Absence of a $T^{2/3}$ specific heat anomaly in a $U(1)$ spin liquid with a large spinon Fermi surface

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(Received 13 August 2021; revised 1 October 2021; accepted 4 October 2021; published 11 October 2021)

Effective gauge theories based on slave particle construction are widely used to describe quantum number fractionalization in strongly correlated electron systems. However, even setting aside the debates on the confinement issue of the slave particles, there are still significant conflicts between theory and experiment. In particular, a $T^{2/3}$ specific heat anomaly has been predicted to be the smoking-gun signature of low-lying gauge fluctuation in a $U(1)$ spin liquid with a large spinon Fermi surface, which has, however, never been observed. Here we show that such an anomaly is actually an artifact of a Gaussian approximation and is absent when the no-double-occupancy constraint on the slave particles is strictly enforced. We also show that projective construction based on slave particle representation provides a unified understanding of the mechanism of spin fractionalization in one- and two-dimensional spin liquids.

DOI: 10.1103/PhysRevB.104.165123

I. INTRODUCTION

Quantum spin liquids are exotic states of matter hosting fractionalized excitation [1,2], a novel object that has been suggested to provide an exotic interpretation of the anomalous dynamical behavior observed in many quantum magnets, which are hard to explain within the traditional spin wave theory framework [3–7]. It also offers a novel mechanism for the non-Fermi liquid behavior observed in cuprate superconductors [8]. Effective gauge theory based on slave particle construction is the most widely used theoretical tool to describe quantum number fractionalization in strongly correlated electron systems.

The $U(1)$ spin liquid with a large spinon Fermi surface is a particular example of a system showing quantum number fractionalization. Such a state can be understood roughly as the descendant of a metallic state near a Mott transition, in which electron correlation has already opened a charge gap while leaving the electron Fermi surface intact. An insulator with a large Fermi surface is exotic in the sense that the gapless excitation on the Fermi surface should carry only the spin and not the charge quantum number of an electron and is intrinsically fractionalized. Indeed, in organic Mott insulators with a triangular lattice, people do find evidence of the existence of such a quantum spin liquid near the Mott transition [9–12]. Magnetic susceptibility and specific heat measurement at low temperature on such systems exhibit typical behavior of a Fermi liquid metal with a finite density of states on the Fermi surface. Such a picture is also supported by theoretical studies. Variational studies find that when the multispin exchange is strong enough, which is expected near a Mott transition, a $U(1)$ spin liquid state with a large spinon Fermi surface is the best variational ground state of a quantum antiferromagnet defined on the triangular lattice [13]. An effective field theory study based on slave particle construction also arrived at the same conclusion in the saddle point approximation [14].

However, one encounters serious problems when trying to go beyond the saddle point approximation. The effective theory of the above $U(1)$ spin liquid has the form of a compact $U(1)$ gauge field coupled to fermionic slave particles that form a large Fermi surface [12]. It is well known that in 2+1 dimensions a pure compact $U(1)$ gauge field is always confining as a result of the proliferation of the singular gauge field configuration called an instanton [15]. Whether the instanton effect can be suppressed by dissipative coupling to a gapless fermion system and whether the gauge non-neutral slave particle can appear in a physical spectrum [8,16–26] have been strongly debated. Even if the instanton effect can, indeed, be suppressed, there are still strong conflicts between theory and experiment. The noncompact $U(1)$ gauge field in the Gaussian effective theory, which has no intrinsic dynamics of its own, will acquire a relaxational dynamics with a dynamical exponent $z = 3$ as a result of the dissipative coupling to the current of the gapless fermionic slave particles [8,27]. In two dimensions, such an ultraslow dynamics in the gauge fluctuation will result in a $T^{2/3}$ anomaly in the low-temperature specific heat [28]. This smoking-gun signature of the Gaussian effective theory, however, has never been observed in any serious experimental investigation [12]. These unresolved issues cast serious doubt on our identification of the organic Mott insulators as $U(1)$ spin liquid materials [29–33].

We note, however, confinement of slave particles does not necessarily imply the instability of a $U(1)$ spin liquid and the forbiddance of spin fractionalization. For example, it is well known that in one dimension, in which gauge non-neutral particles are always confined, fractionalized spin excitations can emerge as domain walls in the spin correlation pattern. Most theorists think this mechanism of spin fractionalization is fundamentally different from the mechanism of deconfinement of slave particles since the slave particles are local objects, while the domain wall excitations are topological in nature [19,34,35]. However, one still cannot help wondering whether there is any unrevealed connection between the slave particles and the physical spinons in this special case. After all, the two share the same Fermi surface in the $U(1)$ spin liquid state.

In this paper, we reinvestigate these issues by combing effective field theory analysis and variational construction. We show that the Gaussian approximation to the gauge fluctuation in the effective theory of a $U(1)$ spin liquid is invalid as singular gauge field configurations always proliferate. We find that the dissipative coupling between the transverse $U(1)$ gauge field and the current of the slave particles is prohibited when the time component of the $U(1)$ gauge field is exactly integrated out. We find further that the dynamics of the transverse gauge fluctuation in the $U(1)$ spin liquid is determined by its coupling to the scalar spin chirality, which features a large characteristic energy throughout the Brillouin zone. The $T^{2/3}$ specific heat anomaly predicted by the Gaussian effective theory is thus absent. We also show that the Gutzwiller projection will transform the slave particle into a genuine nonlocal object, which a physical spinon should be, thanks to the Friedel sum rule and Anderson's theorem of orthogonality catastrophe. This unifies our understanding of spin fractionalization in one-dimensional (1D) and two-dimensional (2D) spin liquids.

This paper is organized as follows. In the next section, we review the effective gauge theory of the $U(1)$ spin liquid with a large spinon surface constructed from a slave particle representation of the spin operator. We show that the Gaussian approximation on the time component of the $U(1)$ gauge field always fails as a result of proliferation of the singular gauge fluctuation configuration. In Sec. III, we construct an effective theory for the gauge fluctuation around the $U(1)$ spin liquid saddle point with the time component of the $U(1)$ gauge field exactly integrated out. We show that the $T^{2/3}$ specific heat anomaly predicted by the Gaussian effective field theory is absent in our theory. In Sec. IV, we present a unified picture of the mechanism of spin fractionalization in one- and two-dimensional spin liquids. We show that Gutzwiller projection on slave particle excitation can recover correctly the nonlocal nature of a physical spinon. We conclude our work in Sec. V and discuss possible extensions of this work to other strongly correlated systems in which effective field theory based on slave particle construction is employed. Some technical details of the paper can be found in the three Appendixes.

II. THE FAILURE OF THE GAUSSIAN EFFECTIVE GAUGE THEORY OF A $U(1)$ SPIN LIQUID

We start from the standard $U(1)$ gauge field formulation of a quantum antiferromagnet. For illustrative purposes, we consider the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg model on the triangular lattice,

$$H = 2J \sum_{\langle i,j\rangle} \vec{S}_i \cdot \vec{S}_j. \tag{1}$$

Here the sum $\sum_{\langle i,j\rangle}$ is over nearest-neighbor bonds. In real materials, additional terms are needed to stabilize the $U(1)$ spin liquid state. Such terms will not change the discussion that follows, and we will include them at a later time.

To introduce the gauge field formulation of the problem, we represent the spins in terms of the fermionic slave particles as

$$\vec{S}_i = \frac{1}{2}\sum_{\alpha,\beta} f^\dagger_{i,\alpha}\vec{\sigma}_{\alpha,\beta} f_{i,\beta}. \tag{2}$$

To preserve the spin algebra, the slave particle should satisfy the constraint of no double occupancy of the form

$$\sum_\alpha f^\dagger_{i,\alpha} f_{i,\alpha} = 1. \tag{3}$$

This representation has a built-in $U(1)$ gauge redundancy since the spin operator is unaffected when we perform a $U(1)$ gauge transformation of the form

$$f_{i,\alpha} \to e^{i\phi_i} f_{i,\alpha}, \tag{4}$$

where ϕ_i is an arbitrary $U(1)$ phase.

