Quantum diffusion of massive Dirac fermions induced by symmetry breaking

Ting Zhang [1,2] Chushun Tian [3,*] and Ping Sheng [1,2]
[1]Department of Physics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
[2]Institute for Advanced Study, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
[3]CAS Key Laboratory of Theoretical Physics and Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

(Received 25 May 2020; revised 2 July 2021; accepted 26 July 2021; published 16 August 2021)

We show that when a two-dimensional (2D) Dirac fermion moves in disordered environments, the weak time-reversal symmetry breaking by a small mass gives rise to the diffusive wave propagation, i.e., that the wave-packet spread obeys the diffusive law of Einstein, up to a—practically inaccessible—exponentially large length. Strikingly, the diffusion constant is larger than that given by the Boltzmann kinetic theory, and grows unboundedly as the energy-to-mass ratio increases. This diffusive phenomenon is of quantum nature and different from weak antilocalization. It implies a new type of transport in topological insulators at zero temperature.

DOI: 10.1103/PhysRevB.104.075427

I. INTRODUCTION

A common wisdom brought by the discovery of localization in quantum disordered [1,2] and quantum chaotic [3,4] systems is that the diffusive law established by Einstein in 1905, i.e., the mean squared displacement grows linearly in time:

$$\langle \boldsymbol{r}^2 \rangle_t \sim Dt, \tag{1}$$

with D the diffusion constant, is not favored in the low-dimensional quantum world. The loss of the memory of particle's momentum due to collisions with scatterers can be remedied by various quantum ingredients, such as interference of quantum waves [5–7], quantum chaoticity of dynamics [8,9], and the exchange interaction between particles [10]. The memory recovery corrupts the foundation of Boltzmann kinetic theory for diffusion [11,12]. As a result, the linear scaling (1) is violated and the (normal) diffusion is suppressed. Should the temperature be finite, the quantum phase coherence is destroyed by thermal noises and diffusion can thus appear [13].

However, recent progresses achieved in very different research areas, ranging from quantum transport of superconducting films [14,15] to wave-packet dynamics of 2D quantum chaotic systems [16–18], have posed a fundamental challenge for the common wisdom. In particular, by using the field theory and the mathematical spectral theory, it is established that, when some canonical quantum chaotic systems are endowed with spin, the diffusive law (1) can persist in the entire course of wave-packet propagation at the critical point of topological phase transitions [16–18]. It is thus suggested that the arising of the irreversible diffusion can go far beyond the canonical Einstein-Boltzmann paradigm and may have novel quantum origin in the presence of spin.

In reality an important spin system is the 2D massive Dirac fermion, described by a two-component spinor $\psi(\boldsymbol{r}, t)$, which propagates according to the $(2+1)$D Dirac equation. It models a number of electronic materials including anomalous Hall systems [19], graphene in the spin-$\uparrow$ K valley gapped by a spin-orbit interaction [20]; and magnetically doped surface states of topological insulators [21–23]. These Dirac fermions carry rich spin properties [24]. It is natural to expect that when they move in a disordered environment, strong interplay between their spin properties and multiple random wave scattering may give rise to rich wave propagation phenomena. In particular, whether the diffusive law (1) emerges—at zero temperature—is of fundamental interest, and may find practical applications. It has been a long-term interest to generalize the Boltzmann equation to investigate the interplay between diffusion and various wave effects [25,26]. Recent nonperturbative studies [11,12,16–18] have suggested that to address the emergence of quantum diffusion it is crucial to go beyond traditional ladder and maximally crossing diagrams [2,5–8,21]. This turns out to be a highly nontrivial task, as general disordered Dirac systems are concerned. To the best of our knowledge, such task has been undertaken only for the limiting massless case [27]. But in that case, instead of Eq. (1), a superdiffusive propagation $\langle \boldsymbol{r}^2 \rangle_t \sim t \ln t$ was found, leading to topological metallic behaviors [28,29].

