


Composite fermion nonlinear sigma models

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We study particle-hole symmetry at the integer quantum Hall plateau transition using composite fermion mean-field theory. Because this theory implicitly includes some electron-electron interactions, it also has applications to certain fractional quantum Hall plateau transitions. Previous work [P. Kumar *et al.*, *Phys. Rev. B* **100**, 235124 (2019)] using this approach showed that the diffusive quantum criticality of this transition is described by a nonlinear sigma model with topological $\theta = \pi$ term. This result, which holds for both the Dirac and Halperin, Lee, and Read composite fermion theories, signifies an emergent particle-hole (reflection) symmetry of the integer (fractional) quantum Hall transition. Here we consider the stability of this result to various particle-hole symmetry-violating perturbations. In the presence of quenched disorder that preserves particle-hole symmetry, we find that finite longitudinal conductivity at this transition requires the vanishing of a symmetry-violating composite fermion effective mass, which if present would generally lead to $\theta \neq \pi$ and a corresponding violation of particle-hole symmetric electrical transport $\sigma_{xy} \neq \frac{1}{2} \frac{e^2}{h}$. When the disorder does not preserve particle-hole symmetry, we find that θ can vary continuously within the diffusive regime. Our results call for further study of the universality of the quantum Hall plateau transition.

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I. INTRODUCTION

A. Overview

Electrons in two spatial dimensions exhibit quantized Hall plateaus when their density is comparable to the strength of a transverse magnetic field [1,2]. Within the picture of *disordered noninteracting electrons*—appropriate for the integer quantum Hall effect [3]—these plateaus are due to the localization of all electronic states, except for those lying at a single energy level within each Landau level [4]. An integer quantum Hall plateau transition occurs as the electron chemical potential is tuned through this critical energy, typically by varying the magnetic field at fixed electron density [5]. While accounting for the basic phase structure, this description has some unsatisfactory features, in relation to the observed quantum critical phenomena [6]. Because the set of delocalized states is of measure zero, interactions must be included for nonzero finite-temperature longitudinal conductivity [7]. The numerically calculated localization length critical exponent ν_{calc} (varying from about 2.48–2.62 with the specific lattice model and calculation details [8–16]) do not appear to lie within the error bars of the measured exponent $\nu \approx 2.38$ [17,18]. (Some recent works are critical of these theoretical results: inclusion of additional types of disorder may be relevant [19] and/or the critical scaling regime may require significantly larger systems sizes [20]; see also [21,22].) Furthermore, the

difference of the calculated and measured dynamical critical exponents $z_{\text{calc}} - z \approx 1$ [23]. These disagreements are not entirely surprising since the Coulomb interaction is expected to be a relevant perturbation of the noninteracting fixed point [24,25]. Thus, a faithful model of the integer quantum Hall transition is thought to be interacting.

Composite fermions [26,27] provide a powerful dual framework for the quantum Hall effect. Intuitively, this duality proceeds by an exact transformation of the electron wave function by “flux attachment” [28]. A $(2 + 1)$ d effective field theory implementing this transformation consists of a composite fermion coupled to a Chern-Simons gauge field whose level is determined by the electron filling fraction [29–31]. A large variety of integer and fractional quantum Hall effects can then be understood within a *composite fermion mean-field theory*—in which fluctuations of this Chern-Simons gauge field and all other interactions are ignored—as integer quantum Hall states of composite fermions. Although inadequate to fully account for the effects of electron-electron interactions, e.g., $\nu_{\text{calc}} \approx 2.56$ and $z_{\text{calc}} = 2$ [32], semiclassical reasoning and numerical calculations suggest that composite fermion mean-field theory provides a unified framework for the quantum Hall plateau transitions [33–35], hosting delocalized states for a range of critical energies [36,37].

This paper is concerned with an *analytical* description of the plateau transition using composite fermions and, in

particular, any symmetry it might exhibit. We focus on the integer quantum Hall transition, however, our results can be readily adapted to certain fractional quantum Hall plateau transitions [38]. The relevant symmetry is therefore particle-hole symmetry [39]; the generalization of this symmetry to appropriate fractional quantum Hall transitions is known as reflection symmetry [38,40–42]. In the limit of infinite cyclotron frequency, the particle-hole symmetry transformation relates states at electron filling fractions $\nu \leftrightarrow 1 - \nu$ (in lowest Landau level). This symmetry is not present in the microscopic electron Hamiltonian. Nevertheless, evidence for an emergent particle-hole symmetry is seen in electrical transport measurements that find the DC Hall conductivity at $\nu = 1/2$ to equal $\sigma_{xy} = \frac{1}{2} \frac{e^2}{h}$ (see [43,44] and references therein). In the composite fermion theory of Halperin, Lee, and Read (HLR) [31], this electrical Hall conductivity requires [45] (and occurs [37,46–48] in the presence of weak disorder $\rho_{xx} \neq 0$) a composite fermion Hall conductivity $\sigma_{xy}^{\text{cf}} = -\frac{1}{2} \frac{e^2}{h}$ [49]. From the point of view of composite fermion mean-field theory, particle-hole symmetry helps to explain the diffusive quantum criticality of the plateau transition: Because localized free fermions necessarily exhibit integral Hall conductivity [3], the half-integer value of σ_{xy}^{cf} (in units of e^2/h) implies the critical states are delocalized [50]. Our interest in this paper is to better understand whether particle-hole symmetric σ_{xy}^{cf} is necessary for the diffusive quantum criticality. In particular, how universal is particle-hole symmetry to the integer quantum Hall plateau transition?

B. Detailed summary

We now provide a more detailed overview of this paper. There are two types of composite fermions theories for electrons at $\nu = 1/2$: the HLR theory [31], which involves a nonrelativistic composite fermion coupled to a Chern-Simons gauge field; and the Dirac composite fermion theory, introduced by Son [46] and further developed in a number of subsequent works [47,51–53], in which the nonrelativistic composite fermion is replaced by a Dirac fermion. Using a standard Dirac notation, the mean-field Lagrangians for HLR ($\eta = 1$) and Dirac ($\eta = 0$) composite fermion theories can be expressed as (see Sec. II)

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ + \phi)\Psi - \Psi^\dagger P_2(i\eta\partial_0 + a_0)\Psi + m_1\Psi^\dagger\Psi + m_2\bar{\Psi}\Psi. \quad (1.1)$$

Here Ψ is a two-component Dirac fermion; $a_0(\mathbf{x})$ and $a_j(\mathbf{x})$ for $j \in \{1, 2\}$ are possible uncorrelated quenched scalar and vector potential disorders (probing electromagnetic fields are set to zero); P_2 projects onto the second component of Ψ ; m_1 is a chemical potential that fixes the nonzero composite fermion density; and m_2 is a possible mass term. In this formulation (1.1), particle-hole symmetry acts as time-reversal invariance. Consequently, the terms involving P_2 and m_2 violate particle-hole symmetry. The vector potential disorder $a_j(\mathbf{x})$ preserves particle-hole symmetry for an appropriate choice of disorder ensemble (see Sec. II A).

The numerical works [36,37] that found the HLR composite fermion mean-field theory to exhibit an integer quantum Hall plateau transition with particle-hole symmetric transport at $\nu = 1/2$, assumed a particle-hole symmetric ensemble

for $a_j(\mathbf{x})$ and set $a_0(\mathbf{x})$ to zero. Since m_2 is generally nonzero in the HLR theory, this result is consistent with an emergent particle-hole symmetry. The Dirac composite fermion mean-field theory (1.1) manifestly has particle-hole symmetry so long as time-reversal invariance is preserved. Significant progress towards an analytical description of this transition was then made in [54], where it was shown that, for particle-hole symmetric disorder and when $m_2 = 0$, the DC zero-temperature electrical conductivity of composite fermions is encoded in the nonlinear sigma model (NLSM),

$$S_{\text{NLSM}} = \int d^2x \text{Tr} \left(\frac{1}{2g} (\partial_j Q)^2 + i \frac{\theta}{16\pi} \epsilon_{ij} Q \partial_i Q \partial_j Q \right), \quad (1.2)$$

with $\theta = \pi$. Here the matrix boson $Q \in U(2n)/U(n) \times U(n)$ parametrizes composite fermion charge density fluctuations with the replica limit $n \rightarrow 0$ to be taken at the end of any calculation using S_{NLSM} . The marginally relevant coupling [55] (in the renormalization group sense), $g \propto 1/\sigma_{xx}^{\text{cf}}$, characterizes the evolution from the ballistic to the diffusive regime of finite (impurity-averaged) conductivity σ_{ij}^{cf} (henceforth measured in units of e^2/h). If only the first term in S_{NLSM} is present, the marginal relevancy of g implies “weak” localization $\sigma_{xx}^{\text{cf}} \rightarrow 0$ in the thermodynamic limit. The topological θ term [56] is believed to prevent this localization when $\theta = \pi$ and (if unrenormalized) thereby *explain* the quantum critical diffusion [57–63]. As such, θ and σ_{xy}^{cf} are related [54,57,61]:

$$\theta = \pi + 2\pi \left(\sigma_{xy}^{\text{cf}} - \frac{\eta}{2} \right) \text{mod } 2\pi. \quad (1.3)$$

Here σ_{xy}^{cf} denotes either the HLR or Dirac DC composite fermion Hall conductivity, with particle-hole symmetric values $\sigma_{xy}^{\text{cf}} = -\frac{\eta}{2}$. Two features of this NLSM are worth emphasizing: first, it is identical in form to that argued by Pruisken *et al.* from the (dual) electron perspective [57,58]; second, $\theta = \pi$ regardless of whether or not η is nonzero. [Intuitively, this occurs because the dependence on η drops out in the DC limit in which the time derivative in (1.1) can be ignored.] Thus, in the presence of particle-hole symmetric disorder, the HLR and Dirac composite fermion mean-field theories are controlled by the *same* IR fixed point.

In this paper we generalize [54] to include uniform m_2 and random $a_0(\mathbf{x})$. First for vanishing $a_0(\mathbf{x})$ and when particle-hole symmetric disorder $a_j(\mathbf{x})$ is present, we find that finite σ_{xx}^{cf} requires the renormalized $m_2 = 0$. The resulting NLSM reduces to that found in [54] in which a $\theta = \pi$ term appears. We interpret this as an emergent particle-hole symmetry and the irrelevance of m_2 near the diffusive quantum critical point. When $a_0(\mathbf{x})$ is included, we find that the topological θ term varies continuously with the strength of this particle-hole symmetry-violating disorder, consistent with the simultaneous generation of nonzero symmetry-violating m_2 . Because the NLSM formulation is controlled only for $\sigma_{xx}^{\text{cf}} \gg 1$ and the beta functions that determine the two-parameter flow of $1/g$ and θ are nonperturbative [64], we are unable here to determine whether the ultimate low-energy fixed point of the sigma model with $\theta \neq \pi$ is an insulator or a quantum critical metal with varying Hall conductivity.

