Ground-state energy of the polarized dilute gas of interacting spin-$\frac{1}{2}$ fermions

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The effective field theory approach simplifies the perturbative computation of the ground-state energy of the diluted gas of repulsively interacting fermions allowing in the case of the unpolarized system to easily rederive the classic results up to the $(k_F a_0)^2$ order (where k_F is the system's Fermi momentum and a_0 the s-wave scattering length) and (with more labor) to extend it up to the order $(k_F a_0)^4$. The analogous expansion of the ground-state energy of the polarized gas of spin 1/2 fermions is known only up to the $k_F a_0$ order (where k_F stands for $k_{F\uparrow}$ or $k_{F\downarrow}$); the order $(k_F a_0)^2$ contribution has been computed (analytically) only using a specific (hard-core type) interaction potential. Here we show that the effective field theory method also allows to easily obtain the order $(k_F a_0)^2$ correction to the ground state of the polarized gas in a way applicable to all repulsive interactions.

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I. INTRODUCTION

Effective field theories are used in high energy physics since already forty years (for a recent review of their applications to nuclear physics problems see Ref. [1]). Their applications rely on the separation of energy scales involved in physical problems which makes reliable the expansion of the computed quantities in powers of their ratio and on the possibility of fixing values of the parameters, which cannot be obtained by matching onto the underlying more fundamental theory, by directly extracting them from low energy data. Relatively more recent are applications of the effective field theory methods to nonrelativistic many body problems [2–6]. A particularly instructive is the application of this technique [3] to the classic problem of computing the energy E_Ω of the ground state of the system of N fermions (enclosed in the volume V) interacting through a two-body spin independent potential which may not be specified explicitly but is, instead, characterized by the (in principle infinite) set of the scattering lengths a_ℓ and the effective radii r_ℓ ($\ell = 0, 1, \dots$) parameterizing the expansion of the resulting partial amplitudes of elastic scattering of two fermions in powers of their relative momentum. Stated in this form the problem is ideally suited for handling it within the framework of an effective theory, because the information on the fundamental dynamics (on the two-body potential) is traded from the beginning for the (infinite) set of low energy data. If the underlying interaction potential is natural in the sense that the magnitudes of the scattering lengths a_ℓ and the effective radii r_ℓ it gives rise to are set by some common scale $1/\Lambda$ (this excludes from the considerations attractive potentials which can lead to bound states and formation of resonances), this scale is, if the gas of fermions is sufficiently diluted, well separated from its characteristic momentum scale set by the Fermi momentum (wave vector) k_F of the gas. The expansion of the ground-state energy E_Ω/N or E_Ω/V in powers of $k_F a_\ell \propto k_F/\Lambda$ and $k_F r_\ell \propto k_F/\Lambda$ naturally provided by the effective theory methods is then reliable. This allows to drastically simplify the classic treatment (Refs. [7–12]) of the problem summarized in Ref. [13]. Owing to its simplicity the effective theory method allowed to obtain [14] recently the complete fourth ($\mathcal{O}(k_F^4/\Lambda^4)$ and $\mathcal{O}((k_F^4/\Lambda^4)\ln(k_F/\Lambda))$ order terms which complement the third, $\mathcal{O}(k_F^3/\Lambda^3)$, order result obtained earlier [15] by more conventional (semianalytic) methods.

The interest in properties of a diluted gas of fermions stemmed originally from the study of nuclear matter, although this model obviously cannot capture all realistic features of systems of nucleons interacting through (mostly) attractive potentials. More recently models of diluted gases (of fermions and bosons) find their more natural application as continuum models of interacting cold atomic gases bound in optical or harmonic traps, complementing more traditional ways of investigating properties of such systems based on lattice models known generally as (paradigmatic for condensed matter physics) Hubbard models which, despite of more than 60 years of development, still leave many problems without clear answers [16]. One of them is the mechanism of possible emergence of ferromagnetism in systems of mutually repelling atoms which has been also investigated experimentally [17]. Theoretical investigations of many questions of interest related to this result, like the problem of itinerant ferromagnetism in lattice models of mutually repelling spin 1/2 fermions as well as the possibility of spontaneous separation of magnetic and nonmagnetic phases begun already in the sixtieth of the XX century, but a successful explanation of these phenomena with the help of the so-called dynamical mean field theory (DMFT) approach [18] has been achieved only some twenty years ago.

Clearly, any study of the emergence of magnetism in systems of N mutually repelling spin 1/2 fermions must start with the computation of the ground-state energy of such a system for $N_\uparrow \neq N_\downarrow$, where $N_\uparrow$ and $N_\downarrow$ ($N_\uparrow + N_\downarrow = N$) are the

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(conserved by the assumed interaction) numbers of spin up and spin down fermions. Within the continuum model applying the ordinary Rayleigh-Schrödinger perturbative expansion to the Hamiltonian (see Ref. [13,19])

$$H = \sum_{\mathbf{p},\sigma=\uparrow,\downarrow} \frac{\hbar^2\mathbf{p}^2}{2m_f} a^\dagger_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} + \frac{1}{2V}\sum_{\mathbf{q}} \tilde{V}_{\rm pot}(\mathbf{q}) \sum_{\mathbf{p}_1,\mathbf{p}_2} \sum_{\sigma_1,\sigma_2=\uparrow,\downarrow} a^\dagger_{\mathbf{p}_1+\mathbf{q},\sigma_1} a^\dagger_{\mathbf{p}_2-\mathbf{q},\sigma_2} a_{\mathbf{p}_2,\sigma_2} a_{\mathbf{p}_1,\sigma_1},$$

it is easy to obtain the relevant expression in the first order in the s-wave scattering length a_0 (this result can be given also a mathematically more rigorous foundation [20,21]):

$$\begin{aligned} E_\Omega &= \frac{V}{6\pi^2}\frac{3}{5}\frac{\hbar^2}{2m_f}\left(p^5_{\mathrm{F}\uparrow} + p^5_{\mathrm{F}\downarrow}\right) + V\frac{4\pi\hbar^2}{m_f} a_0 \frac{p^3_{\mathrm{F}\uparrow}}{6\pi^2}\frac{p^3_{\mathrm{F}\downarrow}}{6\pi^2} \\ &= V\frac{3}{5}\frac{\hbar^2}{2m_f}(6\pi^2)^{2/3}\left(\rho^{5/3}_\uparrow + \rho^{5/3}_\downarrow\right) + V\frac{\hbar^2}{2m_f} 8\pi a_0 \rho_\uparrow \rho_\downarrow, \end{aligned} \tag{1}$$

where $\rho_{\uparrow/\downarrow} = N_{\uparrow/\downarrow}/V$ and $p^3_{\mathrm{F}\uparrow/\downarrow} = 6\pi^2\rho_{\uparrow/\downarrow}$. The next order term has been computed analytically using the traditional approach only within the hard spheres model interaction [22]. This approach does not allow, however, to easily recognize the universality of this result (in the class of natural spin-independent repulsive potentials). Apart from these result, there are also Monte Carlo simulations [23,24] which—while providing quite reliable numerical estimates of the exact (non-perturbative) ground-state energy—must necessarily employ concrete model potentials and therefore suffer from the lack of universality.

