Tilt-induced many-body corrections to optical conductivity of tilted Dirac cone materials

Z. Jalali-Mola[1,2] and S. A. Jafari[1,*]

[1]*Department of Physics, Sharif University of Technology, Tehran 11155-9161, Iran*

[2]*Independent Researcher, Personal Research Laboratory, West Ferdows Blvd, Tehran, 1483633987, Iran*

(Received 14 February 2021; revised 10 July 2021; accepted 13 August 2021; published 30 August 2021)

Katsnelson has shown that, within the Fermi-liquid approach, the optical conductivity of Dirac electrons in graphene is not affected by many-body interactions [M. I. Katsnelson, Eur. Phys. Lett. **84**, 37001 (2008)]. We show that, when the Dirac cone is tilted, the Fermi-liquid corrections arise in the optical conductivity in a manner that the correction depends on the angle between the electric field of the incident light and the tilt direction. Therefore the mapping of the optical conductivity for various directions of the incident light enables a determination of the many-body effect in the optical conductivity spectrum of the two-dimensional tilted Dirac cone materials.

DOI: 10.1103/PhysRevB.104.085152

I. INTRODUCTION

Dirac fermions are now everywhere in the quantum matter in various dimensions [1–3]. The salient example is a two-dimensional sheet of graphene [4,5]. In two space dimensions, the Dirac Hamiltonian is simply given by $v_F(\sigma_x k_x + \sigma_y k_y)$, where σ_x and σ_y are 2×2 Pauli matrices that encode the fact that two bands are touching in certain points in the Brillouine zone [5]. The velocity scale v_F, which in graphene is nearly 300 times smaller than the speed of light, signifies the fact that the underlying Lorentz symmetry in graphene is "emergent."

Starting from the above Hamiltonian, how many ways are there to modify the Dirac theory? In the space of 2×2 matrices, there are only two matrices left: the Pauli matrix σ_z and the unit matrix σ_0. The Pauli matrix σ_z squares to $\mathbb{1}$ and anticommutes with both σ_x and σ_y, and therefore, it gives rise to a "Dirac mass" [6] that corresponds to a gap in the one-particle spectrum and is realized in graphene samples grown on SiC [7] or in boron nitride [8]. However, if we do not want to generate a gap, the only remaining possibility is to add a term proportional to σ_0 to the Dirac theory of graphene. If one wants to maintain the linear cone-shaped band dispersion, the coefficients of σ_0 must be linear in the momentum $\boldsymbol{k}$, and therefore, from a mathematical point of view, the only way to deform the $2+1$ dimensional Dirac theory, without opening a gap, is to deform it to [9,10]

$$H_{0\boldsymbol{k}} = v_F(\sigma_x k_x + \sigma_y k_y) + v_F \boldsymbol{k} \cdot \boldsymbol{\zeta} \sigma_0, \tag{1}$$

where a vector looking dimensionless set of parameters $\boldsymbol{\zeta} = (\zeta_x, \zeta_y)$ (known as the tilt parameters) has been introduced to form a scalar product with $\boldsymbol{k}$.

Indeed there are material examples where the above "tilted" Dirac theory is their effective band structure at the band touching point. Perhaps the oldest example is the organic material [11], and the most recent example is the so-called $8Pmmn$ borophene [12,13], which has been predicted to be a stable structure. When the tilt parameter is zero, the "Dirac" fermions enjoy an emergent Lorentz symmetry, and hence the effective Minkowski space-time emerges, which can be described by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$ in two space dimensions. This metric allows one to express the dispersion relation of Dirac fermions in a covariant form, $\eta_{\mu\nu} k^\mu k^\nu = 0$, where k^μ is an appropriate three-vector that combines energy and the wave vector $\boldsymbol{k} = (k_x, k_y)$ [14]. Once the smallest nonzero tilt $\boldsymbol{\zeta}$ is introduced, the Lorentz symmetry and, hence, the emergent Minkowsi structure of the effective space-time disappears. But it turns out that still one can find a modified metric $g_{\mu\nu}$ [15],

$$g_{\mu\nu} = \begin{pmatrix} -1+\zeta^2 & \zeta_x & \zeta_y \\ \zeta_x & 1 & 0 \\ \zeta_y & 0 & 1 \end{pmatrix}, \tag{2}$$

that is a natural extension of the Minkowski metric, and $\zeta^2 \equiv \zeta_x^2 + \zeta_y^2$. This metric not only allows a covariant description of the dispersion relation of tilted Dirac fermions but also allows one to express the polarization (fermion loop) of the quantum matter described by the Hamiltonian (1) in a covariant form that further satisfies the Ward identity [16].

The emergence of the metric (2) in the description of the quantum states of the materials with tilted Dirac cone has far-reaching consequences. It allows for the formation of an undamped transverse electric mode in undoped tilted Dirac cone materials (TDCMs) [17]. The atomic scale manipulations can change the tile parameter [18]. If this effect can be made space dependent, the emergent geometry of the space-time can be manipulated. In this way, non-Abelian gauge structures emerge that in some limits reduce to interesting forms of (pseudo) spin-orbit coupling [19]. The redshift factors $1/\sqrt{1-\zeta^2}$ connecting the Minkowski space-time and the above space-time naturally show up in the nuclear magnetic relaxation rates of (the three-dimensional) TDCMs [20]. Such materials in combination with superconductors show interesting Andreev reflection behavior, where irrespective of the angle of the incident electron the Andreev reflected hole comes closer to the perpendicular reflection upon increasing

*jafari@physics.sharif.edu

the tilt parameter ζ towards 1 [21]. Furthermore, the charge-neutral Andreev mode arising from the specular Andreev reflection (that is possible only for the Dirac electrons) acquires an electric charge when a nonzero $\boldsymbol{\zeta}$ is present [22]. The tilt can also generate kinks in the plasmon dispersion [23,24] whose connection to the metric (2) has not been explored yet. Last, but not least, the hydrodynamic formulation of interacting Dirac fermions in a background metric given by Eq. (2) shows that not only the spatial gradients ∇T of temperature can derive currents but also, in quantum materials described by the tilted Dirac cone in Eq. (1), even a temporal gradient $\partial_t T$ of temperature can lead to electron currents. Needless to say, this effect is controlled by $\boldsymbol{\zeta}$ that mixes "space" and "time" in Eq. (2).

Therefore, it is of extreme importance to examine the fate of various solid-state phenomena when a background metric of the form (2) exists. In a neat work [25], Katsnelson has used the Fermi-liquid (FL) form of energy functional to show that the interactions (within the FL theory) do not modify the optical absorption of the free Dirac fermions. In this work, we apply the same logic to two-dimensional TDCMs, with an additional assumption of the background metric (2) as the starting point, to study how the optical absorption of free fermions in such a background metric is modified by introducing the FL interactions.

This paper is organized as follows. We start in Sec. II with a discussion of the tilted Dirac cone model and investigation of optical conductivity in the presence of the external electric field without consideration of Coulomb interaction. Inclusion of interaction in the spirit of the Fermi-liquid theory by considering both tilt and pseudospin degree of freedom is the subject of Sec. III. Finally, in Sec. IV we summarize the findings of this paper. The details of calculation are given in Appendix A.

II. MODEL

The most generic form of a low-energy effective theory for a two-dimensional TDCM is given by [9,26]

$$H_0 = \sum_k H_{0\boldsymbol{k}} = \sum_k \hbar v_F(\boldsymbol{\sigma}\cdot\boldsymbol{k} + \boldsymbol{\zeta}\cdot\boldsymbol{k}), \tag{3}$$

where the subscript 0 emphasizes the noninteracting nature of the above Hamiltonian, v_F is the (isotropic) Fermi velocity, σ_i with $i = 1$ and 2 are the Pauli matrices that act on the space of the two bands, and $\boldsymbol{k} = (k_1, k_1) \equiv (k_x, k_y)$ is a momentum wave vector in two dimensions with the polar angle θ_k.