In terms of the slave particles, the Hamiltonian can be rewritten as

$$H = -J\sum_{\langle i,j\rangle} \hat{\chi}^\dagger_{i,j}\hat{\chi}_{i,j}, \tag{5}$$

with

$$\hat{\chi}_{i,j} = \sum_\alpha f^\dagger_{i,\alpha} f_{j,\alpha}. \tag{6}$$

After the standard Hubbard-Stratonovich transformation on $\hat{\chi}_{i,j}$ and assuming a uniform saddle point value χ for the magnitude of $\chi_{i,j}$ (namely, assuming $|\chi_{i,j}| = \chi$), which is believed to be gapped, the partition function of the system can be written as

$$Z = Z_0 \int \prod_{i,\mu,\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) Da^\mu_i(\tau) e^{-S}, \tag{7}$$

where

$$S = \int_0^\beta d\tau \left[\sum_{i,j,\alpha} f^\dagger_{i,\alpha}(\tau) G^{-1}_{i,j}(\tau) f_{j,\alpha}(\tau) - i\sum_i a^0_i(\tau)\right]. \tag{8}$$

Z_0 is an unimportant constant.

$$G^{-1}_{i,j}(\tau) = \left[\partial_\tau + i a^0_i(\tau)\right]\delta_{i,j} - J\chi e^{i a^\mu_i(\tau)} \tag{9}$$

is the inverse propagator of the slave particles in the presence of the auxiliary fields a^μ_i and a^0_i, which should be interpreted as the spatial and temporal components of a compact $U(1)$ gauge field. We note that $a^0_i(\tau)$ is a Lagrange multiplier introduced to enforce the no-double-occupancy constraint. The above form involves integration over a huge number of pure gauge degree of freedoms. We can fix the gauge for a^μ_i and rewrite the partition function as

$$Z = Z'_0 \int \prod_{i,\mathrm{x},\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) D\Phi_{\mathrm{x}}(\tau) e^{-S} \tag{10}$$

(Z'_0 differs from Z_0 by powers of the gauge volume). Here $\Phi_{\mathrm{x}}(\tau)$ is the gauge flux enclosed in a triangle centered at x and at the imaginary time τ. It is related to the scalar spin chirality on the triangle by

$$\sin\Phi_{\mathrm{x}} \propto \langle \hat{C}_{\mathrm{x}}\rangle = \langle \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)\rangle. \tag{11}$$

In the Gaussian approximation, we approximate $ia_i^0(\tau) = \lambda$. The fluctuation of $a_i^0(\tau)$ around λ is argued to be screened by the density response of the fermion system and is neglected at low energy [8]. We are then left with a free-fermion system coupled to the transverse $U(1)$ gauge field at low energy. When the fermion degree of freedom is integrated out, the transverse gauge field will acquire a relaxational dynamics with a dynamical exponent $z = 3$ at low energy as a result of its dissipative coupling to the spinon current. In two dimensions, such an ultraslow dynamics would imply a $T^{2/3}$ anomaly in the low-temperature specific heat [28].

However, the treatment of $a_i^0(\tau)$ outlined above is not justified from either a physical or a mathematical point of view. When the no-double-occupancy constraint is strictly enforced by the integration over $a_i^0(\tau)$, the spinon current should vanish identically. Thus, the coupling between the spinon current and the transverse gauge field is unphysical. At the same time, the projection to the subspace of no double occupancy is achieved by *destructive* interference between the contributions to Z from different gauge paths $a_i^0(\tau)$. One thus should not expect any single gauge path to dominate the partition function. To illustrate this point, we have calculated the contributions to Z from different gauge paths by discretizing the imaginary time into N_τ slices. We find such contributions are unbounded in magnitude and strongly fluctuating in phase. For example, the contribution from the gauge path $a_i^0(\tau) = N_\tau\phi/\beta$ to Z is found to be given exactly by

$$C = e^{iN_sN_\tau\phi}\prod_{\mathrm{k}}\left[1+\left(1+i\phi-\frac{\beta\epsilon_{\mathrm{k}}}{N_\tau}\right)^{N_\tau}\right]^2, \qquad (12)$$

where ϵ_{k} denotes the mean-field eigenvalue of the saddle point Hamiltonian and N_s is the number of lattice sites. In the large-N_τ limit(with ϕ kept finite), we find

$$C \simeq e^{iN_sN_\tau\phi}(1+i\phi)^{2N_\tau N_s}. \qquad (13)$$

Such a contribution obviously diverges in the large-N_τ limit(see Appendix B for more details on this point). The saddle point approximation in such unbounded contributions is thus meaningless.

III. THE DYNAMICS OF GAUGE FLUCTUATION ON A $U(1)$ SPIN LIQUID WITH A LARGE SPINON FERMI SURFACE

Anticipating the inadequacy of the Gaussian approximation, we now integrate out $a_i^0(\tau)$ exactly. This leaves us with an effective theory for the gauge flux $\Phi_{\mathrm{x}}(\tau)$, which takes the form of

$$Z = \int\prod_{\mathrm{x},\tau}D\Phi_{\mathrm{x}}(\tau)e^{-\tilde{S}[\Phi]}, \qquad (14)$$

where

$$e^{-\tilde{S}[\Phi]} = Z_0'\int\prod_{i,\tau,\alpha}Df_{i,\alpha}^\dagger(\tau)Df_{i,\alpha}(\tau)Da_i^0(\tau)e^{-S}. \qquad (15)$$

Thus, the effective action of the transverse gauge field is determined by the response of a projected fermion system. To make further progress, we apply the saddle point approximation to the *physical* gauge flux $\Phi_{\mathrm{x}}(\tau)$ and assume $\Phi_{\mathrm{x}}(\tau) = 0$ at the saddle point. This saddle point corresponds to the $U(1)$ spin liquid state with a large spinon Fermi surface. To study the fluctuation effect around such a saddle point, we expand $\tilde{S}[\Phi]$ around $\Phi_{\mathrm{x}}(\tau) = 0$ to the second order. The expansion reads

$$\tilde{S}[\Phi] \simeq \tilde{S}[0] + \int d\tau d\tau'\sum_{\mathrm{x},\mathrm{x}'}\Phi_{\mathrm{x}}(\tau)K_{\mathrm{x},\mathrm{x}'}(\tau,\tau')\Phi_{\mathrm{x}'}(\tau'). \qquad (16)$$

As will be shown below, to the lowest order in χ, what survives the exact integration over $a_i^0(\tau)$ is a linear coupling between the $U(1)$ gauge flux $\Phi_{\mathrm{x}}(\tau)$ and the scalar spin chirality. We thus have

$$K_{\mathrm{x},\mathrm{x}'}(\tau,\tau') \propto -\langle \mathrm{T}_\tau\hat{C}_{\mathrm{x}}(\tau)\hat{C}_{\mathrm{x}'}(\tau')\rangle, \qquad (17)$$

where

$$\hat{C}_{\mathrm{x}} = \vec{\mathrm{S}}_i\cdot(\vec{\mathrm{S}}_j\times\vec{\mathrm{S}}_k) = (P_{ijk}-P_{ikj})/4i \qquad (18)$$

is the scalar spin chirality on the triangle centered at x. i, j, and k are the three sites of the triangle, and

$$P_{i,j,k} = \sum_{\alpha,\beta,\gamma}(f_{i,\alpha}^\dagger f_{j,\alpha})(f_{j,\beta}^\dagger f_{k,\beta})(f_{k,\gamma}^\dagger f_{i,\gamma}) \qquad (19)$$

is the three-spin ring exchange operator on the triangle. This form is drastically different from that derived from the Gaussian effective theory, in which the gauge dynamics is determined by the current response of a free-fermion system.

To see this more clearly, we again discretize the imaginary time into N_τ slices and rewrite the fermion path integral representation of $e^{-\tilde{S}[\Phi]}$ in the form of a trace over a series of fermion Fock bases, which is given by

$$e^{-\tilde{S}[\Phi]} = Z_0'\,\mathrm{Tr}\prod_{i_\tau=1}^{N_\tau}\langle\{n_{i_\tau+1}\}|P_{\mathrm{G}}e^{-\Delta\tau H_{i_\tau}^\chi}P_{\mathrm{G}}|\{n_{i_\tau}\}\rangle. \qquad (20)$$

Here $|\{n_{i_\tau}\}\rangle$ denotes a fermion Fock basis at time $\tau = i_\tau\Delta\tau$, and Tr indicates summation over all possible fermion Fock bases $|\{n_{i_\tau}\}\rangle$ that satisfy the condition $|\{n_{N_{\tau+1}}\}\rangle = |\{n_1\}\rangle$. The integration over the Lagrange multiplier $a_i^0(\tau)$ has been replaced by the Gutzwiller projection P_{G} on the Fock bases. $H_{i_\tau}^\chi$ is given by

$$H_{i_\tau}^\chi = -J_2\chi\sum_{\langle i,j\rangle,\alpha}\left(e^{ia_i^\mu(\tau)}f_{i,\alpha}^\dagger f_{j,\alpha} + \mathrm{H.c.}\right). \qquad (21)$$

Here we emphasize that an effective theory for $\Phi_{\mathrm{x}}(\tau)$ is meaningful only when the magnitude of the Resonating Valence Bond order parameter $\chi_{i,j}$ becomes well defined, namely, only for excitation energy smaller than the characteristic energy of the fluctuation in $|\chi_{i,j}|$, which is of the order of J_2. We thus should have $J_2\Delta\tau \geqslant 1$.