In this paper, we compute the second-order correction to the ground-state energy of the polarized gas of spin 1/2 fermions with the help of the effective theory method. It makes it clear from the outset that this correction can only depend on the s-wave scattering length a_0 and, therefore, that the result of Ref. [22] is universal. Nevertheless, it is interesting to recover it using this new method as this may pave the way to extend the computation to yet higher orders. From the conceptual point of view, this task reduces to only a minor modification of the computation performed in [3] for $N_\uparrow = N_\downarrow$, but it is a little bit more involved from the technical point of view. While the order a_0^2 correction to the ground-state energy is in Ref. [3] given in a completely analytic form, and, as the result of Ref. [22] shows, the same is possible also in the case of $N_\uparrow \neq N_\downarrow$, we do not attempt to perform the resulting integrals analytically and content ourselves with providing the formulas which involve integrals which can be easily evaluated numerically with the help of a three-line Mathematica code.

Our computation parallels that of Ref. [3] but instead of using the dimensional reduction as the regularization method, we cut off divergent integrals over the wave vectors at the scale Λ. While being technically more troublesome (but only in higher orders) this regularization prescription seems more in line with the main idea of the effective theory method and moreover it allows to partly control the correctness of the calculation, which is not possible with the dimensional regularization which automatically sets to zero all powerlike divergences. We begin in Sec. II by recalling the relevant effective Lagrangian and how one relates its couplings to the "low energy data," that is, to the scattering lengths a_ℓ and effective radii r_ℓ. In Sec. III, we compute the second-order correction to the ground-state energy using the effective theory method, demonstrate explicitly cancellation of the dependence on the cutoff Λ and give the result in the form dependent on a single function of the ratio of Fermi momenta of spin up and spin down fermions which is evaluated numerically. We also compare our perturbative result with the existing nonperturbative estimates based on Monte Carlo simulations. We summarize the results in Sec. IV and speculate about perspectives of generalizing them.

II. EFFECTIVE FIELD THEORY APPROACH

A convenient method of computing the ground-state energy based on the effective theory has been proposed in Ref. [3]. The most general Lagrangian density consistent with the Galileo, parity and time-reversal symmetries with respect to which the dynamics of the spinor field is assumed to be invariant, has the form (spinor indices of field in the brackets are implicitly contracted)

$$\mathcal{L} = \psi^\dagger\left(i\hbar\partial_t + \frac{\hbar^2\boldsymbol{\nabla}^2}{2m_f}\right)\psi - \frac{C_0}{2}{:}(\psi^\dagger\psi)^2{:} + \frac{C_2}{16}{:}[(\psi^\dagger\psi^\dagger)(\psi\overleftrightarrow{\boldsymbol{\nabla}}^2\psi) + \mathrm{H.c.}]{:} + \cdots \tag{2}$$

It consists of infinitely many local operator structures of increasing dimensions. Their coefficients C_0, C_2, etc. have to be determined by comparing the scattering amplitude of two fermions[1] computed using this effective theory with the one known from the potential scattering (i.e., the one parametrized by the scattering lengths etc.), or—if it is explicitly given—by matching onto the "fundamental" theory in which fermions interact through a well-defined two-body potential.

The local (i.e., singular) nature of the interaction terms of the Lagrangian density (2) results in short-distance (i.e., ultraviolet) divergences (absent in the "fundamental" theory) in various quantities computed with the help of it. These should be regularized and removed by applying the standard renormalization procedure. Since we are interested here only in a directly measurable quantity (the ground-state energy E_Ω), renormalization can be straightforwardly carried out by simply computing (using the same regularization) an appropriate set of observables other than E_Ω itself (in this case the scattering lengths a_ℓ and effective radii r_ℓ in the expansion of the elastic scattering amplitude of two fermions) and expressing the computed quantity, E_Ω, in terms of them. Any consistent regularization can be used for this purpose because when the computed physical quantities (such as E_Ω) are expressed in terms of other observables (a_ℓ's and r_ℓ's) they become independent of it (in the limit of removed regularization). The most popular in high energy physics computations

[1]The proliferation of possible operator structures of higher dimensions has the effect that also amplitudes of three-body scattering must be used as the input to determine all independent coefficients in (2).

(in which preservation of gauge invariance is the main concern) is the dimensional regularization and it is the one which was used in the seminal paper [3]. Here we implement regularization by cutting off all integrals over the wave vectors $\mathbf{k}$ at the scale Λ.[2] Once energy E_Ω is expressed in terms of observables as a_ℓ, r_ℓ the cutoff can be safely removed by formally taking the limit $\Lambda \to \infty$.

The procedures of expressing the scattering lengths in terms of the coefficients C_i of the Lagrangian density (2) is well known (see, e.g., Refs. [1,3]). One way is to compute the S-matrix element corresponding to the elastic scattering of two fermions (of oppositely oriented spins, in order to directly extract the relevant part of the amplitude) in their center of mass system (CMS) using the formula [25]

$$S_{\beta\alpha} \equiv \delta_{\beta\alpha} - \frac{i}{\hbar}(2\pi)^4\delta^{(4)}(k_1' + k_2' - k_1 - k_2)\mathcal{A}$$
$$= \langle \mathbf{k}_1'\uparrow, \mathbf{k}_2'\downarrow | \mathrm{T}\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt V_{\mathrm{int}}^I(t)\right)|\mathbf{k}_1\uparrow, \mathbf{k}_2\downarrow\rangle \quad (3)$$

(T is here the symbol of the chronological ordering and $V_{\mathrm{int}}^I(t)$ is the interaction picture counterpart of the minus interaction term of the Lagrangian density (2); employed is the four-vector notation in which $k^0 = \omega_{\mathbf{k}} \equiv \hbar\mathbf{k}^2/2m_f$, $k = |\mathbf{k}|$) and to apply the rule

$$-\frac{m_f}{4\pi\hbar^2}\mathcal{A}(k,\theta) = f(k,\theta), \quad (4)$$

to obtain the standard scattering amplitude $f(k,\theta)$.

