Strain-induced spin vortex and Majorana Kramers pairs in doped topological insulators with nematic superconductivity

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Using the Ginzburg-Landau approach, we show that the strain of the nematic superconductor can generate a specific (nematic) vorticity. In the case of doped topological insulators that vorticity forms a spin vortex. We find two types of topologically different spin vortices that either enhance (type I) or suppress (type II) superconductivity far from the vortex core. We apply Bogoliubov–de Gennes equations to study electronic states in the nematic superconductor with spin vortices. We find that in the case of the type I vortex, zero-energy states are localized near the vortex core. These states can be identified as Majorana Kramers pairs. In the case of the type II vortex, there are no localized zero-energy states. Thus, we establish a nontrivial connection between the strain and Majorana fermions in the doped topological insulators with nematic superconductivity.

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I. INTRODUCTION

Nematic superconductivity in doped topological insulators attracts a great deal of attention nowadays [1–8]. In these systems, the superconducting order parameter is time reversal invariant, which corresponds to the E_u representation that breaks inversion symmetry and couples electrons with the same spin projections but from different orbitals [9,10]. NMR measurements confirmed the triplet nature of the nematic topological superconductivity in doped topological insulators [11].

Exotic quasiparticles with non-Abelian statistics such as Majorana fermions can exist in topological superconductors [12,13]. The Majorana fermions can be localized on various types of topological defects [14]. One way to induce Majorana fermions is to generate vorticity in the mass term. For example, the Majorana fermions can be localized in the cores of Abrikosov vortices [12,15–18]. If time-reversal symmetry is present, then, the Majorana fermions arise as Kramers pairs [14,19].

Superconducting order in doped topological insulators belongs to the DIII symmetry class [20]. An analog of the nematic superconductor of class DIII is the superfluid B phase in ^{3}He [21]. An interesting property of such a phase of helium is a possible realization of the spin vortex that preserves the time-reversal symmetry. The spin vortices in the B phase of ^{3}He have been observed experimentally [22].

The spin vortices in the context of superconductivity have been briefly discussed for $(p_x + ip_y)_\uparrow(p_x - ip_y)_\downarrow$ superconductors [19]. The spin vortex (referred to as the nematic vortex) was studied in a superconductor with the nematic order parameter in Ref. [23]. A single-orbital Hamiltonian with a quadratic dispersion and k-dependent order parameter was considered. It has been argued that the spin vortex brings Majorana Kramers pairs into the system that form a Majorana flat band.

A distinct feature of the nematic superconductivity is strong coupling of the superconductivity with strain [24]. In particular, the strain is responsible for a twofold symmetry of the second critical field that has been observed in the experiments [25,26]. The strain can be either spontaneous or external [7]. We show that the applied centrosymmetric strain can generate spin vortices in doped topological insulators with nematic superconductivity.

We assume that a local force is applied to a sample of the doped topological insulator, which has the form of a disk. The force generates a centrosymmetric strain that couples with the superconductivity and forms a (nematic) spin vorticity. Depending on the sample properties, two types of topologically different spin vortices can exist. Such spin vortices have a normal core. We solve Bogoliubov–de Gennes (BdG) equations and show that one type of spin vortices localizes the Majorana Kramers pairs. Near the core of another type of spin vortex there are no localized zero-energy states.

II. GINZBURG-LANDAU APPROACH

The Ginzburg-Landau (GL) free energy of the E_u topological superconductor with D_{3d} crystal symmetry can be written as [10]

$$F_0 = A(|\Delta_1|^2 + |\Delta_2|^2) + B_1(|\Delta_1|^2 + |\Delta_2|^2)^2 + B_2|\Delta_1^*\Delta_2 - \Delta_1\Delta_2^*|^2. \tag{1}$$

Here $\vec{\Delta} = (\Delta_1, \Delta_2)$ is the vector order parameter, and $A_1 \propto T - T_c < 0$ and $B_1 > 0$ are the GL coefficients. We suppose that B_2 is positive, which corresponds to the nematic superconductivity with a real order parameter $\vec{\Delta} = \Delta(\cos\alpha, \sin\alpha)$. The vector $\vec{n} = (\cos\alpha, \sin\alpha)$ shows the nematicity direction. The free energy (1) is degenerate with respect to α. The nematicity direction can be fixed by the strain [10,24].

We assume that the sample is deformed by some local external force and the corresponding strain tensor has components u_{xx}, u_{yy}, and u_{xy}, which depend on the coordinate $\vec{r}$. The strain tensor couples with the superconducting order parameter. This coupling is described by an additional term in the GL free energy [10,24],

$$F_u = g_N(u_{xx} - u_{yy})(|\Delta_1|^2 - |\Delta_2|^2) + 2g_N u_{xy}(\Delta_1^*\Delta_2 + \Delta_1\Delta_2^*), \quad (2)$$