Here we ignore the spin degeneracy. Note that we have only considered one valley, corresponding to which there is another valley with the opposite tilt direction $-\boldsymbol{\zeta}$ if the system respects the parity. Although, compared to the upright Dirac cone (e.g., $\boldsymbol{\zeta} = 0$), the energy spectrum $E_{0,k,\pm} = \hbar v_F(\pm k + \boldsymbol{\zeta}\cdot\boldsymbol{k})$ has an additional $\boldsymbol{\zeta}\cdot\boldsymbol{k}$ term, its eigenstates are same as those of the upright Dirac cone case,

$$\langle\psi_\pm| = \frac{1}{\sqrt{2}}(1 \pm e^{-i\theta_k}), \quad \theta_k = \arctan(k_y/k_x). \tag{4}$$

Equation (3) is associated with some basis $\{|\Phi_A\rangle, |\Phi_B\rangle\}$, where, in the case of graphene, A and B correspond to the two sublattices forming the honeycomb lattice. The relation between the eigenstates (4) and the above basis is

$$|\Phi_A\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle), \quad |\Phi_B\rangle = \frac{e^{-i\theta_k}}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle). \tag{5}$$

In this section, following Katsnelson [25], we use the pseudospin precession formulation to obtain the optical conductivity of the noninteracting tilted Dirac cone system first. In the next section we build upon the noninteracting formulation to formulate the optical conductivity of the interacting fermions in the TDCM system. The Hamiltonian of the noninteracting fermions in the presence of the time-dependent electric field can be written as

$$\tilde{H}_0 = H_0 + V_{\rm ext}(\boldsymbol{r}), \quad V_{\rm ext}(\boldsymbol{r}, t) = -\boldsymbol{E}(t)\cdot\boldsymbol{r}, \tag{6}$$

where the time dependence of $\boldsymbol{E}$ is assumed to have a monochromatic form, $e^{-i\omega t}$. In the rest of the paper, without loss of generality, we consider the electric field along the x axis for the sake of simplicity. Therefore in our final results, ζ_x (ζ_y) will mean the component of the tilt vector $\boldsymbol{\zeta}$ along (perpendicular to) the electric field.

Let us define the (2×2) single-particle density matrix by

$$\rho_k(t) = \rho_k(0) + \delta\rho_k e^{-i\omega t}, \tag{7}$$

where $\rho_k(0)$ is the density matrix operator before coupling to the external electric field, and therefore, $\delta\rho_k$ denotes the modification arising from coupling to the external electric field. In general, the compact representation of the density matrix $\langle\Phi_\alpha^\dagger\Phi_\beta\rangle$ can be expanded as

$$\rho_k(0) = \begin{pmatrix} \langle\Phi_A^\dagger\Phi_A\rangle & \langle\Phi_A^\dagger\Phi_B\rangle \\ \langle\Phi_B^\dagger\Phi_A\rangle & \langle\Phi_B^\dagger\Phi_B\rangle \end{pmatrix} = \sum_{\alpha=0}^{3} m_{\alpha,k}\sigma_\alpha, \tag{8}$$

where the rightmost term indicates expansion in terms of a basis composed of four 2×2 matrices, σ_α. The $\sigma_{\alpha=0}$ is the 2×2 identity matrix, while $\sigma_{i=1,2,3}$ are the usual Pauli matrices. The Latin indices such as i are counted from 1, while the Greek indices such as α start from 0. Sometimes in the expansion of the density matrix, i runs on 1 and 2 only, in which case we explicitly indicate it by $i = 1$ and 2. Therefore, the parameters m_α are a possible way of parametrizing the density matrix and are given by

$$m^0_{\alpha,k} = \tfrac{1}{2}\mathrm{tr}[\sigma_\alpha\rho_{\rm k}]. \tag{9}$$

Employing Eq. (5), the static density matrix (i.e., the one before coupling to the electric field emphasized by superscript 0) components are given by

$$m^0_{0,k} = \frac{1}{2}(f_+ + f_-), \quad m^0_{3,k} = 0, \tag{10}$$

$$m^0_{i,k} = \frac{k_i}{2k}(f_+ - f_-), \quad i = 1, 2, \tag{11}$$

where the Fermi occupation numbers $f_\pm$ for the upper and lower bands of a system at the chemical potential μ (as measured from the node of the conic spectrum) are defined by $f_\pm = \langle\psi_\pm^\dagger\psi_\pm\rangle = \Theta(\mu \mp \hbar v_F k - \hbar v_F\boldsymbol{\zeta}\cdot\boldsymbol{k})$. In the following, we are interested in the conductivity of TDCMs and the effect of the tilt parameter on its optical absorption. The time evolution of the density matrix in the presence of a time-dependent

external electric field is determined by

$$i\hbar\frac{d\rho_k}{dt} = [\tilde{H}_0, \rho_k]. \tag{12}$$

The current operator for a single particle is given by

$$\boldsymbol{r} = i\hbar\nabla_k \Rightarrow \boldsymbol{j} = e\frac{d\boldsymbol{r}}{dt} = \frac{e}{\hbar}\partial_k H, \tag{13}$$

which can be directly written in terms of parameters $m_{i,k} = m^0_{i,k} + \delta m_{i,k}e^{-i\omega t}$ and gives the following total current:

$$J_i = 2ev_F\sum_k (m_{i,k} + \zeta_i m_{0,k}) = \sigma(\omega)E_i, \quad i = 1, 2. \tag{14}$$

The last equation defines the optical conductivity $\sigma(\omega)$.

Equation of motion (12) when decomposed according to Eq. (8) gives

$$\omega\delta m_{i,k} = 2iv_F(\boldsymbol{k}\times\delta\boldsymbol{m}_k)_i - \frac{ie}{\hbar}(\boldsymbol{E}\cdot\nabla_k)m^0_{i,k}, \tag{15}$$

$$\omega\delta m_{0,k} = -\frac{ie}{\hbar}(\boldsymbol{E}\cdot\nabla_k)m^0_{0,k}. \tag{16}$$

As can be seen the $i = 1, 2$, and 3 components in Eq. (15) are decoupled from the $\alpha = 0$ component in Eq. (16). The above equations are formally similar to the case of graphene [5]. But one has to note that the $m^0_{\alpha,k}$ in Eqs. (10) and (11) involve the Fermi distribution function for the upper band f_+ which is tilt dependent (ζ dependent).

Now let us evaluate the current. Although the current operator in Eq. (14) is different from that of an upright Dirac cone in that the last term contains an explicit ζ_i, but owing to the presence of a relative polar angle between tilt and the momentum wave vector, the $\alpha = 0$ component satisfying

$$\delta m_{0,k} = \frac{ie}{\hbar\omega}\delta(\mu/\hbar v_F - k - \boldsymbol{\zeta}\cdot\boldsymbol{k})\left(\frac{k_i}{k} + \zeta_i\right)E_i \tag{17}$$

gives zero once the summation over $\boldsymbol{k}$ is performed. Therefore at the noninteracting electrons level, the contribution of the tilt parameters ζ_i to optical conductivity comes only through $\delta m_{i,k}$ and is encoded into the appropriate Fermi occupation functions as follows:

$$J_i = 2ev_F\sum_k \delta m_{i,k}. \tag{18}$$

Let us now derive an analytic expression for the above current in two-dimensional tilted Dirac materials and compare it with the results of the Kubo formula.

Since the electric field is directed along the x axis, we need to derive an expression for $\delta m_{1,k}$. By exploiting Eq. (15) we obtain

$$\begin{aligned}\left(\omega^2 - 4v_F^2k_y^2\right)\delta m_{x,k} + 4v_F^2k_xk_y\delta m_{y,k} &= \frac{e\omega}{i\hbar}(\boldsymbol{E}\cdot\nabla_k)m^0_{x,k},\\ \left(\omega^2 - 4v_F^2k_x^2\right)\delta m_{y,k} + 4v_F^2k_xk_y\delta m_{x,k} &= \frac{e\omega}{i\hbar}(\boldsymbol{E}\cdot\nabla_k)m^0_{y,k}.\end{aligned} \tag{19}$$

Solving the above equations, $\delta m_{k,x}$ becomes

$$\begin{aligned}&i\hbar\omega\left(\omega^2 - 4v_F^2k^2\right)\delta m_{x,k}\\ &\quad = eE_x\left[(\omega^2 - 4v_F^2k_x^2)\frac{\partial m^0_{x,k}}{\partial k_x} - 4v_F^2k_xk_y\frac{\partial m^0_{y,k}}{\partial k_x}\right],\end{aligned} \tag{20}$$

which after being plugged into Eq. (18) can be used to read the longitudinal conductivity $\sigma(\omega)$,

$$\begin{aligned}\frac{i\sigma(\omega)}{2ev_F} &= -\frac{e}{\hbar\omega}\sum_k\frac{\partial m^0_{k,x}}{\partial k_x}\\ &\quad + \frac{2ev_F^2}{\hbar\omega}\sum_k\frac{k_y^2}{k\left(\omega^2 - 4v_F^2k^2\right)}(f_+ - f_-).\end{aligned} \tag{21}$$