We now seek a quadratic approximation for $\tilde{S}[\Phi]$ around the $U(1)$ spin liquid saddle point $\Phi_{\mathrm{x}}(\tau) = 0$. Denoting $Z[\Phi] = e^{-\tilde{S}[\Phi]}$, the kernel of the quadratic approximation for $\tilde{S}[\Phi]$ is given by

$$K_{\mathrm{x},\mathrm{x}'}(\tau,\tau') = -\frac{\delta^2\ln Z[\Phi]}{\delta\Phi_{\mathrm{x}}(\tau)\delta\Phi_{\mathrm{x}'}(\tau')}. \qquad (22)$$

To find the kernel K, we expand $e^{-\Delta\tau H^{\chi}_{i\tau}}$ in $\Phi_{\rm x}(\tau)$. The lowest-order term in the expansion is given by

$$H_1 = -J_2\Delta\tau\chi\sum_{i,\mu} a_i^{\mu}(\tau) j_i^{\mu}, \tag{23}$$

in which

$$j_i^{\mu} = -i\sum_{\alpha}(f^{\dagger}_{i,\alpha} f_{j,\alpha} - \text{H.c.}) \tag{24}$$

is the fermion current. However, such a term does not survive the Gutzwiller projection $P_{\rm G}$ in Eq. (20). We find that when the Gutzwiller projection is taken into account, to the lowest order in χ, the expansion of $P_{\rm G}e^{-\Delta\tau H^{\chi}_{i\tau}}P_{\rm G}$ in $\Phi_{\rm x}(\tau)$ is given by

$$H_3 = -\frac{2(J_2\Delta\tau\chi)^3}{3}\sum_{\rm x}\Phi_{\rm x}(\tau)\hat{C}_{\rm x}, \tag{25}$$

with the scalar spin chirality $\hat{C}_{\rm x}$ defined in Eq. (18). We note that while the second-order term in the expansion survives the Gutzwiller projection, it does not experience the gauge flux. This reasoning leads us to Eq. (17).

A computation of the full spectrum of $\hat{C}_{\rm x}$ for the projected fermion system is difficult. However, the center of gravity of the spectrum can be obtained easily from a sum rule analysis and is given *exactly* by

$$E_{\rm q} = \frac{1}{2}\frac{\langle G|[[\hat{C}_{\rm q}, H], \hat{C}^{\dagger}_{-\rm q}]|G\rangle}{\langle G|\hat{C}_{\rm q}\hat{C}^{\dagger}_{-\rm q}|G\rangle}, \tag{26}$$

where

$$\hat{C}_{\rm q} = \frac{1}{N}\sum e^{iq\cdot{\rm x}}\hat{C}_{\rm x} \tag{27}$$

is the density of scalar spin chirality at momentum q and $|G\rangle$ is the ground state of the system in the saddle point approximation, which is nothing but the Gutzwiller projected Fermi sea state. We note that with $E_{\rm q}$ we can already judge the validity of the Gaussian effective theory, which predicts that the characteristic energy for long-wavelength gauge fluctuation should vanish like q^3. If the Gaussian theory is indeed valid, one should expect $E_{\rm q}$ to vanish in the same way. More generally, in the Gaussian effective theory $E_{\rm q}$ should always vanish in the $q \to 0$ limit as a result of the $U(1)$ gauge symmetry of the Gaussian effective action.

To check this point, we calculate $E_{\rm q}$ for the projected Fermi sea state on the triangular lattice assuming the following Hamiltonian:

$$H = J_2\sum_{\langle i,j\rangle}P_{ij} + J_4\sum_{[i,j,k,l]}(P_{ijkl} + P_{ilkj}). \tag{28}$$

Here $P_{ij} = 2\vec{\rm S}_i\cdot\vec{\rm S}_j + 1/2$ is the Heisenberg exchange coupling. P_{ijkl} is the four-spin ring exchange around a rhombus $[i, j, k, l]$. $\sum_{[i,j,k,l]}$ denotes the sum over all elementary rhombi of the triangular lattice. As found by Motrunich [13], when $J_4 \geqslant 0.3J_2$, the projected Fermi sea state is the best variational state of the model. Here we set $J_4 = 0.3J_2$.

The result of $E_{\rm q}$ is shown in Fig. 1. In stark contrast to the prediction of the Gaussian effective theory, $E_{\rm q}$ is found to be strongly gapped throughout the Brillouin zone. This result can be understood from an inspection of the structure factor of the scalar spin chirality, which is shown in Fig. 2. We find that the correlation of $\hat{C}_{\rm x}$ in real space is extremely short range and the corresponding structure factor is almost featureless around $q = 0$. We note that the short-range nature of the correlation in $\hat{C}_{\rm x}$ was also mentioned by Motrunich [13].

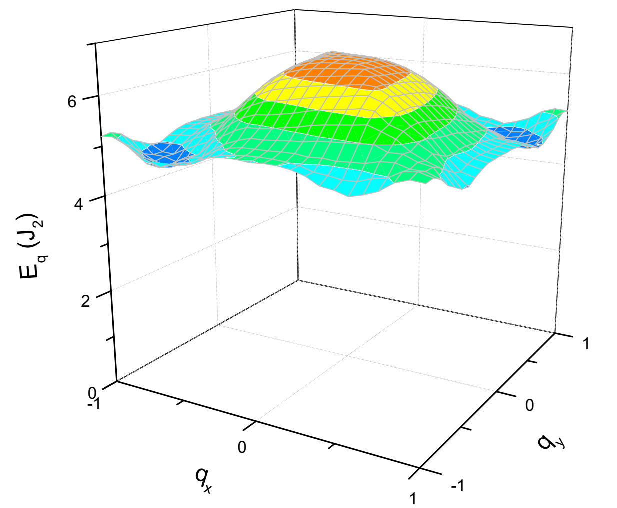

FIG. 1. The dispersion of $E_{\rm q}$ in the projected Fermi sea state on the triangular lattice. Shown here is the result for the acoustic mode in which the scalar spin chiralities in the up and down triangles fluctuate in phase. We have adopted the convention $\vec{\rm q} = q_x\vec{\rm G}_1/2 + q_y\vec{\rm G}_2/2$ for momentum, where $\vec{\rm G}_{1,2}$ are the two reciprocal vectors of the triangular lattice.

A nonzero $E_{\rm q}$ does not necessarily imply a gapped gauge fluctuation spectrum. In Appendix C, we present a mean-field analysis of the fluctuation spectrum of the scalar spin chirality operator $\hat{C}_{\rm x}$ in the $U(1)$ spin liquid state. It is found that the scalar spin chirality operator can excite either one, two, or, at most, three pairs of particle-hole excitations on the spinon Fermi sea, whose spectral weights vanish as ω, ω^3, and ω^5 at low energy. However, as is shown there, such local excitations

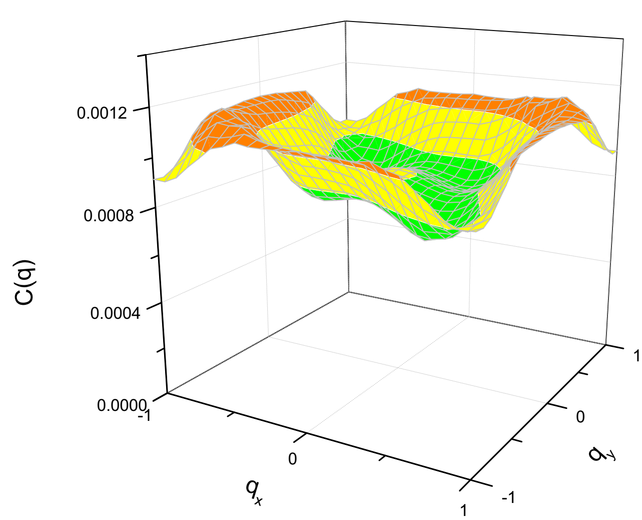

FIG. 2. The structure factor of the scalar spin chirality in the projected Fermi sea state on the triangular lattice. Shown here is the result for the acoustic mode.

can contribute, at most, a T^2 correction to the specific heat at low temperature. The $T^{2/3}$ specific heat anomaly predicted by the Gaussian effective theory is absent.

IV. A UNIFIED PICTURE OF SPIN FRACTIONALIZATION IN 1D AND 2D SPIN LIQUIDS

The above result implies that the Gaussian effective theory for the $U(1)$ spin liquid is invalid and the physical spinon cannot be understood as a deconfined slave particle. A natural question is then how the two are related. After all, they share the same Fermi surface in this $U(1)$ spin liquid state. This question was addressed by Mudry and Fradkin more than two decades ago [34,35]. They argued that, at least in one dimension, the two are fundamentally different objects since the physical spinon is then a topological object that corresponds to an antiphase domain wall in the spin correlation pattern, while the slave particle is a local object. Here we show that while the physical spinon should not be understood perturbatively as a dressed slave particle as we do in the Gaussian approximation, the Gutzwiller projection will transform the slave particle into a nonlocal object that corresponds just to such an antiphase domain wall. We note that it is not our purpose here to prove the generally accepted opinion that a physical spinon should be understood as a nonlocal object, but rather to show how the Gutzwiller projection will transform the slave particle into such a nonlocal object.

The Gutzwiller projected Fermi sea state, namely,

$$|G\rangle = P_{\rm G}\prod_{|\mathbf{k}|<\mathbf{k}_{\rm F}} f^{\dagger}_{\mathbf{k},\uparrow}f^{\dagger}_{\mathbf{k},\downarrow}|0\rangle = P_{\rm G}|{\rm FS}\rangle, \tag{29}$$

is known to be a very accurate description of the ground state of the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain model [36]. For example, the relative error in the ground state energy calculated from $P_{\rm G}|{\rm FS}\rangle$ is smaller than 0.2%. In fact, one should not be surprised by such accuracy from the gauge field theory perspective since the only gauge field component in the case, $a^0_i(\tau)$, has been exactly integrated out through Gutzwiller projection (we note that the fluctuation in the amplitude of the bond variable $|\chi_{i,j}|$, which is believed to be unimportant for long-wavelength physics, is still only treated at the saddle point level).

