Disorder-robust *p*-wave pairing with odd-frequency dependence in normal metal–conventional superconductor junctions

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(Received 26 April 2020; revised 11 July 2021; accepted 16 August 2021; published 2 September 2021)

We investigate the induced superconducting pair correlations in junctions between a conventional spin-singlet *s*-wave superconductor and a disordered normal metal. Decomposing the pair amplitude based on its symmetries in the time domain, we demonstrate that the odd-time, or equivalently odd-frequency, spin-singlet *p*-wave correlations are both significant in size and entirely robust against random nonmagnetic disorder. We find that these odd-frequency correlations can even be generated by disorder. Our results show that anisotropic odd-frequency pairing represent an important fraction of the proximity-induced correlations in disordered superconducting hybrid structures.

DOI: 10.1103/PhysRevB.104.094503

I. INTRODUCTION

Superconductivity arises from the formation and condensation of a macroscopic number of electron pairs, or Cooper pairs. The superconducting state is characterized by the Cooper-pair amplitude, which, due to the fermionic nature of electrons, is antisymmetric under the exchange of all degrees of freedom describing the paired electron states. These symmetries include the positions, spin, and relative time separation of the two electrons.

In BCS theory of superconductivity the interparticle interaction is instantaneous and, hence, only static, equal-time, pair amplitudes contribute to the order parameter. Such equal-time pair symmetries are then constrained to be either even in space and odd in spin, as in the conventional spin-singlet *s*-wave state, or odd in space and even in spin (e.g., spin-triplet *p* wave). However, even for BCS superconductors, a growing body of evidence suggests that dynamical pair correlations that are odd in the relative time [1–5] can play a role in the physics of superconducting heterostructures [6–8]. Due to their odd time dependence, these dynamical pair correlations must be either even in space and even in spin or odd in space and odd in spin. Such exotic odd-time pairs are equivalently also referred to as odd-frequency (odd-ω) pairs [1].

A well-established host of odd-time/odd-ω pairs are ferromagnet-superconductor (FS) junctions. In these systems, the combination of spatial invariance breaking due to the interface and spin-singlet to spin-triplet conversion due to the ferromagnet, generates finite odd-ω spin-triplet *s*-wave pairs in the F region even for conventional spin-singlet *s*-wave BCS superconductors [9,10]. The presence of odd-ω *s*-wave pairs explains several unconventional features of FS junctions, such as the long-range proximity effect [11–19], zero-bias peaks [20,21], and paramagnetic Meissner effect [22–24].

Odd-ω pairing can also emerge in normal metal-conventional superconductor (NS) junctions. Here the interface allows for both the original even-ω spin-singlet *s*-wave (isotropic) and an induced odd-ω spin-singlet *p*-wave (anisotropic) symmetry in the N region [8,25–28]. However, unlike FS junctions, in disordered NS systems odd-ω pairing has so far been largely ignored. One often-cited reason for this neglect is the argument that *p*-wave amplitudes are killed in the presence of disorder [29–31]. For example, in the quasiclassical Usadel formulation for dirty systems, only isotropic *s*-wave solutions are explicitly considered, while any anisotropic part is assumed to be negligible [32]. Additionally, Anderson's theorem [33] has been invoked to defend the assertion of fragility of *p*-wave pairs in disordered NS junctions, even though it does not technically concern proximity-induced effects.

In this paper we explicitly show that in NS junctions odd-ω spin-singlet anisotropic *p*-wave pair correlations are not only robust against disorder but as disorder strength increases, the odd-ω amplitudes constitute a growing fraction of the proximity-induced pair correlations in the N region. Remarkably, we even find odd-ω pairing generated by disorder. We do this by considering a NS junction between a disordered N region and a conventional spin-singlet *s*-wave superconductor, see Fig. 1, explicitly calculating the fully quantum mechanical time evolution of the proximity-induced superconducting correlations. By showing that odd-ω *p*-wave pair correlations are ubiquitous and robust, our results provide a completely different view of superconducting pair correlations in disordered heterostructures, with potentially far reaching consequences for the physics, such as a paramagnetic Meissner effect.

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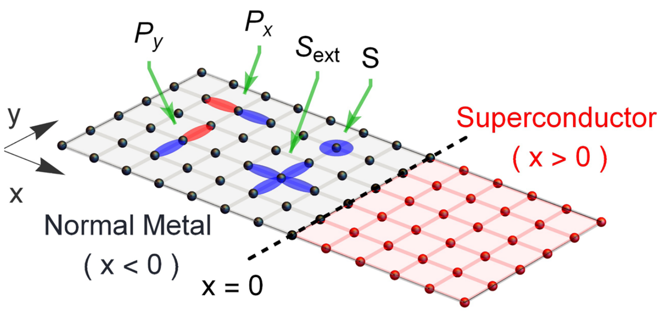

FIG. 1. Sketch of a 2D NS junction with interface at $x = 0$, where filled blue and red circles represent sites in N ($x < 0$) and S ($x > 0$) regions. The relevant spatial orbital symmetries, s wave, extended s wave, and $p_{x,y}$ wave, are depicted in blue (red) for positive (negative) amplitudes.