To extract the real part of the conductivity we substitiute $\omega \to \omega + i0^+$ (in the clean limit). The first term in the first line, $\sigma_{1\mathrm{st}}(\omega)$, contributes to the Drude peak,

$$\begin{aligned}\mathrm{Re}\left[\frac{\sigma_{1\mathrm{st}}(\omega)}{2ev_F}\right] &= \mathrm{Im}\left[\frac{e}{\hbar(\omega + i0^+)}\sum_k\frac{\partial m^0_{k,x}}{\partial k_x}\right]\\ &= \frac{e\pi}{\hbar}\delta(\omega)\sum_k\frac{\partial m^0_{k,x}}{\partial k_x} = \frac{e}{4\hbar\pi}\delta(\omega)\int kdkd\theta\frac{\partial m^0_{k,x}}{\partial k_x}\\ &= \frac{e}{4\hbar\pi}\delta(\omega)\int kdkd\theta\left\{\frac{k_y^2}{2k^3}(f_+ - f_-)\right.\\ &\quad \left. - \frac{k_x}{2k}\delta(\mu/\hbar v_F - k - \boldsymbol{\zeta}\cdot\boldsymbol{k})\left(\frac{k_x}{k} + \zeta_x\right)\right\}\\ &= -\frac{e\Lambda}{8\hbar}\delta(\omega),\end{aligned} \tag{22}$$

where Λ is the momentum cutoff. Here we assume the area of system is $A = 1$. There is yet another contribution to the Drude peak that comes from the second line of Eq. (21). Denoting it by $\sigma_{2\mathrm{nd}}(\omega)$, after doing interaction in momentum space we obtain

$$\begin{aligned}&\mathrm{Re}\left[\frac{\sigma_{2\mathrm{nd}}(\omega)}{2ev_F}\right]\\ &\quad = \mathrm{Im}\left[\frac{2ev_F^2}{\hbar(\omega + i0^+)}\sum_k\frac{k_y^2}{k\left[(\omega + i0^+)^2 - 4v_F^2k^2\right]}(f_+ - f_-)\right].\end{aligned} \tag{23}$$

The real part of the integral over $\boldsymbol{k}$ gives the second contribution to the Drude peak, namely,

$$\left[\frac{e\Lambda}{8\hbar} - \frac{e\mu}{4\hbar^2v_F\zeta^2}(1 - \sqrt{1 - \zeta^2})\right]\delta(\omega). \tag{24}$$

Adding up the contributions of $\sigma_{1\mathrm{st}}$ and $\sigma_{2\mathrm{nd}}$ in Eqs. (22) and (24), the total Drude peak becomes

$$\mathrm{Re}\left[\frac{\sigma_{\mathrm{Drude}}(\omega)}{2ev_F}\right] = -\frac{e\mu\delta(\omega)}{4\hbar^2v_F\zeta^2}(1 - \sqrt{1 - \zeta^2}). \tag{25}$$

In the limit of vanishing tilt, when one expands $\sqrt{1 - \zeta^2} \approx 1 - \zeta^2/2$, the ζ^2 factors in the numerator and denominator cancel each other and one recovers the Drude peak of the upright Dirac cone [27,28].

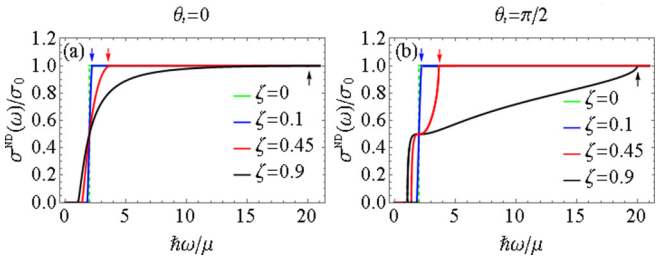

FIG. 1. The optical conductivity of noninteracting fermions in the tilted Dirac matter for various values of tilt parameter ζ in two different directions: $\theta_t = 0$ in panel (a) and $\theta_t = \pi/2$ in panel (b). Here, the upright Dirac cone (graphene) $\zeta = 0$ is shown with green color. In both panels, the blue, red, and black colors correspond to $\zeta = 0.1$, 0.45, and 0.9, respectively, and arrows indicate the energy scales at which the optical conductivity value equals to that of the universal optical conductivity per valley and spin, i.e., $\sigma_0 = \pi e^2/4h$. For every given value of ζ, the arrows correspond to $\hbar\omega_0/\mu = 2/(1-\zeta)$. Furthermore, the onset of Pauli blocking for optical absorption is at $\hbar\omega_{\rm onset}/\mu = 2/(1+\zeta)$, beyond which the absorption takes place.

The second term, $\sigma_{2\rm nd}$, contains a non-Drude contribution ($\sigma_{2\rm nd}^{\rm ND}$) as well, which is given by

$$-{\rm Re}\left[\frac{\sigma_{2\rm nd}^{\rm ND}(\omega)}{2ev_F}\right] = \frac{2ev_F^2\pi}{\hbar\omega}\sum_k \frac{k_y^2}{4v_F k^2}\delta(\omega - 2v_F k)\left[\Theta\left(\frac{\mu}{\hbar v_F} - k - \boldsymbol{\zeta}\cdot\boldsymbol{k}\right) - 1\right] = \frac{e}{32\pi\hbar v_F}\int d\theta \sin^2\theta\left[\Theta\left(\frac{2\mu}{\hbar} - \omega - \boldsymbol{\zeta}\omega\cos(\theta-\theta_t)\right) - 1\right]. \tag{26}$$

This invites us to define $X_- = (2\mu - \hbar\omega)/\hbar\omega\zeta$, whereby

$${\rm Re}\left[\frac{\sigma_2^{\rm ND}(\omega)}{2ev_F}\right] = -\frac{e}{32\pi\hbar v_F}\int d\theta \sin^2(\theta+\theta_t)[\Theta(X_- - \cos\theta) - 1]. \tag{27}$$

Doing the integration, the conductivity per valley per spin is

$${\rm Re}[\sigma^{\rm ND}(\omega)] = \frac{e^2}{16\pi\hbar}\begin{cases}\theta_- - \frac{1}{2}\cos 2\theta_t \sin 2\theta_- & \Theta(1-X_-^2),\\ \pi & \Theta(-X_- - 1),\end{cases} \tag{28}$$

where $\theta_- = \arccos(X_-)$. These results agree with a computation based on the Kubo formula in the clean limit [29]. In the high-frequency limit, $\omega \to \infty$, one has $-X_- \to \zeta^{-1}$, and therefore, the second piece of the above piecewise function is chosen which gives a tilt-independent optical absorption. For the finite ω, the effect of the tilt is to broaden the sharp steplike absorption of the noninteracting upright Dirac fermions. This can be intuitively explained as follows: The Pauli blocking giving rise to the steplike absorption in the upright case will become different for different values of the angle θ of the wave vector $\boldsymbol{k}$. This is shown in Fig. 1.

Now that within the pseudospin precession formulation we have reproduced the optical conductivity of noninteracting tilted Dirac fermions, the stage is set for the main computation of our paper and we are ready to consider the role of many-body interactions.

III. INTERACTING TILTED DIRAC FERMIONS

In the previous section we investigated the role of the tilt parameter in the optical conductivity in the absence of Coulomb interaction. We will follow Katsnelson [5,25] in order to incorporate the effects of interactions within the Landau Fermi-liquid phenomenology. In the case of upright Dirac cones of graphene, Katsnelson generalized the density-density interaction ($\rho\rho$) of the ordinary Fermi liquids by considering an additional pseudospin degree of freedom $\boldsymbol{\sigma}$ to include $(\boldsymbol{\sigma}\cdot\boldsymbol{k})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}')$ and $(\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}')$ types of terms in the free energy of the system. The rationale behind the above forms is that in graphene the emergent space-time structure has the Minkowski form that includes the ordinary rotations as a subgroup of its emergent Lorentz symmetry. Therefore the dot product ensures that the energy is a scalar with respect to rotations of the two-dimensional space. But in the presence of a nonzero tilt, $\boldsymbol{\zeta}$, the rotational invariance is lost. So what is the guiding principle to ensure us that we enumerate all possible terms allowed by the symmetries of the emergent geometry?