where g_N is a coupling coefficient. The order parameter becomes coordinate dependent, and in general, we have to take into account the corresponding gradient terms in the GL free energy. However, deformation and superconductivity are different phenomena with their spatial scales. The scale of the superconductivity is the effective coherence length ξ_{eff}. It is a microscopic value, while the strain characteristic scale l_u is a macroscopic value of the order of the sample sizes. Thus, it is reasonable to suppose that $l_u \gg \xi_{\text{eff}}$. In this case, away from the center of the vortex $r \gg \xi_{\text{eff}}$, we can neglect the gradient terms and assume that $\vec{\Delta}$ depends on the coordinate parametrically, that is, $\vec{\Delta}(\vec{r}) = \vec{\Delta}[u_{ik}(\vec{r})]$, and the order parameter can be found from minimization of $F_0 + F_u$ with respect to $\vec{\Delta}$. To obtain the correct behavior of the order parameter near the center of the vortex, $r \sim \xi_{\text{eff}}$, we should take into account the gradient terms in the GL functional. This procedure is performed in Appendix A. We show that the spin vortex has a normal core with the size $\sim\xi_{\text{eff}}$ similar to the Abrikosov vortices [27]. This normal core can be considered a topological defect. The values of the coherent length for vortices of type I, ξ_{I}, and type II, ξ_{II}, are different.

We suppose that the force, and hence the strain, has rotational symmetry and the strain tensor components can be written in the cylindrical coordinates (r, φ, z) as (see Ref. [28] and Appendix A)

$$u_{xx} - u_{yy} = u(r, z)\cos(2\varphi), \quad 2u_{xy} = u(r, z)\sin(2\varphi), \quad (3)$$

where $u(r, z)$ depends on the applied force, sample sizes, and boundary conditions.

After substitution of expressions for $\vec{\Delta}$ and u_{ik} in Eqs. (1) and (2) we obtain

$$F_0 + F_u = A\Delta^2 + B_1\Delta^4 + g_N u\Delta^2\cos[2(\alpha - \varphi)]. \quad (4)$$

When $g_N u(r, z) > 0$, the minimization of $F_0 + F_u$ by α gives

$$\alpha = \varphi + \pi\left(n + \tfrac{1}{2}\right). \quad (5)$$

When $g_N u(r, z) < 0$, the minimum of $F_0 + F_u$ is attained if

$$\alpha = \varphi + \pi n. \quad (6)$$

Here n is an integer or zero. The value of $\Delta(r, z)$ is obtained from minimization of the GL free energy with respect to Δ. Taking into account Eqs. (5) and (6), we derive

$$\Delta(r, z) = \sqrt{\frac{-A + g_N u(r, z)}{2B_1}}. \quad (7)$$

Thus, the external force not only affects the value of the order parameter but also forms a vortex in the nematicity $\alpha \propto \varphi$. We have two types of vorticity depending on the sign of $g_N u(r)$. If $g_N u(r, z) > 0$ [see Eq. (5)], we call the corresponding solution a type I spin vortex, and

$$\vec{\Delta}_{\text{I}} = \Delta(r, z)(\cos\varphi, \sin\varphi). \quad (8)$$

In the case with $g_N u(r, z) < 0$ [see Eq. (6)], we have a type II spin vortex:

$$\vec{\Delta}_{\text{II}} = \Delta(r, z)(-\sin\varphi, \cos\varphi). \quad (9)$$

The superconducting order parameter away from the vortex core is either enhanced (type I) or reduced (type II). A schematic picture of the local nematicity direction and the order parameter behavior for these spin vortices are shown in Fig. 1. The spin vortices are topologically different, the vector field of the type I spin vortex looks like a hedgehog, while the nematicity vector field for the type II spin vortex has the form of a curl. We can calculate the winding number of the nematicity vector $\vec{n}(\mathbf{r})$ around the vortex core,

$$P = \oint_C \vec{n}\cdot d\mathbf{r}/2\pi,$$

where $d\mathbf{r} = (dx, dy)$ and C is the closed contour around the vortex core with unit radius. In the case of the type I spin vortex the winding number vanishes, $P = 0$, while for the type II spin vortex the winding number is nonzero, $P = 1$. Further, we show that different topologies of the vortices result in different quasiparticle spectra.

III. BOGOLIUBOV–DE GENNES EQUATIONS

Now we seek zero-energy solutions of the BdG equations assuming that $l_u \to +\infty$ (for more details see also Appendix B). For the doped topological insulators these equations can be presented as [10,29–31]

$$H_{\text{BdG}}(\mathbf{k}) = H_0(\mathbf{k})\tau_z + \vec{\Delta}\tau_x, \quad (10)$$

where the single-electron Hamiltonian H_0 is

$$H_0(\mathbf{k}) = -\mu + m\sigma_z + v\sigma_x(s_x k_y - s_y k_x) + v_z k_z\sigma_y. \quad (11)$$

Here σ, $\mathbf{s}$, and τ are the Pauli matrices acting in orbital, spin, and electron-hole spaces, respectively, the superconducting order parameter is $\vec{\Delta} = \Delta(r)\sigma_y\mathbf{s}\cdot\vec{n}$ (the symmetry of the order parameter corresponds to E_u pairing [10,29–31]), $\mathbf{k}$ is the momentum, μ is the chemical potential, m is a single electron gap, and v and v_z are the in-plane and transverse Fermi velocities. According to the GL consideration, we choose $\vec{n}$ as $\vec{n} = [\cos(\varphi + \nu\pi/2), \sin(\varphi + \nu\pi/2)]$, where $\nu = 0$ corresponds to the type I spin vortex, Eq. (8), and $\nu = 1$ corresponds to the type II vortex, Eq. (9). The strain induces a so-called pseudomagnetic field in the system with Hamiltonian (11); however, this field is negligible in the doped Bi_2Se_3 materials [32].