The remainder of this paper is organized as follows. In Sec. II we present the model and outline the method to obtain the superconducting pair amplitudes. In Sec. III we present the results for the clean and disordered regimes in 1D and also show some consequences in 2D. We conclude in Sec. IV. For completeness, in the Appendices we analytically obtain the induced pair amplitudes in 1D for the clean regime analytically using perturbation theory. Furthermore, here we also present further numerical calculations for 2D.

II. MODEL AND METHODS

To demonstrate the robustness of odd-time/odd-ω pair correlations we study the proximity-induced pair amplitudes in both one-dimensional (1D) and 2D NS junctions between a disordered N region and a conventional spin-singlet s-wave S region, with the interface at $x = 0$, see Fig. 1. The length of the N (S) regions are $L_{\mathrm{N(S)}}$, measured in units of the lattice spacing $a = 1$. The Hamiltonian for these junctions can be written as

$$H = -t\sum_{\langle ij\rangle} c^\dagger_{i\sigma}c_{j\sigma} - \mu\sum_i c^\dagger_{i\sigma}c_{i\sigma} + H_\Delta + H_{\mathrm{V}}, \qquad (1)$$

where $c_{j\sigma}$ is the destruction operator for a particle with spin σ at site $j = (j_x, j_y)$ in 2D or $j = j_x$ in 1D, $\langle\ldots\rangle$ denotes nearest-neighbor sites, t the nearest-neighbor hopping amplitude, and μ the chemical potential that controls the band filling. The conventional spin-singlet s-wave superconductor is captured by $H_\Delta = \sum_i \theta(x)\Delta c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + \mathrm{H.c.}$, with Δ being the order parameter and θ the Heaviside step function. We further introduce random scalar, i.e., nonmagnetic, potential disorder on the normal side of the junction using the Anderson disorder model: $H_{\mathrm{V}} = \sum_i \theta(-x)V_i c^\dagger_{i\sigma}c_{i\sigma}$, where the uncorrelated potentials V_i are drawn from the uniform box distribution on the interval $[-\delta, \delta]$ and added to every site of our discrete lattice model [34]. The disorder strength is controlled by the width of the distribution δ/t, which allows us to tune the disorder from the clean limit or weakly disordered regimes (all sites have none or weak scatterers) to extremely strong disorder (some sites host strong scatterers, while other stays weak), where the disorder dominates over the kinetic energy. This type of disorder can arise from nonmagnetic charge impurities or fluctuations in the chemical potential, both inevitable in real experimental samples.

Pair amplitudes

We are interested in the superconducting correlations in NS junctions. These are characterized by the pair amplitudes, which can be obtained from the time-ordered anomalous Green's function $F_{ij\alpha\beta}(t_2, t_1) = \langle\mathcal{T}c_{i\alpha}(t_2)c_{j\beta}(t_1)\rangle$, where $\mathcal{T}$ is the time-ordering operator. The pair amplitude $F_{ij\alpha\beta}(t_2, t_1)$ accounts for the propagation of quasiparticles from the $(i\alpha)$ state at time t_1 to the $(j\beta)$ state at time t_2. Since $F_{ij\alpha\beta}(t_2, t_1)$ only depends on the relative time difference $\tau = t_2 - t_1$, we focus on the forward time propagation, where the even- (e) and odd-time (o), or equivalently even- and odd-ω, components are extracted as $F^{\mathrm{e(o)}}_{ij\alpha\beta}(\tau > 0) = (\langle c_{i\alpha}(\tau)c_{j\beta}(0)\rangle \mp \langle c_{j\beta}(\tau)c_{i\alpha}(0)\rangle)/2$. Moreover, with no spin-active fields in NS junctions, the pair amplitudes exhibit the same spin symmetry as the parent superconductor. Hence, the pair amplitudes $F^{\mathrm{e(o)}}_{ij\alpha\beta}(\tau)$ can be decomposed into the pair wave amplitudes [35],

$$d^{\mathrm{e(o)}}_w(\tau, R) = \frac{1}{2}\sum_r w(r)\sum_{\alpha\beta}[i\sigma_y]^\dagger_{\alpha\beta}F^{\mathrm{e(o)}}_{R+r\beta R-r\alpha}(\tau), \qquad (2)$$

where we focus our study on the lowest rotation symmetric wave representation projections with $|r| \leqslant 1$: $w_S(r) = \delta_{|r|=0}$, $w_{S_{ext}}(r) = (1/z)\delta_{|r|=1}$, $w_{P_i}(r) = [(\hat{i}\cdot r)/2]\delta_{|r|=1}$ are the s-, extended s-wave, and p_i-wave amplitudes. Here z is the coordination number, σ_y the y-Pauli matrix, and the center of mass $R = (i + j)/2$, and relative $r = (i - j)/2$ spatial coordinates. While we expect that higher order representations also have a finite magnitude [25], to simplify the presentation we focus on $|r| \leqslant 1$, which allows for a targeted and controlled investigation of the interplay between disorder and anisotropic odd-ω pair correlations.