It has been shown that, in the presence of the tilt parameter $\boldsymbol{\zeta}$, the structure of the emergent Lorentz group is deformed and the rotation generators will also include the effect of nonzero $\boldsymbol{\zeta}$ [15]. Therefore, we need to generalize the above line of thought in order to construct the appropriate scalar free energy functional in such a background metric. So the line of reasoning is to promote the dot products into the contraction of covariant and contravariant entities. In fact, the emergent symmetry of the TDCMs is given by an appropriate generalization of the Lorentz group [15], which includes modified rotation and modified Lorentz boosts. The following steps will ensure that we have generated all possible couplings between the momenta and pseudospins that include the effect of tilt parameters $\boldsymbol{\zeta}$ in a systematic way: First of all note that the density-density interaction is actually $\rho\rho'\mathbf{1}\otimes\mathbf{1}'$, which in the Lorentz indexed notation is given by $\sigma_0\otimes\sigma'^0$. Boosting this expression will give $\sigma_\mu\otimes\sigma^\mu$, where use has been made of the fact that the 3-current operator has the following matrix part: $j^\mu \sim \sigma^\mu$. Combining it with the Lorentz scalar $k_\mu k'^\mu$ and other possiblities such as $k_\mu\sigma^\mu$, etc., we obtain the following possible contributions to the Landau parameters $\hat{f}_{k,k'}$:

$$k_\mu\sigma^\mu\otimes k'_\nu\sigma^\nu, \quad k_\mu k'^\mu\sigma_\nu\otimes\sigma'^\nu. \tag{29}$$

The above summations generate all possible kinds of terms allowed by the principle of covariance that can possibly arise in the Landau Fermi-liquid expansion of the free energy functional of the interacting system. The hat on $\hat{f}$ emphasizes that it has a matrix structure in the pseudospin space of the two-particle Landau parameters. The above expressions generalize the expressions used by Katsnelson [25] into arbitrary background metrics and give us clues to write down all the possible combinations of vectors and pseudospin degrees of freedom that contribute to the Landau free energy functionals in such space-times. In the case of upright Dirac fermions, the Lorentz indices μ, ν, etc., will be raised or lowered by the Minkowski metric $\eta_{\mu\nu} = {\rm diag}(-1,1,1)$. However, for

the tilted Dirac materials, the space-time acquires a different space-time structure given by the following metric, $g^{\mu\nu}$:

$$g^{\mu\nu} = \begin{pmatrix} -1 & -\zeta_x & -\zeta_y \\ -\zeta_x & 1-\zeta_x^2 & -\zeta_x\zeta_y \\ -\zeta_y & -\zeta_x\zeta_y & 1-\zeta_y^2 \end{pmatrix}. \tag{30}$$

Once the Landau parameters are given, an effective one-particle Hamiltonian can be constructed in the spirit of Landau's Fermi-liquid theory as [30]

$$H_k = H_{0k} + \sum_{k'} \hat{f}_{kk'}\delta\rho_{k'}, \tag{31}$$

where the first term is a 2×2 matrix given by Eq. (3) and the second term contains the effects of interactions with other particles within the Landau Fermi-liquid theory. When the space-time metric is given by $\eta_{\mu\nu}$, Katsnelson [25] has found that the above forms of interactions do not lead to any many-body corrections to the optical absorption of free Dirac fermions. What we show in this paper is that, once the replacement $\eta_{\mu\nu} \to g_{\mu\nu}$ where $g_{\mu\nu}$ is given by Eq. (2) [or equivalently $g^{\mu\nu}$ is given by Eq. (30)] is performed, the many-body corrections start to play a decisive role in the optical absorption of the system.

Landau parameters

For the interacting theory, we need an equation of motion of the form

$$i\partial_t\delta\rho_k = [H_0, \delta\rho_k] + \left[V_{\text{ext}}, \rho_k^0\right] + \left[\delta H_k, \rho_k^0\right], \tag{32}$$

where δH_k is the second term in Eq. (31) that encodes the effects of the interactions and the emergent geometry of space-time into the 4×4 matrix (two-particle) Landau parameters $\hat{f}_{kk'}$, and $\delta\rho_{k'}$ is a 2×2 density matrix of a single particle. To consider the most generic form of the Landau parameters, we build upon Eq. (29) and make use of the fact that the lower Lorentz indices such as ν in σ_ν are brought up by the metric $g^{\mu\nu}$ of Eq. (30) as $\sigma^\mu = g^{\mu\nu}\sigma_\nu$, etc. In this way, a multitude of terms are generated that include the tilt parameters encoded in various components of the metric $g^{\mu\nu}$. The structure of the space-time for noninteracting theory is definitely given by the metric (30). If the presence of Coulomb interactions does not destroy the isometry of the space-time (30), then the covariance principle will assign a *single coefficient* to every term in Eq. (29).

In order to allow for a more generic possibility of the breaking of the space-time structure (30), we expand each of the terms appearing in Eq. (29) to figure out all possible sorts of terms and then allow the coefficients of various terms (as a result of Coulomb interactions) to be different. This generalization accounts for the possibility that in the course of the formation of a stable Fermi-liquid state, various terms given in Eq. (A2) can undergo a renormalization. At the end, when a stable Fermi-liquid state is formed, the coefficients of various terms appearing in the two-quasiparticle channel might be different [31]. In this case, the Coulomb interactions break the emergent space-time symmetry (the Lorentz symmetry in the case of graphene and the modified Lorentz symmetry in the case of TDCMs). The most generic forms of such Landau parameters are given in Eq. (A2).

Plugging the most generic form of $\hat{f}_{k,k'}$ into the equation of motion, Eq. (32), and dropping terms that give zero commutators, we end up with the following subset of Landau parameters,

$$\begin{aligned}\hat{f}_{k,k'} \supset\ & \mathcal{A}_{k,k'}(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}_0' + \mathbb{A}_{k,k'}(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) \\ & + \mathfrak{B}_{k,k'}(\sigma\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}_0'(\boldsymbol{k}'\cdot\boldsymbol{\zeta}) + \mathbb{B}_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}_0' \\ & + \mathcal{B}_{k,k'}(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}') + H_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}')] \\ & + D_{k,k'}[(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta})] \\ & + \mathcal{D}_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta})] \\ & + \mathfrak{G}_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}_0'(\boldsymbol{k}'\cdot\boldsymbol{\zeta})],\end{aligned} \tag{33}$$

where the notation $\supset$ means that the left-hand side includes the right-hand side. Now we are ready to compute the equation of motion: The time evolution for each component $\delta m_{k,i}$ becomes

$$\hbar\omega\delta m_{k,x} = 2i\hbar v_F k_y\delta m_{k,z} - ie\boldsymbol{E}\cdot\boldsymbol{\nabla}_k m_{k,x}^0, \tag{34}$$

$$\hbar\omega\delta m_{k,y} = -2i\hbar v_F k_x\delta m_{k,z} - ie\boldsymbol{E}\cdot\boldsymbol{\nabla}_k m_{k,y}^0, \tag{35}$$

$$\hbar\omega\delta m_{k,z} = 2i\hbar v_F(k_x\delta m_{y,k} - k_y\delta m_{x,k}) + \Delta_k, \tag{36}$$

where the many-body corrections within Landau's Fermi-liquid approach are encoded into

$$\begin{aligned}\Delta_k = \frac{2i}{k}(\boldsymbol{\zeta}\times\boldsymbol{k})_z\sum_{k'}\ & \mathcal{A}_{k,k'}\delta m_{0,k'} + \mathbb{A}_{k,k'}\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta} \\ & + \mathfrak{B}_{k,k'}(\boldsymbol{k}'\cdot\boldsymbol{\zeta})\delta m_{0,k'} + \mathbb{B}_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})\delta m_{0,k'} \\ & + \mathcal{B}_{k,k'}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{k}') + H_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\delta\boldsymbol{m}_{k'}\cdot k') \\ & + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{k}'\cdot\boldsymbol{\zeta})[\mathcal{D}_{k,k'}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta}) + \mathfrak{G}_{k,k'}\delta m_{0,k'}] \\ & + D_{k,k'}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta})[(\boldsymbol{k}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})].\end{aligned} \tag{37}$$