The above implies that the behaviors of massless Dirac materials and electronic materials with spin-orbit interaction [30,31] in disordered environments are completely different in the scaling characteristic of their conductance. This is the case even though both systems are in the same sympletic symmetry class and the same dimension (2D). The difference is especially striking in that for the former system the scaling law is found to be one-loop type even in the nonperturbative regime [27–29], and thus no localization transition occurs; this is in sharp contrast to the presence of localization transition in the latter system [31]. A problem arising thereby is what happens to massive Dirac particles, which no longer belongs to the sympletic symmetry class; this is the problem addressed in the present paper.

*ct@itp.ac.cn

II. SUMMARY OF MAIN RESULTS AND PHYSICAL PICTURE

In this paper we formulate a density response theory for 2D Dirac fermions of mass m moving in a disordered scalar potential, which allows us to address the propagation of wave packet. We focus on the case where the particle energy ε satisfies $\varepsilon/m \gg 1$ and $\varepsilon\tau \gg 1$, with τ the elastic scattering time, which is order of the Boltzmann collision time. (Both the velocity parameter and the Planck constant $\hbar$ are set to unity.) The two inequalities quantify the conditions of small mass and weak disorder, respectively. We find that, at zero temperature, quantum diffusion occurs at the time scale $\tau_m \equiv (\frac{\varepsilon}{m})^2\frac{\tau}{2}$, much larger than the time scale τ when the Boltzmann kinetic theory applies. Specifically, fluctuations in the particle number density relax according to the diffusion equation, that gives rise to the scaling law (1); however, the diffusion constant D is determined by a highly nonlinear equation:

$$\frac{D_0}{D} = 1 - \frac{1}{2\pi^2\nu}\int_0^{\frac{1}{\tau}} dQ Q\frac{1}{\tau_m^{-1} + DQ^2}, \qquad (2)$$

where the second term on the right hand side is of quantum origin, with ν the local density of state. It exceeds the Boltzmann diffusion constant $D_0 = \tau$, and grows unboundedly with the ratio $\frac{\varepsilon}{m}$. In two limiting regimes: (I) $\frac{\varepsilon}{m} \ll e^{\pi\varepsilon\tau}$ and (II) $\frac{\varepsilon}{m} \gg e^{\pi\varepsilon\tau}$, the explicit analytical expression of D reads:

$$D = \begin{cases} D_0 + \frac{1}{2\pi^2\nu}\ln\frac{\varepsilon}{\sqrt{2}m}, & \text{for regime I ;} \\ \frac{1}{2\pi^2\nu}\ln\frac{\varepsilon}{\sqrt{2}m}, & \text{for regime II.} \end{cases} \qquad (3)$$

In the regime I, the quantum term $\frac{1}{2\pi^2\nu}\ln\frac{\varepsilon}{\sqrt{2}m}$ is $\ll D_0$ implying weak quantum diffusion; in the regime II, it is $\gg D_0$ implying strong quantum diffusion. The quantum term is completely determined by ν and ε/m, independent of the disorder parameter τ. Interestingly, it is reproduced when one replaces the logarithm $\ln\frac{L}{\tau}$ (with L being a length scale) in the conductivity for $m = 0$ [27–29] by $\ln\sqrt{\frac{\tau_m}{\tau}}$ (and uses the Einstein relation between the conductivity and the diffusion constant). Note that the one-loop weak antilocalization correction [30,32,33] corresponds only to the first line of Eq. (3), but not to the second. The latter is for the regime where the "correction" is $\gg D_0$, i.e., beyond the expected validity of the one-loop calculation as shown in Fig. 1, where quantum diffusion is shown to exist at $D \gg D_0$. We further show that the length scale to develop unitary class localization is exponentially large in $(\nu D)^2$, and thus all localization effects are invisible in practice. In contrast, the results shown in this paper are in the relevant length scales that are experimentally accessible.