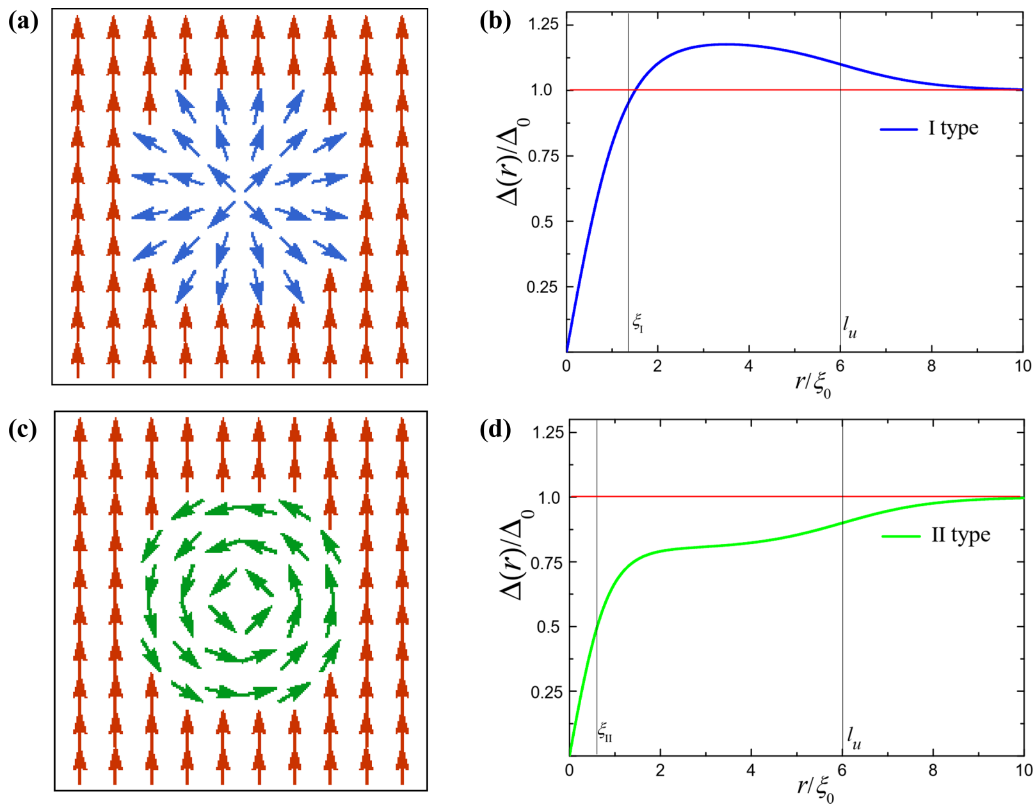

FIG. 1. (a) A schematic picture of the nematicity direction $\vec{n}$ and (b) the function $\Delta(r)/\Delta_0$ for the type I vortex. (c) and (d) The same as (a) and (b), but for the type II vortex. Here $\xi_{\rm I}$, $\xi_{\rm II}$, and ξ_0 are effective coherence lengths for the type I vortex, for the type II vortex, and for the undeformed sample, respectively [formulas for ξ_i are presented in Appendix A, Eqs. (A7)]. The size of the vortex core is of the order of the corresponding coherence length, which is a microscopic value. The size of the spin vortex is of the order of the macroscopic scale l_u of the strain. We assume that for $r > l_u$ the nematicity direction (1,0) is fixed.

The spin vortex can be induced in Hamiltonian (10) by the transformation [19,21]

$$e^{-is_z[\varphi+(\nu-1)\pi/4]}\Delta\sigma_y s_x \tau_x e^{is_z[\varphi+(\nu-1)\pi/4]}. \tag{12}$$

The vortex generates vorticity in the spin space $\mathbf{s}$, while the Abrikosov vortex generates vorticity in the mass space τ [21].