When we substitute Eq. (36) in Eqs. (34) and (35) and define $i\tilde{\Delta}_{i,k} = \zeta_i\Delta_k/(2\boldsymbol{\zeta}\times\boldsymbol{k})_z$, the equations for $\delta m_{x,k}$ and $\delta m_{y,k}$ become

$$\begin{aligned}&\left(\hbar^2\omega^2 - 4\hbar v_F^2k_y^2\right)\omega\delta m_{x,k} + 4\hbar^2v_F^2k_xk_y\delta m_{y,k} \\ &\quad = -ie\boldsymbol{E}\cdot\boldsymbol{\nabla}_k m_{x,k}^0 - 4\hbar v_Fk_y(k_y\tilde{\Delta}_{k,x} - k_x\tilde{\Delta}_{k,y}), \\ &\left(\hbar^2\omega^2 - 4\hbar v_F^2k_x^2\right)\omega\delta m_{y,k} + 4\hbar^2v_F^2k_xk_y\delta m_{x,k} \\ &\quad = -ie\boldsymbol{E}\cdot\boldsymbol{\nabla}_k m_{y,k}^0 + 4\hbar v_Fk_x(k_y\tilde{\Delta}_{k,x} - k_x\tilde{\Delta}_{k,y}).\end{aligned} \tag{38}$$

As before, if we assume $\boldsymbol{E} = E\hat{x}$, the Landau Fermi-liquid correction to the current can be written as

$$j_x^{\text{corr}} = \sum_k \frac{4\hbar v_Fk_y}{\left(\hbar^2\omega^2 - 4\hbar^2v_F^2k^2\right)}\sum_{k'}(k_y\tilde{\Delta}_{x,k} - k_x\tilde{\Delta}_{y,k}). \tag{39}$$

For the calculation of the above correction, we need $\delta m_{\alpha,k'}$. Excluding the contribution of the Drude peak, we have

$$\begin{aligned}\delta m_{k,x} &= -i\frac{2ev_F^2k_y^2E_x}{\hbar k\omega\left(\omega^2 - 4v_F^2k^2\right)}(f_+ - f_-), \\ \delta m_{k,y} &= i\frac{2ev_F^2k_xk_yE_x}{\hbar k\omega\left(\omega^2 - 4v_F^2k^2\right)}(f_+ - f_-), \\ \delta m_{k,z} &= \frac{2iv_F}{\omega}(k_x\delta m_{k,y} - k_y\delta m_{k,x}), \\ \delta m_{0,k} &= \frac{-ie}{\hbar\omega}E_x\frac{\partial m_{0,k}^0}{\partial k_x}.\end{aligned} \tag{40}$$

Doing some straightforward computations, we find that the only nonzero terms arise from summation on the first term, namely, $k_y\tilde{\Delta}_{k,y}$. Hence,

$$\frac{j_x^{\rm corr}}{2ev_F}=\sum_{k,k'}\frac{4\hbar v_F k_y^2\zeta_x^2}{k\left(\hbar^2\omega^2-4\hbar^2v_F^2k^2\right)}\times[\mathbb{A}_{k,k'}\delta m_{x,k'}+\mathfrak{B}_{k,k'}k'_x\delta m_{0,k'}]. \quad (41)$$

As is evident in Eq. (41), the nonzero corrections to the current solely depend on the existence of the tilt parameter ζ. For the upright Dirac cone where $\boldsymbol{\zeta}=0$, this correction vanishes, and our result reduces to that of Katsnelson [5,25], stating that the many-body corrections within the Fermi-liquid theory do not alter the optical conductivity of the upright Dirac fermions. From the current corrections in Eq. (41), we can obtain the following two types of contributions to the many-body corrections to the optical conductivity:

$$i\sigma_A^{\rm corr}(\omega)=\frac{e^2v_F^4\zeta_x^2}{\hbar^2\pi^4(\omega+i0^+)}\int dkd\theta\frac{k_y^2}{\left[(\omega+i0^+)^2-4v_F^2k^2\right]}\times\int dk'd\theta'k_y'^2\mathbb{A}_{k,k'}\frac{(f_+-f_-)}{\left[(\omega+i0^+)^2-4v_F^2k'^2\right]}, \quad (42)$$

and

$$i\sigma_B^{\rm corr}(\omega)=\frac{e^2v_F^2\zeta_x^2}{2\hbar^2\pi^4(\omega+i0^+)}\times\int dkd\theta\frac{k_y^2}{\left[(\omega+i0^+)^2-4v_F^2k^2\right]}\times\int k'dk'd\theta'\mathfrak{B}_{k,k'}k'_x\frac{\partial m^0_{0,k'}}{\partial k'_x}, \quad (43)$$

where subscripts A and B above refer to the first and second terms in Eq. (41), respectively. The Drude part [imaginary part of $1/(\omega+i0^+)$] will be given by the real part of the integrals in the above equations. Both equations include an integral over (k,θ), which vanishes after the (dimensional) regularization [32]. Therefore, *the Drude part does not receive any many-body corrections within the present phenomenological Fermi-liquid approach.*

Having proven that the Drude part receives zero correction, now the imaginary part of $i\sigma_{A,B}^{\rm corr}$ can be picked from either the k integral or the k' integral. Let us analyze the $\mu=0$ (undoped) case first. In this case, both k and k' integrals in the $i\sigma_A^{\rm corr}$ term contribute equally: one of them contributes its own imaginary part, and the other contributes its real part. The real part vanishes again upon dimensional regularization when the cutoff is sent to infinity. As for the second term, $i\sigma_B^{\rm corr}$, the k' integral is zero when $\mu=0$. This is because the partial derivative of m_0^0 as in Eq. (10) involves Fermi occupation numbers that are constant all over the k' space.

When the system is doped, namely, $\mu\neq0$, the nonzero corrections can only be picked when the k integral contributes an imaginary part *and* the k' integral contributes a real part. In this way, a dependence on the Fermi energy enters the theory via the Fermi occupation numbers involved in the k' integral.

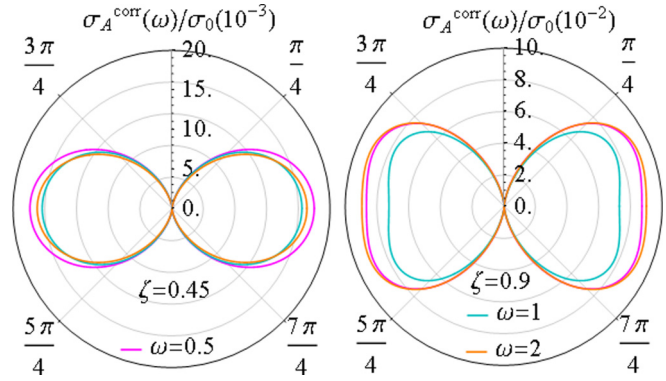

FIG. 2. The polar plot of the $m=0$ contribution (see the text) of the first correction term, $\sigma_A^{\rm corr}(\omega)$, in the unit of universal optical conductivity per valley and spin, i.e., $\sigma_0=\pi e^2/4h$ as a function of the angle θ_t of $\boldsymbol{\zeta}$ with respect to the electric field direction $\hat{x}$. Panel (a) corresponds to the tilt parameter $\zeta=0.45$, while panel (b) corresponds to $\zeta=0.9$. Various values of the frequency, $\omega=0.5$, 1, and 2 (in units of μ), are denoted with magenta, cyan, and orange, respectively. The strength of the Landau parameter A_0 in Eq. (46) is assumed to be 1 in units of $(\hbar v_F)^2/\mu$.

