Moving vortices in anisotropic superconductors

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The magnetic field of moving vortices in anisotropic superconductors is considered in the framework of the time-dependent London approach. It is found that, at distances large relative to the core size, the field may change sign that alludes to a nontrivial intervortex interaction which depends on the crystal anisotropy and on the speed and direction of motion. These effects are caused by the electric fields and corresponding normal currents which appear due to the moving vortex magnetic structure. We find that the motion related part of the magnetic field attenuates at large distances as $1/r^3$ unlike the exponential decay of the static vortex field. The electric field induced by the vortex motion decreases as $1/r^2$.

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I. INTRODUCTION

The problem of interaction of vortices in anisotropic superconductors has been studied extensively in early 90s both theoretically [1–3] and experimentally [4]. For vortices parallel to one of the principal crystal directions the problem is solved just by rescaling the isotropic results. In particular, the interaction is repulsive for any position of the second vortex relative to the first. However, the force direction in general is not along the vector $\boldsymbol{R}$ connecting the vortices, in other words, for an arbitrary position of the pair there is a torque, unless $\boldsymbol{R}$ is directed along principal directions [5].

The situation is different if parallel vortices are tilted out of principal directions [1–3]. Then, at distances of the order of London penetration depth λ, the magnetic field $\boldsymbol{h}(\boldsymbol{R})$ of a single tilted vortex may change sign and approach zero for $R \to \infty$ being negative. In other words, the vortex-vortex interaction being repulsive at short distances may turn attractive at large distances. This leads to formation of chains of vortices in tilted fields [4].

In this paper we consider the magnetic field and current distributions of *moving* vortices in anisotropic materials. Commonly, moving vortices are considered as static but displaced as a *whole*. It was argued, however, that an out-of-core moving vortex structure differs from the static case due to out-of-core dissipation [6,7]. The moving vortex magnetic field $h(r,t)$ generates the electric field and currents of normal excitations, which in turn distort the field h. We show that, at large distances, the distortion is not small and is even able to change field sign. Unexpectedly, this distortion attenuates with distance as a power law $1/R^3$, i.e., much slower than the standard decay of undistorted field $\sim e^{-R/\lambda}$.

At distances large in comparison with the core size of interest in this work, one can use the time-dependent London approach based on the assumption that the current consists of the normal and superconducting parts:

$$\boldsymbol{J} = \sigma \boldsymbol{E} - \frac{2e^2|\Psi|^2}{mc}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\nabla\chi\right), \tag{1}$$

where $\boldsymbol{A}$ is the vector potential, Ψ is the order parameter, χ is the phase, ϕ_0 is the flux quantum, $\boldsymbol{E}$ is the electric field, and σ is the conductivity associated with normal excitations.

The conductivity σ approaches the normal-state value σ_n when the temperature T approaches the critical one; in *s*-wave superconductors it vanishes with decreasing temperature along with the density of normal excitations. This is not the case, however, for strong pair breaking when superconductivity is gapless while the density of states approaches the normal-state value at all temperatures. Unfortunately, not much experimental information about the T dependence of σ is available. Theoretically, this question is still debated, e.g., Ref. [8] discusses the possible enhancement of σ due to inelastic scattering. Experimentally, the interpretation of the microwave absorption data is not yet settled either [9].

At distances large in comparison with the vortex core size, $|\Psi|$ is a constant Ψ_0 and Eq. (1) becomes

$$\frac{4\pi}{c}\boldsymbol{J} = \frac{4\pi\sigma}{c}\boldsymbol{E} - \frac{1}{\lambda^2}\left(\boldsymbol{A} + \frac{\phi_0}{2\pi}\nabla\chi\right), \tag{2}$$

where $\lambda^2 = mc^2/8\pi e^2|\Psi_0|^2$ is the London penetration depth. Acting on this by curl one obtains

$$\boldsymbol{h} - \lambda^2\nabla^2\boldsymbol{h} + \tau\frac{\partial \boldsymbol{h}}{\partial t} = \phi_0\hat{z}\sum_\nu \delta(\boldsymbol{r} - \boldsymbol{r}_\nu), \tag{3}$$

where $\boldsymbol{r}_\nu(t)$ is the position of the νth vortex which may depend on time t, $\hat{z}$ is the direction of vortices, and the relaxation time

$$\tau = 4\pi\sigma\lambda^2/c^2. \tag{4}$$

Equation (3) can be considered as a general form of the time-dependent London equation (TDL). The anisotropic generalization of this equation was given in Ref. [10] and is reproduced here in Sec. III.

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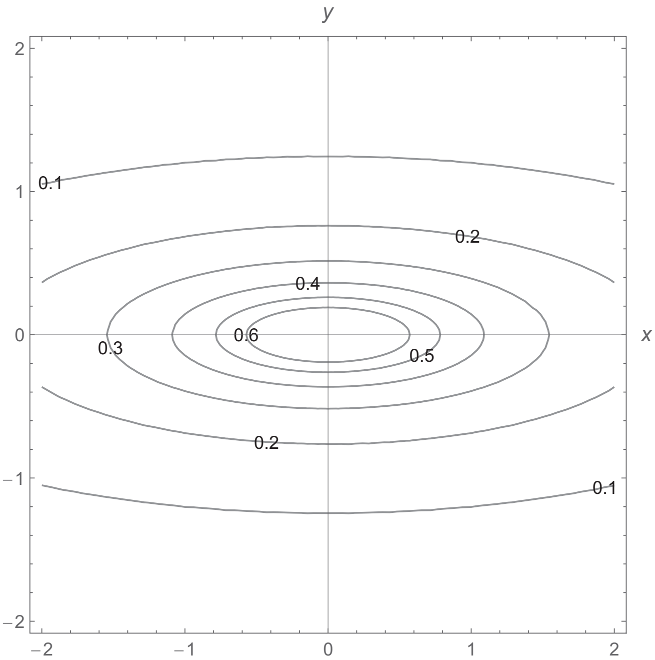

FIG. 1. The stream lines of the current for $\gamma = \lambda_2/\lambda_1 = 3$ or, which is the same, contours of constant $h_z(x, y)$. λ_1 is taken as the unit length.

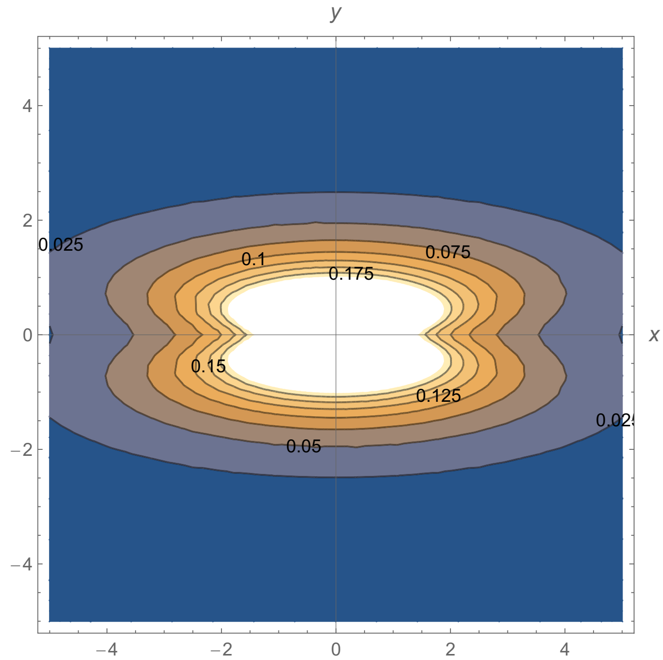

FIG. 2. The contours of constant current *values* $J(x, y) = (J_x^2 + J_y^2)^{1/2}$ for $\lambda_2/\lambda_1 = 3$. x and y are in units of λ_1.

