higher-order NH TSCs, and then study the possible physical realization of the NH MZMs in these NH topological systems.

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