The nonzero contributions are

$$-{\rm Re}\left[\sigma_A^{\rm corr}(\omega)\right]\left(\frac{e^2v_F^4\zeta_x^2}{\hbar^2\pi^3\omega}\right)^{-1}=\int d^2\boldsymbol{k}\frac{P(k)}{k}\delta(\omega-2v_Fk)\times\int d^2\boldsymbol{k}'\mathbb{A}_{k,k'}\frac{Q(k')}{k'}(f_+-f_-) \quad (44)$$

and

$${\rm Re}\left[\sigma_B^{\rm corr}(\omega)\right]=\frac{e^2v_F^2\zeta_x^2}{2\pi^3\omega\hbar^2}\int dkd\theta\frac{k_y^2}{4v_Fk}\delta(\omega-2v_Fk)\times\int dk'd\theta'\mathcal{B}_{k,k'}k'k'_x(\cos\theta_{k'}+\zeta_x)\times\delta(\mu/\hbar v_F-k-\boldsymbol{\zeta}\cdot\boldsymbol{k}), \quad (45)$$

where $P(k)=k_y^2/(\omega+2v_Fk)$ and $Q(k)=k_y^2/(\omega^2-4v_F^2k^2)$. In general, the scattering amplitudes $\mathbb{A}_{k,k'}$ and $\mathfrak{B}_{k,k'}$ can have arbitrary dependence on the wave vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$, which can be expanded in $e^{im\varphi}$ harmonics, where φ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. The isotropic contribution arises from the $m=0$ channel. The evaluation of the corresponding integrals gives,

$${\rm Re}\left[\sigma_{A,m=0}^{\rm corr}(\omega)\right]\left(\frac{e^2A_0\zeta_x^2\mu}{32\pi^2\hbar^3v_F^2}\right)^{-1}=D_0+D_1(\omega)+\Theta(X_--1)D_2(\omega)+\Theta(-1-X_-)D_3(\omega), \quad (46)$$

where the definitions of functions D_0, D_1, D_2, and D_3 in the above relation are given in Appendix B. This correction has been plotted in Fig. 2, where the strength of the Landau parameter A_0 has been taken to be 1 in its natural unit $(\hbar v_F)^2/\mu$. Furthermore the value of $\zeta=0.45$ in Fig. 2(a), nearly corresponds to the estimated values of ζ for 8*Pmmn* borophene [12]. The dipolar dependence on the angle θ_t between $\boldsymbol{\zeta}$ and the electric field arises from ζ_x^2 in the above equation.

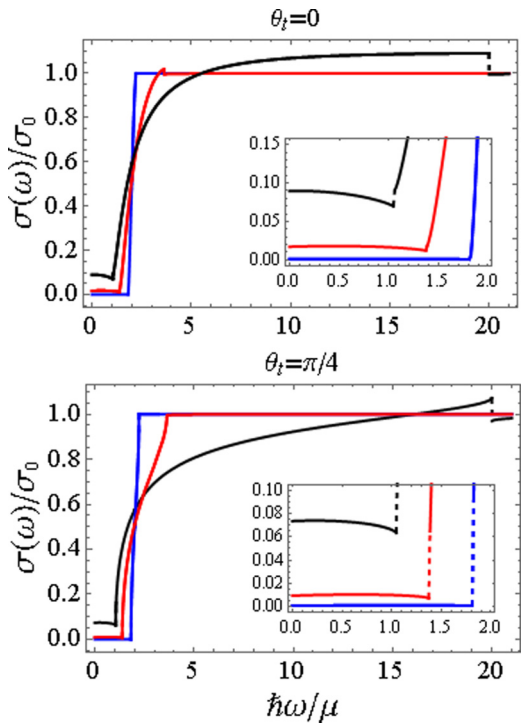

FIG. 3. The total optical conductivity of tilted Dirac matter in the presence of interaction has been shown. The red, blue, and black curves correspond to $\zeta = 0.1$, 0.45, and 0.9, respectively, and panels (a) and (b), the tilt direction is $\theta_t = 0$ and $\theta_t = \pi/4$, respectively. At $\theta_t = \pi/2$ there will be no interaction-induced corrections (see the text). Here, Fermi-liquid parameters are taken as $A_0 = 1$ and $B_0 = 0.1$ in their natural units $(\hbar v_F)^2/\mu$ and $(\hbar v_F)^3/\mu^2$, respectively.

Similarly the term $\sigma_B^{\rm corr}$ at the $m = 0$ channel will give

$$\mathrm{Re}\big[\sigma^{\rm corr}_{B,m=0}\big] = \frac{e^2\mu^2\zeta_x^2\mathfrak{B}_0}{16\hbar^4\pi v_F^4(1-\zeta^2)^{3/2}}. \tag{47}$$

It can be seen by symmetry that the odd values of m give zero contribution. This holds for both Eqs. (44) and (45). Therefore, the next nonzero contribution arises from the $m = 2$ channel. The total optical absorption as a function of energy has been plotted in Fig. 3 for two values of $\theta_t = 0$ and $\theta_t = \pi/4$, respectively.

As can be seen, the correction $\sigma_B^{\rm corr}$ in Eq. (47) has no frequency dependence and amounts to a constant shift in the absorption in the whole frequency range. Note that this interaction-induced absorption takes place even in the low-energy region before the $\omega_{\rm onset}$ that marks the onset of Pauli blocking in doped Dirac cone systems (see Fig. 1). *Such a substantial absorption, in a frequency range where according to free tilted Dirac theory there should be no absorption, can be a hallmark of many-body interactions within the Landau Fermi-liquid approach.* This contribution depends on the angle between $\boldsymbol{\zeta}$ and the electric field polarization of the incident light. The angular dependence according to Eq. (47) is given by $\cos^2\theta_t$. This angular dependence is the same as $\sigma_A^{\rm corr}$. Therefore, the entire many-body corrections to optical absorption vanish when the electric field is aligned along the $\boldsymbol{\zeta}$ direction. This property can be employed as a method to optically characterize the direction of the tilt vector $\boldsymbol{\zeta}$ in interacting tilted Dirac fermion systems in $2+1$ dimensions. The direction of tilt is determined as a direction of the electric field for which no absorption for $\omega < \omega_{\rm onset}$ takes place. Since both $\sigma_A^{\rm corr}$ and $\sigma_B^{\rm corr}$ have the same polar angle dependence, rotating the sample with respect to the incident light can provide information about the combined effects of A_0 and B_0 Landau parameters. Furthermore, such a unique dependence on the polar angle can be employed to disentangle the tilt-related many-body corrections from the temperature-related effects [33,34].

It might appear that the identical polar angle dependence of A- and B-type corrections prevents the experimental separation of the two effects. But the dependence on the chemical potential comes to the rescue: Therefore, a nice way to separate the contribution of A-type corrections from that of B-type corrections is to note that $\sigma_A^{\rm corr} \sim \mu^1$, while $\sigma_B^{\rm corr} \sim \mu^2$. Such a distinct dependence on the chemical potential can be easily mapped by a gate voltage during the optical absorption experiment that serves to disentangle the A- and B-type corrections.

IV. SUMMARY AND DISCUSSIONS

In the first part of this work we used the equation of motion for the pseudospin precession to evaluate the optical conductivity of noninteracting tilted Dirac fermions in $2+1$ dimensions. Our results agree with the earlier calculations based on the Kubo formula.

In order to incorporate the many-body corrections to the above picture, we used Landau's Fermi-liquid theory. To exhaust all possible terms that might appear in the Landau free energy functional, we started from the principle of covariance to form all possible scalars in the background metric (2) of tilted Dirac cone materials. Then, to consider the most generic case, we accounted for the possible breaking of the modified Lorentz symmetry by allowing generic coefficients for various terms in the expansion. In this way we obtained the most generic form of the Landau functional that combines pseudospin degrees of freedom $\boldsymbol{\sigma}$ of electrons with their momentum $\boldsymbol{k}$ and the tilt parameter $\boldsymbol{\zeta}$.

Our main finding is that while for noninteracting tilted Dirac fermions there is a frequency scale $\omega_{\rm onset}$, below which the Pauli blocking of free fermions does not permit any absorption of light, the many-body corrections give rise to the light absorption even for frequencies below $\omega_{\rm onset}$. The dipolar dependence of the two different types of nonzero corrections obtained in this paper allows a full optical determination of the direction of the tilt. This is because the $\cos^2\theta_t$ corrections entirely vanish when the electric field is perpendicular to the tilt direction. The distinct dependence of the above corrections on the chemical potential can be employed to disentangle the corrections arising from the above two types of terms, if a gate voltage is allowed to tune the chemical potential μ.

Both corrections include the factor $(1-\zeta^2)^{-1/2}$, which resembles the gravitational redshift factor of the space-time (2) with respect to the Minkowski geometry [20] (defined by $\zeta = 0$ limit). Therefore the optical absorption measurements, in addition to containing information about the effects of interactions, do provide information about the metric of the underlying space-time structure of the noninteracting electrons.

The effect of nonzero temperature [33,34] can be disentangled from the tilt-related effects in the following ways: (i) monitoring the evolution of spectral lines upon decreasing temperature, one can extrapolate to the $T \to 0$ limit of the present paper; (2) rotating the sample and employing the fact that tilt-related many-body corrections have $\cos^2\theta_t$ dependence on the angle θ_t between the electric field and tilt direction; (3) substitution of some of the boron atoms with carbon can increase the tilt parameter up to $\zeta = 0.69$ [18], meaning that the tilt parameter is acquiring the status of a more or less tunable parameter. Comparision of the low-energy sector of the optical spectra for two different values of ζ at the same temperature can also help to disentangle the tilt-related effects from the temperature-related effects. Furthermore, the temperature T at which the experiment is being done will be modified by the redshift factor and, therefore, will be "seen" by electrons as $\bar{T} = T\sqrt{1-\zeta^2}$. Therefore choosing materials with ζ closer to 1 will diminish the temperature-related effects.