Evaluating to the first nontrivial order the matrix element (3) for $\mathbf{k}_2 = -\mathbf{k}_2 \equiv \mathbf{k}$ one finds

$$f(k,\theta) = -\frac{m_f}{4\pi\hbar^2}C_0\left\{1 + \left(\frac{C_0}{i\hbar}\right)\left(\frac{m_f}{i\hbar}I_0\right) + \cdots\right\}, \quad (5)$$

where

$$I_0 = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\mathbf{q}^2 - \mathbf{k}^2 - i0}. \quad (6)$$

This integral is divergent and requires regularization. Imposing the UV cutoff Λ on $q = |\mathbf{q}|$, one obtains

$$I_0(k,\Lambda) = \frac{1}{4\pi^2}\int_0^\Lambda dqq\left[\frac{1}{q-k-i0} + \frac{1}{q+k+i0}\right]$$
$$= \frac{i}{4\pi}k + \frac{1}{2\pi^2}\Lambda - \frac{1}{2\pi^2}\frac{k^2}{\Lambda} + \ldots \quad (7)$$

upon using the standard Sochocki formula $1/(x \pm i0) = \mathrm{P}(1/x) \mp i\pi\delta(x)$ (P stands for principal value). Matching then the amplitude (5) onto the expansion $f(k,\theta) = -a_0 + \mathcal{O}(k)$ of the scattering amplitude one obtains

$$C_0 = \frac{4\pi\hbar^2}{m_f}a_0\left(1 + \frac{2}{\pi}a_0\Lambda + \ldots\right). \quad (8)$$

In general, the procedure leading to C_0 is slightly more complicated when the cut-off regularization is used (instead of dimensional regularization) and the resulting C_0 is given in the form of an infinite power series in $a_0\Lambda$ (this is why the dimensional regularization is more convenient) but, as will be seen, allows to better control cancellation of divergences in physical quantities.

[2]It is this regularization which (in conjunction with the counterterm technique) has been used in the recent computation [14] of the fourth-order corrections to E_Ω of the unpolarized gas of fermions.

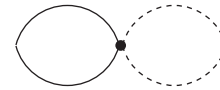

FIG. 1. The effective theory connected vacuum diagram of order C_0 reproducing the first-order correction $E_\Omega^{(1)}$. Solid and dashed lines represent propagators of fermions with opposite spin projections.

III. CORRECTIONS TO THE GROUND-STATE ENERGY

We now compute the corrections $E_\Omega^{(1)}$ and $E_\Omega^{(2)}$ to the ground-state energy diagrammatically, treating the terms of the Lagrangian (2) as the interaction vertices. The calculation closely parallels that of Ref. [3] except that we do not assume that the numbers $N_\uparrow$ and $N_\downarrow$ of spin up and spin down fermions are equal. The basic formula (relying on the adiabatic principle) employed for this purpose reads[3]

$$\lim_{T\to\infty}\exp(-iT(E_\Omega - E_{\Omega_0})/\hbar)$$
$$= \lim_{T\to\infty}\langle\Omega_0|\mathrm{T}\exp\left(-\frac{i}{\hbar}\int_{-T/2}^{T/2} dt V_{\mathrm{int}}^I(t)\right)|\Omega_0\rangle. \quad (9)$$

In other words, $-iT(E_\Omega - E_{\Omega_0})/\hbar$ is (in the limit $T \to \infty$, $V \to \infty$) given by $(2\pi)^4\delta^{(4)}(0)$ times the sum of the momentum space connected vacuum diagrams (diagrams without external lines). The factor $(2\pi)^4\delta^{(4)}(0)$ arising in evaluating diagrams in position space (expressing the overall four-momentum conservation) is interpreted as VT. It follows that $i\hbar$ times the expression arising from summing the momentum space connected vacuum diagrams is just $(E_\Omega - E_{\Omega_0})/V$.

As explained in Ref. [3], to compute the order $(k_\mathrm{F}a_0)^2$ correction to the ground-state energy only the interaction term proportional to C_0 of (2) is needed. It simplifies considerably if there are only two possible spin projections, because $\psi_\alpha^\dagger\psi_\alpha^\dagger = \psi_\alpha\psi_\alpha = 0$ (one can treat fields as anticommuting), and reads[4]

$$V_{\mathrm{int}} = C_0\int d^3\mathbf{x}{:}(\psi_+^\dagger\psi_+)(\psi_-^\dagger\psi_-){:}. \quad (10)$$

The right-hand side of the formula (9) can be evaluated using the standard rules of quantum field theory. In the momentum space, lines of Feynman diagrams correspond to the propagators (see, e.g., Ref. [13])

$$i\tilde{G}_\pm^{(0)}(\omega,\mathbf{k}) = i\left[\frac{\theta(|\mathbf{k}| - p_{\mathrm{F}\pm})}{\omega - \omega_{\mathbf{k}} + i0} + \frac{\theta(p_{\mathrm{F}\pm} - |\mathbf{k}|)}{\omega - \omega_{\mathbf{k}} - i0}\right] \quad (11)$$

[3]The symbol T of the chronological ordering should not be confused with T denoting time.

[4]In what follows, we will denote by $+$ $(-)$ quantities and operators pertaining to the spin projection of larger (smaller) density; thus we will use the Fermi momenta $p_{\mathrm{F}+}$ and $p_{\mathrm{F}-}$ $(p_{\mathrm{F}+} \geqslant p_{\mathrm{F}-})$ understanding that $p_{\mathrm{F}+} = p_{\mathrm{F}\uparrow}$ when $p_{\mathrm{F}\uparrow} \geqslant p_{\mathrm{F}\downarrow}$.

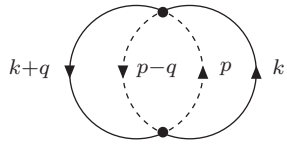

FIG. 2. The only nonvanishing three loop connected vacuum diagram contributing to the ground-state energy of the diluted gas of spin-1/2 fermions. The solid and dashed lines correspond to propagators of fermions having opposite spin projections. The two kinds of propagators differ by the values of the Fermi momenta: we assume that $p_{\mathrm{F}+} \geqslant p_{\mathrm{F}-}$.

and, to account for the normal ordered form of the interaction, one has only to add the rule [13] that if a line originates from and ends up in one and the same vertex, the propagator (11) corresponding to this line has to be multiplied by $e^{i\omega\eta}$ with the limit $\eta \to 0^+$ taken at the end.

In the first order in C_0, there is only one connected vacuum graph shown in Fig. 1. Applying the Feynman rules one immediately obtains ($iG^{(0)}_{\pm}(0)$ are "position" space propagators)

$$TE^{(1)}_{\Omega} = C_0 VTiG^{(0)}_{+}(0)iG^{(0)}_{-}(0) = C_0 VT\frac{p^3_{\mathrm{F}-}}{6\pi^2}\frac{p^3_{\mathrm{F}+}}{6\pi^2},$$

recovering, after using (up to the first order in a_0) the result (8), the second-order term of (1).