On a 1D ring with $N = 4l + 2$ sites and with the periodic boundary condition (the boundary condition is so chosen to guarantee a closed-shell structure at half filling), the wave function of $|{\rm FS}\rangle$ is given by

$$\psi_{\rm FS}(\{i_m, j_n\}) = \psi_s \prod_{m<m'}(Z_{i_m} - Z_{i_{m'}})\prod_{n<n'}(Z_{j_n} - Z_{j_{n'}}), \tag{30}$$

where $\{i_m, j_n\}$ indicates the sets of coordinates for the up and down spin electrons, $Z_{i_m} = \exp(\frac{i2\pi i_m}{N})$ is the chord coordinate of a lattice site on the ring [37] (see Fig. 3 for an illustration), and $\psi_s = (\prod_{m,n} Z^*_{i_m} Z^*_{j_n})^l$. For this wave function, it can be shown that the change in phase when we exchange an up spin electron at site i_1 and a down spin electron at site j_1 is given by $N_c\pi$, where N_c is the total electron number between site i_1 and site j_1 [38]. When $|{\rm FS}\rangle$ is projected to the subspace of no double occupancy, this phase structure reproduces the Marshall sign rule structure of the antiferromagnetic Heisenberg chain [39,40]. More specifically, the change in the phase of

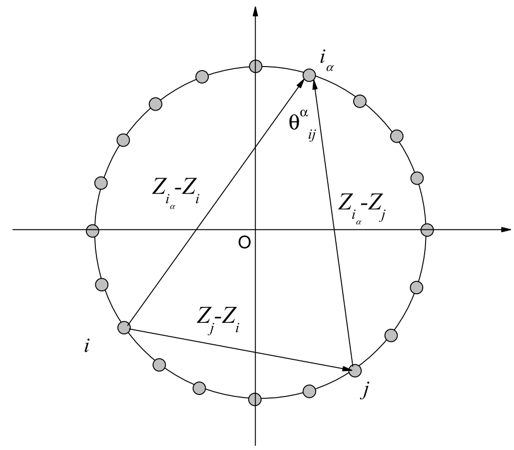

FIG. 3. The chord coordinate on a ring and the meaning of $\theta^{\alpha}_{i,j}$.

the wave function during such an exchange process is given by

$$\Delta\Phi = \arg\left[\prod_{\alpha>1}\frac{Z_{i_\alpha} - Z_{j_1}}{Z_{i_\alpha} - Z_{i_1}}\prod_{l>1}\frac{Z_{j_l} - Z_{i_1}}{Z_{j_l} - Z_{j_1}}\right]. \tag{31}$$

Since $|Z_{i_m}| = 1$, the chord coordinates are complex numbers living on a unit circle. Then

$$\theta^{\alpha}_{i_1,j_1} = \arg\left[\frac{Z_{i_\alpha} - Z_{j_1}}{Z_{i_\alpha} - Z_{i_1}}\right] \tag{32}$$

is nothing but the angle in the segment $Z_{i_1} - Z_{j_1}$ in the unit circle (see Fig. 3 for an illustration). Noting the fact that in a circle the angles in the same segment equal one another and the sum of the opposite angles of quadrilaterals equals π, we easily find that $\Delta\Phi = N_c\pi$, where N_c denotes the number of electrons between site i_1 and site j_1. Taking into account the sign due to fermion exchange, we find the change in the phase of the wave function is in accordance with the Marshall sign rule, which claims that the phase of the wave function should change by π if we exchange two spins in different sublattices.

We now excite a pair of spinons in the ground state. Since the ground state of the system is constructed by Gutzwiller projection of the mean-field ground state, one would naturally expect that Gutzwiller projection of the mean-field excited state to provide a reasonable description of the physical excited state. Such logic has been followed successfully by many groups in the literature [3,41–55]. Following this logic, the variational state for a pair of spinons excited at sites i and j should have the form

$$|i, j\rangle = P_{\rm G} f^{\dagger}_{i,\uparrow} f_{j,\downarrow}|{\rm FS}\rangle. \tag{33}$$

As a result of the Gutzwiller projection, the wave function of $|i, j\rangle$ in the Fock basis is given by the amplitude in $|{\rm FS}\rangle$ with site i empty, site j doubly occupied, and all other sites singly occupied. In other words, a spinon acts effectively as an impurity that generates either one more or one less available state than the singly occupied background. According to the phase structure we proved for $\psi_{\rm FS}$, spin exchange across site i or site j (but not both) in the spin chain would pick up an additional phase shift of π. This π phase shift corresponds just to an antiphase domain wall in the spin chain.

To extend this reasoning to two dimensions, we note that the above π phase shift can actually be understood as the manifestation of the Friedel sum rule in one dimension [56], which claims that with the appearance of each additional available fermion state within a 1D region, the phase of the scattering amplitude across the region will change by π. In two dimensions, the Friedel sum rule equates the scattering phase shift on the Fermi surface with π times the number of additional fermion states generated by the impurity potential below the Fermi energy. Thus, each spinon will contribute a phase shift of π on the spinon Fermi surface and exert nonlocal influence on the surrounding spin state. More specifically, according to Anderson's orthogonality theorem [57–59], we expect the spin state surrounding a spinon to be orthogonal to the ground state in the thermodynamic limit. This can be checked by computing the overlap between the two states. Since spinons can be excited only in pairs whose total contribution to the phase shift on the Fermi surface is zero, we expect the overlap to vanish only when the separation between the two spinons is infinite. This is what we call the orthogonality catastrophe upon the excitation of a pair of spinons.

Now we calculate such an overlap. We first rewrite the state with a pair of spinons excited at sites i and j more explicitly as

$$|i,j\rangle = f^{\dagger}_{i,\uparrow} f_{j,\downarrow} P_0^i P_2^j \prod_{i'\neq i,j} P_G^{i'}|\mathrm{FS}\rangle = f^{\dagger}_{i,\uparrow} f_{j,\downarrow} P_0^i P_2^j |\mathrm{FS}'\rangle, \quad (34)$$

where P_0^i, P_2^i, $P_\uparrow^i$, and $P_\downarrow^i$ are the projection operators into the subspace of the empty, doubly occupied, up spin, and down spin states on site i. P_G^i is the Gutzwiller projection operator on site i:

$$|\mathrm{FS}'\rangle = \prod_{i'\neq i,j} P_G^{i'}|\mathrm{FS}\rangle. \quad (35)$$

We note that what concerns us here is the overlap between the spin state surrounding sites i and j in $|i,j\rangle$ and $P_G|\mathrm{FS}\rangle$, rather than $|i,j\rangle$ and $P_G|\mathrm{FS}\rangle$ themselves. As a result of the conservation of total S^z, only two components in $P_G|\mathrm{FS}\rangle$, namely, $|\uparrow,\downarrow\rangle = P_\uparrow^i P_\downarrow^j|\mathrm{FS}'\rangle$ and $|\downarrow,\uparrow\rangle = P_\downarrow^i P_\uparrow^j|\mathrm{FS}'\rangle$, can have a nonzero overlap with $|i,j\rangle$ in the region surrounding sites i and j. Using inversion symmetry of $P_G|\mathrm{FS}\rangle$, it is easy to show that these two components generate the same spin state surrounding sites i and j. The overlap that we are looking for can thus be expressed as

$$O(i,j) = \frac{\langle\uparrow,\downarrow|\, f_{j,\uparrow} f^{\dagger}_{j,\downarrow}|i,j\rangle}{\sqrt{\langle\uparrow,\downarrow|\uparrow,\downarrow\rangle}\sqrt{\langle i,j|i,j\rangle}} = \frac{\langle \mathrm{FS}'|\, P_\uparrow^i P_\downarrow^j\, f^{\dagger}_{i,\uparrow} f_{j,\uparrow}\, P_0^i P_2^j|\mathrm{FS}'\rangle}{\sqrt{\langle \mathrm{FS}'|P_\uparrow^i P_\downarrow^j|\mathrm{FS}'\rangle}\sqrt{\langle \mathrm{FS}'|P_0^i P_2^j|\mathrm{FS}'\rangle}}. \quad (36)$$

Using the identities $f^{\dagger}_{i,\uparrow}P_0^i = P_G{}^i f^{\dagger}_{i,\uparrow}$ and $f_{j,\uparrow}P_2^j = P_G{}^i f_{j,\uparrow}$ and the conservation of total S^z, the numerator in Eq. (36) can be simplified to

$$\langle \mathrm{FS}|P_G f^{\dagger}_{i,\uparrow} f_{j,\uparrow}|\mathrm{FS}\rangle. \quad (37)$$

Using the translational symmetry of the system, it reduces further to

$$G(i,j)\times\langle \mathrm{FS}|P_G|\mathrm{FS}\rangle. \quad (38)$$

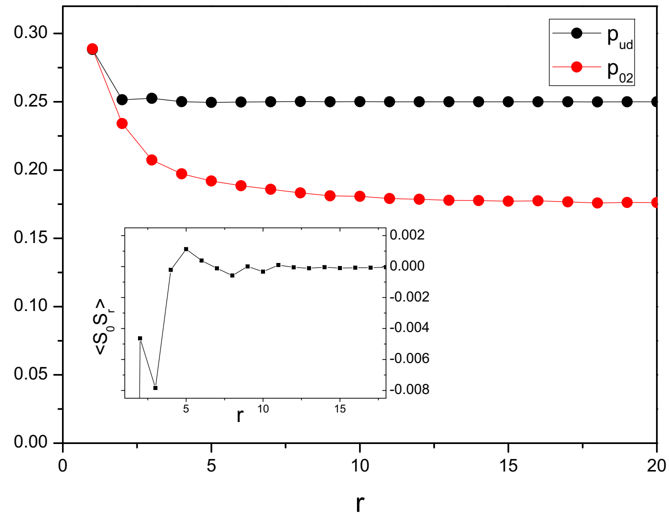

FIG. 4. The behavior of the function $p_{0,2}$ and $p_{\uparrow,\downarrow}$. The calculation is done on a 36×36 lattice. The inset shows the $2k_F$ oscillation in $p_{\uparrow,\downarrow}$ on a magnified scale.