ACKNOWLEDGMENTS

S.A.J. was supported by research deputy of the Sharif University of Technology, Grant No. G960214. S.A.J. acknowledges post doctoral funds by Iran Science Elites Foundation (ISEF).

APPENDIX A: LANDAU'S FERMI LIQUID INTERACTION

As discussed in the main text, all possible correction terms can be generated as

$$\begin{aligned} k_\mu k'^{\mu}\sigma_\nu \otimes \sigma'^{\nu} \to\ & [A'_{k,k'} + \mathcal{A}'_{k,k'} k\cdot\boldsymbol{\zeta} + \mathfrak{A}'_{k,k'} k'\cdot\boldsymbol{\zeta} + \mathbb{A}'_{k,k'}(\boldsymbol{\zeta}\cdot\boldsymbol{k})(\boldsymbol{\zeta}\cdot\boldsymbol{k}')]\sigma_0\otimes\sigma'_0 \\ & + [B'_{k,k'} + \mathcal{B}'_{k,k'} k\cdot\boldsymbol{\zeta} + \mathfrak{B}'_{k,k'} k'\cdot\boldsymbol{\zeta} + \mathbb{B}'_{k,k'}(\boldsymbol{\zeta}\cdot\boldsymbol{k})(\boldsymbol{\zeta}\cdot\boldsymbol{k}')]\sigma_0\otimes\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta} \\ & + [C'_{k,k'} + \mathcal{C}'_{k,k'} k\cdot\boldsymbol{\zeta} + \mathfrak{C}'_{k,k'} k'\cdot\boldsymbol{\zeta} + \mathbb{C}'_{k,k'}(\boldsymbol{\zeta}\cdot\boldsymbol{k})(\boldsymbol{\zeta}\cdot\boldsymbol{k}')]\boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\otimes\sigma'_0 \\ & + [D'_{k,k'} + \mathcal{D}'_{k,k'} k\cdot\boldsymbol{\zeta} + \mathfrak{D}'_{k,k'} k'\cdot\boldsymbol{\zeta} + \mathbb{D}'_{k,k'}(\boldsymbol{\zeta}\cdot\boldsymbol{k})(\boldsymbol{\zeta}\cdot\boldsymbol{k}')]\sigma_i\otimes\sigma'_i \end{aligned} \tag{A1}$$

and

$$\begin{aligned} k_\mu\sigma^\mu \otimes k'_\nu\sigma'^{\nu} \to\ & A_{k,k'}\boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}'_0 + \mathcal{A}_{k,k'}[\boldsymbol{\sigma}_0\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}'_0] + \mathfrak{A}_{k,k'}[\boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}'\cdot\boldsymbol{k}' + \boldsymbol{\sigma}\cdot\boldsymbol{k}\otimes\boldsymbol{\sigma}'_0] + \mathbb{A}_{k,k'}(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) \\ & + B_{k,k'}[\boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}'_0(\boldsymbol{k}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}'_0] + \mathfrak{B}_{k,k'}[(\sigma\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}'_0(\boldsymbol{k}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{\sigma}_0\otimes(\sigma'\cdot\boldsymbol{\zeta})] \\ & + \mathbb{B}_{k,k'}[\boldsymbol{\sigma}_0\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}'_0] + \mathcal{B}_{k,k'}[(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}') + (\boldsymbol{\sigma}\cdot\boldsymbol{k})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta})] \\ & + D_{k,k'}[(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta})] + \mathcal{D}_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta})] \\ & + G_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}'_0(\boldsymbol{k}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{\sigma}\cdot\boldsymbol{k})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}')] + \mathcal{G}_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{\sigma}_0\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}') + (\boldsymbol{\sigma}\cdot\boldsymbol{k})\otimes\boldsymbol{\sigma}'_0(\boldsymbol{k}'\cdot\boldsymbol{\zeta})] \\ & + \mathfrak{G}_{k,k'}[(\boldsymbol{k}\cdot\boldsymbol{\zeta})\boldsymbol{\sigma}_0\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes\boldsymbol{\sigma}'_0(\boldsymbol{k}'\cdot\boldsymbol{\zeta})] \\ & + H_{k,k'}[(\boldsymbol{\sigma}\cdot\boldsymbol{k})\otimes(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}'\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\otimes(\boldsymbol{\sigma}'\cdot\boldsymbol{k}')]. \end{aligned} \tag{A2}$$

Clearly Eq. (A2) includes all possible terms. Even the terms generated in Eq. (A1) are contained in Eq. (A2). This implies that we consider that Eq. (A2) is sufficient to account for the role of all possible interaction terms that involve the tilt parameter $\boldsymbol{\zeta}$ and $\boldsymbol{\sigma}$ and momentum $\boldsymbol{k}$.

In the equation of motion we will need the commutation of $[\hat{f}_{k,k'}, \rho^0_k]$, which is given by

$$\begin{aligned} \frac{1}{2k}[\delta H_k, \boldsymbol{\sigma}\cdot\boldsymbol{k}] = \ & \frac{2i}{k}\mathcal{A}_{k,k'}\delta m_{0,k'}(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma} + \frac{2i}{k}\mathbb{A}_{k,k'}(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta}) \\ & + \frac{2i}{k}\mathfrak{B}_{k,k'}(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\boldsymbol{k}'\cdot\boldsymbol{\zeta})\delta m_{0,k'} + \frac{2i}{k}\mathbb{B}_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}\delta m_{0,k'} + \frac{2i}{k}\mathcal{B}_{k,k'}(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{k}') \\ & + \frac{2i}{k}D_{k,k'}[(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta}) + (\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta})] \\ & + \frac{2i}{k}\mathcal{D}_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\boldsymbol{k}'\cdot\boldsymbol{\zeta})(\delta\boldsymbol{m}_{k'}\cdot\boldsymbol{\zeta}) + \frac{2i}{k}\mathfrak{G}_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\boldsymbol{k}'\cdot\boldsymbol{\zeta})\delta m_{0,k'} \\ & + \frac{2i}{k}H_{k,k'}(\boldsymbol{k}\cdot\boldsymbol{\zeta})(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}(\delta\boldsymbol{m}_{k'}\cdot k'). \end{aligned} \tag{A3}$$

The commonality among all the above terms is the presence of $(\boldsymbol{\zeta}\times\boldsymbol{k})\cdot\boldsymbol{\sigma}$. Additionally, both $\boldsymbol{\zeta}$ and $\boldsymbol{k}$ are in the xy plane, which implies the contribution of σ_z.

APPENDIX B: FIRST PART CORRECTION

In order to derive the correction term that arises from the first term, i.e., Eq. (44), we need to do integration of the $\boldsymbol{k}$ space in two dimensions,

$$-\mathrm{Re}\big[\sigma_A^{\mathrm{corr}}(\omega)\big]\left(\frac{e v_F^3 \zeta_x^2}{2\hbar^2\pi^3\omega}\right)^{-1} = \int d^2\boldsymbol{k}\frac{P(k)}{k}\delta(\omega-2v_F k)\int d^2\boldsymbol{k}'\mathbb{A}_{k,k'}\frac{Q(k')}{k'}(f_+-f_-),$$

in which $P(k)=k_y^2/(\omega+2v_F k)$ and $Q(k)=k_y^2/(\omega^2-4v_F^2k^2)$. The isotropic correction is proportional to $A_{k,k'}=A_{m=0}=$ constant with the following final result:

$$\mathrm{Re}\big[\sigma_{A,m=0}^{\mathrm{corr}}(\omega)\big]\left(\frac{e^2A_0\zeta_x^2\mu}{32\pi^2\hbar^3v_F^2\omega}\right)^{-1} = D_0(\omega)+D_1(\omega)+\Theta(X_--1)D_2(\omega)+\Theta(-1-X_-)D_3(\omega), \tag{B1}$$

subject to the following definitions:

$$D_0(\omega) = 2\pi[\lambda\cos^2\theta_t-(\lambda-1)\zeta^{-2}\cos 2\theta_t],$$