Several connected vacuum diagrams of the next order can be drawn but, as explained in [3], nonzero is only the one shown in Fig. 2. The contribution of this diagram is given by ($d^4q = d^3\mathbf{q}dq_0$)

$$\frac{E^{(2)}_{\Omega}}{V} = -\frac{i}{2!}\frac{C_0^2}{\hbar}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}i\tilde{G}^{(0)}_{-}(p)i\tilde{G}^{(0)}_{-}(p-q)i\tilde{G}^{(0)}_{+}(k)i\tilde{G}^{(0)}_{+}(k+q). \tag{12}$$

Evaluating the integrals over the frequencies q_0, p_0, and k_0 using the standard residue method, one obtains the sum of two terms each involving three integrals over $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{k}$. These two terms become manifestly equal after making in the second one the substitutions $\mathbf{k}+\mathbf{q} = -\mathbf{k}'$, $\mathbf{p}-\mathbf{q} = -\mathbf{p}'$. In this way, one arrives at the expression

$$\frac{E^{(2)}_{\Omega}}{V} = \frac{C_0^2}{\hbar}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\theta(p_{\mathrm{F}+}-|\mathbf{k}|)\theta(p_{\mathrm{F}-}-|\mathbf{p}|)\theta(|\mathbf{k}+\mathbf{q}|-p_{\mathrm{F}+})\theta(|\mathbf{p}-\mathbf{q}|-p_{\mathrm{F}-})}{\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{k}+\mathbf{q}}-\omega_{\mathbf{p}-\mathbf{q}}+i0}.$$

The last step [3] is to pass to the integrations over the variables $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ defined by the relations (the Jacobian $J = 8$)

$$\mathbf{k} = \mathbf{s}-\mathbf{t}, \qquad \mathbf{p} = \mathbf{s}+\mathbf{t}, \qquad \mathbf{q} = \mathbf{t}-\mathbf{u}.$$

The denominator of the integrand then becomes equal $(\mathbf{t}^2-\mathbf{u}^2+i0)/m_f$. Defining

$$I = \int d^3\mathbf{s}\int d^3\mathbf{t}\int d^3\mathbf{u}\frac{\theta(p_{\mathrm{F}-}-|\mathbf{t}+\mathbf{s}|)\theta(p_{\mathrm{F}+}-|\mathbf{t}-\mathbf{s}|)\theta(|\mathbf{u}+\mathbf{s}|-p_{\mathrm{F}-})\theta(|\mathbf{u}-\mathbf{s}|-p_{\mathrm{F}+})}{\mathbf{t}^2-\mathbf{u}^2+i0},$$

one can write the combined first- and second-order contributions in the form

$$\frac{E^{(1)}_{\Omega}+E^{(2)}_{\Omega}}{V} = C_0\frac{p^3_{\mathrm{F}-}p^3_{\mathrm{F}+}}{36\pi^4}+\frac{8C_0^2}{\hbar^2}m_f\frac{I}{(2\pi)^9}.$$

After using (8), i.e., replacing C_0 by $(4\pi\hbar^2/m_f)a_0(1+2a_0\Lambda/\pi)$, one gets

$$\frac{E^{(1)}_{\Omega}+E^{(2)}_{\Omega}}{V} = \frac{p^3_{\mathrm{F}-}-p^3_{\mathrm{F}+}}{9\pi^3}\frac{\hbar^2}{m_f}a_0+\frac{2p^3_{\mathrm{F}-}p^3_{\mathrm{F}+}}{9\pi^4}\frac{\hbar^2}{m_f}a_0^2\Lambda + \frac{\hbar^2}{m_f}32a_0^2\frac{I}{(2\pi)^7}. \tag{13}$$

The regions of integrations over $d^3\mathbf{u}$ and over $d^3\mathbf{t}$ in I are determined by the intersections of two Fermi spheres of unequal radii, $p_{\mathrm{F}-}$ and $p_{\mathrm{F}+}$, the centers of which are displaced from the origin of the $\mathbf{u}$ (of the $\mathbf{t}$) space by the vectors $-\mathbf{s}$ ($\mathbf{s}$ will be taken to determine the z-axes of the $\mathbf{u}$ and $\mathbf{t}$ spaces in the integrals over $d^3\mathbf{u}$ and $d^3\mathbf{t}$) and $\mathbf{s}$, respectively. The integral over $\mathbf{u}$ is over the infinite exterior of both spheres and is, therefore, divergent; the integration over $\mathbf{t}$ covers the intersection of the two spheres. For this reason, the outermost integration over $s \equiv |\mathbf{s}|$ is restricted to $s \leqslant s_{\max} = \frac{1}{2}(p_{\mathrm{F}+}+p_{\mathrm{F}-})$ because if $s > \frac{1}{2}(p_{\mathrm{F}+}+p_{\mathrm{F}-})$, the two spheres which determine the region of integration over $\mathbf{t}$ are disjoint. It will be convenient to write $I = 8(2\pi)^3 J(p_{\mathrm{F}-}, p_{\mathrm{F}+})$ with

$$J(p_{\mathrm{F}-}, p_{\mathrm{F}+}) = \int_0^{s_{\max}} ds s^2\frac{1}{4\pi}\int d^3\mathbf{t}\theta(p_{\mathrm{F}-}-|\mathbf{t}+\mathbf{s}|)\theta(p_{\mathrm{F}+}-|\mathbf{t}-\mathbf{s}|)g(t,s),$$
$$g(t,s) \equiv g(|\mathbf{t}|,s) = \frac{1}{4\pi}\int d^3\mathbf{u}\frac{\theta(|\mathbf{u}+\mathbf{s}|-p_{\mathrm{F}-})\theta(|\mathbf{u}-\mathbf{s}|-p_{\mathrm{F}+})}{\mathbf{t}^2-\mathbf{u}^2+i0}. \tag{14}$$

As far as the integral $g(t,s)$ is concerned, the range of the variable s splits into two domains: $0 \leqslant s \leqslant s_0 = \frac{1}{2}(p_{\mathrm{F}+}-p_{\mathrm{F}-})$ and $s_0 \leqslant s \leqslant s_{\max}$. Correspondingly, the integral (14) will be written as the sum $J = J_1 + J_2$.