Now we explain in a pictorial way that a small but nonvanishing m is the key to the emergence of quantum diffusion. First of all, when the particle moves in a disordered environment, random scattering by impurities renders the memory of momentum lost at the time scale τ, like in the canonical Einstein-Boltzmann paradigm. However, at longer times the memory gets recovered by constructive interference between different propagating paths of quantum waves. In combination with the helicity, that introduces strong spin-momentum locking, the memory recovery enhances the relaxation time of momentum and renormalizes τ: The more the memory is recovered, the slower the momentum relaxes. Then, as we will implement by a systematic analytical theory below, it turns out that on one hand the quantum interference rests on system's invariance under some time-reversal operation $\hat{T}$, while on the other hand this $\hat{T}$ symmetry is weakly broken by small m. As a result, the constructive interference and the ensuing memory recovery can persist only up to some time scale, which is τ_m. After that the particle undergoes random scattering again. So at the time scale of τ_m the wave-packet propagation is diffusive, but with the diffusion constant enhanced from D_0 by the memory recovery (Fig. 1).

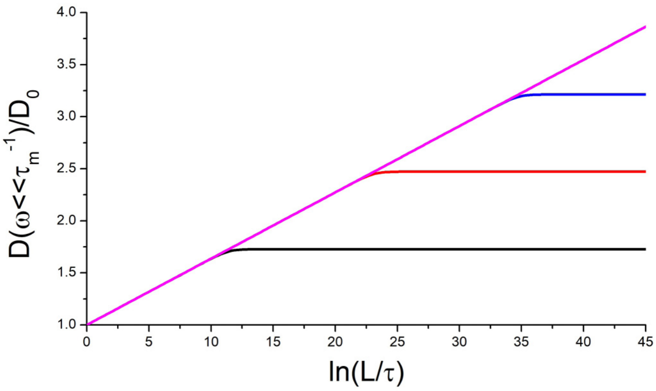

FIG. 1. Solving Eq. (19) numerically shows that as the length scale increases, the low-frequency diffusion constant increases from the Boltzmann value D_0 and levels off at a larger value—the quantum diffusion constant D obeying Eq. (2). Here $\varepsilon\tau = 5$ and from the bottom to the top $\varepsilon/m = 10^5, 10^{10}, 10^{15}, +\infty$.

III. OUTLINE OF ANALYTICAL THEORY

Now we outline the analytical derivations. The complete theory is given in the supplemental material [34] written in a self-contained and article style. The quantum wave propagates according to $\partial_t\psi = \hat{H}\psi$, $\hat{H} \equiv \boldsymbol{\sigma}\cdot\boldsymbol{p} + m\sigma^z + V(\boldsymbol{r})$ with $\boldsymbol{\sigma} \equiv (\sigma^x, \sigma^y)$, where $\sigma^{x,y,z}$ are the Pauli matrices and $\boldsymbol{p} \equiv -i\nabla$, $\boldsymbol{r}$ are the momentum and the position operator, respectively. The disordered potential $V(\boldsymbol{r})$ has a zero mean everywhere, and its fluctuations are spatially independent, i.e., $\langle V(\boldsymbol{r})V(\boldsymbol{r}')\rangle = U_0\delta(\boldsymbol{r}-\boldsymbol{r}')$ with $\langle\cdot\rangle$ denoting the average over disorder configurations. Here U_0 is the disorder strength and can be expressed as $U_0 = 1/(\pi\nu\tau)$. Note that U_0, ε, m are renormalized at short scales [19]. But the renormalized values enter into the large-scale physics discussed below merely as parameters. So we will not discuss this further.

Introducing the time-reversal operation $\hat{T} := -i\sigma^y\hat{C}$, where $\hat{C}$ stands for the complex conjugation and applying it to $\hat{H}$, we find that

$$for\, m = 0:\ \hat{T}\hat{H}\hat{T}^{-1} = \hat{H};$$
$$for\, m \neq 0:\ \hat{T}\hat{H}\hat{T}^{-1} \neq \hat{H}. \qquad (4)$$

That is, for $m = 0$ the system has the time-reversal symmetry, otherwise the symmetry is broken. For $m \ll \varepsilon$, in which we are interested, the breaking is weak, and the field theory developed for strong $\hat{T}$-symmetry breaking (i.e., the unitary class) [35] does not apply.