$$D_1(\omega) = \frac{\pi\cos 2\theta_t}{4\omega\zeta^2\lambda\mu}\{\omega^2+\lambda[4\mu^2+4\mu\omega-\zeta\omega^2G_+(\omega)X_+]\}+\frac{\pi\omega}{4}\ln\left[\frac{\lambda(G_+(\omega)+\zeta X_+)}{(1+\lambda)}\right],$$

$$D_2(\omega)\left(\frac{\pi\omega\cos 2\theta_t}{4\zeta^2\lambda\mu}\right)^{-1} = 1+\lambda\omega\zeta^4X_-G_-(\omega)+\lambda[-4\mu^2-\zeta^4\,\mu\omega+\zeta^2(1+2\mu\omega)]\ln[\zeta\omega]+\zeta^4\omega\ln\{\zeta\omega[X_-+G_-(\omega)]\}$$
$$-\zeta^2\ln[\omega(-1+\lambda^{-1})]+4\ln[\zeta\omega\lambda^{-1}]+\left(\frac{\cos 2\theta_t}{\zeta^2\lambda\mu}\right)^{-1}\left(\zeta^{-2}\lambda^{-2}\cos 2\theta_t(\lambda-1)+\ln\left[\frac{\lambda-1}{\zeta\lambda}\right]\right),$$

$$D_3(\omega) = \frac{\pi}{4\omega\zeta^2\lambda\mu}\omega^2\cos 2\theta_t+\frac{\pi\mu\cos 2\theta_t}{\omega\zeta^2}(\mu-\omega)-\frac{\pi\omega}{4\zeta^2\mu}(\zeta^2+2\cos 2\theta_t)\ln\left[\frac{-\lambda(G_-(\omega)+\zeta X_-)}{-1+\lambda}\right]$$
$$+\frac{\pi\cos 2\theta_t}{4\omega\zeta^2\mu}\left(\zeta\omega^2X_-G_--2\omega^2\ln\left[\frac{\lambda(G_-(\omega)-\zeta X_-)}{1+\lambda}\right]\right),$$

$$G_\pm(\omega)=\sqrt{|X_\pm^2-1|},\quad \lambda=(1-\zeta^2)^{-1/2},\quad X_\pm=\frac{2\mu\pm\hbar\omega}{\hbar\zeta\omega}.$$

[1] V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea, and A. H. Castro Neto, Electron-electron interactions in graphene: Current status and perspectives, Rev. Mod. Phys. **84**, 1067 (2012).

[2] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and dirac semimetals in three-dimensional solids, Rev. Mod. Phys. **90**, 015001 (2018).

[3] E. Fradkin, *Field Theories of Condensed Matter Physics* (Cambridge University, Cambridge, England, 2003).

[4] A. K. Geim and K. S. Novoselov, The rise of graphene, Nat. Mater. **6**, 183 (2007).

[5] M. I. Katsnelson, *Graphene: Carbon in Two Dimensions* (Cambridge University, Cambridge, England, 2012).

[6] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (CRC Press, Boca Raton, FL, 1995).

[7] X. Peng and R. Ahuja, Symmetry breaking induced bandgap in epitaxial graphene layers on sic, Nano Lett. **8**, 4464 (2008).

[8] M. Yankowitz, Q. Ma, P. Jarillo-Herrero, and B. J. LeRoy, Van der Waals heterostructures combining graphene and hexagonal boron nitride, Nat. Rev. Phys. **1**, 112 (2019).

[9] M. O. Goerbig, J.-N. Fuchs, G. Montambaux, and F. Piéchon, Tilted anisotropic Dirac cones in quinoid-type graphene and α-(BEDT-TTF)$_2$I$_3$, Phys. Rev. B **78**, 045415 (2008).

[10] Y. Suzumura, I. Proskurin, and M. Ogata, Effect of tilting on the in-plane conductivity of Dirac electrons in organic conductor, J. Phys. Soc. Jpn. **83**, 023701 (2014).

[11] S. Katayama, A. Kobayashi, and Y. Suzumura, Pressure-induced zero-gap semiconducting state in organic conductor α-(BEDT-TTF)$_2$I$_3$, J. Phys. Soc. Jpn. **75**, 054705 (2006).

[12] X.-F. Zhou, X. Dong, A. R. Oganov, Q. Zhu, Y. Tian, and H.-T. Wang, Semimetallic Two-Dimensional Boron Allotrope with Massless Dirac Fermions, Phys. Rev. Lett. **112**, 085502 (2014).

[13] T. Farajollahpour, Z. Faraei, and S. A. Jafari, Solid-state platform for space-time engineering: The 8*Pmmn* borophene sheet, Phys. Rev. B **99**, 235150 (2019).

[14] The presence of mass m does not affect the metric structure. It only replaces the right-hand side of the above equation by $-(mv_F)^2$.

[15] S. A. Jafari, Electric field assisted amplification of magnetic fields in tilted Dirac cone systems, Phys. Rev. B **100**, 045144 (2019).

[16] Z. Jalali-Mola and S. A. Jafari, Polarization tensor for tilted Dirac fermion materials: Covariance in deformed Minkowski spacetime, Phys. Rev. B **100**, 075113 (2019).

[17] Z. Jalali-Mola and S. A. Jafari, Undamped transverse electric mode in undoped two-dimensional tilted Dirac cone materials, Phys. Rev. B **102**, 245148 (2020).

[18] Y. Yekta, H. Hadipour, and S. A. Jafari, How to tune the tilt of a Dirac cone by atomic manipulations? arXiv:2108.08183.

[19] T. Farajollahpour and S. A. Jafari, Synthetic non-Abelian gauge fields and gravitomagnetic effects in tilted Dirac cone systems, Phys. Rev. Research **2**, 023410 (2020).

[20] A. Mohajerani, Z. Faraei, and S. A. Jafari, Fast nuclear spin relaxation rates in tilted cone Weyl semimetals: Redshift factors from Korringa relation, J. Phys.: Condens. Matter **33**, 215603 (2021).

[21] Z. Faraei and S. A. Jafari, Perpendicular andreev reflection: Solid-state signature of black-hole horizon, Phys. Rev. B **100**, 245436 (2019).

[22] Z. Faraei and S. A. Jafari, Electrically charged andreev modes in two-dimensional tilted Dirac cone systems, Phys. Rev. B **101**, 214508 (2020).

[23] Z. Jalali-Mola and S. A. Jafari, Tilt-induced kink in the plasmon dispersion of two-dimensional Dirac electrons, Phys. Rev. B **98**, 195415 (2018).

[24] Z. Jalali-Mola and S. A. Jafari, Kinked plasmon dispersion in borophene-borophene and borophene-graphene double layers, Phys. Rev. B **98**, 235430 (2018).

[25] M. I. Katsnelson, Optical properties of graphene: The Fermi-liquid approach, Europhys. Lett. **84**, 37001 (2008).

[26] K. Kajita, Y. Nishio, N. Tajima, Y. Suzumura, and A. Kobayashi, Molecular Dirac fermion systems theoretical and experimental approaches, J. Phys. Soc. Jpn. **83**, 072002 (2014).

[27] S. H. Abedinpour, G. Vignale, A. Principi, M. Polini, W.-K. Tse, and A. H. MacDonald, Drude weight, plasmon dispersion, and ac conductivity in doped graphene sheets, Phys. Rev. B **84**, 045429 (2011).

[28] S. A. Mikhailov and K. Ziegler, New Electromagnetic Mode in Graphene, Phys. Rev. Lett. **99**, 016803 (2007).

[29] Y. Suzumura, I. Proskurin, and M. Ogata, Dynamical conductivity of Dirac electrons in organic conductors, J. Phys. Soc. Jpn. **83**, 094705 (2014).

[30] D. Pines and P. Noziéres, *Theory of Quantum Liquids: Normal Fermi Liquids* (CRC Press, Boca Raton, FL, 1966).

[31] These coefficients are denoted by A, B, C, D, etc. in Eq. (A2).

[32] A. Zee, *Quantum Field Theory in a Nutshell* (Princeton University, Princeton, NJ, 2010).

[33] N. M. R. Peres, T. Stauber, and A. H. C. Neto, The infrared conductivity of graphene on top of silicon oxide, Europhys. Lett. **84**, 38002 (2008).

[34] K. F. Mak, M. Y. Sfeir, Y. Wu, C. H. Lui, J. A. Misewich, and T. F. Heinz, Measurement of the Optical Conductivity of Graphene, Phys. Rev. Lett. **101**, 196405 (2008).