Evaluation of the pair wave amplitudes in each symmetry channel in Eq. (2) only requires computation of time-dependent expectation values of the form $\langle c_{i\alpha}(\tau)c_{j\beta}(0)\rangle$. We perform these calculations numerically using the EPOCH (Equilibrium Propagator by Orthogonal polynomial CHain) method for computing the equilibrium propagators, i.e., Green's functions, directly in the time domain that we have developed in Ref. [36]. In the EPOCH method the equilibrium propagator is expanded in Legendre polynomial mode transients that are recursively computed, allowing us to study both the even- and odd-ω pair wave amplitudes directly in real space as well as the time domain. As such, this analysis notably goes beyond previous studies of odd-ω pairing [8,25,26,28], since the treatment is explicitly in the more natural time domain and also fully quantum mechanical.

III. PAIR AMPLITUDES IN CLEAN AND DISORDERED NS JUNCTIONS

To ultimately understand the pair wave amplitude behavior in the presence of disorder in NS junctions, we first study its time evolution without disorder, $\delta/t = 0$. Then, we analyze the disordered regime by varying the disorder strength δ/t.

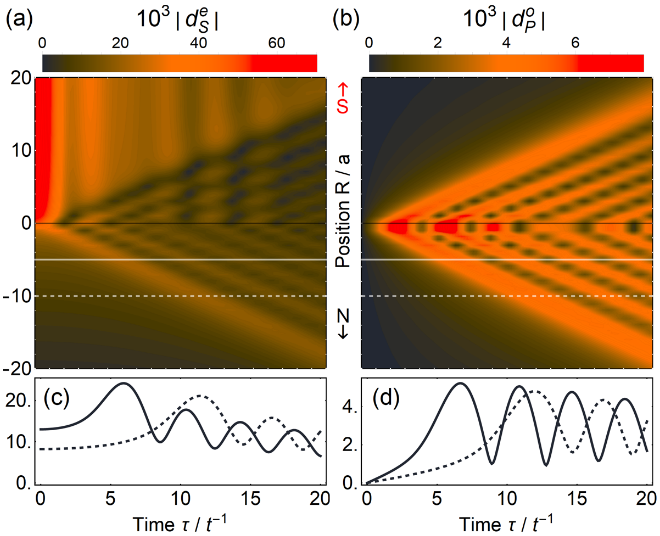

FIG. 2. Even-ω s-wave (a) and odd-ω p-wave (b) pair magnitudes as a function of space and time in clean regime 1D NS junction with interface at $x = 0$. [(c),(d)] Time evolution of the even- and odd-ω pair magnitudes, respectively, with solid (dashed) curve corresponding to the position in N indicated by solid (dashed) lines in [(a),(b)]. Parameters: $\mu = 0.25t$, $L_N = 250$, $L_S = 250$, $\Delta = 0.01t$.

A. Clean regime

In Fig. 2 we show space and time evolution of the magnitude of the even-ω s-wave and odd-ω p-wave pair wave amplitudes, $|d^{e,o}(\tau, R)|$ in a clean 1D NS junction. We explicitly find no other finite amplitudes with $|r| \leqslant 1$, apart from even-ω extended s-wave correlations, which show similar behavior as the s-wave pairing. Notably, the p-wave term is not only nonlocal and odd in space, but also nonlocal and odd in the relative time coordinate τ, fully consistent with previous results studying the time dependence of odd-ω pair amplitudes [37]. At $\tau = 0$ the even-ω s-wave magnitude exhibits a large and constant value in S, due to the finite order parameter Δ in S, as see in Fig. 2(a). The odd-ω pair magnitude in Fig. 2(b), on the other hand, is necessarily zero at $\tau = 0$. For finite τ both pair wave amplitudes develop a fan-shaped profile in both S and N regions, albeit with smaller odd-ω magnitudes. The fan profile is due to wave packet propagation in time and space, with a decay as we move away from the NS interface, since the superconducting proximity effect diminishes with distance [38]. There is also an oscillatory pattern, seen as resonances in Figs. 2(c) and 2(d), due to interference between incident and reflected quasiparticle states at the NS interface, as noted before [28,37,39,40]. In particular, the resonance peaks in the N region are a result of the following time evolution process: an electron at x_1 in N has to first propagate towards the NS interface, where it is reflected back as a hole through Andreev reflection, and then propagate back to x_2 in N in order to contribute to the pair wave amplitude [41]. A simple analytic calculation using a 1D continuum model yields the expected time of this process to $\tau_0 = d/v_F$, where $d = |x_1| + |x_2|$ is the round trip distance and v_F the Fermi velocity, see Supplemental Material (SM) [42] for a complete derivation. This value of τ_0 is fully consistent with the results in Figs. 2(c) and 2(d). The overall result is that large pair magnitudes disperse linearly away from the interface in both the N and S regions, in agreement with the role of the NS interface as the generator of even- and odd-ω correlations [8,25,26,28].