Here

$$G(i,j) = \sum_{|\mathbf{k}|<\mathbf{k}_F} e^{ik\cdot(R_i-R_j)} \quad (39)$$

is the correlator of the free fermion, which satisfies

$$G(i,j) \sim \frac{1}{|R_i-R_j|^2} \quad (40)$$

in the large-distance limit in two dimensions.

Thus, the overlap we are seeking can be expressed as

$$O(i,j) = \frac{G(i,j)}{\sqrt{p_{0,2}\, p_{\uparrow,\downarrow}}}, \quad (41)$$

where

$$p_{0,2} = \frac{\langle \mathrm{FS}'|\, P_0^i P_2^j|\mathrm{FS}'\rangle}{\langle \mathrm{FS}|\, P_G|\mathrm{FS}\rangle}, \quad p_{\uparrow,\downarrow} = \frac{\langle \mathrm{FS}'|\, P_\uparrow^i P_\downarrow^j|\mathrm{FS}'\rangle}{\langle \mathrm{FS}|\, P_G|\mathrm{FS}\rangle}. \quad (42)$$

Since the spin correlation approaches zero in $P_G|\mathrm{FS}\rangle$ in the large-distance limit, $p_{\uparrow,\downarrow}$ should approach $\frac{1}{4}$ in the same limit. What is less obvious is the long-range behavior of $p_{0,2}$. At the mean-field level, we find $p_{0,2} = p_{\uparrow,\downarrow} = \frac{1}{4} + G(i,j)$, and both approach $\frac{1}{4}$ in the large-distance limit. To go beyond the mean-field treatment, we have calculated $p_{0,2}$ and $p_{\uparrow,\downarrow}$ by the variational Monte Carlo method. The result is shown in Fig. 4. One finds both $p_{0,2}$ and $p_{\uparrow,\downarrow}$ approach a finite (but now different) value in the large-distance limit. Thus, the overlap we are seeking is proportional to $G(i,j)$ and will vanish as $|R_i-R_j|^{-2}$ in the large-distance limit. This proves the claimed orthogonality catastrophe upon spinon excitation in the $U(1)$ spin liquid state. We note that according to our construction, $p_{0,2}$ can actually be interpreted as the probability to separate a pair of spinons to the distance $|R_i-R_j|$. A nonvanishing value of $p_{0,2}$ in the large-distance limit is thus consistent with the existence of free spinons.

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that the Gaussian approximation of the gauge fluctuation is, in general, invalid in effective gauge theories of spin liquids based on slave particle construction. In particular, we found that the $U(1)$ spin liquid state with a large spinon Fermi surface on the triangular lattice is robust and the fluctuation in the transverse gauge field on this state features a large characteristic energy throughout the Brillouin zone. The $T^{2/3}$ anomaly in the specific heat predicted by Gaussian effective theories simply does not exist. We also found that projective construction based on the slave particle representation provides a unified understanding of the mechanism of spin fractionalization in 1D and 2D spin liquids and of the nonlocal nature of a physical spinon.

Since effective gauge theory based on slave particle construction is so widely used in the study of strongly correlated electron systems, there are many extensions of the current work to other problems. In a recent work [60], we showed that a bosonic spin liquid state can emerge continuously from a collinear Néel ordered phase and can be locally stable with respect to gauge fluctuation [61]. Other possible extensions of our work range from the fate of the Dirac spin liquid state with an internal $U(1)$ gauge symmetry [62] and the nature of a half-filled Landau level system [63] to the origin of the non-Fermi liquid behavior in the normal state of optimally doped cuprates [8].

ACKNOWLEDGMENTS

The author acknowledges the support from NSFC Grant No. 11674391, National Basic Research Project No. 2016YFA0300504, and the research fund of Renmin University of China.

APPENDIX A: DERIVATION OF THE $U(1)$ EFFECTIVE GAUGE THEORY OF THE SPIN-$\frac{1}{2}$ QUANTUM ANTIFERROMAGNET ON THE TRIANGULAR LATTICE

We now derive an effective gauge field theory for the model introduced in the main text, which is given by [13]

$$H = J_2 \sum_{\langle i,j\rangle} P_{ij} + J_4 \sum_{[i,j,k,l]} (P_{ijkl} + P_{ilkj}). \tag{A1}$$

Here $P_{ij} = 2\vec{S}_i \cdot \vec{S}_j + 1/2$ is the Heisenberg exchange coupling, and P_{ijkl} is the four-spin ring exchange around a rhombus. $\sum_{[i,j,k,l]}$ denotes the sum over all elementary rhombi on the triangular lattice. In terms of the slave particles, the Hamiltonian can be written as

$$\begin{aligned} H = {} & J_2 \sum_{\langle i,j\rangle} (f^\dagger_{i,\alpha} f_{i,\beta})(f^\dagger_{j,\beta} f_{j,\alpha}) \\ & + J_4 \sum_{[i,j,k,l]} [(f^\dagger_{i,\alpha} f_{i,\beta})(f^\dagger_{j,\beta} f_{j,\gamma})(f^\dagger_{k,\gamma} f_{k,\delta})(f^\dagger_{l,\delta} f_{l,\alpha}) + \text{H.c.}]. \end{aligned} \tag{A2}$$

Here and in the following, summation over repeated indices is assumed. The slave particles should be subjected to the no-double-occupancy constraint to be a faithful representation of the spin algebra.

In the coherent state path integral formulation, the partition function of the system can be written as

$$Z = \int \prod_{i,\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) e^{-S}, \tag{A3}$$

in which the action S is given by

$$\begin{aligned} S = \int_0^\beta d\tau \{ & f^\dagger_{i,\alpha}(\tau)\partial_\tau f_{i,\alpha}(\tau) \\ & + H + i a^0_i(\tau)[f^\dagger_{i,\alpha}(\tau) f_{i,\alpha}(\tau) - 1]\}. \end{aligned} \tag{A4}$$

Here $a^0_i(\tau)$ is a Lagrange multiplier introduced to enforce the no-double-occupancy constraint.

We define the bond variable $\hat{\chi}_{i,j} = f^\dagger_{i,\alpha} f_{j,\alpha}$ and decouple the Heisenberg exchange term with the standard Hubbard-Stratonovich transformation on $\hat{\chi}_{i,j}$. The partition function after the transformation reads

$$Z = \int \prod_{i,\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) D\chi_{i,j}(\tau) e^{-S}, \tag{A5}$$

in which the action S is given by

$$\begin{aligned} S = \int_0^\beta d\tau \Big[& f^\dagger_{i,\alpha}(\tau) G^{-1}_{i,j}(\tau) f_{j,\alpha}(\tau) \\ & + H_4 - i \sum_i a^0_i(\tau) - J_2 |\chi_{i,j}(\tau)|^2 \Big]. \end{aligned} \tag{A6}$$

Here

$$G^{-1}_{i,j}(\tau) = [\partial_\tau + i a^0_i(\tau)]\delta_{i,j} - J_2 \chi_{i,j}(\tau) \tag{A7}$$

is the inverse propagator of the slave particle in the presence of the auxiliary field $\chi_{i,j}(\tau)$ and $a^0_i(\tau)$, and H_4 is the four-spin exchange term that is left untouched. In the $U(1)$ spin liquid state, we can assume that the fluctuation in the amplitude of $\chi_{i,j}$ is gapped and can be neglected in low-energy physics. It is then reasonable to assume $\chi_{i,j} \simeq \chi e^{i a^\mu_i}$, where χ is a constant and a^μ_i is the phase of $\chi_{i,j}$. We thus have

$$Z = Z_0 \int \prod_{i,\mu,\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) Da^\mu_i(\tau) e^{-S}, \tag{A8}$$

where

$$S = \int_0^\beta d\tau \Big[f^\dagger_{i,\alpha}(\tau) G^{-1}_{i,j}(\tau) f_{j,\alpha}(\tau) + H_4 - i \sum_i a^0_i(\tau) \Big]. \tag{A9}$$

The above form involves integration over a huge number of pure gauge degrees of freedom. We can fix the gauge for the transverse gauge field and rewrite the partition function as

$$Z = Z_0' \int \prod_{i,\mathrm{x},\tau,\alpha} Df^\dagger_{i,\alpha}(\tau) Df_{i,\alpha}(\tau) Da^0_i(\tau) D\Phi_\mathrm{x}(\tau) e^{-S}, \tag{A10}$$

where Φ_x is the $U(1)$ gauge flux enclosed in a triangle centered at x. According to a well-known identity [64], Φ_x is related to the expectation value of the scalar spin chirality $\hat{C}_\mathrm{x} = \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$ on a triangle by

$$\sin \Phi_\mathrm{x} \propto \langle\, \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)\,\rangle, \tag{A11}$$

where i, j, and k are the three sites of the triangle.