The remainder of this paper is organized as follows. In Sec. II we review the HLR and Dirac composite fermion

mean-field theories with the aim of deriving the mean-field Lagrangian (1.1). In Sec. III A we construct the generating functional for disorder-averaged products of retarded and advanced composite fermion Green's functions. Encoded in this generating functional are observables such as the composite fermion density of states and conductivity. In Sec. III B we derive the NLSM for composite fermion mean-field theory, focusing on the topological θ term. This NLSM contains the effects of particle-hole preserving and particle-hole breaking disorders. In Sec. IV we conclude with a discussion of our results and of possibilities for future work. Appendices supplement arguments in the main text. Unless stated otherwise we take $e^2 = \hbar = 1$.

II. COMPOSITE FERMION MEAN-FIELD THEORY

We begin with a derivation of the mean-field Lagrangian (1.1) for the HLR and Dirac composite fermion theories.

A. HLR mean-field theory

The HLR Lagrangian is [31]

$$\mathcal{L}_{\text{HLR}} = \psi^\dagger \left(i\partial_0 + A_0 + V + a_0 - \frac{(i\partial_j + A_j + a_j)^2}{2m} \right) \psi + \frac{\epsilon^{\mu\nu\rho}}{8\pi} a_\mu \partial_\nu a_\rho + \dots \quad (2.1)$$

Here $\psi^\dagger(t, \mathbf{x})$ creates an HLR composite fermion; $A_\mu(t, \mathbf{x})$ is the external electromagnetic field; $a_\mu(t, \mathbf{x})$ is a dynamical $(2+1)$ d Chern-Simons gauge field; m is an effective mass, $\epsilon^{\mu\nu\rho}$ with $\mu, \nu, \rho \in \{0, 1, 2\}$ is the antisymmetric symbol with $\epsilon^{012} = 1$; and the " \dots " include all other possible couplings and interactions, which we set to zero in the remainder. A nonzero uniform magnetic field $\epsilon_{ij}\partial_i A_j = B > 0$ is assumed such that the electron filling fraction $\nu = 1/2$. Variation of \mathcal{L}_{HLR} with respect to A_0 implies the electron and composite fermion densities are equal. $V(\mathbf{x})$ represents possible quenched electron scalar potential disorder. In this paper we assume that $V(\mathbf{x})$ is particle-hole symmetric, i.e., the disorder ensemble for $V(\mathbf{x})$ has vanishing odd moments $\overline{V(\mathbf{x}_1) \cdots V(\mathbf{x}_{2p+1})} = 0$.

The mean-field approximation can be divided into two steps. The first is to impose the a_0 equation of motion, i.e., flux attachment,

$$\psi^\dagger \psi = -\frac{1}{4\pi} \epsilon^{ij} \partial_i a_j. \quad (2.2)$$

This results in an effective magnetic field $B + \epsilon_{ij}\partial_i a_j$ that vanishes on average at half-filling, which we accommodate by shifting $A_i + a_i \rightarrow a_i$. Flux attachment (2.2) also implies that weak (electron) scalar potential disorder $V(\mathbf{x})$ is realized as anticorrelated vector and scalar potential randomness [48,65]:

$$\epsilon_{ij} \partial_i a_j(\mathbf{x}) = -2mV(\mathbf{x}). \quad (2.3)$$

The second (and final) step of the mean-field approximation is to set all dynamical fluctuations of $a_\mu(t, \mathbf{x})$ to zero. Thus, $a_\mu(\mathbf{x})$ is a background field in the mean-field approximation.

We choose a disorder ensemble for $V(\mathbf{x})$ such that the resulting vector potential $a_j(\mathbf{x})$ determined by (2.3) is a transverse Gaussian random variable with zero mean and variance

$W > 0$:

$$\overline{a_i(\mathbf{x})} = 0, \quad \overline{a_i(\mathbf{x})a_j(\mathbf{x}')} = W\delta_{ij}\delta(\mathbf{x} - \mathbf{x}'). \quad (2.4)$$

Such delta-function correlated vector potentials (2.4) arise from power-law correlated scalar potentials $V(\mathbf{x})V(\mathbf{x}') \propto |\mathbf{x} - \mathbf{x}'|^{-4}$ [66]. We remark that nonrelativistic fermions coupled to power-law correlated vector potential disorder without any scalar potential disorder obtain singular single-particle properties; two-particle properties remain regular and appear to coincide with delta-function correlated vector potential disorder [67–69].

Quenched disorder satisfying (2.3) appears to be necessary for particle-hole symmetric composite fermion Hall conductivity $\sigma_{xy}^{\text{cf}} = -1/4\pi$ [36,37,48]. To further explore this, we will include additional quenched disorder $a_0(\mathbf{x})$ coupling to $\psi^\dagger \psi$ that is independent of $a_i(\mathbf{x})$. For simplicity, we will take $a_0(\mathbf{x})$ to be a Gaussian random variable with zero mean and variance $W_0 > 0$. We refer to $a_0(\mathbf{x})$ as particle-hole violating disorder; as we will see, this terminology is justified in Dirac composite fermion mean-field theory and also in the nonlinear sigma model associated with the HLR and Dirac composite fermion theories. How such disorder $a_0(\mathbf{x})$ might arise if $V(\mathbf{x})$ also violates particle-hole symmetry and/or when interactions ignored in composite fermion mean-field theory are included are interesting questions we leave for future work.

Having included these effects, we can write the mean-field Lagrangian in a Dirac form by factorizing the spatial derivative terms as

$$\mathcal{L}_{\text{HLR}}^{\text{mf}} = \psi^\dagger (i\partial_0 + E_F - a_0) \psi - \psi^\dagger i v (D_1 + iD_2) \chi - \chi^\dagger i v (D_1 - iD_2) \psi + 2mv^2 \chi^\dagger \chi \quad (2.5)$$

using the auxiliary fermion [46]

$$\chi = \frac{i}{2mv} (D_1 - iD_2) \psi, \quad \chi^\dagger = -\frac{i}{2mv} (D_1^* + iD_2^*) \psi^\dagger, \quad (2.6)$$

where $D_j = \partial_j - ia_j$, the Fermi energy $E_F > 0$ fixes $\psi^\dagger \psi = B/4\pi$ on average, and $a_\mu(\mathbf{x})$ denotes the possible quenched randomness discussed above. We set the arbitrary velocity $v > 0$ to unity by rescaling χ . Notice that only ψ is dynamical in this Dirac formulation of HLR mean-field theory; χ is nondynamical. Introducing $\Psi = (\psi \ \chi)^T$ and the gamma matrices $(\Gamma^0, \Gamma^1, \Gamma^2) = (\sigma^3, i\sigma^1, i\sigma^2)$, where σ^i are the usual Pauli matrices, the resulting mean-field Lagrangian is

$$\mathcal{L}_{\text{HLR}}^{\text{mf}} = \bar{\Psi} (i\partial + \phi) \Psi - \Psi^\dagger P_2 (i\partial_0 + a_0) \Psi + \left(\frac{E_F}{2} + m \right) \Psi^\dagger \Psi + \left(\frac{E_F}{2} - m \right) \bar{\Psi} \Psi, \quad (2.7)$$

where $\bar{\Psi} = \Psi^\dagger \Gamma^0$, $i\partial + \phi = \Gamma^0 (i\partial_0 + a_0) + \Gamma^j (i\partial_j + a_j)$, and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projects onto the second component of Ψ . This Lagrangian coincides with (1.1) at $\eta = 1$ upon identifying $m_1 = E_F/2 + m$ and $m_2 = E_F/2 - m$.

B. Dirac mean-field theory

The Dirac composite fermion Lagrangian is [46,47,51,52]

$$\begin{aligned} \mathcal{L}_D = & \bar{\Psi}(i\cancel{\partial} + \cancel{a})\Psi - m_D \bar{\Psi}\Psi - \frac{1}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho \\ & + \frac{1}{8\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \dots, \end{aligned} \quad (2.8)$$

where the Dirac notation is defined below (2.7); $a_\mu(t, \mathbf{x})$ is a dynamical $(2+1)$ d gauge field; $A_\mu(t, \mathbf{x})$ is the external electromagnetic field; m_D is a $(2+1)$ d Dirac mass; and the “...” refer to all other possible couplings and interactions that will be ignored. The Fermi velocity has been set to unity. A uniform magnetic field $\epsilon_{ij}\partial_i A_j = B > 0$ and an electron filling fraction $\nu = 1/2$ are assumed. Variation of \mathcal{L}_D with respect to A_0 gives the electron density $n_e = \frac{B - \epsilon_{ij}\partial_i a_j}{4\pi}$. Consequently, fluctuations about $\nu = 1/2$ are controlled by $\epsilon_{ij}\partial_i a_j$. In particular, quenched electron scalar potential disorder induces randomness in the vector potential a_i . We will take this induced randomness in a_i to be the same as in (2.4), i.e., a_j will be a transverse Gaussian random variable with zero mean and variance $W > 0$ in the mean-field approximation.

Similar to the HLR theory, the mean-field approximation consists of first imposing the a_0 equation of motion,

$$\Psi^\dagger \Psi = \frac{B}{4\pi}. \quad (2.9)$$

Here we see that the composite fermion density is fixed, not by the electron density as is the case in the HLR theory, but by the external magnetic field. While at half-filling there is no difference between the two, differences can appear when deviations from half-filling are considered. After (2.9) is imposed, we then set all dynamical fluctuations of $a_\mu(t, \mathbf{x})$ to zero to achieve the mean-field approximation. In the remainder, $a_\mu(\mathbf{x})$ is understood to be a quenched random variable.

An important distinction between the Dirac and HLR composite fermion Lagrangians is that particle-hole symmetry has a simple implementation [70] in \mathcal{L}_D [46,47]. In particular, particle-hole symmetry acts as time-reversal invariance in Dirac composite fermion mean-field theory: $i \rightarrow -i$, $t \rightarrow -t$, $\Psi \rightarrow i\sigma^2\Psi$. [We see that the P_2 term in the Dirac formulation (2.7) of HLR mean-field theory explicitly violates this definition of particle-hole symmetry.] As such, the Dirac composite fermion mass m_D breaks particle-hole symmetry. Vector potential disorder $a_j(\mathbf{x})$ satisfying (2.4) preserves particle-hole symmetry on average. As in the HLR theory, we consider the effects of particle-hole violating disorder by including quenched scalar potential randomness $a_0(\mathbf{x})$ that couples only to the top component of Ψ . This term explicitly violates particle-hole symmetry. We will again take $a_0(\mathbf{x})$ to be a Gaussian random variable with zero mean and variance $W_0 > 0$.

The resulting mean-field Lagrangian is

$$\mathcal{L}_D^{\text{mf}} = \bar{\Psi}(i\cancel{\partial} + \cancel{a})\Psi - \Psi^\dagger P_2 a_0 \Psi + E_F \Psi^\dagger \Psi - m_D \bar{\Psi}\Psi, \quad (2.10)$$

where $E_F > 0$ fixes (2.9) on average and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that in contrast to the HLR theory, (2.9) involves both components of Ψ and so E_F fixes the total Dirac fermion density; further comparing (2.7) with (2.10), both components of Ψ are

dynamical in $\mathcal{L}_D^{\text{mf}}$. We recover the general composite-fermion mean-field Lagrangian (1.1) at $\eta = 0$ upon identifying $m_1 = E_F$ and $m_2 = -m_D$.