B. Disorder regime

Next we turn to our main topic on how the induced pair correlations are affected by disorder in the N region. Our goal is to ascertain if the odd-ω p-wave correlations are very fragile to disorder as previously assumed [29–31]. To this end, we investigate how the odd-ω p-wave correlations are affected by random scalar potential disorder. As a control, we also investigate how both the on-site and extended s-wave pair correlations are affected, since both of these isotropic wave states are assumed to be disorder robust. The extended s wave also provides additional control in that it has the same spatial extent as the odd-ω p wave, both being nearest-neighbor correlations. Our main results are presented in Fig. 3, where we plot the induced pair correlations in a 1D NS system for a wide range of disorder strength δ/t, from the clean limit $\delta/t = 0$ to the strongly disordered limit $\delta/t = 2$. For any finite δ/t, the disorder produces a finite elastic mean free path $l_e \approx 4.3a(t/\delta)^2$, see SM [42]. Our range of disorder therefore spans both the regimes where the mean free path is significantly shorter and longer than the superconducting coherence length $\xi = \hbar v_F/\Delta \sim 20a$.

We first present the time evolution of the disorder-averaged pair magnitudes with the on-site even-ω s wave in Fig. 3(a) and the odd-ω p wave in Fig. 3(b). Both magnitudes are taken at the fixed spatial position indicated by the solid-white line in Fig. 2. At the vanishing disorder strength, we recover the same situation as discussed in Fig. 2, where both the even and odd-ω pair magnitudes are finite in the N region and exhibit strong oscillations in time. For both magnitudes, the oscillatory patterns in time gradually diminish with increased disorder, but the changes in the magnitudes are clearly different between the even and odd-ω correlations. Extracting the pair magnitudes at their first resonance peak, indicated by the circle markers in Figs. 3(a) and 3(b) and plotted in Fig. 3(c) as a function of disorder strength, we show that, while the on-site and extended s-wave correlations are both monotonically decreasing with the disorder strength, the odd-ω p-wave correlations are nonmonotonic with an initial enhancement with increasing disorder. Thus, as we present in Fig. 3(d), the ratio of odd-ω p-wave to even-ω s-wave correlations actually *increases* with the disorder strength. In particular, the odd to even ratio undergoes a rapid growth around the crossover $l_e \sim \xi$, appearing near $\delta/t \approx 0.5$, and is monotonically increasing for all l_e/ξ, all the way to the dirty limit $l_e \ll \xi$. This is in complete contrast to the expected disorder fragility of all anisotropic non-s-wave states [29–31] and clearly shows that the p-wave pairs cannot be ignored even in the dirty limit. The relative increase of the odd-ω p-wave correlations is sustained throughout the N region, even beyond the superconducting coherence length $\xi \sim 20a$, as is clearly evident from the ratios calculated at multiple distances away from the NS junction in Fig. 3(d). This rules out a mere interface effect.

To examine the stability of these disorder effects for different parameter values, we present in Figs. 3(e) and 3(f)