APPENDIX B: FAILURE OF THE SADDLE POINT APPROXIMATION ON $a_i^0(\tau)$

To begin, we first demonstrate the failure of the saddle point approximation on $a_i^0(\tau)$ for a two-site toy model of the form $H = 2\vec{S}_1 \cdot \vec{S}_2$. Following the general rule outlined above, we find the action of the system is given by

$$S = \int_0^\beta d\tau \{ f_{1,\alpha}^\dagger(\tau)[\partial_\tau + ia_1^0(\tau)] f_{1,\alpha}(\tau) + f_{2,\alpha}^\dagger(\tau)[\partial_\tau + ia_2^0(\tau)] f_{2,\alpha}(\tau) - \chi[e^{ia^1(\tau)} f_{1,\alpha}^\dagger(\tau) f_{2,\alpha}(\tau) + \text{H.c.}] - ia_1^0(\tau) - ia_2^0(\tau)\}. \quad \text{(B1)}$$

In the saddle point approximation, $ia_i^0(\tau)$ plays the role of a chemical potential. As a result of the particle-hole symmetry of the action, the chemical potential is always zero at half filling. For our toy model, the spatial component of the gauge field $a^1(\tau)$ can be gauged away [note that this is also true for a 1D spin chain with an open boundary, for which $a_i^0(\tau)$ is the only gauge field component that we need to consider]. Thus, the partition function of the toy model in the saddle point approximation is simply that of a two-level free-fermion system with eigenvalues χ and $-\chi$.

Now we discretize the imaginary time into N_τ segments and calculate the contributions to the partition function from different gauge paths $a_i^0(\tau)$. For illustrative purposes, we will calculate the contributions to Z from gauge paths of the form $a_i^0(\tau) = z(i,\tau)N_\tau\pi/\beta$, where $z(i,\tau) = 0$ or 1 is a random integer defined on the sites of the space-time lattice. The reason to choose such a special form can be understood as follows. As a result of the Pauli principle, the total number of fermions on a given site can only be 0, 1, or 2. Thus, the projection into the singly occupied subspace can also be achieved by a discrete sum over all possible $z(i,\tau)$ configurations, rather than by an integration over the continuous Lagrange multiplier $a_i^0(\tau)$.

The contribution of a given gauge path $a_i^0(\tau)$ to the partition function is given by [65]

$$C[a_i^0(\tau)] = \eta[\text{Det}S]^2, \quad \text{(B2)}$$

where

$$S = \begin{pmatrix} S_1 & S_\chi \\ S_\chi & S_2 \end{pmatrix} \quad \text{(B3)}$$

is a $2N_\tau \times 2N_\tau$ matrix and $\eta = \pm 1$ is a sign determined by the parity of the sum $A = \sum_{i,\tau} z(i,\tau)$. The submatrices S_1, S_2, and S_χ are given by

$$S_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{i,1} \\ -a_{i,2} & 1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -a_{i,N_\tau} & 1 \end{pmatrix}$$

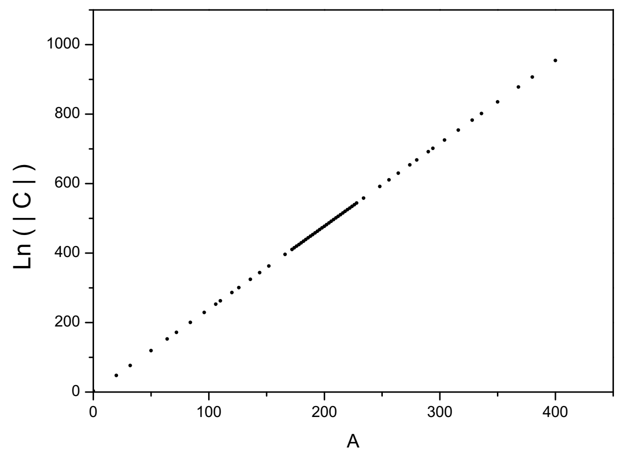

FIG. 5. The dependence of $\ln(|C|)$ on the sum $A = \sum_{i,i_\tau} z(i,\tau)$ for 1000 randomly chosen gauge paths. Here we set $N_\tau = 200$, $\beta = 1$, $\chi = 1$.

for $i = 1, 2$ and

$$S_\chi = \frac{-\beta\chi}{N_\tau} \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

where $a_{i,i_\tau} = 1 + iz(i,\tau)\pi$.

When $z(i,\tau) = 0$, we should recover the contribution to the partition function from the saddle point, which is given by

$$C[z(i,\tau) = 0] = 4[1 + \cosh(\beta\chi)]^2. \quad \text{(B4)}$$

For $N_\tau = 200$, we find the truncation error in $C[z(i,\tau)]$ is about 5×10^{-3} at $\beta\chi = 1$. For a random gauge path $z(i,\tau)$, we find the contribution to the partition function is strongly fluctuating in phase and unbounded in magnitude. In fact, we find that the amplitude of such contributions increases almost exponentially with the sum $A = \sum_{i,\tau} z(i,\tau)$, as is illustrated in Fig. 5 for 1000 randomly chosen gauge paths. The maximum of $|C[z(i,\tau)]|$ is found to be achieved at $z(i,\tau) = 1$, which is more than 400 orders of magnitude larger than the saddle point contribution for $N_\tau = 200$. In fact, we can show that the contribution from this gauge path is given exactly by

$$C = (-1)^A \left[1 + \left(1 + \frac{\beta\chi}{N_\tau} + i\pi\right)^{N_\tau}\right]^2 \times \left[1 + \left(1 - \frac{\beta\chi}{N_\tau} + i\pi\right)^{N_\tau}\right]^2, \quad \text{(B5)}$$

which diverges as $(1 + i\pi)^{4N_\tau}$ for large N_τ. More generally, we note that in the $N_\tau \to \infty$ limit, the details in S_χ becomes immaterial to the value of the determinant Det$S$, which can then be approximated by

$$\text{Det}\, S \simeq \prod_{i=1,2} \left\{1 + \prod_{i_\tau=1,N_\tau} [1 + iz(i,\tau)\pi]\right\}. \quad \text{(B6)}$$

This explains the approximate exponential increase of $|\text{Det}S|$ with A shown in Fig. 5.

The same reasoning can easily be extended to the case of a general lattice model. For example, the contribution from the gauge path $a_i^0(\tau) = N_\tau\phi/\beta$ to Z is found to be given exactly by

$$C = e^{iN_sN_\tau\phi}\prod_{\mathrm{k}}\left[1+\left(1+i\phi-\frac{\beta\epsilon_{\mathrm{k}}}{N_\tau}\right)^{N_\tau}\right]^2, \tag{B7}$$

where ϵ_{k} denotes the mean-field eigenvalue of the saddle point Hamiltonian and N_s is the number of lattice sites. In the large-N_τ limit, we find $C \simeq e^{iN_sN_\tau\phi}(1+i\phi)^{2N_\tau N_s}$. Such a contribution also diverges in the large-N_τ limit. More generally, for an arbitrary gauge path $a_i^0(\tau) = N_\tau\phi_i(\tau)/\beta$, the details of the Hamiltonian are again immaterial if $\phi_i(\tau)$ remain finite in the $N_\tau \to \infty$ limit. We thus find

$$C\left[a_i^0(\tau)\right] \simeq e^{i\sum_{i,i_\tau}\phi_i(\tau)}\prod_i\left\{1+\prod_{i_\tau=1,N_\tau}[1+i\phi_i(\tau)]\right\}^2. \tag{B8}$$

This is obviously unbounded in magnitude and strongly fluctuating in phase. The saddle point approximation on such contributions is meaningless.

In the $N_\tau \to \infty$ limit, a gauge path with finite $\phi_i(\tau)$ is singular. Such singular gauge field configurations are related (but not equivalent) to instantons of the $U(1)$ gauge field. For example, a gauge path of the form $a_i^0(\tau) = \frac{2\pi N_\tau}{\beta}\delta(\tau-\tau_0)\theta(y-y_0)$ corresponds to a Dirac string of strength 2π running in the x direction, which can be understood as the remnant of a pair of oppositely charged instantons when they are annihilated after traversing the x circumference of the system once.