If $0 \leqslant s \leqslant s_0 = \frac{1}{2}(p_{\mathrm{F}+}-p_{\mathrm{F}-})$, the sphere of radius $p_{\mathrm{F}-}$ is entirely contained inside the one of radius $p_{\mathrm{F}+}$ (see Fig. 3) and plays no role in determining the domain of integration over $\mathbf{u}$: this domain is then just the (infinite) exterior of the sphere of

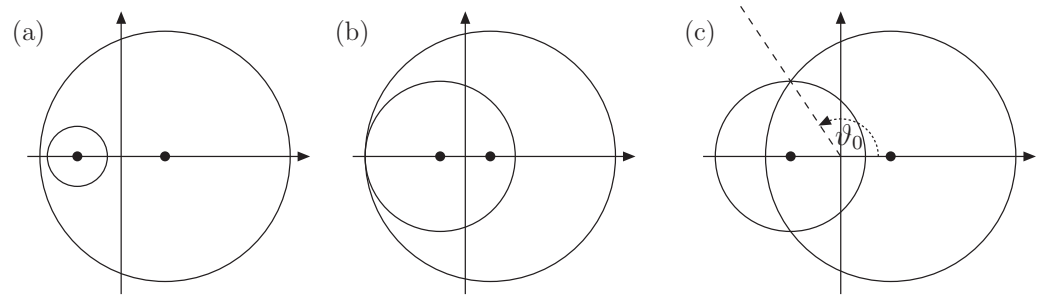

FIG. 3. Configurations of the Fermi spheres. Dots mark their centers shifted by $\mp s$ from the origin of the space. (a) $p_{\mathrm{F}-}/p_{\mathrm{F}+} < 1/3$; in this case, if $p_{\mathrm{F}-} < s < (p_{\mathrm{F}+} - p_{\mathrm{F}-})/2$, the smaller sphere is entirely in the left half of the space. (b) $p_{\mathrm{F}-}/p_{\mathrm{F}+} > 1/3$; in this case, part of the smaller sphere is always in the right half of the space. (c) The spheres intersect for $(p_{\mathrm{F}+} - p_{\mathrm{F}-})/2 < s < (p_{\mathrm{F}+} + p_{\mathrm{F}-})/2$. Marked is the "critical" polar angle ϑ_0.

radius $p_{\mathrm{F}+}$ the center of which is at $u_z = 0$, when $s = 0$ and moves to the right as s increases but the origin of the $\mathbf{u}$ space always remains inside this sphere. The details of computing the integral over $d^3\mathbf{u}$ are given in Appendix. The resulting formula for the function $g(t, s)$ in the range $0 \leqslant s \leqslant s_0$ is

$$g(t,s) = -\Lambda + \frac{1}{2}p_{\mathrm{F}+} + \frac{t}{4}\ln\frac{(p_{\mathrm{F}+}-t)^2 - s^2}{(p_{\mathrm{F}+}+t)^2 - s^2} + \frac{p_{\mathrm{F}+}^2 - s^2 - t^2}{8s}\ln\frac{(p_{\mathrm{F}+}+s)^2 - t^2}{(p_{\mathrm{F}+}-s)^2 - t^2}. \tag{15}$$

Integrated over t, η and then over s from 0 to s_0 this function gives the corresponding contribution to the ground-state energy density. The integration over t and η is in this case over the interior of the (smaller) sphere of radius $p_{\mathrm{F}-}$. Since the origin of the $\mathbf{t}$ space remains inside this sphere [see Fig. 3(b)] in the entire range $0 < s < s_0$ only when $p_{\mathrm{F}-}/p_{\mathrm{F}+} > 1/3$, performing the remaining integrations in the original variables would, if $p_{\mathrm{F}-}/p_{\mathrm{F}+} < 1/3$, require further splitting the integration over s into two parts: over $0 \leqslant s \leqslant p_{\mathrm{F}-}$ and over $p_{\mathrm{F}-} \leqslant s \leqslant s_0$. This can be avoided by shifting the variable $\mathbf{t}$, that is by writing $\mathbf{t} = \mathbf{t}' - \mathbf{s}$ and introducing the spherical coordinate system in the $\mathbf{t}'$ space with the t'_z axis taken in the direction of the vector $\mathbf{s}$. Then[5]

$$J_1 = \frac{1}{2}\int_0^{s_0} ds s^2 \int_{-1}^{1} d\eta \int_0^{p_{\mathrm{F}-}} dt' t'^2 g\big(\sqrt{t'^2 - 2t's\eta + s^2}, s\big).$$

Since when $g(t, s) \equiv 1$ the integrations over η and t give $(2/3)p_{\mathrm{F}-}^3$, the divergent part of J_1 is

$$J_1^{\mathrm{div}} = \frac{1}{3}s_0^3\frac{1}{3}p_{\mathrm{F}-}^3(-\Lambda) = -\frac{1}{72}(p_{\mathrm{F}+} - p_{\mathrm{F}-})^3 p_{\mathrm{F}-}^3\Lambda. \tag{16}$$

We now compute the function $g(t, s)$ for $s_0 \leqslant s \leqslant s_{\max}$ and the corresponding contribution J_2 to the integral (14). In this regime, the two Fermi spheres which determine the ranges of integrations over $\mathbf{u}$ and over $\mathbf{t}$ intersect one another. In the $\mathbf{u}$ space, the z coordinate u_z of the intersection and its distance u_0 from the origin are determined by solving the equations

$$u_\perp^2 + (u_z - s)^2 = p_{\mathrm{F}+}^2,$$
$$u_\perp^2 + (u_z + s)^2 = p_{\mathrm{F}-}^2,$$

which give

$$u_z^0 = -\frac{p_{\mathrm{F}+}^2 - p_{\mathrm{F}-}^2}{4s}, \qquad u_0^2 = \frac{1}{2}\big(p_{\mathrm{F}+}^2 + p_{\mathrm{F}-}^2\big) - s^2. \tag{17}$$

In the spherical system, the "critical" angle ϑ_0 corresponding to the intersection of the spheres [marked in Fig. 3(c)] is given by

$$\cos\vartheta_0 = \xi_0 = \frac{u_z^0}{u_0} = -\frac{p_{\mathrm{F}+}^2 - p_{\mathrm{F}-}^2}{4s\sqrt{\frac{1}{2}\big(p_{\mathrm{F}+}^2 + p_{\mathrm{F}-}^2\big) - s^2}}. \tag{18}$$

Therefore, if $s_0 \leqslant s \leqslant s_{\max}$ is given by[6]

$$g(t,s) = \frac{1}{2}\int_{\xi_0}^{1} d\xi \int_{u_+(\xi,s)}^{\infty} du \frac{u^2}{t^2 - u^2 + i0} + \frac{1}{2}\int_{-1}^{\xi_0} d\xi \int_{u_-(\xi,s)}^{\infty} du \frac{u^2}{t^2 - u^2 + i0}, \tag{19}$$

[5]Another way of evaluating (the finite part of) J_1 without distinguishing the cases $p_{\mathrm{F}-} > \frac{1}{3}p_{\mathrm{F}+}$ and $p_{\mathrm{F}-} < \frac{1}{3}p_{\mathrm{F}+}$ is to use the MATHEMATICA package instruction 0.5NIntegrate[$s^2t^2g[t, s]$Boole[$t^2 + 2tsx + s^2 < p_{\mathrm{F}-}^2$], $\{s, 0, s_0\}$, $\{x, -1, 1\}$, $\{t, 0, \infty\}$]].