III. EFFECTIVE ACTION FOR CHARGE DIFFUSION

In this section we first construct the generating functional for disorder-averaged composite fermion Green's functions for the mean-field Lagrangian (1.1). We then derive the NLSM for composite fermion charge diffusion in the presence of particle-hole symmetric and particle-hole violating disorders.

A. Generating functional

The generating functional of disorder-averaged products of retarded/advanced composite fermion Green's functions (that determine observables such as the conductivity) is a Euclidean two-dimensional (2D) field theory. This is a 2D rather than three-dimensional (3D) field theory because, in the absence of inelastic scattering, each frequency component of the composite fermion evolves independently. Within the replica approach (see, e.g., [71,72]), this generating functional is

$$Z = \frac{1}{\mathcal{N}} \int \mathcal{D}[\Psi] \mathcal{D}[\Psi^\dagger] \mathcal{D}[a_\mu] e^{-S}, \quad (3.1)$$

with action $S = S_f + S_a$,

$$S_f = \int d^2x \Psi^\dagger \left(\gamma^5 \gamma^j D_j - \frac{1 - \gamma^5}{2} a_0 - M_1 - M_2 \gamma^5 \right) \Psi, \quad (3.2)$$

$$S_a = \int d^2x \left(\frac{1}{2W_0} a_0^2 + \frac{1}{2W} a_i^2 \right), \quad (3.3)$$

and normalization \mathcal{N} . Here S_f takes the form of an Euclidean 2D action [73] for $2n$ Dirac fermions,

$$\Psi_I = \begin{pmatrix} \psi_I \\ \chi_I \end{pmatrix}, \quad \Psi_I^\dagger = (\psi_I^\dagger \quad \chi_I^\dagger), \quad (3.4)$$

where $I \in \{1, \dots, n\}$ indexes the n replicas of retarded fermions and $I \in \{n+1, \dots, 2n\}$ indexes the n replicas of advanced fermions. The gamma matrices are

$$\gamma^1 = \sigma^2 \otimes \tau^0, \quad \gamma^2 = \sigma^1 \otimes \tau^0, \quad \gamma^5 = -\sigma^3 \otimes \tau^0, \quad (3.5)$$

where σ^j are the usual Pauli matrices that act on the spinor indices and τ^0 is the $2n \times 2n$ identity matrix acting on the replica indices. The replica limit $n \rightarrow 0$ is to be taken at the end of any calculation using Z [74]. Up to an overall sign, S_f coincides with $2n$ copies of the (spatial integral of the) composite fermion mean-field Lagrangian (1.1) with the replacements $\Psi(t, \mathbf{x}) \rightarrow \Psi_I(\omega, \mathbf{x})$ and $i\partial_t \rightarrow E_F \pm (\omega + i\epsilon)$, where the \pm sign distinguishes between retarded/advanced fermions. The mass matrices depend on the specific composite fermion theory: in the HLR composite fermion theory,

$$\begin{aligned} M_1 &= \left(\frac{E_F}{2} + m \right) \tau^0 + \frac{\omega + i\epsilon}{2} \tau^3, \\ M_2 &= -\left(\frac{E_F}{2} - m \right) \tau^0 + \frac{\omega + i\epsilon}{2} \tau^3; \end{aligned} \quad (3.6)$$

in the Dirac composite fermion theory,

$$M_1 = E_F \tau^0 + (\omega + i\epsilon)\tau^3, \quad M_2 = m_D \tau^0. \quad (3.7)$$

The diagonal matrix $\tau_{IJ}^3 = \delta_{IJ}$ for $I, J \in \{1, \dots, n\}$ and $\tau_{IJ}^3 = -\delta_{IJ}$ for $I, J \in \{n+1, \dots, 2n\}$; the mass matrices are diagonal in the gamma matrix indices. The Fermi energy E_F and frequency ω carried by Ψ enter M_1 and M_2 differently in the HLR and Dirac theories because only the top component of Ψ is dynamical in the HLR theory. The frequency of the composite fermion is merely a parameter in this 2D theory. We will set $\omega = 0$ henceforth to focus on the DC conductivity. We identify

$$m_1 = \frac{1}{2n} \text{Tr} M_1, \quad m_2 = -\frac{1}{2n} \text{Tr} M_2. \quad (3.8)$$

S_f has a $U_R(n) \times U_A(n) \subset U(2n)$ symmetry with the infinitesimal $\epsilon > 0$ distinguishing between retarded and advanced fermions.

S_a defines the Gaussian ensembles for the independent random variables $a_0(\mathbf{x})$ and $a_i(\mathbf{x})$. The interaction between a_μ and Ψ in this 2D theory is responsible for the diffusion of composite fermions. Analyzing this interaction is the subject of the next section. We will assume a regularization preserving the $U(2n)$ vector symmetry (present at $\omega = \epsilon = 0$) that identically rotates the top $\psi = \frac{1}{2}(1 - \gamma^5)\Psi$ and bottom $\chi = \frac{1}{2}(1 + \gamma^5)\Psi$ spinor components of Ψ .

B. NLSM derivation

From the generating functional (3.1), we now derive the NLSM for composite fermion DC charge diffusion, focusing on the θ term and its physical interpretation. The derivation of the leading nontopological term in this NLSM and additional

details for the calculation of the topological term can be found in Appendix B.

1. Saddle-point analysis

Performing the Gaussian integrals over a_0 and a_i , we obtain the “disorder-averaged action” $\bar{S} = \bar{S}^{(1)} + \bar{S}^{(2)} + \bar{S}^{(3)}$:

$$\bar{S}^{(1)} = \int d^2x \Psi^\dagger (\gamma^5 \gamma^j \partial_j - M_1 - M_2 \gamma^5) \Psi, \quad (3.9)$$

$$\bar{S}^{(2)} = \frac{W}{2} \int d^2x (\Psi^\dagger \gamma^5 \gamma^j \Psi) (\Psi^\dagger \gamma^5 \gamma^j \Psi), \quad (3.10)$$

$$\bar{S}^{(3)} = -\frac{W_0}{2} \int d^2x \left(\Psi^\dagger \frac{1 - \gamma^5}{2} \Psi \right) \left(\Psi^\dagger \frac{1 - \gamma^5}{2} \Psi \right). \quad (3.11)$$

The parentheses around fermion bilinears in $\bar{S}^{(2)}$ and $\bar{S}^{(3)}$ indicate contraction of the spinor and replica indices, e.g., $(\Psi^\dagger \gamma^5 \gamma^j \Psi) \equiv \Psi_{a,I}^\dagger \gamma_{ab}^5 \gamma_{bc}^j \Psi_{c,I}$ for $a, b, c \in \{1, 2\}$ and $I \in \{1, \dots, 2n\}$. The disorder-averaged action has the form of a massive $SU(2n)$ Thirring model [26] (with coupling $-W$) plus an additional chiral interaction $\bar{S}^{(3)}$. This interacting Euclidean 2D theory is translated to Minkowski signature in Appendix A.

We perform a semiclassical analysis of the interactions in \bar{S} . To this end, we first rewrite $\bar{S}^{(2)}$ as

$$\bar{S}^{(2)} = \frac{W}{2} \int d^2x \left[(\Psi_I^\dagger \Psi_J) (\Psi_J^\dagger \Psi_I) - (\Psi_I^\dagger \gamma^5 \Psi_J) (\Psi_J^\dagger \gamma^5 \Psi_I) \right] \quad (3.12)$$

and then we decouple each of the interactions using the Hubbard-Stratonovich fields X_{IJ} , Y_{IJ} , and Z_{IJ} as

$$e^{-\bar{S}^{(2)} - \bar{S}^{(3)}} = \int \mathcal{D}[X] \mathcal{D}[Y] \mathcal{D}[Z] e^{-\int d^2x \left[\frac{1}{2W} \text{Tr}(X^2 + Y^2) + \frac{1}{2W_0} \text{Tr} Z^2 - i(X_{IJ} + \frac{Z_{IJ}}{2}) \Psi_J^\dagger \Psi_I - (Y_{IJ} - i \frac{Z_{IJ}}{2}) \Psi_J^\dagger \gamma^5 \Psi_I \right]}, \quad (3.13)$$

where $\text{Tr} X^2 \equiv X_{IJ} X_{IJ}$, $\text{Tr} Y^2 \equiv Y_{IJ} Y_{IJ}$, and $\text{Tr} Z^2 \equiv Z_{IJ} Z_{IJ}$. Each of the fields X_{IJ} , Y_{IJ} , and Z_{IJ} transform in the adjoint rep of $U(2n)$. Next, we look for solutions to the saddle-point equations (at $\epsilon = 0$) for $\langle X_{IJ} \rangle$, $\langle Y_{IJ} \rangle$, and $\langle Z_{IJ} \rangle$:

$$X = \frac{W}{2} \text{Tr}_\sigma \left(\int \frac{d^2k}{(2\pi)^2} \frac{i}{i\gamma^5 \gamma^j k_j - (M_1 + iX + i\frac{Z}{2}) - (M_2 + Y - i\frac{Z}{2})\gamma^5} \right), \quad (3.14)$$

$$Y = \frac{W}{2} \text{Tr}_\sigma \left(\int \frac{d^2k}{(2\pi)^2} \frac{\gamma^5}{i\gamma^5 \gamma^j k_j - (M_1 + iX + i\frac{Z}{2}) - (M_2 + Y - i\frac{Z}{2})\gamma^5} \right), \quad (3.15)$$

$$Z = \frac{W_0}{4} \text{Tr}_\sigma \left(\int \frac{d^2k}{(2\pi)^2} \frac{1 - \gamma^5}{i\gamma^5 \gamma^j k_j - (M_1 + iX + i\frac{Z}{2}) - (M_2 + Y - i\frac{Z}{2})\gamma^5} \right), \quad (3.16)$$

where the trace Tr_σ is only taken over the spinor indices; and we write X for $\langle X_{IJ} \rangle$ and similarly for Y and Z . We consider the $U_R(n) \times U_A(n)$ -preserving ansatz:

$$\langle X \rangle = (\Gamma_+ + \Gamma_-) \tau^3 + iX_0 \tau^0, \quad (3.17)$$

$$\langle Y \rangle = Y_0 \tau^0, \quad (3.18)$$

$$\langle Z \rangle = -2\Gamma_- \tau^3 + iZ_0 \tau^0, \quad (3.19)$$

where Γ_\pm , X_0 , Y_0 , and Z_0 are real. Γ_\pm parametrize nonzero scattering rates of the composite fermions, while X_0 , Y_0 , and Z_0 parametrize possible renormalization of m_1 and m_2 [recall Eq. (3.8)]. We have set to zero a possible term in the ansatz for $\langle Y \rangle$ proportional to τ^3 that appears to prevent a $U_R(n) \times U_A(n)$ -preserving solution to these equations; preliminary analysis indicates that such a term requires a more general ansatz that only preserves a subgroup of $U_R(n) \times U_A(n)$.