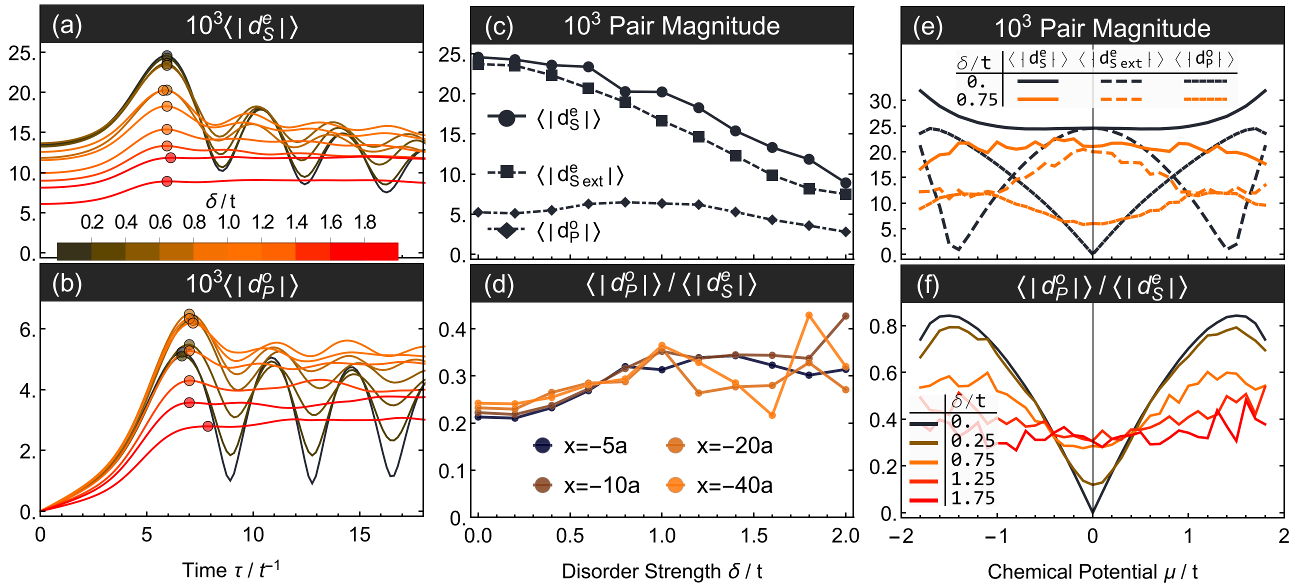

FIG. 3. Even- (a) and odd-ω (b) pair magnitudes at $x = -5a$ in disordered N region in a 1D NS junction, averaged over 300 disorder realizations. Height of the first correlation peak, marked with circles in [(a),(b)], as a function of disorder strength (c) and chemical potential (e) for s, extended-s, and p waves. Ratio between p-wave and s-wave magnitudes as a function of disorder strength (d) and chemical potential (f). Unless otherwise stated, same parameters as in Fig. 2.

the disordered averaged pair magnitudes as a function of the chemical potential μ, again at the first resonance peak [circles in Figs. 3(a) and 3(b)]. In the clean regime the even-ω s-wave pair magnitude is almost constant for all values of μ, while the odd-ω p-wave amplitude identically vanishes at half-filling, $\mu = 0$. The latter is due to particle-hole symmetry in the normal state at $\mu = 0$, resulting in a cancellation of odd-time correlations [43]. While this cancellation of the odd-ω amplitude at $\mu = 0$ in the clean limit occurs in a fine-tuned situation, what is highly relevant for the general understanding of p-wave pair correlations, is that they still become finite at $\mu = 0$ when introducing disorder. Thus disorder can even *generate* p-wave pairing. Furthermore, as seen in Fig. 3(f), where we plot the p wave to s wave ratio, the p-wave pair correlations are large, at around 40%, and essentially constant for disorder ranging from moderately weak to very strong. These results highlight the crucial role of nonmagnetic disorder, with disorder not suppressing odd-ω p-wave pairing, but even generating it. Additionally, we have also verified that qualitatively both the clean and disordered behaviors are independent of the size of the superconducting order parameter for all $\Delta < t$.

The exceptional disorder robustness of the odd-ω p-wave correlations is a consequence of the specific mechanism by which they appear. In the clean limit, p-wave pair correlations in NS junctions are generated due to the spatial translation symmetry breaking at the interface. Fermion statistics dictates that these p-wave correlations must have an odd-ω symmetry. Introducing nonmagnetic disorder in the N region cause two diametrically opposing effects. First, disorder reduces the propagation of Cooper pairs, especially anisotropic pairs. Secondly, disorder further disrupts the translation symmetry and thus generates more odd-ω p-wave correlations. The exact balance between these two effects can only be handled numerically and our results show that the generation even outweighs the suppression in propagation of p-wave pairs. Thus odd-ω spin-singlet p-wave pairing is remarkably robust to disorder in dirty NS junctions. This result is in stark contrast to the dirty limit, or Usadel, formulation of quasiclassical theory, where only isotropic s-wave pairing states are explicitly considered, while any anisotropic pairing terms are assumed to vanish in the dirty limit $l_e/\xi \ll 1$ [29–31]. It hence follows that any proper quasiclassical treatment has to include anisotropic pair channels using, e.g., the Eilenberger equations [44,45]. We also stress that our results are not in contradiction to Anderson's theorem [32], as the pair amplitudes in N are proximity-induced, while Anderson's theorem only formally applies to bulk superconductors.

C. Beyond 1D

In the above we focused on the results for 1D NS junctions, but our main findings also hold for NS junctions in higher dimensions. In the SM [42] we show plots that are equivalent to Figs. 2–3 but for a 2D NS junction and that are analogously obtained by calculating the time evolution of Eq. (1). We have likewise verified similar results for 3D junctions. The fact that we find similar results across multiple spatial dimensions means that our main results are independent of any localization effects that can otherwise be pronounced in low-dimensional systems [46], since the localization lengths are of different orders of magnitude in different dimensions [47].