II. VORTEX AT REST IN ANISOTROPIC CASE

For an arbitrary oriented vortex in anisotropic material this problem has been considered in Refs. [1,11]. In general, the results are cumbersome, so here we consider a simple situation of an orthorhombic superconductor in a field along the c axis. The London equation in this case is

$$h_z(x, y) - \lambda_1^2 \frac{\partial^2 h_z}{\partial y^2} - \lambda_2^2 \frac{\partial^2 h_z}{\partial x^2} = \phi_0 \delta(\boldsymbol{r}). \tag{5}$$

Here, the frame x, y, z is chosen to coincide with a, b, c of the crystal, $\boldsymbol{r} = (x, y)$, and $\lambda_{xx}^2 = \lambda_1^2$ and $\lambda_{yy}^2 = \lambda_2^2$ are the diagonal components of the tensor $(\lambda^2)_{ik}$. The solution of this equation is

$$h_z(x, y) = \frac{\phi_0}{2\pi\lambda_1\lambda_2} K_0(\rho), \qquad \rho^2 = \frac{x^2}{\lambda_2^2} + \frac{y^2}{\lambda_1^2}. \tag{6}$$

Current densities follow

$$J_x = -\frac{c\phi_0}{8\pi^2\lambda_1^3\lambda_2} \frac{y K_1(\rho)}{\rho}, \qquad J_y = \frac{c\phi_0}{8\pi^2\lambda_1\lambda_2^3} \frac{x K_1(\rho)}{\rho}, \tag{7}$$

where K_0 and K_1 are modified Bessel functions.

It is easy to see that the contours $h_z(x, y) =$ const. coincide with the stream lines of the current; an example is shown in Fig. 1. The current lines have the expected ellipse-like shape.

This is, however, not the case for the distribution of the current *values* $J(x, y) = (J_x^2 + J_y^2)^{1/2}$. An example is shown in Fig. 2. Hence, the geometry of the streamlines of the vector $\boldsymbol{J}$ differs from that of contours $|J(x, y)| =$ const., unlike the isotropic case where they are in fact the same.

III. MOVING VORTEX

The anisotropic generalization of the isotropic Eq. (2) for the current is straightforward:

$$J_k = \sigma_{kl} E_l - \frac{c}{4\pi} (\lambda^{-2})_{kl} \left(A_l + \frac{\phi_0}{2\pi} \frac{\partial \chi}{\partial x_l} \right). \tag{8}$$

Here, σ_{kl} and $(\lambda^{-2})_{kl}$ are tensors of the conductivity due to normal excitations and of the inverse square of the penetration depth.

Having in mind to derive an equation for the magnetic field $\boldsymbol{h}$ we first have to get rid of the vector potential. To this end, multiply both sides by $4\pi(\lambda^2)_{k\mu}/c$ where $(\lambda^2)_{k\mu}$ is the tensor inverse to $(\lambda^{-2})_{k\mu}$ and sum up over k. Then apply the curl to both sides and use the relation

$$\text{curl}(\boldsymbol{A} + \phi_0 \boldsymbol{\nabla} \chi / 2\pi) = \boldsymbol{h} - \phi_0 \hat{\boldsymbol{z}} \delta(\boldsymbol{r} - \boldsymbol{r}_\nu), \tag{9}$$

where $\boldsymbol{r}_\nu$ are vortex cores positions.

It is convenient to use in the following the notation $\text{curl}_\nu \boldsymbol{V} = \epsilon_{\nu s\mu} \partial V_\mu / \partial x_s$ where $\epsilon_{\nu s\mu}$ is Levi-Civita unit antisymmetric tensor: $\epsilon_{xyz} = 1$ and so do all components with even number of transpositions of indices, it is -1 for odd numbers, and zero otherwise.

Hence, applying $\epsilon_{\nu s\mu} \partial / \partial x_s$ to

$$\frac{4\pi}{c} \lambda_{k\mu}^2 J_k = \frac{4\pi}{c} \lambda_{k\mu}^2 \sigma_{kl} E_l - \left(A_\mu + \frac{\phi_0}{2\pi} \frac{\partial \chi}{\partial x_\mu} \right), \tag{10}$$

one obtains the anisotropic version of the TDL [10]:

$$h_\nu + \frac{4\pi}{c} \epsilon_{\nu s\mu} \lambda_{k\mu}^2 \frac{\partial J_k}{\partial x_s} - \frac{4\pi}{c} \epsilon_{\nu s\mu} \lambda_{k\mu}^2 \sigma_{kl} \frac{\partial E_l}{\partial x_s} = \phi_0 \hat{z}_\nu \delta(\boldsymbol{r} - \boldsymbol{v}t). \tag{11}$$

In this form, the equation is valid for an arbitrary oriented vortex and any crystal anisotropy.

For an orthorhombic crystal in which the vortex and its field are along one of the principal directions (call it z), this cumbersome equation takes the form

$$h_z - \frac{4\pi}{c}\left(\lambda_{xx}^2\frac{\partial J_x}{\partial y} - \lambda_{yy}^2\frac{\partial J_y}{\partial x}\right) + \frac{4\pi\sigma}{c}\left(\lambda_{xx}^2\frac{\partial E_x}{\partial y} - \lambda_{yy}^2\frac{\partial E_y}{\partial x}\right) = \phi_0\delta(\boldsymbol{r}-\boldsymbol{v}t). \quad (12)$$

Here we further simplified the problem assuming isotropic conductivity of normal excitations $\sigma_{xx} = \sigma_{yy} = \sigma$. This should be solved together with quasistationary Maxwell equations curl$\boldsymbol{E} = -\partial_t \boldsymbol{h}/c$ and div$\boldsymbol{E} = 0$ [12,13], which can be done in two-dimensional (2D) Fourier space:

$$E_{\boldsymbol{k}x} = -\frac{k_y}{k_x}E_{\boldsymbol{k}y} = -\frac{ik_y}{ck^2}\frac{\partial h_{\boldsymbol{k}z}}{\partial t}, \quad (13)$$

so that we obtain the 2D Fourier transform of Eq. (12):

$$h_{\boldsymbol{k}}\left(1+k_x^2\lambda_{yy}^2+k_y^2\lambda_{xx}^2\right)+\frac{4\pi\sigma}{c^2}\frac{\lambda_{yy}^2k_x^2+\lambda_{xx}^2k_y^2}{k^2}\frac{\partial h_{\boldsymbol{k}}}{\partial t}=\phi_0 e^{-i\boldsymbol{k}\boldsymbol{v}t}, \quad (14)$$