We consider here the states with $k_z = 0$ and rewrite Eq. (10) in the coordinate space, substituting $k_{x(y)} = -i\nabla_{x(y)}$. In the polar coordinates the Hamiltonian reads

$$\begin{aligned} H_{\rm BdG} = &-\mu\tau_z + m\sigma_z\tau_z + iv\sigma_x s_x\tau_z \\ &\times\left[e^{i(\varphi+\pi/2)s_z}\nabla_r - \frac{1}{r}e^{i\varphi s_z}\nabla_\varphi\right] + \Delta\sigma_y s_x\tau_x e^{i(\varphi+\nu\pi/2)s_z}. \end{aligned} \tag{13}$$

We are interested in the solutions of the BdG equations with zero energy: $H_{\rm BdG}\Psi = 0$, where the eight-component spinor is $\Psi = (f_{1\uparrow}, f_{1\downarrow}, f_{2\uparrow}, f_{2\downarrow}, h_{1\uparrow}, h_{1\downarrow}, h_{2\uparrow}, h_{2\downarrow})^T$. Here 1 and 2 are orbital indices, $\uparrow$ and $\downarrow$ are spin projections, and f and h represent electron and hole states. We will seek such a solution in the form

$$\Psi(r,\varphi) = \exp\left[i(l - s_z/2)\varphi\right]\frac{\psi(r)}{\sqrt{r}}, \tag{14}$$

where l is the orbital number. The wave function is single valued if l is a half-integer $l = \pm 1/2, \pm 3/2, \ldots$.

In the case with $k_z = 0$, the Hamiltonian (10) conserves a spin-orbital index, that is, $[H, \sigma_z s_z] = 0$. We decompose the spinor basis in two spin-orbital blocks with $\sigma_z s_z \hat{\Psi}_\pm = \pm\hat{\Psi}_\pm$. Here $\hat{\Psi}_+ = (\Psi_+, 0)^T$, and $\hat{\Psi}_- = (0, \Psi_-)^T$, where $\Psi_{+(-)} = (f_{1\uparrow(\downarrow)}, f_{2\downarrow(\uparrow)}, h_{1\downarrow(\uparrow)}, h_{2\uparrow(\downarrow)})^T$. After the transformation given by Eq. (14), we obtain

$$H_\rho = \left(c_z m - \mu - \frac{lv}{r}c_x - \rho i v c_y \nabla_r\right)\tau_z - \Delta c_y \tau_x (ic_z)^\nu. \tag{15}$$

Here $\rho = \pm 1$ corresponds to different spin-orbital blocks, Pauli matrices c_i act in the spin-orbital space $(1\uparrow, 2\downarrow)$ for $\rho = +1$ and $(1\downarrow, 2\uparrow)$ for $\rho = -1$, and τ_i acts in the particle-hole space.

The Hamiltonian (10) has time-reversal, $T = is_y K$, and particle-hole conjugation, $\Xi = \sigma_y\tau_y K$, symmetries that combine into a chiral symmetry $U_c = \Xi T = i\tau_y$. The latter symmetry anticommutes with the Hamiltonian, $\{H, U_c\} = 0$. In the basis where the chiral operator U_c is diagonal, the Hamiltonian transforms to

$$H_t = \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix},$$

$$H_\mp = \mu + m\kappa_{\mp z} - \kappa_{\mp x}\frac{lv}{r} + i[\rho v\nabla_r \mp \Delta(i\kappa_{\mp z})^\nu]\kappa_{\mp y}, \tag{16}$$

where $\kappa_{\mp i}$ are the Pauli matrices that act in the basis $\vec{L} = (L_1, L_2)^T = (h_{2\uparrow(\downarrow)} \mp i f_{1\downarrow(\uparrow)}, h_{1\downarrow(\uparrow)} \mp i f_{1\uparrow(\downarrow)})^T$ for $\rho = +1(-1)$. As a result, we decompose an 8×8 system into four blocks of 2×2 equations. We can see that H_+ differs from H_- by only the sign before Δ.

IV. ZERO-ENERGY SOLUTIONS

First, we solve equations $H_\mp \vec{L} = 0$ for the type I vortex ($\nu = 0$). The order parameter is eliminated by the substitution $L_{1,2} = l_{1,2} \exp(\mp\rho \int dr \Delta/v)$. We get

$$v^2 l_1'' + \left[\mu^2 - m^2 - \frac{v^2 l(l-\rho)}{r^2}\right] l_1 = 0,$$
$$(\mu - m) l_2 = v\left(\rho l_1' - \frac{l}{r} l_1\right), \quad (17)$$

where the prime means differentiation over r. Solutions regular at $r = 0$ are

$$\vec{L} = N e^{\mp\rho\int dr\Delta/v} \sqrt{r} \times \begin{pmatrix} \sqrt{\mu - m} J_{l+\rho/2}(r\sqrt{\mu^2 - m^2}/v) \\ \sqrt{\mu + m} J_{l-\rho/2}(r\sqrt{\mu^2 - m^2}/v) \end{pmatrix}, \quad (18)$$

where $J_\alpha(x)$ are the Bessel functions and N is a constant. We take into account that the strain and, consequently, the order parameter can be coordinate dependent. As we can see, H_+ has a normalized solution if $\rho = +1$, and H_- has such a solution when $\rho = -1$.