[6]Actually this way of computing $g(t, s)$ in this regime ($s_0 < s < s_{\max}$) is justified geometrically only for s not greater than some critical value (depending on the ratio $p_{\mathrm{F}-}/p_{\mathrm{F}+}$) which is smaller than $s_{\max}$. For s greater than critical, the dashed line in Fig. 3(c) passes through the interior of the smaller sphere and the justification of the formula (19) hinges on the fact that, as can be shown, the integral of an *even* function of $u = |\mathbf{u}|$ over the interior of the sphere not involving the origin of the $\mathbf{u}$ space, which should in principle be done as if the origin remained inside the interior.

where now $u_+(\xi, s) = s\xi + \sqrt{p_{\mathrm{F}+}^2 - s^2(1-\xi^2)}$, $u_-(\xi, s) = -s\xi + \sqrt{p_{\mathrm{F}-}^2 - s^2(1-\xi^2)}$; of course $u_+(\xi_0, s) = u_-(\xi_0, s) \equiv u_0$. Details of the evaluation of these integrals are given in Appendix. The resulting function $g(t, s)$ takes, in the range $s_0 \leqslant s \leqslant s_{\max}$ the form

$$g(t,s) = -\Lambda + \frac{1}{4}(p_{\mathrm{F}+} + p_{\mathrm{F}-} + 2s) + \frac{t}{4}\ln\frac{p_{\mathrm{F}+}+s-t}{p_{\mathrm{F}+}+s+t} + \frac{t}{4}\ln\frac{p_{\mathrm{F}-}+s-t}{p_{\mathrm{F}-}+s+t} + \frac{p_{\mathrm{F}+}^2 - t^2 - s^2}{8s}\ln\frac{(p_{\mathrm{F}+}+s)^2 - t^2}{u_0^2 - t^2} + \frac{p_{\mathrm{F}-}^2 - t^2 - s^2}{8s}\ln\frac{(p_{\mathrm{F}-}+s)^2 - t^2}{u_0^2 - t^2}, \tag{20}$$

in which u_0^2 is given by (17).

It is now straightforward to compute J_2^{div} and to check the cancellation of Λ. Indeed, the integral

$$\frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(p_{\mathrm{F}-} - |\mathbf{t}+\mathbf{s}|)\theta(p_{\mathrm{F}+} - |\mathbf{t}-\mathbf{s}|)(-\Lambda),$$

can be done using the already computed integrals: shifting the origin of the coordinate system of the $\mathbf{t}$ space so that the intersection of the two Fermi spheres occurs at $t_z' = 0$, one readily obtains the result

$$-\frac{\Lambda}{2}\left\{\left[\frac{1}{3}p_{\mathrm{F}-}^3 - \frac{1}{2}p_{\mathrm{F}-}^2(s-\varepsilon) + \frac{1}{6}(s-\varepsilon)^3\right] + \left[\frac{1}{3}p_{\mathrm{F}+}^3 - \frac{1}{2}p_{\mathrm{F}+}^2(s+\varepsilon) + \frac{1}{6}(s+\varepsilon)^3\right]\right\},$$

with $\varepsilon = (p_{\mathrm{F}+}^2 - p_{\mathrm{F}-}^2)/4s \equiv -u_z^0$. This should be integrated from $s_0 = \frac{1}{2}(p_{\mathrm{F}+} - p_{\mathrm{F}-})$ to $s_{\max} = \frac{1}{2}(p_{\mathrm{F}+} + p_{\mathrm{F}-})$ with the weight s^2. MATHEMATICA does it readily and the result is

$$J_2^{\mathrm{div}} = -\Lambda\left(\frac{p_{\mathrm{F}+}^2 p_{\mathrm{F}-}^4}{24} - \frac{p_{\mathrm{F}+}p_{\mathrm{F}-}^5}{24} + \frac{p_{\mathrm{F}-}^6}{72}\right).$$

Combining this with the divergent part (16) of J_1 one gets

$$J_1^{\mathrm{div}} + J_2^{\mathrm{div}} = -\Lambda\frac{p_{\mathrm{F}+}^3 p_{\mathrm{F}-}^3}{72},$$

which is precisely what is needed to cancel in (13) the term explicitly proportional to Λ which comes from expressing C_0 in terms of the scattering length in the first-order result. Thus, as expected, the divergences disappear when observable quantities (the ground state energy) computed within the effective theory are expressed in terms of other observable quantities (the scattering lengths).

In the limit $p_{\mathrm{F}-} = p_{\mathrm{F}+} = k_{\mathrm{F}}$, the expression (20) this goes over into the function

$$g(t,s) = -\Lambda + \frac{1}{2}(k_{\mathrm{F}} + s) + \frac{t}{2}\ln\frac{k_{\mathrm{F}}+s-t}{k_{\mathrm{F}}+s+t} + \frac{k_{\mathrm{F}}^2 - s^2 - t^2}{4s}\ln\frac{(k_{\mathrm{F}}+s)^2 - t^2}{k_{\mathrm{F}}^2 - s^2 - t^2}.$$

arising in the case of equal densities of spin up and spin down fermions. In this case, in which the integral J_1 is zero, the remaining integrals over s, η and t which give $J_2 = J$ can be even worked out explicitly [3] with the result

$$J(k_{\mathrm{F}}, k_{\mathrm{F}}) = -\frac{1}{72}\Lambda k_{\mathrm{F}}^6 + k_{\mathrm{F}}^7\frac{11 - 2\ln 2}{24\cdot 35}.$$

If $p_{\mathrm{F}-} < p_{\mathrm{F}+}$, the integrals over s, η and t can be easily evaluated numerically.[7] It is convenient to write the complete function $J(p_{\mathrm{F}-}, p_{\mathrm{F}+})$ defined in (14), setting $\Lambda = 0$ as $p_{\mathrm{F}+}^7 J(r, 1)$ with $0 \leqslant r \equiv p_{\mathrm{F}-}/p_{\mathrm{F}+} \leqslant 1$. The function $J(r, 1)$ is shown in Fig. 4. The complete result can be therefore written as

$$\frac{E_\Omega}{V} = \frac{\hbar^2 p_{\mathrm{F}+}^2}{m_f}\frac{p_{\mathrm{F}+}^3}{6\pi^2}\left\{\frac{3}{10}(1+r^5) + \frac{2}{3\pi}r^3(p_{\mathrm{F}+}a_0) + \frac{96}{\pi^2}(p_{\mathrm{F}+}a_0)^2 J(r,1) + \ldots\right\}. \tag{21}$$

[7]The simplest way is to use the MATHEMATICA numerical integration routine to integrate over the domain $s_0 \leqslant s \leqslant s_{\max}$, $-1 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant \infty$, imposing the conditions $t^2 + 2tsx + s^2 < p_{\mathrm{F}-}^2$ and $t^2 - 2tsx + s^2 < p_{\mathrm{F}+}^2$ but we have also evaluated it using other methods always with the same results.