While the results in higher dimensions are strikingly similar to those in 1D, and also more stable due to increased self-averaging, there are some importance differences. In addition to the on-site and extended even-ω s-waves and the odd-ω p_x-wave correlations previously seen in 1D, there exist in 2D also a odd-ω p_y-wave correlations that is oriented along the junction interface, see Fig. 1. In Fig. 4 we plot the induced disorder-averaged pair magnitudes versus disorder strengths

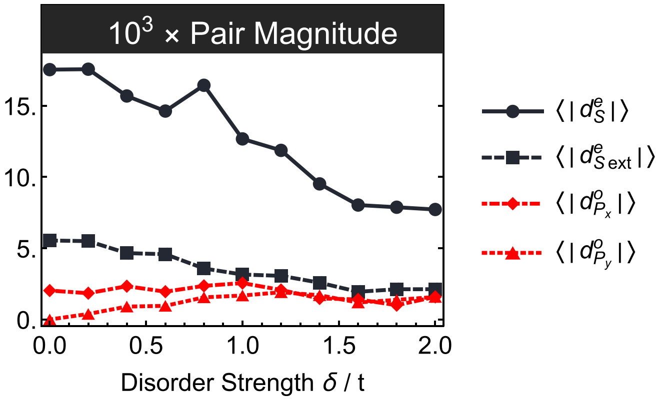

FIG. 4. Disorder-averaged *s*-wave (black) and *p*-wave (red) pair magnitudes at $x = -5a$ as a function of disorder strength in a 2D SN junction, taken as the height of the first peak in the time evolution, averaging over 15 disorder realizations. Parameters: $\mu = 0.5t$, $L_{N_{x,y}} = 50$, $L_{S_{x,y}} = 50$, $\Delta = 0.01t$.

in a 2D SN junction, now also with the p_y-wave correlations. The plot can be directly compared to the analogous plot in Fig. 3 for 1D, given the very similar elastic mean free path in the 2D, $l_e \approx 4.7a(t/\delta)^2$, see SM [42]. In the clean limit $\delta/t = 0$, the p_y-wave magnitude is completely absent, because the interface only breaks the translation invariance along the x direction and therefore no correlations with a p_y-wave symmetry can be generated. Still, at finite disorder, we see that finite odd-ω p_y-wave pair amplitudes are generated alongside the initial enhancement of the already finite odd-ω p_x-wave amplitude. In contrast, the presumed disorder robust isotropic *s*-wave states instead show a pronounced monotonic suppression with increasing disorder. This demonstrates again that the odd-ω *p*-wave pair amplitudes are highly disorder robust and always constitute a significant portion of the induced pairing in disordered NS junctions. The emergence of odd-ω correlations along the y axis with disorder can generally be understood as arising from the additional breaking of translational invariance along the y axis by the disorder and its resulting scattering of pair wave amplitudes into the p_y-wave channel, although we point out that this alone is not enough. In fact, random charge potential disorder cannot by itself generate odd-ω pair correlations, as shown in Ref. [37]. What is needed in addition is the NS junction and its associated spatial gradient in the superconducting order parameter, which therefore plays an important role in the disorder generation of the odd-ω p_y-wave pair amplitudes.

IV. CONCLUSIONS

In conclusion, we have demonstrated that odd-ω spin-singlet *p*-wave pair correlations are robust against random scalar, nonmagnetic, disorder in conventional and dirty NS junctions. In fact, the *p*-wave pair correlations display a nonmonotonic dependence on the disorder strength, with an initial enhancement and also constitute a growing fraction of the induced pair correlations in the N region, notably even in the highly disordered regime where the elastic mean free path is much smaller than the superconducting coherence length. The *p*-wave pair correlations therefore display an even greater disorder robustness than the isotropic *s*-wave components that are instead monotonically decreasing with an increased disorder. As a consequence, odd-ω *p*-wave pair correlations always constitute a significant portion of the proximity-induced superconductivity in dirty NS junctions. With properties, such as the Josephson and Meissner effects, being directly dependent on the superconducting pair amplitudes, we expect our results to have interesting consequences for various NS systems and superconducting hybrid structures.

ACKNOWLEDGMENTS

We thank A. Balatsky, D. Chakraborty, and T. Löfwander for useful discussions. We acknowledge financial support from the Swedish Research Council (Vetenskapsrådet Grant No. 2018-03488), the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (ERC-2017-StG-757553), the Knut and Alice Wallenberg Foundation through the Wallenberg Academy Fellows program and the EU-COST Action CA-16218 Nanocohybri.