The solutions with different signs of ρ and l are degenerate and form Kramers pairs. In the considered basis

$$\psi_1 = [L_1(l), L_2(l), 0, 0, 0, 0, 0, 0],$$
$$\psi_2 = (-1)^{2l}[0, 0, 0, 0, 0, 0, L_1(-l), L_2(-l)]$$

are the components of such a pair, where L_i is the ith component of the vector given by Eq. (18). We can rewrite the obtained solutions in the original basis,

$$\Psi_1 = [-iL_2(l), 0, 0, -iL_1(l), L_2(l), 0, 0, L_1(l)],$$
$$\Psi_2 = [0, iL_2(-l), iL_1(-l), 0, 0, L_2(-l), L_1(-l), 0].$$

Since $J_n = (-1)^n J_{-n}$ for integer n and l is a half-integer, we obtain $\Psi_2 = i s_y K \Psi_1$. Thus, Ψ_1 and Ψ_2 are the Kramers pair.

We can derive dispersion of the obtained solution in k_z using first-order perturbation theory in $v_z k_z \sigma_y \tau_z$. For this goal, we have to calculate elements of the 4×4 matrix $N = \vec{\Psi}_i v_z k_z \sigma_y \tau_z \vec{\Psi}_j$. Note that the solutions with the same ρ but different signs for the angular momenta, $\Psi_1(l)$ and $\Psi_1(-l)$, have different densities of states. Nevertheless, a majority of states are located at the distance $L \sim l\xi$ from the center of the vortex in both cases, and in the limit $\mu \ll m$, the densities of states for $\Psi_1(l)$ and $\Psi_1(-l)$ coincide. So it is reasonable to consider only the matrix elements for the states with the same absolute value of the angular momenta $|l|$. If $\mu \ll m$, we find that the eigenvalues of N consist of two doubly degenerate branches with $E = \pm v_z k_z$. Thus, the states in the type I spin vortex have linear dispersion in the z direction.

Similarly, we consider the type II vortex. We assume that $\Delta(r) = \Delta$, and for $\vec{L} = (L_1, L_2)$ we have

$$v^2 L_1'' + \left[\Delta^2 + \mu^2 - m^2 \pm 2i\Delta\frac{vl}{r} - \frac{v^2 l(l-\rho)}{r^2}\right] L_1 = 0,$$
$$(\mu - m) L_2 = v\left(\rho L_1' + \frac{l}{r} L_1\right) \pm i\Delta L_1. \quad (19)$$

The solution of this system can be expressed through Whittaker's functions $M_{\beta,\gamma}(z)$ as $L_1 = M_{\beta,\gamma}(2ir\sqrt{\mu^2+\Delta^2-m^2}/v)$, with $\beta = \pm l\Delta/\sqrt{\mu^2 + \Delta^2 - m^2}$ and $\gamma^2 = 1/4 - l(l-\rho)$. The obtained solutions are regular at $r = 0$ but are not regularized at $r \to +\infty$ since $L_1 \propto r^{|\beta|} \exp[ir\sqrt{\mu^2 + \Delta^2 - m^2}/v]$ at large radius. Thus, we conclude that no localized zero-energy solutions exist in the type II vortex core.

The considered system belongs to the DIII symmetry class due to the presence of both time-reversal and particle-hole conjugation symmetries. This class is characterized by the topological invariant Z_2, which is associated with the time-reversal symmetry [19,20]

$$Z_2 = \prod_{\mathbf{K}} \mathrm{Pf}[\,w(\mathbf{K})]/\sqrt{\det w(\mathbf{K})}. \quad (20)$$

Here Pf is a Pfaffian, elements of a skew-symmetric matrix $w_{ij}(k) = \langle u_i(k)|\hat{T}|u_j(k)\rangle$ are calculated at time-reversal-invariant momenta $\mathbf{K}$ in the reduced Brillouin zone, and u_i are eigenvectors of the Hamiltonian (10) at $k_z = 0$. We found (for details see Appendix C) that Z_2 is trivial for the type I vortex and nontrivial for the type II vortex,

$$\text{Type I:} \quad Z_2 = 1, \quad \nu = 0,$$
$$\text{Type II:} \quad Z_2 = -1, \quad \nu = 1. \quad (21)$$

Thus, the spin vortices of different types are topologically different.

V. DISCUSSION

We find that two types of topologically different spin vortices can be induced by strain in the topological superconductors. The localized Majorana solutions of Hamiltonian (10) exist near the core of the type I vortex, while in the case of the type II vortex such solutions are not observed. In Ref. [23] a similar result was obtained for a different Hamiltonian with a k-dependent nematic order parameter. We argue that this similarity arises due to the similarity of the topologies of the spin vortices. That is, properties of the spin vortices are similar in different materials and are governed by the Z_2 topological index. The spin vortex can be created by the application of a mechanical force applied at the center of a circular film of a doped topological insulator. The lattice strain caused by defects or the substrate can also generate spin vortices. Since spin vortices have a normal core, they can be detected by scanning tunneling microscopy or magnetic force microscopy.