where $h_{\boldsymbol{k}}$ is the Fourier transform of $h_z(\boldsymbol{r})$. In the isotropic case we obtain the equation studied in Ref. [7]. We further denote $\lambda_{yy}^2 = \lambda_2^2$, $\lambda_{xx}^2 = \lambda_1^2$, and $\lambda = \sqrt{\lambda_1\lambda_2}$:

$$h_{\boldsymbol{k}}\left(1 + k_x^2\lambda_2^2 + k_y^2\lambda_1^2\right) + \tau\,\frac{\lambda_2^2k_x^2 + \lambda_1^2k_y^2}{\lambda^2k^2}\frac{\partial h_{\boldsymbol{k}}}{\partial t} = \phi_0 e^{-i\boldsymbol{k}\boldsymbol{v}t}, \quad (15)$$

with $\tau = 4\pi\sigma\lambda^2/c^2$. This is a linear differential equation for $h_{\boldsymbol{k}}(t)$ with the solution

$$h_{\boldsymbol{k}} = \frac{\phi_0 e^{-i\boldsymbol{k}\boldsymbol{v}t}}{C - iD\boldsymbol{k}\cdot\boldsymbol{s}}, \quad \boldsymbol{s} = \boldsymbol{v}\tau,$$
$$C = 1 + k_x^2\lambda_2^2 + k_y^2\lambda_1^2, \quad D = \frac{\lambda_2^2k_x^2 + \lambda_1^2k_y^2}{\lambda^2k^2}. \quad (16)$$

Since we are interested in stationary motion with a constant velocity, we can set here $t = 0$.

The dimensionless parameter

$$S = \frac{s}{\lambda} = \frac{4\pi v\sigma\lambda}{c^2} \quad (17)$$

is small even for vortex velocities exceeding the speed of sound presently attainable [14,15]. Although in principle S can take larger values, we restrict this discussion by small S and call this case a "slow motion."

IV. SLOW MOTION

For $s \to 0$ one can expand $h(\boldsymbol{k}, \boldsymbol{s})$ in powers of small s up to $O(s)$:

$$h_{\boldsymbol{k}} = \frac{\phi_0}{C} + i\frac{\phi_0 D}{C^2}\boldsymbol{k}\cdot\boldsymbol{s}. \quad (18)$$

The first term corresponds to the static solution discussed above:

$$h_0(x, y) = \frac{\phi_0}{2\pi\lambda^2}K_0(\rho), \quad \rho^2 = \frac{x^2}{\lambda_2^2} + \frac{y^2}{\lambda_1^2}. \quad (19)$$

The correction due to motion is given by

$$\frac{\delta h_{\boldsymbol{k}}\lambda^2}{\phi_0} = i\frac{\left(\lambda_2^2k_x^2 + \lambda_1^2k_y^2\right)\boldsymbol{k}\cdot\boldsymbol{s}}{k^2\left(1 + \lambda_2^2k_x^2 + \lambda_1^2k_y^2\right)^2}. \quad (20)$$

To separate the part that does not disappear when $\lambda_1 = \lambda_2$, one can use the identity

$$\frac{\lambda_2^2k_x^2 + \lambda_1^2k_y^2}{k_x^2 + k_y^2} = \lambda_2^2 + \frac{k_y^2\left(\lambda_1^2 - \lambda_2^2\right)}{k_x^2 + k_y^2} \quad (21)$$

to obtain

$$\frac{4\pi^2\lambda^2\delta h(\boldsymbol{r})}{i\phi_0} = \lambda_2^2\int\frac{d^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{s})e^{i\boldsymbol{k}\boldsymbol{r}}}{\left(1 + \lambda_2^2k_x^2 + \lambda_1^2k_y^2\right)^2} + \left(\lambda_1^2 - \lambda_2^2\right) \times \int\frac{d^2\boldsymbol{k}k_y^2(\boldsymbol{k}\cdot\boldsymbol{s})e^{i\boldsymbol{k}\boldsymbol{r}}}{k^2\left(1 + \lambda_2^2k_x^2 + \lambda_1^2k_y^2\right)^2}. \quad (22)$$

Evaluation of the first contribution is outlined in Appendix A:

$$h_1 = -\frac{\phi_0}{2\pi\lambda^2}\frac{S_xX + S_yY\gamma^2}{2}K_0\left(\sqrt{\frac{X^2}{\gamma} + Y^2\gamma}\right). \quad (23)$$

Here,

$$\boldsymbol{S} = \frac{\boldsymbol{s}}{\lambda}, \quad X = \frac{x}{\lambda}, \quad Y = \frac{y}{\lambda}, \quad \lambda = \sqrt{\lambda_1\lambda_2}, \quad \gamma = \frac{\lambda_2}{\lambda_1}, \quad (24)$$

so that $\lambda_2^2 = \lambda^2\gamma$ and $\lambda_1^2 = \lambda^2/\gamma$.

It is shown in Ref. [16] that, in the isotropic case for a vortex moving along x,

$$h(\boldsymbol{r}) = \frac{\phi_0}{2\pi\lambda^2}e^{-sx/2\lambda^2}K_0\left(\frac{r}{2\lambda}\sqrt{4 + s^2/\lambda^2}\right) \quad (25)$$

in common units. Expanding this in small s one obtains for a slow motion:

$$\delta h(\boldsymbol{r}) = -\frac{\phi_0}{4\pi\lambda^4}sxK_0\left(\frac{r}{\lambda}\right). \quad (26)$$

Hence, h_1 of Eq. (23) has the correct isotropic limit.

The second integral over two components of $\boldsymbol{k}$ in Eq. (22) can be reduced to integrals over a single variable which are easy to deal with numerically, see Appendix B:

$$\frac{2\pi\lambda^2}{\phi_0}h_2 = \frac{(\gamma^2 - 1)}{4\gamma}\Bigg\{S_xX\int_0^\infty\frac{d\zeta}{(\zeta + \gamma)^{3/2}(\zeta + 1/\gamma)^{3/2}} \times\left[K_0(\mathcal{R}_\zeta) - \frac{Y^2}{(\zeta + 1/\gamma)\mathcal{R}_\zeta}K_1(\mathcal{R}_\zeta)\right] + S_yY\int_0^\infty\frac{d\zeta}{(\zeta + \gamma)^{1/2}(\zeta + 1/\gamma)^{5/2}} \times\left[3K_0(\mathcal{R}_\zeta) - \frac{Y^2}{(\zeta + 1/\gamma)\mathcal{R}_\zeta}K_1(\mathcal{R}_\zeta)\right]\Bigg\},$$
$$\mathcal{R}_\zeta = \sqrt{\frac{X^2}{\zeta + \gamma} + \frac{Y^2}{\zeta + 1/\gamma}}. \quad (27)$$