It is, however better to express it in terms of $k_F = (3\pi^2\rho)^{1/3}$, where $\rho = N/V$—the Fermi wave vector in the case $N_\uparrow = N_\downarrow = N/2$, which does not change when the ratio $r = p_{F-}/p_{F+}$ (i.e., the system's polarization) is varied. Since $p_{F+} = k_F(2/(1+r^3))^{1/3}$,

$$\frac{E_\Omega}{V} = \frac{\hbar^2 k_F^2}{2m_f}\frac{k_F^3}{3\pi^2}\left(\frac{2}{1+r^3}\right)^{5/3}\left\{\frac{3}{10}(1+r^5) + \frac{2}{3\pi}r^3\left(\frac{2}{1+r^3}\right)^{1/3}(k_F a_0) + \frac{96}{\pi^2}\left(\frac{2}{1+r^3}\right)^{2/3}(k_F a_0)^2 J(r,1) + \ldots\right\}. \quad (22)$$

This energy density is plotted in Fig. 5 as a function of the system's polarization $P = (N_+ - N_-)/N$ ($0 \leqslant P \leqslant 1$; it is related to the variable r by $r = [(1-P)/(1+P)]^{1/3}$) for five different values of the expansion parameter $k_F a_0$. We have checked that our result agrees with that of Ref. [22].[8] The computed correction of order $(k_F a_0)^2$ is rather small, ∼1% for $k_F a_0 = 0.2$ and $r = 1$ and decreases with decreasing r. All curves assume at $P = 1$ the same value $2^{2/3} = 1.5874$—due to the Pauli exclusion principle interactions do not induce any corrections to the ground state of a fully polarized ($P = 1, r = 0$) system of fermions.[9] Note also that the prefactor $\hbar^2 k_F^5/6\pi^2 m_f$ in (22) can be written in the form $(N/V)(\hbar^2 k_F^2/2m_f)$. Therefore our Fig. 5 can be directly compared to Fig. 3 of Ref. [23]: our curve for $k_F a_0 = 0.6$ corresponds to the lowest curve in this plot obtained for a model repulsive potential by a numerical estimate of the exact ground state energy. It is seen that the result of the second-order expansion is somewhat lower than the numerical estimate. This agrees with the comparison of the ground-state energies of the unpolarized system ($P = 0$, $r = 1$) performed in Fig. 2 of Ref. [23] which shows that the perturbative expansion of the ground-state energy in powers of $k_F a_0$ remains reliable up to $k_F a_0 \lesssim 0.5$ but is systematically below the numerical estimates of the exact value.

IV. SUMMARY

We have recomputed the order $(k_F a_0)^2$, where a_0 is the s-wave scattering length and $k_F = (3\pi^2 N/V)^{1/3}$, correction to the ground state energy of a polarized gas of (nonrelativistic) fermions of spin 1/2 using the effective theory approach proposed in Ref. [3], which does not require specifying explicitly the (spin independent) interaction potential. We have demonstrated the cancellation of ultraviolet divergences when the result is expressed in terms of the scattering length. Our result obtained by the method applicable to arbitrary repulsive interaction potentials is identical with that of Ref. [22] obtained with the help of traditional methods within the specific model of hard spheres. That it should be so is almost obvious in the effective theory approach but was not such in the old framework.

Second-order corrections to the energy levels of systems of spin 1/2 fermions have been used to investigate the order of the phase transition from the paramagnetic to the ferromagnetic state [26] and of the separation of phases in mixtures of fermions of different masses [27]. While potentially important, such study seem to require, however, better approximations to system's energy levels than the second order one. Since the main technical problem of this approach is only isolating ultraviolet divergences and working out cancellation of imaginary contributions, it seems that with some more labor the computations presented here could be extended to yet higher orders of the expansion, similarly as was done

[8]The precise relation, checked numerically, of the function $I(P)$ used in Ref. [22] (the variable r used there is our polarization P) to the function J defined by (14) evaluated with the cutoff $\Lambda = 0$ and $p_{F+} = 1$ is

$$I(P) = 160(1+P)^{7/3}J(r(P),1), \quad r(P) = ((1-P)/(1+P))^{1/3}.$$

[9]This readily follows from the form of the effective interaction written in terms of the field operators $\psi_\pm$, $\psi_\pm^\dagger$ introduced in (10) and the absence of the "sea" of oppositely polarized fermions.

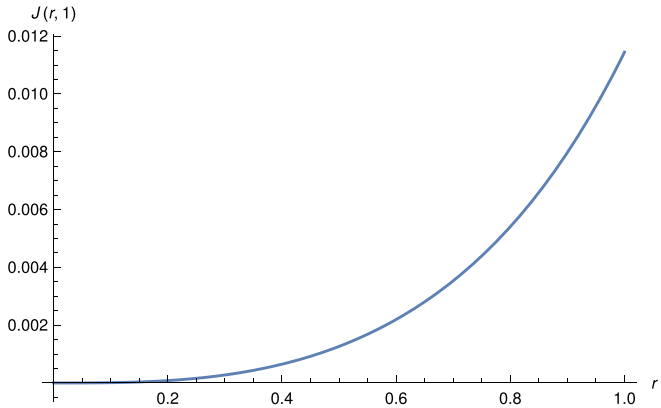

FIG. 4. Plot of the function $J(r,1)$. The value $J(1,1) = 0.0114449 = (11 - 2\ln 2)/840$ is the result of Ref. [3].

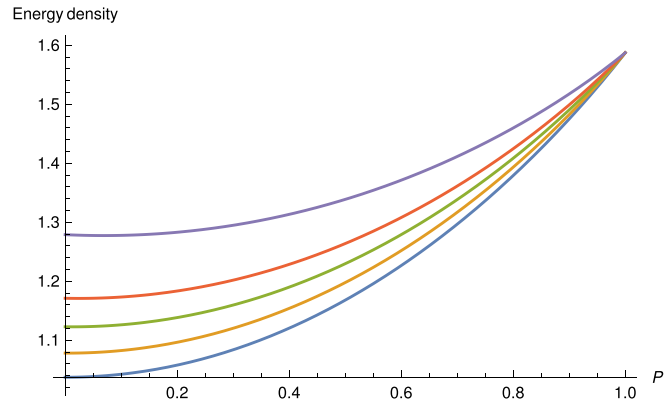

FIG. 5. Energy density E_Ω/V in units $(3/5)\hbar^2 k_F^5/6\pi^2 m_f = (N/V)(\hbar^2 k_F^2/2m_f)(3/5)$ of the polarized gas of spin 1/2 fermions as a function of its polarization $P = (N_+ - N_-)/N$ for different values (from below) 0.1 (blue), 0.2 (yellow), 0.3 (green), 0.4 (red), and 0.6 (blue) of the expansion parameter $k_F a_0$.

in the case of unpolarized system in Refs. [3,14]. A more challenging task would be obtaining a rigorous estimate of the high-order terms of the perturbative expansion which could allow to assess the range of its convergence.