We next present the detailed solution to these equations that results from the ansatz above. The conclusion from this analysis will be that the composite fermions obtain the renormalized mass matrix

$$\begin{pmatrix} (m_1^R + m_2^R)\tau^0 + i(\Gamma_+^R - \Gamma_-^R)\tau^3 & 0 \\ 0 & (m_1^R - m_2^R)\tau^0 + i(\Gamma_+^R + \Gamma_-^R)\tau^3 \end{pmatrix}. \quad (3.20)$$

Here the R superscript (dropped in the next section) indicates renormalized parameters. We will find that the saddle-point solutions allow nonzero m_2^R and unequal scattering rates $\Gamma_-^R \neq 0$ for the two components of Ψ only when particle-hole violating disorder is present ($W_0 \neq 0$).

To this end, we begin by defining the real parameters J_1 and J_2 :

$$iJ_1\tau^3 + J_2\tau^0 \equiv \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + (M_1 + iX + i\frac{Z}{2})^2 + (M_2 + Y - i\frac{Z}{2})^2}. \quad (3.21)$$

By direct evaluation we find

$$J_1 = \frac{\text{sgn}(\Gamma_- \tilde{m}_2 - \Gamma_+ \tilde{m}_1)}{8\pi} \left[\pi - 2 \arctan \left(\frac{\tilde{m}_2^2 - \tilde{m}_1^2 - \Gamma_-^2 + \Gamma_+^2}{2|\Gamma_- \tilde{m}_2 - \Gamma_+ \tilde{m}_1|} \right) \right], \quad (3.22)$$

$$J_2 = -\frac{1}{8\pi} \log \left(\frac{\Lambda^4}{[(\Gamma_+ + \Gamma_-)^2 + (\tilde{m}_1 + \tilde{m}_2)^2][(\Gamma_+ - \Gamma_-)^2 + (\tilde{m}_1 - \tilde{m}_2)^2]} \right), \quad (3.23)$$

Here we have introduced the shifted masses

$$\tilde{m}_1 = m_1 - X_0 - Z_0, \quad (3.24)$$

$$\tilde{m}_2 = -m_2 - Y_0, \quad (3.25)$$

and UV cutoff Λ . We search for a self-consistent solution when $|\Gamma_\pm/\tilde{m}_1| \ll 1$ and $|\tilde{m}_2/\tilde{m}_1| \ll 1$: this corresponds to the small scattering rate limit $\Gamma_\pm/E_F \ll 1$ with approximate particle-hole symmetry. In this regime, $J_1 \approx -1/4\pi$ and $J_2 \approx \frac{1}{2\pi} \log(|\tilde{m}_1|/\Lambda)$. Written in terms of these parameters the saddle-point equations (3.14)–(3.16) become

$$2\Gamma_- = W_0 [J_1(\tilde{m}_1 + \tilde{m}_2) + J_2(\Gamma_+ + \Gamma_-)], \quad (3.26)$$

$$Z_0 = -W_0 [J_1(\Gamma_+ + \Gamma_-) - J_2(\tilde{m}_1 + \tilde{m}_2)], \quad (3.27)$$

$$\Gamma_+ + \Gamma_- = -W (J_1\tilde{m}_1 + J_2\Gamma_+), \quad (3.28)$$

$$X_0 = -W (J_1\Gamma_+ - J_2\tilde{m}_1), \quad (3.29)$$

$$0 = W (J_1\tilde{m}_2 + J_2\Gamma_-), \quad (3.30)$$

$$Y_0 = -W (J_1\Gamma_- + J_2\tilde{m}_2). \quad (3.31)$$

We solve these equations for $X_0, Y_0, Z_0, \Gamma_\pm$, and m_2 :

$$m_2 = \frac{m_1 W_0 [J_2 + W (J_1^2 + J_2^2)]}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}, \quad (3.32)$$

$$Y_0 = \frac{-m_1 W_0 W (J_1^2 + J_2^2)}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}, \quad (3.33)$$

$$\Gamma_- = \frac{m_1 W_0 J_1}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}, \quad (3.34)$$

$$\Gamma_+ + \Gamma_- = \frac{-2m_1 W J_1}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}, \quad (3.35)$$

$$Z_0 = \frac{2m_1 W_0 (J_2 + W (J_1^2 + J_2^2))}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}, \quad (3.36)$$

$$X_0 = \frac{m_1 W [2J_2 + (W_0 + 2W)(J_1^2 + J_2^2)]}{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W) (J_1^2 + J_2^2)}. \quad (3.37)$$

Notice that Y_0, Γ_-, Z_0 , and m_2 vanish as $W_0 \rightarrow 0$. We treat m_2 as a variable in order to avoid an overly constrained set of equations; our interpretation is that W_0 and W disorders cause m_2 to renormalize to a value determined by the saddle-point

equations. To ensure the scattering rates $\Gamma_+ \pm \Gamma_- \geq 0$ of $\frac{1 \pm \gamma^5}{2} \Psi$ are nonnegative, we require

$$J_1[2 + J_2(3W_0 + 4W) + W(3W_0 + 2W)(J_1^2 + J_2^2)] \leq 0. \quad (3.38)$$

The retarded/advanced 2D Euclidean fermion Green's function that obtains from this saddle-point solution is

$$\bar{g}_{R/A}(\omega, \mathbf{x}, \mathbf{x}') = \langle \mathbf{x} | \frac{1}{i\sigma_1 \partial_1 - i\sigma_2 \partial_2 + (m_1 - X_0 - Z_0 + \omega \pm i\Gamma_+) + (m_2 + Y_0 + \omega \mp i\Gamma_-)\sigma_3} | \mathbf{x}' \rangle. \quad (3.39)$$

We have included nonzero ω , parametrizing deviations about the Fermi energy. This Green's function suggests how to absorb the logarithmic divergence J_2 into renormalized parameters. We first introduce the renormalization factors,

$$Z_R \equiv \frac{2 + J_2(3W_0 + 4W) + W(3W_0 + 2W)(J_1^2 + J_2^2)}{(2 + W_0 J_2 + 2W J_2)}, \quad (3.40)$$

$$Z_L \equiv \frac{J_2 + W(J_1^2 + J_2^2)}{J_2}. \quad (3.41)$$

Then the renormalized $m_{1,2}$ and Γ_{\pm} are

$$m_1^R = m_1 - X_0 - Z_0 = \frac{m_1}{Z_R}, \quad (3.42)$$

$$m_2^R = m_2 + Y_0 = \frac{m_2}{Z_L}, \quad (3.43)$$

$$\Gamma_-^R = W_0 \frac{m_1^R J_1}{2 + (W_0 + 2W)J_2}, \quad (3.44)$$

$$\Gamma_+^R = -(W_0 + 2W) \frac{m_1^R J_1}{2 + (W_0 + 2W)J_2}. \quad (3.45)$$

The condition (3.38) ensures the renormalized $\Gamma_+^R \pm \Gamma_-^R$ are nonnegative. Notice that X_0 and Z_0 are absorbed into a renormalization of m_1 , while Y_0 is a renormalization of m_2 . We use these renormalized parameters in the remainder without the R superscript to determine the θ angle.

2. θ term

The saddle-point solutions for $\langle X \rangle$ and $\langle Z \rangle$ —in particular, the nonzero scattering rates Γ_{\pm} —spontaneously break the $U(2n)$ symmetry (present at $\epsilon = 0$) to $U(n)_R \times U(n)_A$. We consider the fluctuations about this saddle-point by writing $Q_1(\mathbf{x}) \equiv [X(\mathbf{x}) + Z(\mathbf{x})]/(\Gamma_+ - \Gamma_-) = U_1(\mathbf{x})\tau^3 U_1^\dagger(\mathbf{x})$ and $Q_2(\mathbf{x}) \equiv X(\mathbf{x})/(\Gamma_+ + \Gamma_-) = U_2(\mathbf{x})\tau^3 U_2^\dagger(\mathbf{x})$ where $U_1(\mathbf{x}), U_2(\mathbf{x}) \in U(2n)$. This parametrization ensures that $Q_{1,2}(\mathbf{x})$ satisfy $Q_{1,2}^2(\mathbf{x}) = 1$. Since $Q_{1,2}(\mathbf{x}) = \tau^3$ for $U_{1,2}(\mathbf{x}) \in U_R(n) \times U_A(n)$, the target manifold of $Q_{1,2}(\mathbf{x})$ is $U(2n)/U_R(n) \times U_A(n)$. Recall that $\langle Y \rangle$ has been absorbed into a renormalized m_2 after which $\langle Y \rangle = 0$. Consequently, fluctuations about $\langle Y \rangle$ are massive and will not be considered further. Since the action is only invariant under global vector $U(2n)$ rotations (under which $\frac{1-\gamma^5}{2}\Psi$ and $\frac{1+\gamma^5}{2}\Psi$ transform identically), the gapless Goldstone bosons correspond to those fluctuations of Q_1 and Q_2 for which $U_1 = U_2$; the ‘‘axial’’ fluctuations $Q_1 \neq Q_2$ are massive (see Appendix B 2 for an explicit demonstration) and can be neglected at low energies. Thus, we have a single light matrix boson $Q = Q_1 = Q_2 \in U(2n)/U_R(n) \times U_A(n)$ that we parametrize as $Q(\mathbf{x}) = U(\mathbf{x})\tau^3 U^\dagger(\mathbf{x})$ with $U(\mathbf{x}) \in U(2n)$.

The NLSM (1.2) for Q is obtained by integrating out the fermions,

$$e^{-S_{\text{NLSM}}} = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger e^{-\int d^2x \Psi^\dagger [\gamma^5 \gamma^j \partial_j - (M_1 + i\Gamma_+ Q) - (M_2 + i\Gamma_- Q) \gamma^5] \Psi}. \quad (3.46)$$

The real part of S_{NLSM} , which describes the longitudinal conductivity of the system $1/g \propto W \gg 1$, is calculated in Appendix B 3. The imaginary part of S_{NLSM} is the topological θ term,

$$S_{\text{top}} = i \frac{\theta}{16\pi} \int d^2x \text{Tr}(\epsilon_{ij} Q \partial_i Q \partial_j Q). \quad (3.47)$$

This term weights Q configurations by the second homotopy group $\Pi_2[U(2n)/U_R(n) \times U_A(n)] = \mathbb{Z}$. Since this classification is independent of the replica index for any positive integer n , we set $n = 1$ and so $Q(\mathbf{x}) = U(\mathbf{x})\tau^3 U^\dagger(\mathbf{x}) \in U(2)/U_R(1) \times U_A(1) = \text{SU}(2)/U(1)$ with $U(\mathbf{x}) \in \text{SU}(2)$. Furthermore, since the classification of Q configurations by \mathbb{Z} arises from the relative $U(1)$ subgroup of $U_R(n) \times U_A(n)$, the result we find for θ should continue to hold in the replica limit $n \rightarrow 0$ [58].