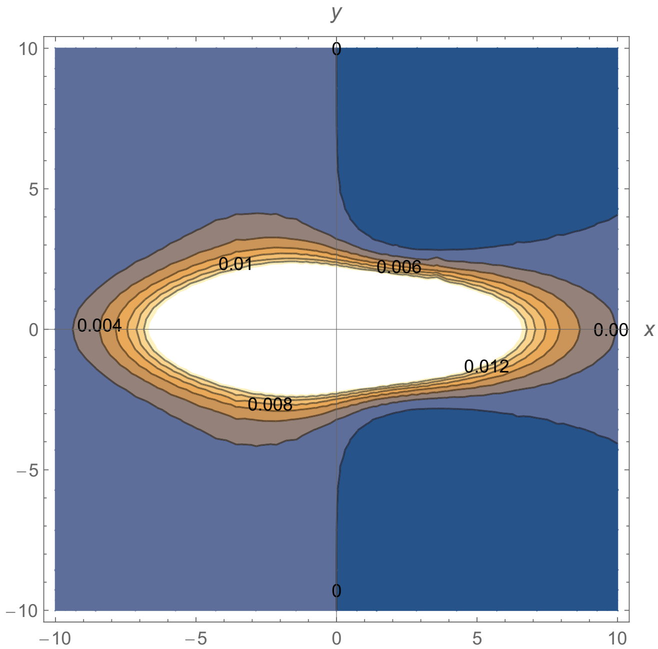

FIG. 3. Contours $h(x, y) = \text{const.}$ for the vortex moving along x axis ($S_x = 0.1,\ S_y = 0$) and $\lambda_2/\lambda_1 = 3$. The motion is directed to $+x$. x and y are in units of $\lambda = \sqrt{\lambda_1\lambda_2}$.

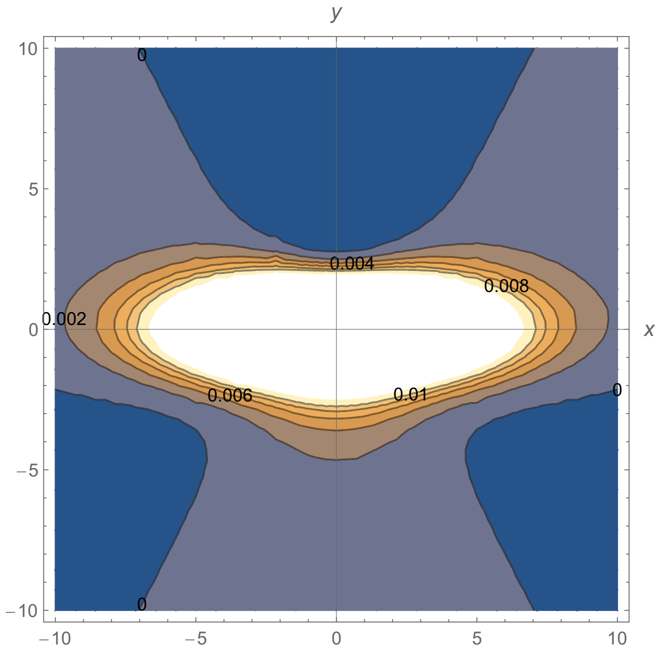

FIG. 4. Contours $h(x, y) = \text{const.}$ for the vortex moving along y axis ($S_x = 0, S_y = 0.1$) and $\lambda_2/\lambda_1 = 3$. The motion is directed to $+y$. x and y are in units of $\lambda = \sqrt{\lambda_1\lambda_2}$.

Thus, the vortex field can be calculated as $h = h_0 + h_1 + h_2$ with h_0 given in Eq. (19), h_1 in Eq. (23), and h_2 in Eq. (27). The results obtained with the help of Wolfram *Mathematica* package are shown below.

One can see in Fig. 3 that the current streamlines [or, what is the same, contours $h(x, y) = \text{const.}$] in the vicinity of the moving vortex core are only weakly distorted relative to the static elliptic shape. The most interesting feature of this distribution is that, at large distances, $h(x, y)$ changes sign in some parts of the (x, y) plane. Since the interaction energy of the vortex at the origin with another one at (x, y) is proportional to $h(x, y)$, the presence of domains with $h < 0$ means that, for the second vortex in these domains, the intervortex interaction is attractive.

The field distribution is different for the motion along y axis shown in Fig. 4. It is seen that the flux in front of the moving vortex is depleted whereas behind it is enhanced, the feature discussed in Ref. [17] for the isotropic case. This feature remains also for a general direction of motion; an example of motion along the line $x = y$ is shown in Fig. 5. Moreover, Figs. 3–5 show that this depletion may even change sign of the field. It is worth mentioning here that the London theory is reliable in the region $r \gg \xi$, ξ being the core size, and so are our predictions of a nontrivial behavior of $h(x, y)$ at large distances.

It is instructive to see how the interaction changes along certain directions. For example, Fig. 6 shows that for $S_x = 0$, $S_y = 0.1$, the motion along the y axis, $h(0, Y)$ is positive if $0 < Y \lesssim 2.5$ [so that the second vortex at $(0, Y)$ in this region is repelled by the vortex at the origin]. If the second vortex is at $2.5 \lesssim Y < \infty$ the interaction is attractive.

The profiles $h_z(X, Y)$ shown in Figs. 3–5 are quite unusual and were obtained as a result of a lengthy analytical procedure. To be confident in the result, we applied the 2D Fast Fourier Transform (FFT) using the original Fourier components of Eq. (18) to obtain $h(\boldsymbol{r})$. As is seen from Fig. 6, the two methods give practically identical results. Although the FFT calculation is convenient and efficient, the analytical

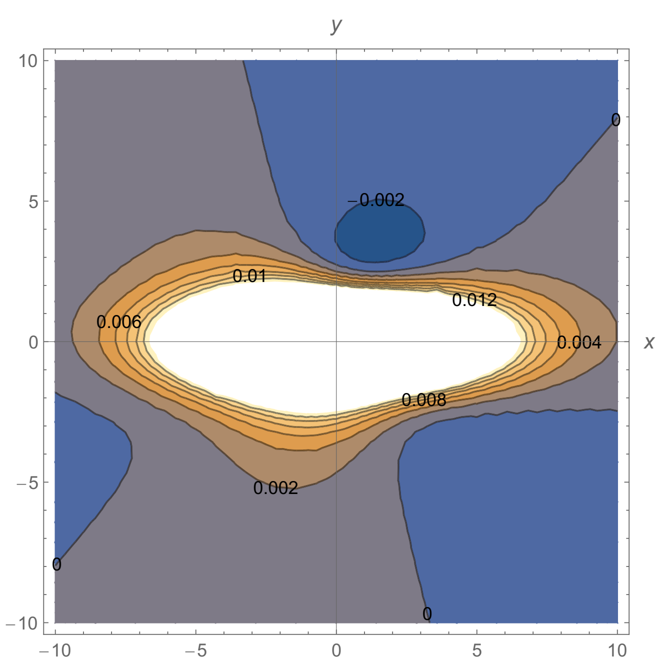

FIG. 5. Contours $h(x, y) = \text{const.}$ for the vortex moving along the diagonal $x = y$ ($S_x = S_y = 0.1$) and $\lambda_2/\lambda_1 = 3$. x and y are in units of $\lambda = \sqrt{\lambda_1\lambda_2}$.