[1] V. L. Berezinskii, New model of the anisotropic phase of superfluid ^{3}He, Zh. Eksp. Teor. Fiz. **20**, 628 (1974) [JETP Lett. **20**, 287 (1974)].
[2] T. R. Kirkpatrick and D. Belitz, Disorder-Induced Triplet Superconductivity, Phys. Rev. Lett. **66**, 1533 (1991).
[3] A. Balatsky and E. Abrahams, New class of singlet superconductors which break the time reversal and parity, Phys. Rev. B **45**, 13125 (1992).
[4] D. Belitz and T. R. Kirkpatrick, Even-parity spin-triplet superconductivity in disordered electronic systems, Phys. Rev. B **46**, 8393 (1992).
[5] E. Abrahams, A. Balatsky, J. R. Schrieffer, and P. B. Allen, Interactions for odd-ω-gap singlet superconductors, Phys. Rev. B **47**, 513 (1993).
[6] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Odd triplet superconductivity and related phenomena in superconductor-ferromagnet structures, Rev. Mod. Phys. **77**, 1321 (2005).
[7] J. Linder and A. V. Balatsky, Odd-frequency superconductivity, Rev. Mod. Phys. **91**, 045005 (2019).
[8] J. Cayao, C. Triola, and A. M. Black-Schaffer, Odd-frequency superconducting pairing in one-dimensional systems, Eur. Phys. J.: Spec. Top. **229**, 545 (2020).
[9] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Long-Range Proximity Effects in Superconductor-Ferromagnet Structures, Phys. Rev. Lett. **86**, 4096 (2001).
[10] A. Kadigrobov, R. I. Shekhter, and M. Jonson, Quantum spin fluctuations as a source of long-range proximity effects in diffusive ferromagnet-super conductor structures, Europhys. Lett. **54**, 394 (2001).
[11] V. T. Petrashov, V. N. Antonov, S. V. Maksimov, and R. S. Shaĭkhaĭdarov, Conductivity of mesoscopic structures with ferromagnetic and superconducting regions, Pis'ma Zh. Eksp. Teor. Fiz. **59**, 523 (1994) [JETP Lett. **59**, 551 (1994)].

[12] M. Giroud, H. Courtois, K. Hasselbach, D. Mailly, and B. Pannetier, Superconducting proximity effect in a mesoscopic ferromagnetic wire, Phys. Rev. B **58**, R11872 (1998).

[13] R. S. Keizer, S. T. B. Goennenwein, T. M. Klapwijk, G. Miao, G. Xiao, and A. Gupta, A spin triplet supercurrent through the half-metallic ferromagnet CrO_2, Nature (London) **439**, 825 (2006).

[14] T. S. Khaire, M. A. Khasawneh, W. P. Pratt, and N. O. Birge, Observation of Spin-Triplet Superconductivity in Co-Based Josephson Junctions, Phys. Rev. Lett. **104**, 137002 (2010).

[15] J. Wang, M. Singh, M. Tian, N. Kumar, B. Liu, C. Shi, J. K. Jain, N. Samarth, T. E. Mallouk, and M. H. W. Chan, Interplay between superconductivity and ferromagnetism in crystalline nanowires, Nat. Phys. **6**, 389 (2010).

[16] J. W. A. Robinson, J. D. S. Witt, and M. G. Blamire, Controlled injection of spin-triplet supercurrents into a strong ferromagnet, Science **329**, 59 (2010).

[17] M. S. Anwar, F. Czeschka, M. Hesselberth, M. Porcu, and J. Aarts, Long-range supercurrents through half-metallic ferromagnetic CrO_2, Phys. Rev. B **82**, 100501(R) (2010).

[18] C. Cirillo, S. Voltan, E. A. Ilyina, J. M. Hernández, A. García-Santiago, J. Aarts, and C. Attanasio, Long-range proximity effect in Nb-based heterostructures induced by a magnetically inhomogeneous permalloy layer, New J. Phys. **19**, 023037 (2017).

[19] Y. Proshin and M. Avdeev, The theory of long-range Josephson current through a single-crystal ferromagnet nanowire, J. Magn. Magn. Mater. **459**, 359 (2018).

[20] M. Alidoust, K. Halterman, and O. T. Valls, Zero-energy peak and triplet correlations in nanoscale superconductor/ferromagnet/ferromagnet spin valves, Phys. Rev. B **92**, 014508 (2015).

[21] A. D. Bernardo, S. Diesch, Y. Gu, J. Linder, G. Divitini, C. Ducati, E. Scheer, M. Blamire, and J. Robinson, Signature of magnetic-dependent gapless odd frequency states at superconductor/ferromagnet interfaces, Nat. Commun. **6**, 8053 (2015).

[22] K. Hashimoto, Meissner effect of odd-frequency superconductors, Phys. Rev. B **64**, 132507 (2001).

[23] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Josephson current in superconductor-ferromagnet structures with a nonhomogeneous magnetization, Phys. Rev. B **64**, 134506 (2001).

[24] A. Di Bernardo, Z. Salman, X. L. Wang, M. Amado, M. Egilmez, M. G. Flokstra, A. Suter, S. L. Lee, J. H. Zhao, T. Prokscha, E. Morenzoni, M. G. Blamire, J. Linder, and J. W. A. Robinson, Intrinsic Paramagnetic Meissner Effect Due to s-Wave Odd-Frequency Superconductivity, Phys. Rev. X **5**, 041021 (2015).

[25] Y. Tanaka, Y. Tanuma, and A. A. Golubov, Odd-frequency pairing in normal-metal/superconductor junctions, Phys. Rev. B **76**, 054522 (2007).

[26] M. Eschrig, T. Löfwander, T. Champel, J. C. Cuevas, J. Kopu, and G. Schön, Symmetries of pairing correlations in superconductor–ferromagnet nanostructures, J. Low Temp. Phys. **147**, 457 (2007).