To extract the topological term, it is convenient to perform the gauge transformation $\Psi \rightarrow U(\mathbf{x})\Psi$ before integrating out the fermions. This allows us to interpret the fluctuations of Q in terms of the $\text{SU}(2)$ gauge field,

$$A_j = iU^\dagger \partial_j U. \quad (3.48)$$

(Note that A_j here and below is not the electromagnetic vector potential, despite our use of the same symbol.) Thus, we compute

$$e^{-S_{\text{NLSM}}} = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger e^{-\int d^2x \Psi^\dagger [\gamma^5 \gamma^j (\partial_j - iA_j) - \varphi_1 - \varphi_2 \gamma^5] \Psi}, \quad (3.49)$$

where

$$\varphi_1 = m_1 \tau^0 + i\Gamma_+ \tau^3, \quad \varphi_2 = -m_2 \tau^0 + i\Gamma_- \tau^3. \quad (3.50)$$

It is sufficient to determine S_{NLSM} to quadratic order in A_i to obtain the topological term, which is cubic in Q . Using (3.48) to relate Q to A_j and the identity $\epsilon_{jk} \partial_j A_k = i\epsilon_{jk} A_j A_k$ for a pure gauge potential, the topological θ term

$$S_{\text{top}} = \frac{1}{4\pi} \int d^2x \text{Tr}[\tau^3 \epsilon_{jk} (\theta^I \partial_j A_k + i\theta^{II} A_j A_k)] \quad (3.51)$$

and the θ angle

$$\theta = \theta^I + \theta^{II}. \quad (3.52)$$

θ^I and θ^I are associated with the ‘‘classical’’ and ‘‘quantum’’ contributions to the Hall conductivity of the system [75].

We will now calculate each of these contributions to θ (see Appendix B 4 for additional details). We will combine a result of Goldstone and Wilczek [76], familiar from work on topological insulators [77], with a direct evaluation; a similar argument in this context can be found in [61].

Because θ^I in (3.51) is only sensitive to the Abelian subgroup of $SU(2)$ generated by τ^3 , we can simplify its determination by focusing on the associated Abelian gauge field $\frac{1}{2}\text{Tr}(\tau^3 A_j)$ under which the retarded and advanced components of Ψ carry opposite charge. We will furthermore treat φ_1 and φ_2 in Eq. (3.50) as smoothly varying complex fields that assume their fixed values at the end of this calculation. Writing $\varphi_{1,2}$ in terms of the complex fields $\chi_{1,2}^{R,A}$,

$$\varphi_1 + \varphi_2 = \begin{pmatrix} e^{i\chi_1^R} & 0 \\ 0 & e^{i\chi_1^A} \end{pmatrix}, \quad \varphi_1 - \varphi_2 = -\begin{pmatrix} e^{-i\chi_2^R} & 0 \\ 0 & e^{-i\chi_2^A} \end{pmatrix}, \quad (3.53)$$

the generalization of [76] to complex $\varphi_{1,2}$ gives the linear in A_j contribution to S_{top} :

$$S_{\text{top}}^I = \frac{1}{8\pi} \int d^2x \epsilon_{jk} \partial_j (\chi_1^R + \chi_2^R - \chi_1^A - \chi_2^A) \text{Tr}(\tau^3 A_k). \quad (3.54)$$

After an integration by parts, we identify

$$\theta^I = -\frac{1}{2}(\chi_1^R + \chi_2^R - \chi_1^A - \chi_2^A). \quad (3.55)$$

We now evaluate θ^I on the saddle-point solution by setting φ_1 and φ_2 to their values in (3.50). To ensure that θ^I is well defined modulo an integer multiple of 2π , we constrain the real parts $\text{Re}[\chi_{1,2}^{R,A}] \in [0, \pi)$. For general $\varphi_{1,2}$ it is necessary to perform some combination of a charge conjugation,

$$\mathcal{C}\Psi\mathcal{C} = \gamma^2\Psi^*, \quad (3.56)$$

and a chiral rotation with $\alpha = \pi/2$ [see (A7)] on each fermion species in order to solve for $\chi_{1,2}^{R,A}$. Note that a flip of the relative sign in $\chi_1^R + \chi_2^R - \chi_1^A - \chi_2^A$ accompanies a charge conjugation on an advanced or retarded fermion. Likewise, any chiral rotation would generally include an additional contribution to the NLSM action due to the anomalous variation of the fermion measure [78] in Eq. (3.49). For the $\varphi_{1,2}$ under consideration, it is only necessary to perform a charge conjugation on the advanced fermions with the result (see Appendix B 4 b)

$$\theta^I = \pi + \arctan\left(\frac{\Gamma_+ - \Gamma_-}{m_1 + m_2}\right) - \arctan\left(\frac{\Gamma_+ + \Gamma_-}{m_1 - m_2}\right). \quad (3.57)$$

In the diffusive regime $0 < \Gamma_{1,2} \ll E_F$ with weak particle-hole symmetry violation $|\Gamma_-| \propto W_0 \ll W$,

$$\theta^I = \pi + O(W_0). \quad (3.58)$$

Equation (3.51) indicates that θ^{II} is sensitive to the non-Abelian nature of $SU(2)$. By direct evaluation of (3.49) of the quadratic in A_j contribution to S_{top} , we find

(see Appendix B 4 a)

$$\theta^{\text{II}} = \frac{m_1\Gamma_- + m_2\Gamma_+}{m_1\Gamma_+ + m_2\Gamma_-} \left[\arctan\left(\frac{\Gamma_+ - \Gamma_-}{m_1 + m_2}\right) + \arctan\left(\frac{\Gamma_+ + \Gamma_-}{m_1 - m_2}\right) \right]. \quad (3.59)$$

Notice that $\theta^{\text{II}} \neq 0$ only when particle-hole violating disorder $W_0 \neq 0$ is present. The θ term only receives a contribution from θ^I when the disorder is particle-hole symmetric because the particle-hole violating mass m_2 vanishes in the saddle-point solution.

IV. DISCUSSION

In this article we studied the effect of particle-hole symmetry violation on the integer quantum Hall plateau transition using the HLR and Dirac composite fermion mean-field theories. For the particle-hole violating perturbations we considered, we showed that the diffusive quantum criticality of this transition is described by the same nonlinear sigma model (NLSM). The derivation of this NLSM proceeded by a semiclassical analysis in which Schwinger-Dyson equations, associated with the ‘‘interactions’’ produced by the various types of quenched disorders in the $(2+0)\text{d}$ generating functional of composite fermion Green’s functions (3.1), are solved via a particular ansatz. (Here we used the replica trick to obtain this generating functional with $2n$ replicas.) Our ansatz preserved the $U_R(n) \times U_A(n)$ symmetry present in this generating functional. For particle-hole symmetric quenched disorder (see Sec. II A for the definition), we found that the solution to the Schwinger-Dyson equations—in particular, finite composite fermion DC conductivity σ_{xx}^{cf} —required the vanishing of a particle-hole breaking (renormalized) mass term corresponding to m_2 in (1.1). In the Dirac composite fermion theory, this term is the usual $(2+1)\text{d}$ Dirac mass; in the HLR theory, this mass term is controlled by the difference of the renormalized Fermi energy and effective mass. The resulting NLSM contains a topological $\theta = \pi$ term. This suggests an emergent particle-hole symmetry of the diffusive quantum critical point. It would be interesting to explore other ansatze, e.g., those associated with flag manifolds [79], with reduced symmetry to better understand this interpretation.

We also considered the effect of particle-hole symmetry breaking quenched disorder. We found such disorder can shift the θ angle away from π . This corresponds to a violation of particle-hole symmetric electrical transport. Because the NLSM that we study is only appropriate for longitudinal conductivities $\sigma_{xx}^{\text{cf}} \gg 1$ (in units of e^2/h), we are not able to determine the identity of the state that obtains for $\sigma_{xx}^{\text{cf}} \sim 1$ towards which the nonlinear sigma model evolves at low energies. [Recall that the longitudinal conductivity is a coupling in the nonlinear sigma model: in the absence of a topological θ term, $\sigma_{xx}^{\text{cf}} \rightarrow 0$ in the IR; with a topological θ term, it is expected that σ_{xx}^{cf} takes an $O(1)$ fixed point value.] It would be interesting to understand if particle-hole symmetry emerges for $\sigma_{xx}^{\text{cf}} \sim 1$, as predicted in [80] and found in a numerical study of noninteracting electrons [81]. Alternatively, if particle-hole symmetry does not emerge, we expect either a gapped insulator or a diffusive metal (at least in the vicinity

of the particle-hole symmetric limit at $\theta = \pi$). It would be interesting to connect our result with recent work [32] that found evidence for a line of extended states with continuously varying critical exponents as particle-hole symmetry is violated by varying the coefficient of electron scalar potential in Eq. (2.3).

In [54] the $\theta = \pi$ term was extracted by a careful study of the chiral anomaly of a 2D theory closely related to the one in (3.49). In this earlier study, only particle-hole symmetric disorder was considered. We have been unsuccessful in applying this chiral anomaly argument to the case studied in this paper, where particle-hole violating disorder is included. The technical issue is that a nonunitary chiral rotation is required to relate the retarded and advanced fermion mass matrices. We are unaware how the gauge field effective action changes under such transformations. Perhaps the methods in [82] can shed light on this question.

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APPENDIX A: MINKOWSKI ACTION

In this Appendix we translate the 2D Euclidean disorder-averaged action $\bar{S} = \bar{S}^{(1)} + \bar{S}^{(2)} + \bar{S}^{(3)}$ in (3.9)–(3.11) into Minkowski signature. We first define the Minkowski spinors [73],

$$\Phi = e^{\frac{\pi}{4}\gamma^2\gamma^5}\Psi, \quad \Phi^\dagger = \Psi^\dagger e^{\frac{\pi}{4}\gamma^2\gamma^5}. \quad (\text{A1})$$

Note the same transformation is used for Φ and Φ^\dagger . The Euclidean-signature gamma matrices are given in (3.5). Next

we introduce the Minkowski-signature $(+1, -1)$ gamma matrices,

$$\Gamma^0 = -\gamma^5, \quad \Gamma^1 = i\gamma^1, \quad \Gamma^5 = i\Gamma^0\Gamma^1 = i\gamma^2, \quad (\text{A2})$$

and Wick rotate $x^0 = -ix^2, x^1 = x^1$. [We caution that we are redefining in this Appendix the uppercase gamma matrices previously employed in the $(2+1)$ d composite fermion mean-field theories in Eqs. (2.7) and (2.10).] Making these replacements in Eqs. (3.9) and (3.10), we find the Minkowski-signature action $\bar{S}_M = \bar{S}_M^{(1)} + \bar{S}_M^{(2)} + \bar{S}_M^{(3)}$:

$$\bar{S}_M^{(1)} = \int dx^0 dx^1 \bar{\Phi} \left[i\Gamma^\mu \partial_\mu + M \left(\frac{1 + \Gamma^5}{2} \right) + M^* \left(\frac{1 - \Gamma^5}{2} \right) \right] \Phi, \quad (\text{A3})$$

$$\bar{S}_M^{(2)} = \frac{W}{2} \int dx^0 dx^1 (\bar{\Phi} \Gamma^\mu \Phi) (\bar{\Phi} \Gamma_\mu \Phi), \quad (\text{A4})$$

$$\bar{S}_M^{(3)} = -\frac{W_0}{2} \int dx^0 dx^1 \left(\bar{\Phi} \frac{1 + i\Gamma^5}{2} \Phi \right)^2, \quad (\text{A5})$$

where $\mu \in \{0, 1\}$, $\bar{\Phi} = \Phi^\dagger \Gamma^0$, and the complex mass matrix $M = M_2 - iM_1$ with the mass matrices for the HLR and Dirac theories given in Eqs. (3.6) and (3.7). We see that M_2 is a conventional Dirac mass and M_1 is an axial mass. $\bar{S}_M^{(2)}$ is a $SU(2n)$ Thirring interaction with coupling $-W$; the chiral interaction $\bar{S}_M^{(3)}$ is a chiral repulsion $W_0 > 0$.