Under the assumptions used above, an arbitrary small strain generates the spin vortex. This is a result of the degeneracy of the free energy of the nematic superconductor with respect to the nematicity direction α. The degeneracy of the nematicity is commonly lifted by the presence of a strain or hexagonal warping [10]. The initial strain u_0, either spontaneous or arising during the crystal growth, is usually observed in samples of doped topological insulators [7,26]. Thus, a large enough force should be applied to generate the spin vortex in a real experiment. In particular, the strain, u, produced by the applied force must be much larger than the initial strain u_0. Hexagonal warping fixes the nematicity direction as well [10] and, consequently, prevents generation of the spin vortex. However, the corresponding terms appears in the GL free energy in the sixth order in the order parameter and are less relevant for fixing the nematicity than the strain. In principal, if the symmetry-breaking field is smaller than spontaneous deformation [7], then, the nematicity direction becomes degenerate, and the spin vortices can nucleate spontaneously. However, preparation of such samples with an unfixed nematicity direction has not been reported so far.

The considered spin vortices have a normal core, and consequently, the usual Caroli–de Gennes–Matricon states with nonzero energies E_n exist near their centers. The spectrum of such quasiparticles for the doped topological insulator was calculated in Ref. [33] in the quasiclassical approximation, $E_n = n\Delta^2/\sqrt{\mu^2 - m^2}$, where $n = 1, 2, \ldots$. However, the Majorana fermions with zero energy discussed here are a special type of the BdG solution, and they do not require a normal vortex core to be localized. This can be seen from Eq. (18) (or from Refs. [17,18] for the case of emergent chiral symmetry). In particular, it is evidence that a particular form of the order parameter spatial dependence near the core of the vortex is not important for the existence of the Majorana solutions, and the assumption that Δ in BdG equations is independent of r is a good approximation for zero-energy solutions.

In conclusion, we showed that the rotational symmetric strain can generate spin vortices in doped topological insulators. These vortices can be either type I or II and have normal cores. The type I spin vortex enhances superconductivity far from its core and generates localized zero-energy Majorana states, while the type II spin vortex suppresses superconductivity and has no zero-energy states near its core. The different types of spin vortices have different topologies. We established a nontrivial relation between the strain and Majorana states in doped topological insulators.

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APPENDIX A

Here we take into account the gradient terms in the GL functional. The main effect which is produced by these terms is the appearance of the normal core at the center of the vortex. The size of this vortex core is different for different types of spin vortices.

The GL free energy of the topological superconductor is a sum [24],

$$F = F_0 + F_u + F_D, \tag{A1}$$

where F_0 is the GL free energy in the absence of the strain, (1), and F_u is the contribution due to strain, (2). The term F_D arises due to the inhomogeneity of the order parameter. We consider here the case of nematic order; that is, $B_2 > 0$ in F_0, and the GL order parameter is real, $\vec{\Delta} = \Delta(\cos\alpha, \sin\alpha)$. In this case Eq. (3) from Ref. [24] can be rewritten as

$$\begin{aligned} F_D = &-(J_1+J_4)\left[\left(\frac{\partial\Delta_1}{\partial x}\right)^2 + \left(\frac{\partial\Delta_2}{\partial y}\right)^2\right] - (J_1-J_4)\left[\left(\frac{\partial\Delta_1}{\partial y}\right)^2 - \left(\frac{\partial\Delta_2}{\partial x}\right)^2\right] \\ &-2(J_4+J_2)\frac{\partial\Delta_1}{\partial x}\frac{\partial\Delta_2}{\partial y} - 2(J_4-J_2)\frac{\partial\Delta_1}{\partial y}\frac{\partial\Delta_2}{\partial x} - J_3\left[\left(\frac{\partial\Delta_1}{\partial z}\right)^2 + \left(\frac{\partial\Delta_2}{\partial z}\right)^2\right]. \end{aligned}$$

The corresponding GL equations are

$$\begin{aligned} &\delta F_i = \frac{\delta F}{\delta\Delta_i^*} = 0, \quad i = 1, 2, \\ &\delta F_1 = A\Delta_1 + 2B_1\left(\Delta_1^2 + \eta_2^2\right)\Delta_1 + g_N[(u_{xx}-u_{yy})\Delta_1 + 2u_{xy}\Delta_2] + J_1\left(\partial_x^2+\partial_y^2\right)\Delta_1 + J_3\partial_z^2\Delta_1 + J_4\left[\left(\partial_x^2-\partial_y^2\right)\Delta_1 + 2\partial_x\partial_y\Delta_2\right], \\ &\delta F_2 = A\Delta_2 + 2B_1\left(\Delta_1^2 + \eta_2^2\right)\Delta_2 + g_N[-(u_{xx}-u_{yy})\Delta_2 + 2u_{xy}\eta_1] + J_1\left(\partial_x^2+\partial_y^2\right)\Delta_2 + J_3\partial_z^2\Delta_2 + J_4\left[-\left(\partial_x^2-\partial_y^2\right)\Delta_2 + 2\partial_x\partial_y\Delta_1\right]. \end{aligned} \tag{A2}$$

Let a central symmetric force along the z direction act on a plate of the topological insulator, which has the form of a disk. We introduce cylindrical coordinates $(x, y, z) = (r\cos\varphi, r\sin\varphi, z)$. The force $\vec{f} = [0, 0, f(r, z)]$ produces an elastic displacement of the material with components $u_r(r, z)$ and $u_z(r, z)$. Elementary algebra allows us to express components of the strain tensor u_{ij} in terms of $u_r(r, z)$ and $u_z(r, z)$ and their derivatives

$$\begin{aligned} &u_{xx} - u_{yy} = u(r, z)\cos 2\varphi, \\ &2u_{xy} = u(r, z)\sin 2\varphi, \quad u(r, z) = \frac{\partial u_r}{\partial r} + \frac{u_r}{r}. \end{aligned} \tag{A3}$$

The value $u(r, z)$ depends on $f(r, z)$ and on the boundary conditions of a particular elastic problem. However, $u_r(0, z) = 0$ in any case due to the problem of central symmetry.