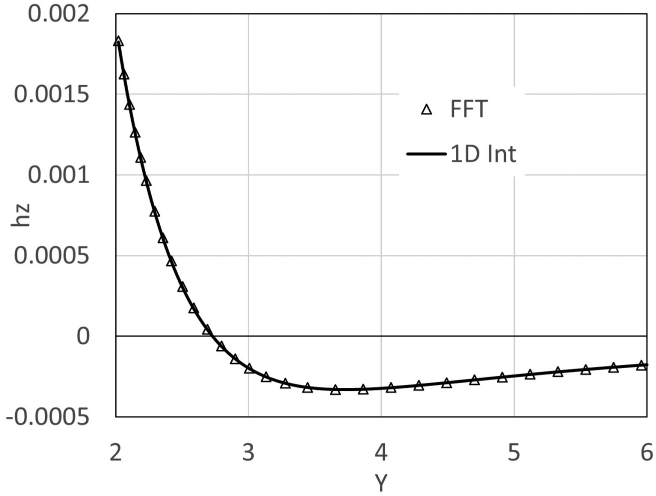

FIG. 6. The field $h_z(0, Y)$ for the vortex moving along Y ($S_x = 0$, $S_y = 0.1$); $\lambda_2/\lambda_1 = 3$. Y is in units of $\lambda = \sqrt{\lambda_1\lambda_2}$. The solid line is according to $h_z = h_0 + h_1 + h_2$ with h_0 given in Eq. (19), h_1 in Eq. (23), and h_2 obtained by numerical integration of Eq. (27). Open triangles are results of FFT of Eq. (18).

formula has an advantage of ensuring numerical accuracy for calculating the far-field behavior of interest.

A. Asymptotic behavior of $h(0, Y)$ for $Y \to \infty$

For $X = 0$, Eq. (27) yields

$$\frac{2\pi\lambda^2}{\phi_0} h_2 = \frac{\gamma^2 - 1}{4\gamma} S_y Y \int_0^\infty \frac{d\zeta\,[3K_0(\eta) - \eta K_1(\eta)]}{(\zeta + \gamma)^{1/2}(\zeta + 1/\gamma)^{5/2}},$$
$$\eta = \frac{|Y|}{\sqrt{\zeta + 1/\gamma}}. \quad (28)$$

Going to the integration variable η, we get

$$\frac{2\pi\lambda^2}{\phi_0} h_2 = \frac{\gamma^2 - 1}{2\gamma} \frac{S_y}{Y^2} \int_0^{Y\sqrt{\gamma}} \frac{d\eta\, \eta^3 [3K_0(\eta) - \eta K_1(\eta)]}{\sqrt{Y^2 + \eta^2(\gamma - 1/\gamma)}}. \quad (29)$$

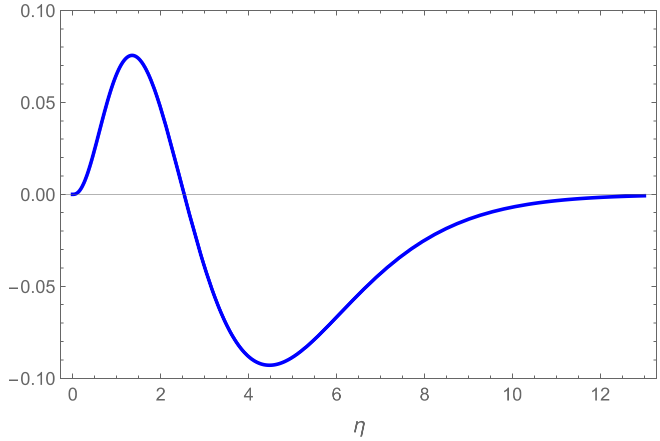

FIG. 7. The integrand of Eq. (29) for $Y = 10$ and $\gamma = 3$.

Figure 7 shows that the integrand here is substantial only in a finite region $0 < \eta \lesssim 10$. Therefore, being interested in the asymptotic behavior for $|Y| \to \infty$, one can replace the integrand denominator by $|Y|$ and the upper limit of integration by ∞:

$$\frac{2\pi\lambda^2}{\phi_0} h_2(0, Y) = \frac{\gamma^2 - 1}{2\gamma} \frac{S_y}{Y^3} \int_0^\infty d\eta\, \eta^3 [3K_0 - \eta K_1]_\eta$$
$$= -\frac{\gamma^2 - 1}{\gamma} \frac{2S_y}{Y^3}. \quad (30)$$

Thus, $h_2(0, Y)$ is negative when $Y \to \infty$ and positive for $Y \to -\infty$. It decays as $1/Y^3$, therefore, the total field $h_0 + h_1 + h_2$ attenuates as a power law as well, since h_0 and h_1 decay exponentially and at large distances can be disregarded. Hence, h_2 can be replaced with h in this region. This conclusion agrees with direct numerical evaluation of $h(0, Y)$ shown in Fig. 6.

In the same manner one can obtain the leading term in the asymptotic behavior if $Y = S_y = 0$ for the motion along the x axis:

$$h(X, 0) \sim \frac{\phi_0}{2\pi\lambda^2} \frac{\gamma^2 - 1}{2\gamma} \frac{2S_x}{X^3}. \quad (31)$$

For the sake of brevity we do not provide other terms in the asymptotic series.

The power-law decay of the field $h(x, y)$ for vortices moving in anisotropic superconductors is a surprising feature. Clearly, this feature disappears for vortices at rest as well as for vortices moving in isotropic materials. Formally, the power-law behavior in real space originates in the factor $1/k^2$ in Fourier transforms, see, e.g., Eq. (22), which, however, cancels out for $\gamma = 1$.

V. ELECTRIC FIELD FOR SLOW MOTION

In the approximation linear in velocity, we have according to Eq. (16)

$$\frac{\partial h_{\boldsymbol{k}}}{\partial t} = -i \frac{\phi_0(\boldsymbol{k} \cdot \boldsymbol{v})}{C}, \quad C = 1 + k_x^2\lambda_2^2 + k_y^2\lambda_1^2. \quad (32)$$

According to Eqs. (13) the electric field is

$$E_{kx} = -\frac{k_y}{k_x} E_{ky} = -\frac{\phi_0}{c\tau} \frac{k_y(\boldsymbol{k} \cdot \boldsymbol{s})}{k^2 C}, \quad \boldsymbol{s} = \boldsymbol{v}\tau. \quad (33)$$

Hence, we have in real space

$$E_x = -\frac{\phi_0}{4\pi^2 c\tau} \int \frac{d^2\boldsymbol{k}\, k_y(\boldsymbol{k} \cdot \boldsymbol{s})}{k^2 C} e^{i\boldsymbol{k}\boldsymbol{r}}, \quad (34)$$

or, using $\lambda = \sqrt{\lambda_1\lambda_2}$ as the unit length,

$$E_x = -\frac{\phi_0}{4\pi^2 c\tau\lambda} \int \frac{d^2\boldsymbol{q}\, q_y(\boldsymbol{q} \cdot \boldsymbol{S})}{q^2 C} e^{i\boldsymbol{q}\boldsymbol{R}}. \quad (35)$$

Here, $\boldsymbol{q} = \boldsymbol{k}\lambda$, $\boldsymbol{R} = (X, Y) = \boldsymbol{r}/\lambda$, and

$$C = 1 + q_x^2\gamma + q_y^2/\gamma, \quad \gamma = \lambda_2/\lambda_1. \quad (36)$$

In the same way we obtain

$$E_y = \frac{\phi_0}{4\pi^2 c\tau\lambda} \int \frac{d^2\boldsymbol{q}\, q_x(\boldsymbol{q} \cdot \boldsymbol{S})}{q^2 C} e^{i\boldsymbol{q}\boldsymbol{R}}. \quad (37)$$

The integrals in Eqs. (35) and (37) are dimensionless.