In Minkowski signature, the chiral transformation is

$$\Phi \rightarrow e^{i\alpha\Gamma^5} \Phi, \quad \Phi^\dagger \rightarrow \Phi^\dagger e^{-i\alpha\Gamma^5}, \quad (\text{A6})$$

where α is a real. Using (A1) and (A6), the Euclidean-signature fermions transform under a chiral transformation as

$$\Psi \rightarrow e^{i\alpha\gamma^5} \Psi, \quad \Psi^\dagger \rightarrow \Psi^\dagger e^{i\alpha\gamma^5}. \quad (\text{A7})$$

Notice again that Hermitian conjugation and Wick rotation (between Euclidean and Minkowski signatures) do not commute. The derivative terms and Thirring interaction are invariant under continuous chiral rotations; the mass terms and interaction proportional to W_0 are only invariant under the $\mathbb{Z}/2$ subgroup for which $\alpha \in \pi\mathbb{Z}$.

APPENDIX B: DETAILED NLSM DERIVATION

In this Appendix we detail the calculation of the NLSM for composite fermion mean-field theory that is sketched in Sec. III B.

1. Setup

Here we have found it convenient to employ a different convention than the one used in the main text. The translation table between the two conventions is given below:

$$\Gamma_+ = \frac{\alpha_2}{2}, \quad (\text{B1})$$

$$\Gamma_- = \frac{-\alpha_1}{2}, \quad (\text{B2})$$

$$\gamma_1 = \sigma_y, \quad \gamma_2 = \sigma_x, \quad \gamma_5 = -\sigma_z. \quad (\text{B3})$$

Above $(\sigma_x, \sigma_y, \sigma_z)$ are the usual Pauli sigma matrices that act on the spinor indices; note that we are leaving implicit the replica indices since the gamma matrices are diagonal in replica space [see (3.5)]. Furthermore, we reflect both coordinates $(x, y) \rightarrow$

$(-x, -y)$ and reverse the overall signs of the fermion and NLSM actions in (3.49):

$$e^{S_{\text{NLSM}}} = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger e^S, \quad (\text{B4})$$

with

$$S = \int d^2x \Psi^\dagger \left(i\sigma_x \partial_x - i\sigma_y \partial_y - M_2 \sigma_z + M_1 \sigma_0 + i \frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} Q \right) \Psi. \quad (\text{B5})$$

Here σ_0 is the 2×2 identity matrix. The masses M_1 and M_2 for the HLR and Dirac theories are given in (3.6) and (3.7); we set $\omega = \epsilon = 0$ in these expressions for the masses. The matrix boson $Q(\mathbf{x}) = U(\mathbf{x}) \tau^3 U^\dagger(\mathbf{x})$ for $U(\mathbf{x}) \in U(2n)$ and $\tau^3 = \sigma_z \otimes \sigma_0^n$. Since all terms except for Q are singlets with respect to the replica indices (i.e., the τ space), we (generally) leave implicit the $2n \times 2n$ identity matrix τ_0 that acts in this subspace.

We partially follow [61] to separately derive the real and imaginary parts of S_{NLSM} . Before proceeding, we define the self-consistent Born approximation (SCBA) Green's functions,

$$g_\pm \equiv \frac{1}{M_1 \sigma_0 + i\sigma_x \partial_x - i\sigma_y \partial_y - M_2 \sigma_z \pm i \left(\frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} \right) \tau^3}, \quad (\text{B6})$$

which are related to the retarded and advanced Green's functions $g_{R,A}$ as follows:

$$(g_+)^{\dagger} = g_-, \quad g_+ = \text{Diag}(g_R, g_A)_\tau, \quad g_- = \text{Diag}(g_A, g_R)_\tau, \quad (\text{B7})$$

$$g_+ - g_- = (g_R - g_A) \tau^3, \quad g_+ + g_- = (g_R + g_A) \tau_0. \quad (\text{B8})$$

Note that the \pm subscript of g_\pm labels the sign of the imaginary part of the Green's function.

2. Goldstone parametrization

In the main text we argued that the massless Goldstone modes correspond to $Q_1 = Q_2$ fluctuations, where the matrix bosons $Q_1, Q_2 \in U(2n)/U(n) \times U(n)$. The $Q_1 = Q_2$ fluctuations correspond to vector gauge transformations of Ψ . Here we show explicitly that axial gauge transformations corresponding to $Q_1 \neq Q_2$ fluctuations are massive.

Consider the two-field sigma model involving Q_1 and Q_2 :

$$S[Q_1, Q_2] \equiv \text{Tr} \ln \left[(M_1 \sigma_0 - M_2 \sigma_z + i\sigma_x \partial_x - i\sigma_y \partial_y) + \begin{pmatrix} i\Gamma_1 Q_1 & 0 \\ 0 & i\Gamma_2 Q_2 \end{pmatrix}_\sigma \right]. \quad (\text{B9})$$

Using the same logic that produced (B22) with $\hat{H} = \left(\frac{\alpha_1 \sigma_z + \alpha_2 \mathbf{1}_\sigma}{2} \right) = i\Gamma_1 P_1 + i\Gamma_2 P_2$ for $P_1 = \frac{1+\sigma_z}{2}$ and $P_2 = \frac{1-\sigma_z}{2}$, and including a test function f , we calculate

$$\begin{aligned} & (g_+^{-1} - i\hat{H}\tau^3 + i\Gamma_1 Q_1 P_1 + i\Gamma_2 Q_2 P_2) (\hat{H})^{-1} (g_-^{-1} + i\hat{H}\tau^3 - i\Gamma_1 Q_1 P_1 - i\Gamma_2 Q_2 P_2) f \\ &= g_+^{-1} \hat{H}^{-1} g_-^{-1} f + \nabla (Q_V \mathbf{1}_\sigma + Q_A \sigma_z) f + (\nabla f) (Q_V \mathbf{1}_\sigma + Q_A \sigma_z) - (Q_V \mathbf{1}_\sigma + Q_A \sigma_z) (\nabla f) \\ &= g_+^{-1} \hat{H}^{-1} g_-^{-1} f + \nabla (Q_V \mathbf{1}_\sigma + Q_A \sigma_z) f - 2Q_A \sigma_z (\nabla f), \end{aligned} \quad (\text{B10})$$

where we have used $Q_1^2 = Q_2^2 = \mathbf{1}$ and introduced the ‘‘vector’’ $Q_V = \frac{1}{2}(Q_1 + Q_2)$ and ‘‘axial’’ $Q_A = \frac{1}{2}(Q_1 - Q_2)$ matrix bosons. The crucial term that leads to a mass for Q_A is $-2Q_A \sigma_z \nabla$. We find

$$\begin{aligned} \text{Re}[S[Q_1, Q_2]] &= \frac{1}{2} \text{Tr} \ln [(g_+^{-1} \hat{H}^{-1} g_-^{-1}) (\mathbf{1} + (g_- \hat{H} g_+) (\nabla Q_V + \nabla Q_A \sigma_z - 2Q_A \sigma_z \nabla))] \\ &= \frac{1}{16} \text{Tr} [(g_R - g_A) (\nabla Q_V + \nabla Q_A \sigma_z - 2Q_A \sigma_z \nabla) (g_R - g_A) (\nabla Q_V + \nabla Q_A \sigma_z - 2Q_A \sigma_z \nabla)]. \end{aligned} \quad (\text{B11})$$

Consider the last term, quadratic in $Q_A \nabla$:

$$\begin{aligned} & \frac{1}{4} \text{Tr} [(g_R - g_A) (Q_A \sigma_z \nabla) (g_R - g_A) (Q_A \sigma_z \nabla)] \\ &= \int_{x, x_1, x_2, x_3} \int_{k_1 k_2 k_3 k_4} \frac{1}{4} \text{Tr} [x | (g_R - g_A) | k_1 \rangle \langle k_1 | x_1 \rangle \langle x_1 | Q_A \sigma_z \nabla | k_2 \rangle \langle k_2 | x_2 \rangle \langle x_2 | (g_R - g_A) | k_3 \rangle \langle k_3 | x_3 \rangle \langle x_3 | Q_A \sigma_z \nabla | k_4 \rangle \langle k_4 | x] \end{aligned} \quad (\text{B12})$$

$$= \frac{-1}{4} \int_{k_1, k_2} \text{Tr} [(g_R - g_A) | k_1 \rangle Q_A (k_2 - k_1) \sigma_z \not{k}_2 (g_R - g_A) | k_2 \rangle Q_A (k_1 - k_2) \sigma_z \not{k}_1] \quad (\text{B13})$$

$$= \frac{-1}{4} \int_p Q_A(p) Q_A(-p) \int_{k_1} \text{Tr} [(g_R - g_A) | k_1 \rangle \sigma_z (\not{p} + \not{k}_1) (g_R - g_A) | k_1 + p \rangle \sigma_z \not{k}_1] \quad (\text{B14})$$

$$\equiv \frac{-1}{4} \int_p Q_A(p) Q_A(-p) F(p) \quad (\text{B15})$$

$$= \frac{-1}{4} \int_p Q_A(p) Q_A(-p) \left(F(p=0) + \frac{\partial F}{\partial p} \Big|_{p=0} p + \frac{1}{2} \frac{\partial^2 F}{\partial p^2} \Big|_{p=0} p^2 + \dots \right), \quad (\text{B16})$$

where we made the change of variables $k_2 - k_1 \equiv p$. Next, examine the two crossing terms containing $Q_A \nabla$ and ∇Q_A :

$$\begin{aligned} & \frac{1}{16} \text{Tr}[(g_R - g_A)(\nabla Q_A \sigma_z)(g_R - g_A)(-2Q_A \sigma_z \nabla) + (g_R - g_A)(-2Q_A \sigma_z \nabla)(g_R - g_A)(\nabla Q_A \sigma_z)] \\ &= \frac{-1}{4} \text{Tr}[(g_R - g_A)(\nabla Q_A \sigma_z)(g_R - g_A)(Q_A \sigma_z \nabla)] \end{aligned} \quad (\text{B17})$$

$$= \frac{-1}{4} \int_{dp} Q_A(p) Q_A(-p) \int_{dk_1} \text{Tr}[(g_R - g_A)|_{k_1} \sigma_z k_1 (g_R - g_A)|_{p+k_1} \sigma_z k_1]. \quad (\text{B18})$$

Combining Eqs. (B16) and (B18), the total mass term for Q_A is $\frac{-1}{2} \int_{dp_1} Q_A(p_1) Q_A(-p_1) F(0)$. The function $F(0) \neq 0$ generally: Q_A is only massless if it is tuned to criticality. This is in sharp contrast to Q_V , which is massless because it is a Goldstone boson. Thus, Q_A is generally massive and we neglect it at low energies.