[27] C. Fleckenstein, N. T. Ziani, and B. Trauzettel, Conductance signatures of odd-frequency superconductivity in quantum spin Hall systems using a quantum point contact, Phys. Rev. B **97**, 134523 (2018).

[28] J. Cayao and A. M. Black-Schaffer, Odd-frequency superconducting pairing in junctions with Rashba spin-orbit coupling, Phys. Rev. B **98**, 075425 (2018).

[29] Y. Tanaka and A. A. Golubov, Theory of the Proximity Effect in Junctions with Unconventional Superconductors, Phys. Rev. Lett. **98**, 037003 (2007).

[30] A. A. Golubov, Y. Tanaka, Y. Asano, and Y. Tanuma, Odd-frequency pairing in superconducting heterostructures, J. Phys.: Condens. Matter **21**, 164208 (2009).

[31] Y. Tanaka, M. Sato, and N. Nagaosa, Symmetry and topology in superconductors -odd-frequency pairing and edge states-, J. Phys. Soc. Jpn. **81**, 011013 (2012).

[32] K. D. Usadel, Generalized Diffusion Equation for Superconducting Alloys, Phys. Rev. Lett. **25**, 507 (1970).

[33] P. Anderson, Theory of dirty superconductors, J. Phys. Chem. Solids **11**, 26 (1959).

[34] P. W. Anderson, Absence of Diffusion in Certain Random Lattices, Phys. Rev. **109**, 1492 (1958).

[35] K. Björnson and A. M. Black-Schaffer, Probing vortex Majorana fermions and topology in semiconductor/superconductor heterostructures, Phys. Rev. B **91**, 214514 (2015).

[36] T. Löthman, C. Triola, J. Cayao, and A. M. Black-Schaffer, Efficient numerical method for evaluating normal and anomalous time-domain equilibrium Green's functions in inhomogeneous systems, Phys. Rev. B **104**, 125405 (2021).

[37] C. Triola and A. M. Black-Schaffer, Odd-frequency superconductivity induced by nonmagnetic impurities, Phys. Rev. B **100**, 144511 (2019).

[38] T. M. Klapwijk, Proximity effect from an Andreev perspective, J. Supercond. **17**, 593 (2004).

[39] W. L. McMillan, Theory of Superconductor—Normal-metal Interfaces, Phys. Rev. **175**, 559 (1968).

[40] P. Burset, B. Lu, G. Tkachov, Y. Tanaka, E. M. Hankiewicz, and B. Trauzettel, Superconducting proximity effect in three-dimensional topological insulators in the presence of a magnetic field, Phys. Rev. B **92**, 205424 (2015).

[41] Normal reflection is also allowed at the NS interface but can be suppressed in ballistic junctions, and, as shown in Ref. [28], the coexistence of even- and odd-ω correlations can be fully understood in terms of Andreev reflection.

[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.104.094503 for additional discussions and figures to support the conclusions, including Refs. [47–55].

[43] With particle-hole symmetry in the normal state the process of traveling in N towards the junction as an electron and the process traveling away from the interface as a hole identically cancel in the anomalous Green's function, resulting in zero odd-time pair correlations.

[44] W. Belzig, C. Bruder, and A. L. Fauchère, Diamagnetic response of a normal-metal–superconductor proximity system at arbitrary impurity concentration, Phys. Rev. B **58**, 14531 (1998).

[45] A. C. Keser, V. Stanev, and V. Galitski, Long range p-wave proximity effect into a disordered metal, Phys. Rev. B **91**, 094518 (2015).

[46] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions, Phys. Rev. Lett. **42**, 673 (1979).

[47] J. Kroha, T. Kopp, and P. Wölfle, Self-consistent theory of anderson localization for the tight-binding model with site-diagonal disorder, Phys. Rev. B **41**, 888 (1990).
[48] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Courier Corporation, New York, 2012).
[49] G. D. Mahan, *Many-Particle Physics* (Springer Science & Business Media, New York, 2013).
[50] S. Roche and D. Mayou, Conductivity of Quasiperiodic Systems: A Numerical Study, Phys. Rev. Lett. **79**, 2518 (1997).
[51] E. N. Economou, *Green's Functions in Quantum Physics* (Springer, Berlin, 2006).
[52] A. Uppstu, Z. Fan, and A. Harju, Obtaining localization properties efficiently using the Kubo-Greenwood formalism, Phys. Rev. B **89**, 075420 (2014).
[53] Z. Fan, V. Vierimaa, and A. Harju, GPUQT: An efficient linear-scaling quantum transport code fully implemented on graphics processing units, Comput. Phys. Commun. **230**, 113 (2018).
[54] A. Datta, A. Banerjee, N. Trivedi, and A. Ghosal, New paradigm for a disordered superconductor in a magnetic field, arXiv:2101.00220.
[55] Z. Fan, J. H. Garcia, A. W. Cummings, J. E. Barrios-Vargas, M. Panhans, A. Harju, F. Ortmann, and S. Roche, Linear scaling quantum transport methodologies, Phys. Rep. **903**, 1 (2021).