We assume that the angular symmetries of the vortex near and far from the core [see Eqs. (8) and (9)] are similar. Thus, we will seek solutions of the GL equations (A2) in the form

$$\vec{\Delta} = \Delta(r, z)[\cos\alpha(\varphi), \sin\alpha(\varphi)], \quad \alpha(\varphi) = m\varphi + \phi_0. \tag{A4}$$

Here m and ϕ_0 are real, and $\Delta(r, z)$ is positive or zero.

We introduce linear combinations

$$\delta F_\alpha = \delta F_1 \sin\alpha - \delta F_2 \cos\alpha, \;\; \delta F_\Delta = \delta F_1 \cos\alpha + \delta F_2 \sin\alpha.$$

In the cylindrical coordinates the equation for δF_α is

$$\delta F_\alpha = \sin[2\phi_0 + 2(m-1)\varphi]\left\{J_4\left[\Delta''(r) - \frac{2m-1}{r}\Delta'(r) + \frac{m(m+2)}{r^2}\Delta(r)\right] + g_N u\Delta(r)\right\} = 0, \tag{A5}$$

where the prime means differentiation by r. This equation is compatible with $\delta F_\Delta = 0$ only if $m = 1$ and $\phi_0 = 0$ (the type I spin vortex) or $\phi_0 = \pi/2$ (the type II spin vortex). The second GL equation then reads

$$\delta F_\Delta = (J_1 \pm J_4)\left(\Delta'' + \frac{1}{r}\Delta' - \frac{1}{r^2}\Delta\right) + J_3\frac{\partial^2\Delta}{\partial z^2} + 2B_1\Delta^3 + (A \pm g_N u)\Delta = 0. \tag{A6}$$

The spatial scale of variation of Δ in the z direction is dictated by the (macroscopic) elastic part of the problem, and it is much larger than the scale in the r direction near the center (core) of the vortex, which is of the order of the (microscopic) coherence length of superconductivity. Therefore, the value $\partial^2\Delta/\partial z^2$ is small, and we ignore the z dependence of the order parameter. Under the latter assumption we rewrite Eq. (A6) in dimensionless variables as

$$f''(\bar{r}) + \frac{1}{\bar{r}}f'(\bar{r}) - \left[1 + \frac{1}{\bar{r}^2}f(\bar{r})\right] + f(\bar{r})^3 = 0,$$

$$f(r) = \frac{\Delta(r)}{\Delta_0}, \quad \Delta_0 = \sqrt{\frac{\pm g_N u - A}{2B_1}},$$

$$\bar{r} = r/\xi_{\mathrm{I,II}}, \quad \xi_{\mathrm{I,II}} = \sqrt{\frac{\pm g_N u - A}{J_1 \pm J_4}}. \tag{A7}$$

Here ξ_{I} and ξ_{II} are effective coherence lengths for the type I and type II vortices, respectively. These values differ from the coherence ξ_0 in the sample without strain.

We neglect the coordinate dependence of Δ_0 since its scale is of the order of the spatial scale of the elastic strain and is much larger than ξ_i. The latter equation is the same as the equation for the order parameter in an "ordinary" superconductor near the core of the Abrikosov vortex [27]. Thus, the behavior of the order parameter in the case of the spin vortex is similar to that in the case of the Abrikosov vortex. The order parameter is zero at $r = 0$, increases linearly in the region $\bar{r} \ll 1$, and is equal to Δ_0 when $\bar{r} \gg 1$. Thus, the spin vortex has a normal core which can be considered a topological defect. The size of this normal core is different for different types of spin vortices.

APPENDIX B

Here we give a derivation of the equations used in Secs. III and IV with more technical detail. We start with the BdG Hamiltonian (10) rewritten for convenience in the form

$$H_{\mathrm{BdG}}(\mathbf{k}) = -\mu + m\sigma_z + v\sigma_x(s_x k_y - s_y k_x) + v_z k_z s_x \sigma_y \tau_z + (\Delta_x s_x + \Delta_y s_y)\sigma_y \tau_x.$$

We put first $k_z = 0$. In the polar coordinate space components of the momentum operator are

$$k_x = -i(\nabla_r \cos\varphi - \sin\varphi/r\,\nabla_\varphi),$$
$$k_y = -i(\nabla_r \sin\varphi + \cos\varphi/r\,\nabla_\varphi).$$