3. Real part of S_{NLSM}

To compute the real part of S_{NLSM} we directly compute the fermion determinant implied by (B4):

$$\text{Re}(S_{\text{NLSM}}) + i\text{Im}(S_{\text{NLSM}}) = \text{Tr} \ln \left((M_1 \sigma_0 - M_2 \sigma_z + i\sigma_x \partial_x - i\sigma_y \partial_y) + i \frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} Q \right). \quad (\text{B19})$$

For any operator \hat{X} with determinant $\det \hat{X} = R e^{i\theta}$, we seek $\ln R$. Let $i\hat{C} \equiv i \frac{(\alpha_1 \sigma_z + \alpha_2 \sigma_0)}{2} Q$ and $\hat{B} \equiv M_1 \sigma_0 - M_2 \sigma_z + i\sigma \partial_x - i\sigma_y \partial_y$, then we can decompose the operator in $2 \times 2\sigma$ space using the identities

$$\det[i\hat{C} + \hat{B}] = \det[i\hat{C}] + \det[\hat{B}] + i \det[\hat{C}] \text{Tr}[\hat{C}^{-1} \hat{B}], \quad (\text{B20})$$

$$\det[(\hat{B} + i\hat{C}) \hat{H}^{-1} (\hat{B} - i\hat{C}) \hat{H}] = |\det[i\hat{C} + \hat{B}]|^2, \quad (\text{B21})$$

where \hat{H} is any constant matrix in σ space that satisfies $[\hat{C}, \hat{H}] = 0$. We choose $\hat{H} = \left(\frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} \right)$. Up to an unimportant constant that we drop, we find

$$\begin{aligned} \text{Re}(S_{\text{NLSM}}) &= \frac{1}{2} \text{Tr} \ln[(g_+^{-1} - i\hat{H} \tau^3 + i\hat{H} Q)(\hat{H})^{-1} (g_-^{-1} + i\hat{H} \tau^3 - i\hat{H} Q)] \\ &= \frac{1}{2} \text{Tr} \left[\left(\frac{i}{2} (g_R - g_A) \nabla Q \right) - \frac{1}{2} \left(\frac{i}{2} (g_R - g_A) \nabla Q \right) \left(\frac{i}{2} (g_R - g_A) \nabla Q \right) \right] \\ &= \frac{1}{16} \text{Tr}_\sigma [(g_R - g_A) \kappa_a (g_R - g_A) \kappa_b] \text{Tr}[\nabla_a Q \nabla_b Q] \\ &\equiv \frac{-S_{jk}}{8} \text{Tr}[\nabla_j Q \nabla_k Q], \end{aligned} \quad (\text{B22})$$

where $\frac{\nabla}{i} \equiv \kappa_x \frac{\partial_x}{i} + \kappa_y \frac{\partial_y}{i}$, $\kappa_x = \sigma_x$, $\kappa_y = -\sigma_y$. We identify $S_{jk} = \sigma_{xx}^{\text{cf}} \delta_{jk}$ as the DC longitudinal conductivity:

$$\begin{aligned} \sigma_{xx}^{\text{cf}} &= -\frac{1}{2} \text{Tr}[(g_R - g_A) \sigma_x (g_R - g_A) \sigma_x] \\ &= 1 + \frac{(4m_1^2 - 4m_2^2 - \alpha_1^2 + \alpha_2^2) \left(\pi + 2 \arccot \left[\frac{|m_1 \alpha_2 - m_2 \alpha_1|}{4m_1^2 - 4m_2^2 + \alpha_1^2 - \alpha_2^2} \right] \right)}{8|m_1 \alpha_2 + m_2 \alpha_1|} \end{aligned} \quad (\text{B23})$$

$$\approx \frac{\pi(m_1^2 - m_2^2)}{|m_1 \alpha_2 - m_2 \alpha_1|} \gg 1. \quad (\text{B24})$$

In these expressions we have used $m_1 = \frac{1}{2n} \text{Tr} M_1$ and $m_2 = -\frac{1}{2n} \text{Tr} M_2$. The last expression uses the weak disorder limit, i.e., $O(E_F) \approx O(|m_1|) \gg \alpha_2, \alpha_1$.

Here and below we use the DC ($\omega \rightarrow 0$) limit of the Kubo formula:

$$\begin{aligned} \sigma_{ij}^{\text{cf}}(q \rightarrow 0, \omega + i\eta) = & - \int \frac{d^2\mathbf{r}'}{V} \int d^2\mathbf{r} \frac{1}{2\pi\omega} \int_{-\infty}^{\infty} dz \\ & \times \left([f(z) - f(z - \omega)] \text{Tr}[\kappa_i G_z^R(r, r') \kappa_j G_{z-\omega}^A(r', r)] + f(z) \text{Tr}[\kappa_i G_{z+\omega}^R(r, r') \kappa_j G_z^R(r', r)] \right. \\ & \left. - f(z) \text{Tr}[\kappa_i G_z^A(r, r') \kappa_j G_{z-\omega}^A(r', r)] \right). \end{aligned} \quad (\text{B25})$$

For the Dirac theory, the SCBA Green's function is

$$G_D(\epsilon + E_F; \omega)_{R/A} = \frac{1}{(\epsilon + E_F + \omega)\sigma_0 - m_D\sigma_z + i\partial_x\sigma_x - i\partial_y\sigma_y \pm i\frac{\alpha_1\sigma_z + \alpha_2\sigma_0}{2}}. \quad (\text{B26})$$

For the linearized HLR theory,

$$g(\epsilon + E_F; \omega)_{R/A} = \frac{1}{(\epsilon + E_F)\sigma_0 + \omega P_1 + 2m P_2 + i\partial_x\sigma_x - i\partial_y\sigma_y \pm i\frac{\alpha_1\sigma_z + \alpha_2\sigma_0}{2}}, \quad (\text{B27})$$

where $P_1 = \frac{1+\sigma_z}{2}$ and $P_2 = \frac{1-\sigma_z}{2}$ and we have inserted the specific masses for the HLR (3.6) and Dirac (3.7) theories using (3.8). The frequency term should be thought of as a ‘‘mass’’ term in the Hamiltonian instead of the physical frequency appearing in (B25). Notice that since we have included the Fermi level in the above Greens function, we only need to perform the energy integral $\int dz f(z)$ up to zero. We comment that a direct calculation of the Hall conductivity using the above SCBA Green's functions agrees with the results below, up to the crucial additive term equal to $\frac{e^2}{2\pi}$.

4. Imaginary part of S_{NLSM}

To calculate the imaginary part of S_{NLSM} in (B4), we first set $n = 1$ and perform the gauge transformation $\Psi(\mathbf{x}) \rightarrow U(\mathbf{x})\Psi(\mathbf{x})$. This introduces the gauge field $A_j(\mathbf{x}) = iU^\dagger(\mathbf{x})\partial_j U(\mathbf{x}) = \sum_{b=1}^3 A_j^b \tau_b$, where (τ_1, τ_2, τ_3) are the Pauli matrices in retarded-advanced space. We determine $\text{Im}(S_{\text{NLSM}})$ from the imaginary part of the expectation value of the current (calculated to linear order in A_j):

$$i \frac{\delta(\text{Im}S_{\text{NLSM}})}{\delta A_j^b} = \frac{\delta S_{\text{top}}^I}{\delta A_j^b} + \frac{\delta S_{\text{top}}^{II}}{\delta A_j^b}, \quad (\text{B28})$$

where S_{top}^I and S_{top}^{II} are the linear (quantum) and quadratic (classical) in A_j contributions to the imaginary part of S_{NLSM} [75].

a. S_{top}^{II}

We begin with the classical contribution S_{top}^{II} . By direct evaluation we find

$$\begin{aligned} S_{\text{top}}^{II} &= \frac{1}{2} \text{Tr}[\kappa_\mu A_\mu g_+ \kappa_\nu A_\nu g_+ - \kappa_\mu A_\mu g_- \kappa_\nu A_\nu g_-] \\ &= \frac{1}{2} \text{Tr}[\kappa_\mu A_\mu (g_+ + g_-) \kappa_\nu A_\nu (g_+ - g_-)] \\ &= \frac{i}{2} \text{Tr}_\sigma [\kappa_\mu (g_R + g_A) \kappa_\nu (g_R - g_A)] \text{Tr}_\tau [A_\mu A_\nu \tau_3] \\ &= \frac{i}{4} \text{Tr}_\sigma [\kappa_x (g_R + g_A) \kappa_y (g_R - g_A)] \text{Tr}_\tau [Q(\partial_x Q)(\partial_y Q)] \\ &\equiv i \frac{\theta^{II}}{2\pi} \text{Tr}_\tau [Q(\partial_x Q)(\partial_y Q)], \end{aligned} \quad (\text{B29})$$

where we used $\text{Tr}_\tau [Q(\partial_x Q)(\partial_y Q)] = 4i(A_i^1 A_j^2 - A_i^2 A_j^1) \epsilon_{ij}$. $\frac{\theta^{II}}{2\pi}$ is the classical contribution σ_{xy}^I to the DC Hall conductivity $\sigma_{xy}^{\text{cf}} = \sigma_{xy}^I + \sigma_{xy}^{II}$.

$$\sigma_{xy}^I = \frac{1}{2\pi} \left(\frac{m_2 \Gamma_+ + m_1 \Gamma_-}{m_1 \Gamma_+ + m_2 \Gamma_-} \right) \left(\arctan \left[\frac{2(\Gamma_+ + \Gamma_-)}{2(m_1 - m_2)} \right] + \arctan \left[\frac{2(\Gamma_+ - \Gamma_-)}{2(m_1 + m_2)} \right] \right). \quad (\text{B30})$$

The quantum contribution σ_{xy}^{II} to the Hall conductivity equals $\frac{\theta^I}{2\pi}$ (modulo 1); it is calculated in the next section.

b. S_{top}^I

S_{top}^I obtains from a result first obtained by Goldstone and Wilczek [76]; below we follow the treatment in [83]. We first introduce the ‘‘mass field’’

$$\Phi_m(0) \equiv M_1 \sigma_0 - M_2 \sigma_z + i \left(\frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} \right) \tau_3. \quad (\text{B31})$$

For this calculation we will treat $\Phi_m(0)$ as a spatially varying field $\Phi_m(\mathbf{x})$ that takes its fixed point value (B31) at the end of the calculation. We write the mass field as

$$\Phi_m(x) \equiv [m_{1a}(x) \tau_3 + m_{1b}(x) \tau_0] \sigma_0 + [m_{2a}(x) \tau_3 - m_{2b}(x) \tau_0] \sigma_z. \quad (\text{B32})$$

Notice that $m_1 = m_{1a}(0)$ and $m_2 = m_{2a}(0)$. The real space Green’s function is then expanded about uniform Φ_m as

$$S(\Phi_m) = \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{i\kappa_\mu D_\mu + \Phi_m(x)} e^{ip \cdot (x-y)} = S_0 + \{-S_0[x_\nu \partial_\nu \Phi_m(0)] S_0\} + \dots, \quad (\text{B33})$$

where we drop the ‘‘...’’ in what follows. S_0 is the Green’s function for uniform Φ_m with $S_0|_{A_j=0} = g_+$, given in (B6).