APPENDIX C: A MEAN-FIELD ANALYSIS OF THE FLUCTUATION SPECTRUM OF THE SCALAR SPIN CHIRALITY OPERATOR IN THE $U(1)$ SPIN LIQUID STATE

Unlike the current fluctuation in a free-fermion system, the fluctuation in the scalar spin chirality has a nonvanishing characteristic energy in the $q \to 0$ limit. To illustrate this point, we have calculated the spectral function of $\hat{C}_{\mathrm{x}}$ in the $U(1)$ spin liquid state at the mean-field level. In general, the scalar spin chirality operator $\hat{C}_{\mathrm{x}}$ can excite, at most, three pairs of particle-hole pairs on the Fermi sea state. This can be seen more directly by rewriting $\hat{C}_{\mathrm{x}}$ as the sum of normal-ordered operators with respect to the Fermi sea state. The expansion is given by

$$\hat{C}_{\mathrm{x}} =: \hat{C}_{\mathrm{x}}^{(1)} : + : \hat{C}_{\mathrm{x}}^{(2)} : + : \hat{C}_{\mathrm{x}}^{(3)} :, \tag{C1}$$

where

$$:\hat{C}_{\mathrm{x}}^{(1)} := \frac{3\chi^2}{16i} : (\hat{\chi}_{i,j} + \hat{\chi}_{j,k} + \hat{\chi}_{k,i} - \mathrm{H.c.}) : \tag{C2}$$

is proportional to the sum of the fermion current around the triangle in the counterclockwise manner.

$$\begin{aligned} :\hat{C}_{\mathrm{x}}^{(2)}: &= \frac{\chi}{4i} : (\hat{\chi}_{i,j}\hat{\chi}_{k,i} + \hat{\chi}_{j,k}\hat{\chi}_{i,j} + \hat{\chi}_{k,i}\hat{\chi}_{j,k} - \mathrm{H.c.}) : \\ &\quad - \frac{\chi}{8i} : (\hat{\chi}_{i,i}\hat{\chi}_{j,k} + \hat{\chi}_{j,j}\hat{\chi}_{k,i} + \hat{\chi}_{k,k}\hat{\chi}_{i,j} - \mathrm{H.c.}) :, \end{aligned} \tag{C3}$$

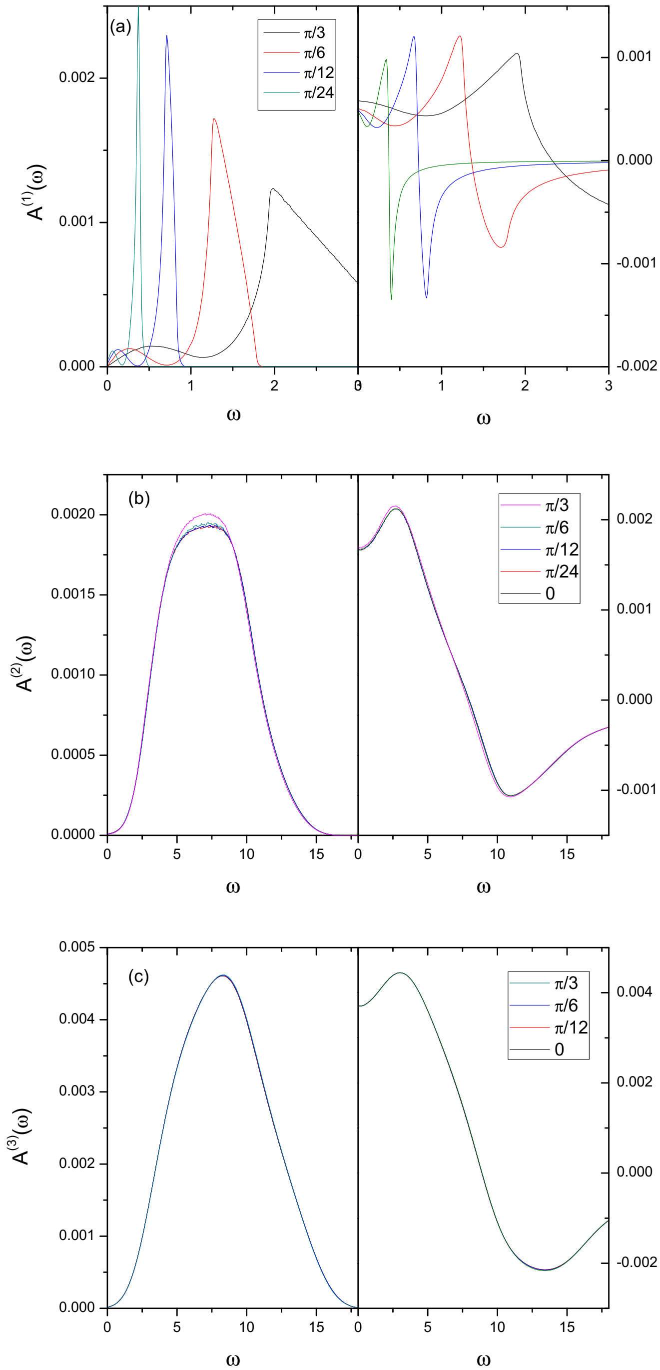

FIG. 6. The spectral weight (left panel) and the real part (right panel) of the response function of $\hat{C}_{\mathrm{x}}$ on the Fermi sea state from the excitation of (a) one, (b) two, and (c) three pairs of particle-hole pairs. Shown here is the result for scalar spin chirality on the up triangles. We set the hopping integral of the fermion between neighboring sites as the unit of energy and adopted the convention $\vec{\mathrm{q}} = q_x\vec{\mathrm{G}}_1/2 + q_y\vec{\mathrm{G}}_2/2$ for momentum. Here $\vec{\mathrm{G}}_{1,2}$ are the two reciprocal vectors of the triangular lattice. The momentum is chosen to be $q = (q_x, 0)$, with $q_x = 0, \pi/24, \pi/12, \pi/6$, and $\pi/3$. The calculation of $A^{(1)}(\omega)$ is done in the thermodynamic limit. The calculation of $A^{(2)}(\omega)$ is done on a 48×48 lattice. The calculation of $A^{(3)}(\omega)$ is done on a 24×24 lattice, and $q_x = \pi/24$ is inaccessible in this case.

where $\hat{\chi}_{i,i} = \sum_\alpha f^\dagger_{i,\alpha} f_{i,\alpha}$ is the particle number operator on site i.

$$: \hat{C}_x^{(3)} := \frac{1}{4i} : (\hat{\chi}_{i,j}\hat{\chi}_{j,k}\hat{\chi}_{k,i} - \text{H.c.}) : . \quad \text{(C4)}$$

These terms excite, respectively, one, two, and three pairs of particle-hole pairs on the Fermi sea state. A simple phase space argument indicates that the spectral weights corresponding to $: \hat{C}_x^{(1)} :$, $: \hat{C}_x^{(2)} :$, and $: \hat{C}_x^{(3)} :$ should vanish as ω, ω^3, and ω^5 at low energy. In particular, the spectral weight corresponding to $: \hat{C}_x^{(1)} :$ should be proportional to $\frac{\omega}{v_F q}$ at low energy and should have an upper cutoff at $v_F q$ in the long-wavelength limit as a result of the Pauli principle. Here v_F is the Fermi velocity on the Fermi surface. On the other hand, the excitations corresponding to $: \hat{C}_x^{(2)} :$ and $: \hat{C}_x^{(3)} :$ do not suffer from as strong a phase space limitation. Their spectral weights can thus extend to large energy even at $q = 0$ and should depend only weakly on q. These arguments are illustrated in Fig. 6, where we plot the spectral weight and the corresponding real part of the response function for $: \hat{C}_x^{(1)} :$, $: \hat{C}_x^{(2)} :$, and $: \hat{C}_x^{(3)} :$ separately. From Fig. 6 we see in the long-wavelength limit the main spectral weight of $\hat{C}_x$ comes from $: \hat{C}_x^{(2)} :$ and $: \hat{C}_x^{(3)} :$, both of which are characterized by a large energy scale and are only weakly momentum dependent. As a result, the real part of the response function of $\hat{C}_x$ is dominated by the contributions from $: \hat{C}_x^{(2)} :$ and $: \hat{C}_x^{(3)} :$ at low energy and is almost momentum and frequency independent.

There is one more detail about the excitation by $: \hat{C}_x^{(1)} :$. On the triangular lattice, there are two inequivalent triangles in each unit cell, namely, the up and down triangles. We thus should consider both the in-phase (acoustic) and out-of-phase (optical) fluctuations of $\hat{C}_x$ on these triangles. We note that the excitation of one particle-hole pair in the acoustic channel is suppressed by an additional factor of q^2 in the long-wavelength limit compared to that in the optical channel since the sum of $: \hat{C}_x^{(1)} :$ over all triangles of the triangular lattice is identically zero.