We substitute these operators in Hamiltonian (10) and come to Eq. (13). As mentioned in Sec. III, this Hamiltonian conserves the spin-orbital index, $[H, s_z\sigma_z] = 0$. In this case, there exists a basis in which eigenvectors $u_i^\pm$ of the Hamiltonian are classified according to this index, that is, $Hu_i^\pm = \varepsilon_i u_i^\pm$, $s_z\sigma_z u_i^+ = +u_i$, and $s_z\sigma_z u_i^- = -u_i$. The Hamiltonian is block diagonal in a basis where the operator of a conserved index is diagonal. Thus, we can choose the basis where each block of the Hamiltonian corresponds to the eigenvectors with the same index (plus or minus) of the spin-orbit operator. The spin orbital operator $s_z\sigma_z$ is already diagonal, and we need simply to rearrange strings of the Hamiltonian to make it block diagonal. This rearrangement can be done by the transformation

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B1}$$

We apply this transformation to the Hamiltonian, $W^\dagger H(r,l)W$, and get

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{B2}$$

where 0 corresponds to a 4×4 zero matrix and $H_\rho = H_\pm$ is determined by Eq. (15).

The Hamiltonian H_ρ anticommutes with $U_c = i\tau_y$, $\{H_\rho, U_c\} = 0$. In the basis where U_c is diagonal, the Hamiltonian H_ρ will be an off-diagonal block matrix. The operator $i\tau_y$ is diagonalized by the transformation

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 & i \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{B3}$$

We apply this transformation to the Hamiltonian, $H_t = R^\dagger H_\rho R$, and obtain Eqs. (16).

APPENDIX C

A nematic superconductor with a spin vortex belongs to the DIII symmetry class that is classified by the Z_2 topological invariant, Eq. (20). Here we calculate this value following Ref. [19]. In formula (20) the elements of a skew-symmetric matrix $w_{ij}(k) = \langle u_i(k)|\hat{T}|u_j(k)\rangle$ are calculated at time-reversal-invariant momenta $\mathbf{K}$ in the reduced Brillouin zone, u_i are eigenvectors of the Hamiltonian given by Eq. (10) for $k_z = 0$, and $\hat{T} = is_yK$ is the operator of the time-reversal symmetry. We use here the basis in which the Hamiltonian has the form (B2). The operator of the time-reversal symmetry in this basis is $\tilde{T} = W^T s_y W iK = ic_z t_y K$, where W is given by Eq. (B1). Explicitly,

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix} iK. \tag{C1}$$

In these terms, the Hamiltonian can be rewritten as

$$H(k) = \begin{pmatrix} H_{\rho=+1} & 0 \\ 0 & H_{\rho=-1} \end{pmatrix}, \quad H_\rho = (c_z m - \mu - \rho v k_x c_y + v k_y c_x)\tau_z + \Delta c_y \tau_x \cos\varphi - \rho\Delta c_x \tau_x \sin\varphi. \tag{C2}$$

Eigenvectors of the Hamiltonian are $P_{+(-)} = (\psi_+, 0)$ and $(0, \psi_-)$, where $\psi_{+(-)}$ are the eigenvectors of $H_{\rho=+1(-1)}$. Matrix elements $P_i\tilde{T}P_j$ for the eigenvectors with the same ρ vanish, and only the states with different ρ contribute to $w(\mathbf{K})$, that is, $\langle P_+\tilde{T}P_-\rangle = \langle\psi_{+i}c_z\psi_{-j}\rangle$ and $\langle P_-\tilde{T}P_+\rangle = -\langle\psi_{-i}c_z\psi_{+j}\rangle$. The skew-symmetric matrix w_{ij} reads

$$w(k_x, k_y) = \begin{pmatrix} 0 & Q(k_x, k_y) \\ -Q^T(k_x, k_y) & 0 \end{pmatrix}, \tag{C3}$$

where $Q_{ij}(k) = \langle\psi_{+i}(k)|c_z K|\psi_{-j}\rangle$, $i, j = 1, \ldots, 4$. Using a well-known formula for the Pfaffian, we get

$$Z_2 = \prod_{\mathbf{K}} \operatorname{Det} Q(\mathbf{k}). \tag{C4}$$

This product is calculated at the time-reversal momenta $\mathbf{K}$ of the reduced Brillouin zone. Explicit calculation for the type I spin vortex gives $\operatorname{Det} Q(k_x, k_y) = 1$. Thus, the topological index is trivial in this case, $Z_2 = +1$. For the type II spin vortex we get $\operatorname{Det} Q(k_x, k_y) = q(k_x, k_y)\mathrm{sgn}\,(k_x^2 v^2 + k_y^2 v^2 + m^2 - \mu^2 - \Delta^2)$. The sign of the latter value is different for small and large momenta. We have to calculate the determinant at the time-reversal-symmetric points of the reduced Brillouin zone. Thus, we have $Z_2 = q(0,0)q(+\infty,0)q(0,+\infty)$, $q(+\infty,+\infty) = -1$. We see that the type II spin vortex has a nontrivial topology, which is different from the topology of the type I spin vortex.

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