The quantum current $\frac{\delta S_{\text{top}}^I}{\delta A_j^b}$ is then computed as

$$\frac{\delta S_{\text{top}}^I}{\delta A_j^b} = - \lim_{y \rightarrow x} \text{Tr}[\langle \kappa_j \tau_b S(x, y) \rangle] \approx \langle x | \text{Tr}[\kappa_j \tau_b S_0 \partial_k \Phi_m(0) i S_0 \kappa_k S_0] | x \rangle, \quad (\text{B34})$$

where we use $\langle x' | S_0 \hat{\tau}_k | x \rangle = \langle x' | i S_0 \kappa_k S_0 | x \rangle$. We set the gauge potential in the Green’s function S_0 to zero to find

$$\begin{aligned} \frac{\delta S_{\text{top}}^I}{\delta A_j^b} &= \int \frac{dq_x dq_y}{(2\pi)^2} \text{Tr}[\kappa_j \tau_b \frac{1}{\not{q} + \Phi_m(0)} \partial_k \Phi_m(0) \frac{1}{\not{q} + \Phi_m(0)} i \kappa_k \frac{1}{\not{q} + \Phi_m(0)}] \\ &= \frac{1}{2\pi} \epsilon_{jk} \delta_{b3} \left[\frac{(m_{1a} + m_{1b}) \partial_k (-m_{2a} + m_{2b}) - (-m_{2a} + m_{2b}) \partial_k (m_{1a} + m_{1b})}{(-m_{2a} + m_{2b})^2 - (m_{1a} + m_{1b})^2} \right. \\ &\quad \left. - \frac{(-m_{1a} + m_{1b}) \partial_k (m_{2a} + m_{2b}) - (m_{2a} + m_{2b}) \partial_k (-m_{1a} + m_{1b})}{(m_{2a} + m_{2b})^2 - (-m_{1a} + m_{1b})^2} \right] \\ &\equiv \frac{i}{2\pi} \delta_{b3} \epsilon_{jk} \partial_k \theta^I, \end{aligned} \quad (\text{B35})$$

where $\not{q} \equiv \kappa_x p_x + \kappa_y p_y$. We now deduce S_{top}^I by coupling this current to A_j^b and then performing an integration by parts:

$$S_{\text{top}}^I = \int d^2x \sum_{b=1}^3 A_j^b \frac{\delta S_{\text{top}}^I}{\delta A_j^b} = \frac{i}{2\pi} \int d^2x A_j^3 \epsilon_{jk} \partial_k \theta^I = -\frac{1}{4\pi} \int d^2x \theta^I \epsilon_{jk} \text{Tr}_\tau[\tau^3 \partial_k A_j]. \quad (\text{B36})$$

It remains to determine θ^I for the specific fixed point value $\Phi_m(0)$ (B31). The detailed discussion below complements the presentation of this in the main text. We replace $\alpha_1/2 = \Gamma_-$ and $\alpha_2/2 = \Gamma_+$, and introduce the superscript $s = R, A$ to label the diagonal retarded and advanced components in τ space of

$$\varphi_1 = m_1 \tau^0 + i\Gamma_+ \tau^3, \quad \varphi_2 = -m_2 \tau^0 + i\Gamma_- \tau^3. \quad (\text{B37})$$

Thus, we may also specify the diagonal components of Φ_m with this superscript,

$$\Phi_m^s(x) = -\varphi_2^s(x) \sigma_z + \varphi_1^s(x) \sigma_0 \equiv \begin{pmatrix} -e^{-i\chi_2^s} & 0 \\ 0 & e^{i\chi_1^s} \end{pmatrix}. \quad (\text{B38})$$

Equation (B35) indicates that the retarded and advanced contributions to θ^I are fully separable, i.e., θ^I only couples to $\frac{1}{2} \text{Tr}(\tau^3 A_j) = A_j^3$. This allows us to determine θ^I as the contribution from the $U(1)$ subgroup of $SU(2)$ associated with τ^3 . Plugging (B38) into (B35), we find

$$-\partial_k \theta^I = \partial_k \frac{\chi_1^R + \chi_2^R - \chi_1^A - \chi_2^A}{2}. \quad (\text{B39})$$

Note that χ_1^s and χ_2^s are generally complex. We restrict $\text{Re}[\chi_{1,2}] \in [0, \pi)$. We now specialize to the masses associated with the HLR theory by setting $m_1 = E_F/2 + m$ and $m_2 = E_F/2 - m$; similar manipulations yield θ^I for the Dirac theory. We use $\varphi_1 + \varphi_2 = e^{i\chi_1}$, $\varphi_1 - \varphi_2 = -e^{-i\chi_2}$, and (B31) to find the equations:

$$2m + i(\Gamma_+ + \Gamma_-) = e^{i\chi_1^R}, \quad E_F + i(\Gamma_+ - \Gamma_-) = -e^{-i\chi_2^R}, \quad (\text{B40})$$

$$2m - i(\Gamma_+ + \Gamma_-) = e^{i\chi_1^A}, \quad E_F - i(\Gamma_+ - \Gamma_-) = -e^{-i\chi_2^A}. \quad (\text{B41})$$

For the retarded component $\chi_{1,2}^R$ we find

$$\text{Re}[\chi_1^R] = \arctan\left[\frac{\Gamma_+ + \Gamma_-}{2m}\right], \quad \text{Im}[\chi_1^R] = \frac{-1}{2} \log[(2m)^2 + (\Gamma_+ + \Gamma_-)^2], \quad (\text{B42})$$

$$\text{Re}[\chi_2^R] = \pi - \arctan\left[\frac{\Gamma_+ - \Gamma_-}{E_F}\right], \quad \text{Im}[\chi_2^R] = \frac{-1}{2} \log[(E_F)^2 + (\Gamma_+ - \Gamma_-)^2]. \quad (\text{B43})$$

Due to the restriction $\text{Re}[\chi_{1,2}^R] \in [0, \pi)$, we need to perform a charge conjugation on the advanced fermions using the definition in (3.56). It is here that we use the fact that θ^I is only sensitive to the Abelian component $\frac{1}{2}\text{Tr}(\tau^3 A_j)$ of A_j . Charge conjugation flips the sign of mass term involving σ_0 as well as the gauge coupling of the advanced fermion to A_j^3 : The part of the action including the advanced fermion only transforms as

$$S_A = \Psi_A^\dagger \left(m_1 \sigma_0 + m_2 \sigma_z + i \sigma_x (\partial_x - i A_x^3) - i \sigma_y (\partial_y - i A_y^3) - i \frac{\alpha_1 \sigma_z + \alpha_2 \sigma_0}{2} \right) \Psi_A, \quad (\text{B44})$$

$$\mathcal{C} S_A \mathcal{C} = \Psi_A^\dagger \left(-m_1 \sigma_0 + m_2 \sigma_z + i \sigma_x (\partial_x + i A_x^3) - i \sigma_y (\partial_y + i A_y^3) - i \frac{\alpha_1 \sigma_z - \alpha_2 \sigma_0}{2} \right) \Psi_A. \quad (\text{B45})$$

After performing this charge conjugation, we find the current $\frac{\delta S_{\text{top}}^I}{\delta A_j^b}$ equals

$$\begin{aligned} \frac{\delta S_{\text{top}}^I}{\delta A_j^b} &= \frac{1}{2\pi} \epsilon_{jk} \delta_{b3} \left[\frac{(m_{1a} + m_{1b}) \partial_k (-m_{2a} + m_{2b}) - (-m_{2a} + m_{2b}) \partial_k (m_{1a} + m_{1b})}{(-m_{2a} + m_{2b})^2 - (m_{1a} + m_{1b})^2} \right. \\ &\quad \left. + \frac{(-1)(-m_{1a} + m_{1b}) \partial_k (m_{2a} + m_{2b}) - (m_{2a} + m_{2b}) (-1) \partial_k (-m_{1a} + m_{1b})}{(m_{2a} + m_{2b})^2 - (-m_{1a} + m_{1b})^2} \right] \\ &\equiv \frac{i}{2\pi} \delta_{b3} \epsilon_{jk} \partial_k \theta^I. \end{aligned} \quad (\text{B46})$$

The retarded contribution to θ^I is unchanged since only the advanced fermions were charge conjugated. The advanced fermion mass becomes

$$\Phi_m^A(x) = -\varphi_2^A(x) \sigma_z + (-1) \varphi_1^A(x) \sigma_0 \equiv \begin{pmatrix} -e^{-i\tilde{\chi}_2^A} & 0 \\ 0 & e^{i\tilde{\chi}_1^A} \end{pmatrix}. \quad (\text{B47})$$

This results in the equations for $\tilde{\chi}_{1,2}^A$:

$$2m - i(\Gamma_+ + \Gamma_-) = e^{i\tilde{\chi}_2^A}, \quad -E_F + i(\Gamma_+ - \Gamma_-) = e^{i\tilde{\chi}_1^A}, \quad (\text{B48})$$

$$\text{Re}[\tilde{\chi}_2^A] = \arctan\left[\frac{\Gamma_+ + \Gamma_-}{2m}\right], \quad \text{Re}[\tilde{\chi}_1^A] = \pi - \arctan\left[\frac{\Gamma_+ - \Gamma_-}{E_F}\right]. \quad (\text{B49})$$

Since charge conjugation also flips relative sign between retarded and advanced contributions in Eq. (B39), we have

$$\theta^I = -\frac{1}{2} \left[(\text{Re}[\chi_1^R] + \text{Re}[\chi_2^R] + \text{Re}[\tilde{\chi}_1^A] + \text{Re}[\tilde{\chi}_2^A]) \right] \quad (\text{B50})$$

$$= \pi + \arctan\left[\frac{\Gamma_+ - \Gamma_-}{m_1 + m_2}\right] - \arctan\left[\frac{\Gamma_+ + \Gamma_-}{m_1 - m_2}\right]. \quad (\text{B51})$$

We have used $\theta^I = 0 \pmod{2\pi}$ to fix the coefficient of π to be unity and replaced the HLR-specific masses by the general masses m_1 and m_2 .

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