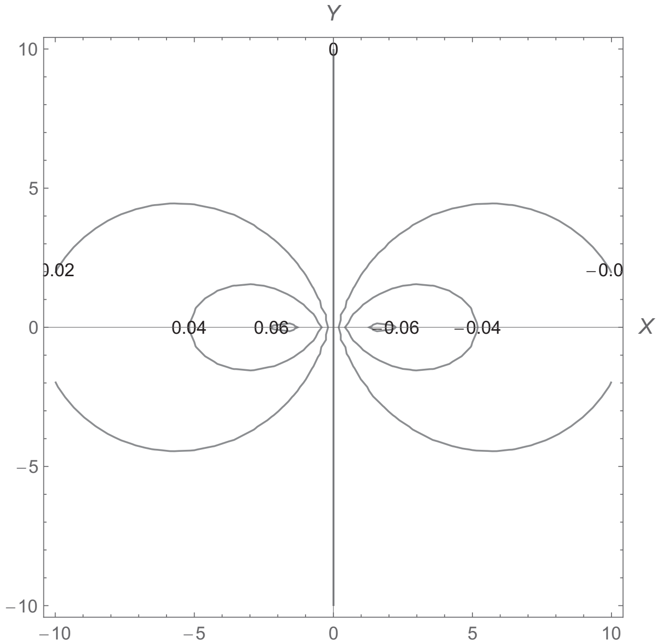

FIG. 8. Streamlines of the field $\boldsymbol{E}$ (or of the normal current $\boldsymbol{J}_n$) for the vortex moving along X ($S_x = 0.1$, $S_y = 0$). $\gamma = \lambda_2/\lambda_1 = 3$. X, Y are in units of $\lambda = \sqrt{\lambda_1\lambda_2}$. Positive constants by the contours correspond to the clockwise current direction, negative otherwise.

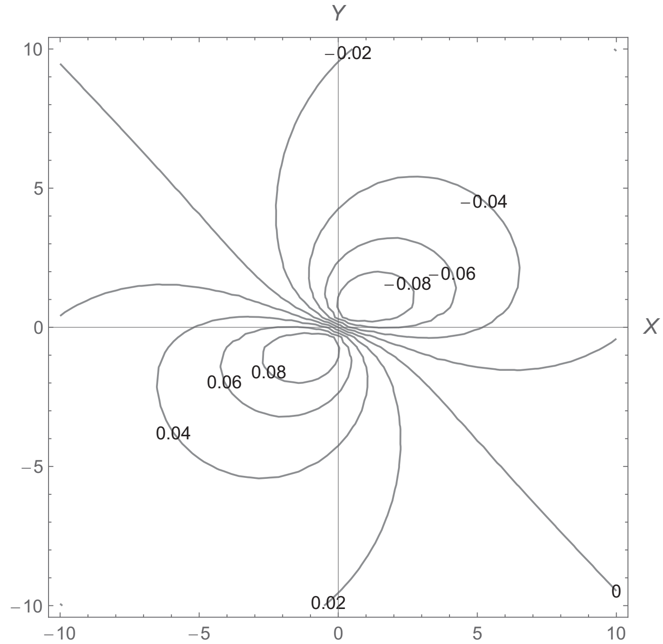

FIG. 9. Streamlines of the normal current for the vortex moving along the line $X = Y$ ($S_x = S_y = 0.1$), $\gamma = \lambda_2/\lambda_1 = 3$. X, Y are in units of $\lambda = \sqrt{\lambda_1\lambda_2}$. Positive constants by the contours correspond to the clockwise current direction, negative otherwise.

It is of interest to see the streamlines of $\boldsymbol{E}$ (or, which is the same, of the normal current $\boldsymbol{J}_n = \sigma\boldsymbol{E}$). To this end, we calculate the stream function $G(x, y)$ such that $E_x = \partial_y G$ and $E_y = -\partial_x G$; the streamlines then are given by contours $G(x, y) = \text{const}$. In Fourier space we have $E_{x\boldsymbol{k}} = ik_y G_{\boldsymbol{k}}$ so that

$$G_{\boldsymbol{k}} = \frac{i\phi_0}{c\tau}\frac{(\boldsymbol{k}\cdot\boldsymbol{s})}{k^2 C}, \quad G(\boldsymbol{r}) = \frac{i\phi_0}{4\pi^2 c\tau}\int\frac{d^2\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{S})e^{i\boldsymbol{q}\boldsymbol{R}}}{q^2 C}. \quad (38)$$

The formal procedure of reducing the double to a single integration is similar to that used for $h(\boldsymbol{r})$ and is outlined in Appendix C. The result is

$$G(\boldsymbol{r}) = -\frac{\phi_0}{4\pi c\tau}\int_0^\infty\frac{d\eta\, K_0(\mathcal{R}\sqrt{\eta})}{\sqrt{\mu\nu}}\left(\frac{S_x X}{\mu} + \frac{S_y Y}{\nu}\right),$$

$$\mu = 1 + \eta\gamma, \quad \nu = 1 + \eta/\gamma, \quad \mathcal{R} = \sqrt{\frac{X^2}{\mu} + \frac{Y^2}{\nu}}. \quad (39)$$

Figures 8 and 9 show two examples of J_n streamlines [or contours $G(X, Y) = \text{const.}$] obtained by numerical integration of Eq. (39).

The electric field is now readily obtained by differentiation of G. We will not write down these cumbersome expressions. Instead we consider the asymptotic behavior of electric fields at large distances in two simple cases using the method employed above for asymptotic behavior of $h(0, y)$ and $h(x, 0)$. Omitting formalities, we give the results:

$$G(X, 0) \sim -\frac{\phi_0}{2\pi c\tau}\frac{S_x}{X}, \quad |X| \to \infty, \quad (40)$$

that yields in common units

$$E_x(X, 0) = 0, \quad E_y(X, 0) \sim \frac{\phi_0}{2\pi c\tau\lambda}\frac{S_x}{X^2} = \frac{\phi_0 v}{2\pi c\, x^2}. \quad (41)$$

Similarly, for the motion along Y axis,

$$E_y(0, Y) = 0, \quad E_x(0, Y) \sim \frac{\phi_0 v}{2\pi c\, y^2}. \quad (42)$$

It is worth noting that the relaxation time τ does not enter the electric-field asymptotic formulas for slow motion. Since τ is the only parameter in Eq. (3) responsible for distortions of the field $\boldsymbol{h}$ by motion, the electric field is present even if the vortex is displaced as a whole with no $\boldsymbol{h}$ distortions. This is, of course, how it should be.