With these understandings in mind, we can write down the asymptotic form of the inverse gauge propagator $K(q, \omega)$ in the low-energy regime as

$$K(q, \omega) \simeq K(q, 0) + \frac{i\alpha(q)\omega}{v_F q}. \quad \text{(C5)}$$

Here $K(q, 0)$ is the response function of $\hat{C}_x$ at zero frequency. According to the discussion above it should be a weakly q dependent real number and can be treated as a constant in the low-energy regime. $\alpha(q)$ is a coupling constant. For the acoustic mode, $\alpha(q) \propto q^2$ in the $q \to 0$ limit. For the optical mode, $\alpha(q)$ should be approximately a constant in the $q \to 0$ limit. We thus expect the gauge fluctuation in the acoustic and optical channels to contribute T^4 and T^2 corrections to the low-temperature specific heat, which are both dominated by the linear in T contribution from single-spinon excitation at low temperature.

We now go beyond the mean-field treatment and consider the fluctuation spectrum of $\hat{C}_x$ on the Gutzwiller projected Fermi sea state. Since $\hat{C}_x$ is a gauge-invariant quantity (it conserves the fermion number on a given site), it commutes with the Gutzwiller projection operator, namely,

$$\hat{C}_x P_G |\text{FS}\rangle = P_G \hat{C}_x |\text{FS}\rangle. \quad \text{(C6)}$$

We thus have

$$\hat{C}_q P_G |\text{FS}\rangle = P_G : \hat{C}_q^{(1)} : |\text{FS}\rangle + P_G : \hat{C}_q^{(2)} : |\text{FS}\rangle + P_G : \hat{C}_q^{(3)} : |\text{FS}\rangle. \quad \text{(C7)}$$

Therefore, the excitation picture of $\hat{C}_x$ on the Gutzwiller projected Fermi sea state is exactly the same as what we have described above in the mean-field treatment, although we should replace the mean-field excited states with their Gutzwiller projected counterparts. Thus, the fluctuation spectrum of scalar spin chirality on the projected Fermi sea state should be qualitatively the same as the mean-field prediction. We note that the mean-field eigenstates will, in general, no longer be orthonormal after the Gutzwiller projection [44]. However, the mean-field energetics will be qualitatively preserved after the projection.

[1] P. A. Lee, Science **321**, 1306 (2008).
[2] L. Balents, Nature (London) **464**, 199 (2010).
[3] B. Dalla Piazza, M. Mourigal, N. B. Christensen, G. J. Nilsen, P. Tregenna-Piggott, T. G. Perring, M. Enderle, D. F. McMorrow, D. A. Ivanov, and H. M. Rønnow, Nat. Phys. **11**, 62 (2015).
[4] A. L. Chernyshev and M. E. Zhitomirsky, Phys. Rev. B **79**, 144416 (2009).
[5] J. Ma, Y. Kamiya, T. Hong, H. B. Cao, G. Ehlers, W. Tian, C. D. Batista, Z. L. Dun, H. D. Zhou, and M. Matsuda, Phys. Rev. Lett. **116**, 087201 (2016).
[6] S. Ito, N. Kurita, H. Tanaka, S. Ohira-Kawamura, K. Nakajima, S. Itoh, K. Kuwahara, and K. Kakurai, Nat. Commun. **8**, 235 (2017).
[7] C. Zhang and T. Li, Phys. Rev. B **102**, 075108 (2020).
[8] P. A. Lee and N. Nagaosa, Phys. Rev. B **46**, 5621 (1992).
[9] Y. Shimizu, K. Miyagawa, K. Kanoda, M. Maesato, and G. Saito, Phys. Rev. Lett. **91**, 107001 (2003).
[10] S. Yamashita, Y. Nakazawa, M. Oguni, Y. Oshima, H. Nojiri, Y. Shimizu, K. Miyagawa, and K. Kanoda, Nat. Phys. **4**, 459 (2008).
[11] M. Yamashita, N. Nakata, Y. Senshu, M. Nagata, H. M. Yamamoto, R. Kato, T. Shibauchi, and Y. Matsuda, Science **328**, 1246 (2010).
[12] Y. Zhou, K. Kanoda, and T.-K. Ng, Rev. Mod. Phys. **89**, 025003 (2017).
[13] O. I. Motrunich, Phys. Rev. B **72**, 045105 (2005).
[14] S.-S. Lee and P. A. Lee, Phys. Rev. Lett. **95**, 036403 (2005).
[15] A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
[16] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **50**, 14048 (1994).
[17] J. Polchinski, Nucl. Phys. B **422**, 617 (1994).

[18] D. H. Kim and P. A. Lee, Ann. Phys. (NY) **272**, 130 (1999).
[19] C. Nayak, Phys. Rev. Lett. **85**, 178 (2000).
[20] X.-G. Wen, Phys. Rev. B **65**, 165113 (2002).
[21] I. F. Herbut, B. H. Seradjeh, S. Sachdev, and G. Murthy, Phys. Rev. B **68**, 195110 (2003).
[22] I. F. Herbut and B. H. Seradjeh, Phys. Rev. Lett. **91**, 171601 (2003).
[23] M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, Phys. Rev. B **70**, 214437 (2004).
[24] S.-S. Lee, Phys. Rev. B **80**, 165102 (2009).
[25] M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010).
[26] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010).
[27] M. Y. Reizer, Phys. Rev. B **40**, 11571 (1989).
[28] J. Gan and E. Wong, Phys. Rev. Lett. **71**, 4226 (1993).
[29] S.-S. Lee, P. A. Lee, and T. Senthil, Phys. Rev. Lett. **98**, 067006 (2007).
[30] T. Grover, N. Trivedi, T. Senthil, and P. A. Lee, Phys. Rev. B **81**, 245121 (2010).
[31] R. R. Biswas, L. Fu, C. R. Laumann, and S. Sachdev, Phys. Rev. B **83**, 245131 (2011).
[32] M. Barkeshli, H. Yao, and S. A. Kivelson, Phys. Rev. B **87**, 140402(R) (2013).
[33] R. V. Mishmash, J. R. Garrison, S. Bieri, and C. Xu, Phys. Rev. Lett. **111**, 157203 (2013).
[34] C. Mudry and E. Fradkin, Phys. Rev. B **50**, 11409 (1994).
[35] C. Mudry and E. Fradkin, Phys. Rev. B **49**, 5200 (1994).
[36] H. Yokoyama and M. Ogata, Phys. Rev. B **53**, 5758 (1996).
[37] R. B. Laughlin, D. Giulianoet, R. Caracciolo, O. L. White, in *Field Theories for Low-Dimensional Condensed Matter Systems: Spin Systems and Strongly Correlated Electrons*, (Springer, Berlin, 2000).
[38] H.-Y. Yang and T. Li, J. Phys.: Condens. Matter **19**, 186218 (2007).
[39] W. Marshall, Proc. R. Soc. London, Ser. A **232**, 48 (1955).
[40] Z. Y. Weng, D. N. Sheng, Y. C. Chen, and C. S. Ting, Phys. Rev. B **55**, 3894 (1997).
[41] S. Yunoki, E. Dagotto, and S. Sorella, Phys. Rev. Lett. **94**, 037001 (2005).
[42] C. P. Nave, D. A. Ivanov, and P. A. Lee, Phys. Rev. B **73**, 104502 (2006).
[43] H.-Y. Yang, F. Yang, Y.-J. Jiang, and T. Li, J. Phys. Condens. Matter **19**, 016217 (2006).
[44] T. Li and F. Yang, Phys. Rev. B **81**, 214509 (2010).
[45] F. Yang and T. Li, Phys. Rev. B **83**, 064524 (2011).
[46] J.-W. Mei and X.-G. Wen, arXiv:1507.03007.
[47] C. Zhang and T. Li, Phys. Rev. B **102**, 195106 (2020).
[48] S.-L. Yu, W. Wang, Z.-Y. Dong, Z.-J. Yao, and J.-X. Li, Phys. Rev. B **98**, 134410 (2018).
[49] F. Ferrari, A. Parola, S. Sorella, and F. Becca, Phys. Rev. B **97**, 235103 (2018).
[50] F. Ferrari and F. Becca, Phys. Rev. B **102**, 014417 (2020).
[51] F. Ferrari and F. Becca, Phys. Rev. X **9**, 031026 (2019).
[52] F. Ferrari and F. Becca, J. Phys.: Condens. Matter **32**, 274003 (2020).
[53] M. Charlebois and M. Imada, Phys. Rev. X **10**, 041023 (2020).
[54] K. Ido, M. Imada, and T. Misawa, Phys. Rev. B **101**, 075124 (2020).
[55] D. Vörös and K. Penc, arXiv:2107.09588.
[56] See, for example, W. A. Harrison, *Solid State Theory* (Dover, New York, 1979).
[57] P. W. Anderson, Phys. Rev. B **78**, 174505 (2008).
[58] P. W. Anderson, Phys. Rev. Lett. **18**, 1049 (1967).
[59] P. Nozieres and C. T. de Dominicis, Phys. Rev. **178**, 1097 (1969).
[60] T. Li, arXiv:1708.02781.
[61] N. Read and S. Sachdev, Phys. Rev. B **42**, 4568 (1990).
[62] W. Rantner and X.-G. Wen, Phys. Rev. B **66**, 144501 (2002).
[63] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
[64] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[65] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).