In this paper, we have considered the polarized diluted gas of (nonrelativistic) interacting spin 1/2 fermions, working in the continuum version of the theory. Our results can be most naturally applied to atomic gases bound in traps. An analogous problem can of course be also formulated using the lattice version, that is within the paradigmatic Hubbard model, with obvious applications to atomic gases bound in periodic laser traps and to the solid state systems. As far as we know, there are no second order results similar to ours in this other version (rigorous first-order results have been given in Refs. [28,29]) and it would be interesting to try to obtain them.

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APPENDIX

We give here details of the calculation of the integrals over the Fermi spheres.

As explained, if $0 \leqslant s \leqslant s_0 = \frac{1}{2}(p_{\mathrm{F}+} - p_{\mathrm{F}-})$ the integration over $d^3\mathbf{u}$ in (14) is just over the exterior of the larger sphere of radius $p_{\mathrm{F}+}$. Thus, in the spherical variables $u_x = u\sin\vartheta\cos\varphi$, $u_y = u\sin\vartheta\sin\varphi$, and $u_z = u\cos\vartheta \equiv u\xi$, the integration over du is bounded from below by the condition $u^2 - 2us\xi - (p_{\mathrm{F}+}^2 - s^2) = 0$:

$$\begin{aligned} g(t,s) &= \frac{1}{2}\int_{-1}^{+1} d\xi \int_{u(\xi,s)}^{\infty} du \frac{u^2}{t^2 - u^2 + i0} \\ &= \int_0^{\infty} du \frac{u^2}{t^2 - u^2 + i0} \\ &\quad + \frac{1}{2}\int_{-1}^{+1} d\xi \int_0^{u(\xi,s)} du \frac{u^2}{u^2 - t^2 - i0}, \end{aligned}$$

where $u(\xi,s) = s\xi + \sqrt{p_{\mathrm{F}+}^2 - s^2(1-\xi^2)}$. The first, divergent, integral is, when regularized by the cutoff Λ, proportional to the integral (6). The second one can be worked out using the trick given in Appendix C of Ref. [5], that is, introducing under the integral over ξ the factor $1 = d\xi/d\xi$, taking this integral by parts and then trading the remaining integration over ξ for the integration over $u(\xi,s)$. This, upon using the result (7), leads to ($v = u^2(\xi,s)$; terms of (7) vanishing in the limit $\Lambda \to \infty$ are omitted)

$$\begin{aligned} g(t,s) = -i\frac{\pi}{2}t - \Lambda + \frac{1}{2}\Bigg\{ &\int_0^{u(1,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &+ \int_0^{u(-1,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &- \frac{1}{4s}\int_{u^2(-1,s)}^{u^2(1,s)} dv \frac{v - \left(p_{\mathrm{F}+}^2 - s^2\right)}{v - t^2 - i0} \Bigg\}. \end{aligned}$$

The sum of imaginary parts of the three integrals should cancel the explicit imaginary contribution which resulted from the divergent integral. In fact, of the three integrals only the first two do develop an imaginary part.[10] They are evaluated using the Sochocki formula and give

$$\int_0^{u_{\max}} du \frac{u^2}{u^2 - t^2 - i0} = i\frac{\pi}{2}t + u_{\max} + \frac{t}{2}\ln\frac{u_{\max} - t}{u_{\max} + t}. \tag{A1}$$

The remaining integral in which one can omit $-i0$ in the denominator is then also easy to evaluate and, taking into account that $u(1,s) = p_{\mathrm{F}+} + s$, $u(-1,s) = p_{\mathrm{F}+} - s$ one obtains the result (15) which is real (all imaginary parts have canceled) as it should be. In the range $s_0 \leqslant s \leqslant s_{\max}$, the function $g(t,s)$ is given by the sum (19) of two integrals. Their divergent parts, extracted as previously, combine to the integral $-2\pi^2 I_0$ and one obtains

$$\begin{aligned} g(t,s) = -\Lambda - i\frac{\pi}{2}t &+ \frac{1}{2}\int_{\xi_0}^{1} d\xi \int_0^{u_+(\xi,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &+ \frac{1}{2}\int_{-1}^{\xi_0} d\xi \int_0^{u_-(\xi,s)} du \frac{u^2}{u^2 - t^2 - i0}. \end{aligned}$$

To work out the imaginary parts of the two remaining integrals, we again use the trick with taking the integral over ξ by parts after inserting into it $1 = d\xi/d\xi$. This gives

$$\begin{aligned} &\int_{\xi_0}^{1} d\xi \int_0^{u_+(\xi,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &= \int_0^{u_+(1,s)} du \frac{u^2}{u^2 - t^2 - i0} - \xi_0 \int_0^{u_+(\xi_0,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &\quad - \frac{1}{4s}\int_{u_+^2(\xi_0,s)}^{u_+^2(1,s)} dv \frac{v - \left(p_{\mathrm{F}+}^2 - s^2\right)}{v - t^2 - i0}, \end{aligned}$$

and, similarly,

$$\begin{aligned} &\int_{-1}^{\xi_0} d\xi \int_0^{u_-(\xi,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &= \xi_0 \int_0^{u_-(\xi_0,s)} du \frac{u^2}{u^2 - t^2 - i0} + \int_0^{u_-(-1,s)} du \frac{u^2}{u^2 - t^2 - i0} \\ &\quad + \frac{1}{4s}\int_{u_-^2(-1,s)}^{u_-^2(\xi_0,s)} dv \frac{v - \left(p_{\mathrm{F}-}^2 - s^2\right)}{v - t^2 - i0}. \end{aligned}$$

Since $u_+(\xi_0,s) = u_-(\xi_0,s) = u_0$, the terms explicitly proportional to ξ_0 mutually cancel out. The first integrals on the right hand sides of these formulas are of the form (A1); the imaginary part of their sum [the above expression enters $g(t,s)$ divided by two] precisely cancels the imaginary part which arose from the divergent integral. The remaining integrals (in which $i0$ in the denominators can be omitted) can be then easily evaluated giving the result (20).

[10] Indeed, it is easy to see geometrically that the maximal value of t reached in the outer integral is $s + p_{\mathrm{F}-}$, whereas the lower limit of the third integral is $(p_{\mathrm{F}+} - s)^2$; since the function $g(t,s)$ computed here is valid only up to $s \leqslant s_0 = (p_{\mathrm{F}+} - p_{\mathrm{F}-})/2$, the variable t^2 never exceeds the lower limit of integration over v.

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