Also, the material anisotropy does not enter these results as well. This means that the power-law decay of the electric field exists also in the isotropic case. In fact, for $\gamma = 1$ one has from Eq. (38)

$$G(X, 0) = \frac{i\phi_0 S_x}{4\pi^2 c\tau}\int\frac{d^2\boldsymbol{q}\, q_x e^{i\boldsymbol{q}\boldsymbol{X}}}{q^2(1 + q^2)}, \quad (43)$$

which is readily done integrating first over the angle between $\boldsymbol{q}$ and $\boldsymbol{X}$. We obtain

$$G(X, 0) = \frac{\phi_0 S_x}{2\pi c\tau}\left[K_1(X) - \frac{1}{X}\right], \quad (44)$$

which gives

$$E_y(X, 0) = -\frac{\phi_0 v}{2\pi c\lambda^2}\left[K_1'(X) + \frac{1}{X^2}\right]. \quad (45)$$

Figure 10 shows that the field $E_y(X, 0)$ changes sign at $x/\lambda \approx 1$, reaches maximum near 2, and slowly decays as a power law λ^2/x^2. This is quite surprising since the electric

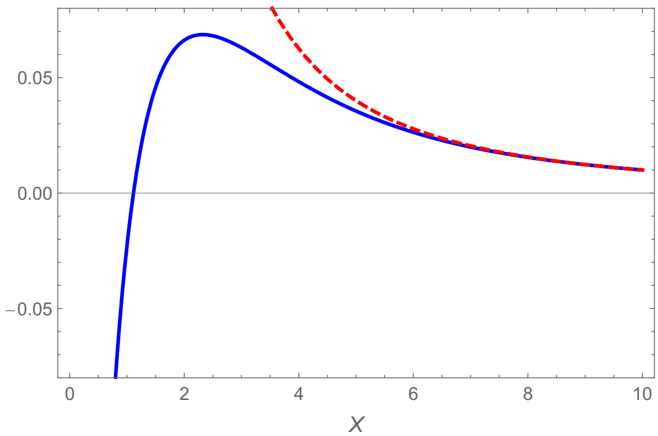

FIG. 10. The solid line is the square brackets in Eq. (45) for $E_y(X,0)$ when the vortex moves along the X axis ($S_y = 0$), $\gamma = 1$. The dashed line shows the power-law term $1/X^2$. X is in units of λ.

field power-law decay means that no Meissner-type screening of $\boldsymbol{E}$ is involved.

VI. DISCUSSION

We have studied effects of vortex motion within time-dependent London theory, which is based on the assumption that, in time-dependent phenomena, the current in superconductors consists of the persistent and normal components, Eq. (2). This approach differs from the common assumption that the vortex magnetic structure moves as a whole, so that in the frame bound to the moving vortex the magnetic field distribution is the same as for a vortex at rest, see, e.g., [18] or multitude of papers describing the flux flow.

Within the TDL approach the field distribution of the moving vortex differs from that of vortex at rest even in the frame moving with the vortex. The physical reason for this is simple: the moving magnetic structure $h(x,y)$ induces the electric field and currents of normal excitations, while the latter distort the moving static field distribution $h_0(x,y)$. This is a general feature of systems with singularities (vortices) moving in dissipative media [6,7].

The equations describing these time-dependent phenomena are diffusion-like, so that solutions $h_{\boldsymbol{k}}$ are obtained in the 2D Fourier space: to recover $h(\boldsymbol{r})$ one has to evaluate double integrals $\int d^2\boldsymbol{k}\cdots$, a heavy numerical procedure. We offer a way to reduce double integrals to a single $\int_0^\infty d\eta\cdots$ which can be evaluated within the Wolfram *Mathematica* package efficiently and fast, which is relevant especially for generating plots of various 2D distributions.

We have investigated the field distribution of moving vortices away of the vortex core whether the time-dependent London theory is reliable. As in the isotropic case [17], the magnetic field h_z of moving vortices in anisotropic materials is distorted relative to the static case, the magnetic flux is redistributed so that it is depleted in front of the moving vortex and enhanced behind it. The depletion could be strong enough to change the sign of h_z in some parts of the xy plane. This suggests that the interaction of two vortices, one at the origin at some moment and another at (x,y), being repulsive at short intervortex distances may turn attractive at large ones.

The physical reason for this change is the induced electric field $\boldsymbol{E}$ (and along with it the currents of normal excitations $\sigma\boldsymbol{E}$). This field is obtained by solving quasistationary Maxwell equations curl$\boldsymbol{E} = -\partial_t\boldsymbol{h}/c$, the condition of quasineutrality div$\boldsymbol{E} = 0$, coupled with the time-dependent London equation (basically, the same procedure as in deriving time-dependent Ginzburg-Landau equations [13]). We find that in anisotropic case the magnetic field of moving vortex has a power-law dependence on distances $r \gg \lambda$: $h \propto (\gamma^2 - 1)v/r^3$ (γ is the anisotropy parameter, v is the vortex velocity). The exponentially decaying part of h is still present, but at large distances it is irrelevant in comparison with the power-law part. In isotropic case, the power law gives way to the standard exponential decay. The electric field, however, goes as $1/r^2$ in both cases.

We note that TDL differs from other approaches to moving vortices. In particular it differs from TDGL which holds only near T_c, which, however, is able to describe also the vortex core. This is of course impossible within TDL, which still has an advantage of applicability at all Ts. It would be of interest to study asymptotic behavior of fields related to moving vortices within TDGL since both TDGL and TDL work near T_c; we are not aware of such studies. Also, TDL differs from the approach of Coffey and Hao [19] who used electrostatics to describe the electric field of moving vortices. Although their approach yields the stream-line pictures reminiscent of our Figs. 8 and 9, we doubt that the electrostatics can be used for description of electric fields in superconductors.

Most of our calculations were done for orthorhombic materials with the in-plane anisotropy parameter $\gamma = 3$ and the vortex along c. Such materials in fact exist; examples are NiBi films [20], or $Ta_4Pd_3Te_{16}$ [21].

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APPENDIX A

Consider the integral

$$\int \frac{d^2\boldsymbol{k}e^{i\boldsymbol{kr}}}{(1+k^2)^2} = \int_0^\infty \frac{k\,dk}{(1+k^2)^2}\int_0^{2\pi} d\varphi e^{ikr\cos(\alpha-\varphi)}$$
$$= 2\pi\int_0^\infty \frac{k\,dk}{(1+k^2)^2}J_0(kr) = \pi r K_1(r). \quad \text{(A1)}$$

$\boldsymbol{k}$ and $\boldsymbol{r}$ are at angles φ and α relative to x. Apply ∂_x to both sides:

$$\int \frac{d^2\boldsymbol{k}k_x e^{i\boldsymbol{kr}}}{(1+k^2)^2} = i\pi x K_0(r), \quad \text{(A2)}$$

The evaluation of the first integral in Eq. (22) is now straightforward.

APPENDIX B

The second contribution in Eq. (22) consists of parts related to x and y projections of the velocity. With the help of the identities

$$\frac{1}{f}=\int_0^\infty du\,e^{-fu},\quad \frac{1}{f^2}=\int_0^\infty du\,u\,e^{-fu}, \tag{B1}$$

one recasts the x part:

$$\begin{aligned}I_x&=\frac{S_x(1-\gamma^2)}{\gamma}\int\frac{d^2\boldsymbol{q}\,q_y^2q_xe^{i\boldsymbol{qR}}}{q^2\left(1+\gamma q_x^2+q_y^2/\gamma\right)^2}\\&=\int_0^\infty d\xi\int_0^\infty du\,ue^{-u}\int d^2\boldsymbol{q}\,q_y^2q_xe^{i\boldsymbol{qR}-(\xi+u\gamma)q_x^2-(\xi+u/\gamma)q_y^2}.\end{aligned} \tag{B2}$$

Here we use λ as a unit length: $\boldsymbol{q}=\boldsymbol{k}\lambda$, $\boldsymbol{R}=\boldsymbol{r}/\lambda$, $\lambda_2^2=\lambda^2\gamma$, $\lambda_1^2=\lambda^2/\gamma$, and $\boldsymbol{S}=\boldsymbol{s}/\lambda$. The anisotropy parameter is $\gamma=\lambda_2/\lambda_1$. We now introduce a new integration variable ζ via $\xi=\zeta u$:

$$\begin{aligned}I_x=&\frac{S_x(1-\gamma^2)}{\gamma}\int_0^\infty d\zeta\int_0^\infty du\,u^2e^{-u}\\&\times\int d^2\boldsymbol{q}\,q_y^2q_xe^{i\boldsymbol{qR}-u(\zeta+\gamma)q_x^2-u(\zeta+1/\gamma)q_y^2}.\end{aligned} \tag{B3}$$

Integrals over q_x, q_y are evaluated with the help of the known Fourier transform of a Gaussian:

$$\int_{-\infty}^\infty dq_x\,e^{iq_xx-aq_x^2}=\sqrt{\frac{\pi}{a}}e^{-x^2/4a}. \tag{B4}$$

Integration over u can be done utilizing relations

$$\begin{aligned}&\int_0^\infty\frac{du}{u}\exp\left(-u-\frac{w^2}{4u}\right)=2\,K_0(w),\\&\int_0^\infty\frac{du}{u^2}\exp\left(-u-\frac{w^2}{4u}\right)=\frac{4}{w}\,K_1(w).\end{aligned} \tag{B5}$$

We obtain after straightforward algebra:

$$\begin{aligned}I_x=&\frac{i\pi(1-\gamma^2)}{2\gamma}S_xX\int_0^\infty\frac{d\zeta}{(\zeta+\gamma)^{3/2}(\zeta+1/\gamma)^{3/2}}\\&\times\left[K_0(R_\zeta)-\frac{Y^2}{(\zeta+1/\gamma)R_\zeta}K_1(R_\zeta)\right],\\R_\zeta^2=&\frac{X^2}{\zeta+\gamma}+\frac{Y^2}{\zeta+1/\gamma}.\end{aligned} \tag{B6}$$

In a similar fashion one obtains the part proportional to S_y and Eq. (27).

APPENDIX C: ELECTRIC FIELD AND NORMAL CURRENTS

We evaluate here the stream function of Eq. (38):

$$G=\frac{i\phi_0}{4\pi^2c\tau}\hat{G},\quad \hat{G}=\int\frac{d^2\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{S})}{q^2C}e^{i\boldsymbol{qR}}. \tag{C1}$$

The following manipulation is similar to that outlined in Appendix B for $h(X,Y)$:

$$\begin{aligned}\hat{G}&=\int d^2\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{S})e^{i\boldsymbol{qR}}\int_0^\infty due^{-uq^2}\int_0^\infty d\xi\,e^{-\xi C}\\&=\int_0^\infty du\int_0^\infty d\xi e^{-\xi}\int d^2\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{S})e^{i\boldsymbol{qR}-uq^2-\xi(q_x^2\gamma+q_y^2/\gamma)}.\end{aligned} \tag{C2}$$

Furthermore, we write the last integral as $\int d^2\boldsymbol{q}\cdots=S_xI_x+S_yI_y$ with

$$I_x=\int_{-\infty}^\infty dq_xq_xe^{iq_xX-q_x^2(u+\xi\gamma)}\int_{-\infty}^\infty dq_ye^{iq_yY-q_y^2(u+\xi/\gamma)}, \tag{C3}$$

and I_y which is obtained from I_x by replacing $x\leftrightarrow y$ and $\gamma\leftrightarrow 1/\gamma$. The integral over q_x and q_y are done using Eq. (B4):

$$\begin{aligned}I_x=&\frac{i\pi X}{2(u+\xi\gamma)^{3/2}(u+\xi/\gamma)^{1/2}}\\&\times\exp\left(-\frac{X^2}{4(u+\xi\gamma)}-\frac{Y^2}{4(u+\xi/\gamma)}\right),\end{aligned} \tag{C4}$$

and the part $\hat{G}$ proportional to S_x takes the form

$$\begin{aligned}\hat{G}_x=&\frac{i\pi XS_x}{2}\int_0^\infty du\int_0^\infty\frac{d\xi\,e^{-\xi}}{(u+\xi\gamma)^{3/2}(u+\xi/\gamma)^{1/2}}\\&\times\exp\left(-\frac{X^2}{4(u+\xi\gamma)}-\frac{Y^2}{4(u+\xi/\gamma)}\right),\end{aligned} \tag{C5}$$

To integrate over u we can use Eq. (B5). To this end we introduce a new integration variable η instead of ξ via $\xi=u\eta$. Then the integral over ξ becomes

$$\begin{aligned}&\frac{1}{u}\int_0^\infty d\eta\frac{e^{-\eta u}}{(1+\eta\gamma)^{3/2}(1+\eta/\gamma)^{1/2}}\exp\left(-\frac{\mathcal{R}_\eta^2}{4u}\right),\\&\mathcal{R}_\eta^2=\frac{X^2}{1+\eta\gamma}+\frac{Y^2}{1+\eta/\gamma}.\end{aligned} \tag{C6}$$

Now, the integration over u is done with the help of Eq. (B5) and we obtain

$$\hat{G}_x=i\pi S_xX\int_0^\infty\frac{d\eta}{(1+\eta\gamma)^{1/2}(1+\eta/\gamma)^{3/2}}K_0(R_\eta\sqrt{\eta}). \tag{C7}$$

The part G_y follows immediately after the replacements $x\leftrightarrow y$ and $(1+\eta\gamma)\leftrightarrow(1+\eta/\gamma)$:

$$\hat{G}_y=i\pi S_yY\int_0^\infty\frac{d\eta}{(1+\eta\gamma)^{3/2}(1+\eta/\gamma)^{1/2}}K_0(\mathcal{R}_\eta\sqrt{\eta}). \tag{C8}$$

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