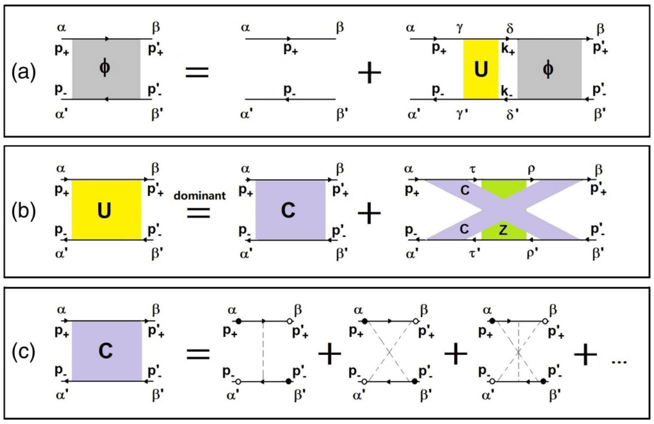

FIG. 2. The diagrammatical representation of the Bethe-Salpeter equation for the response function ϕ (a); the dominant contributions to the two-particle irreducible vertex function U (b); and the singlet cooperon C (c). In (b) the diagrammatical structure of Z is arbitrary. In (c) the solid (respectively empty) circles stand for that two end spins are paired into a singlet state.

The program to be executed below is in spirit parallel to the theory for localization of spinless particles [6,7]. However, some key steps are renovated by generalizing the treatments developed for massless Dirac fermions [27] to the massive case. Most importantly, the final results have totally opposite physical implications. Define the retarded (advanced) 2×2 matrix Green function as $G_\varepsilon^{R(A)} := 1/(\varepsilon - \hat{H} \pm i\delta)$, with δ a positive infinitesimal. Then the motion of Dirac fermions over large length and time scales can be characterized by the response function,

$$\sum_{\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q}} e^{i(\boldsymbol{p}_+\cdot\boldsymbol{r}_+ - \boldsymbol{p}'_+\cdot\boldsymbol{r}'_+ + \boldsymbol{p}'_-\cdot\boldsymbol{r}'_- - \boldsymbol{p}_-\cdot\boldsymbol{r}_-)}\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}(\boldsymbol{q},\omega) := -\frac{1}{2\pi i}\langle (G^R_{\varepsilon_+}(\boldsymbol{r}_+,\boldsymbol{r}'_+))_{\alpha\beta}(G^A_{\varepsilon_-}(\boldsymbol{r}'_-,\boldsymbol{r}_-))_{\beta'\alpha'}\rangle. \quad (5)$$

Here $\boldsymbol{p}_\pm = \boldsymbol{p} \pm \frac{\boldsymbol{q}}{2}$, $\boldsymbol{p}'_\pm = \boldsymbol{p}' \pm \frac{\boldsymbol{q}}{2}$, $\varepsilon_\pm = \varepsilon \pm \frac{\omega}{2}$, and $\alpha, \beta, \ldots$ are spin indices to which the Einstein summation convention applies below. Upon performing the disorder averaging, the translational invariance is restored, and the right-hand side depends only on three independent coordinates: $\frac{\boldsymbol{r}_+ + \boldsymbol{r}_-}{2} - \frac{\boldsymbol{r}'_+ + \boldsymbol{r}'_-}{2}$, $\boldsymbol{r}_+ - \boldsymbol{r}_-$, $\boldsymbol{r}'_- - \boldsymbol{r}'_+$. The Fourier wavenumbers, $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}'$, respectively, conjugate to them.

When we expand $G_\varepsilon^{R,A}$ in V and perform the disorder averaging, each term of $\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}$ is mapped onto a specific diagram. As shown in Fig. 2(a), the backbone of each diagram consists of two particle lines: the upper (lower) particle line corresponds to $G^{R(A)}$. The building blocks of each diagram are the free Green functions $\mathcal{G}_\varepsilon^{R(A)}(\boldsymbol{r}-\boldsymbol{r}') := \langle G_\varepsilon^{R(A)}(\boldsymbol{r},\boldsymbol{r}')\rangle$, represented by a solid line going rightwards (leftwards), and the disorder scattering $U_0\delta(\boldsymbol{r}-\boldsymbol{r}')$, represented by a dashed line. In the Fourier representation, $\mathcal{G}_\varepsilon^{R(A)}$ has the general form: $\mathcal{G}_\varepsilon^{R(A)}(\boldsymbol{p}) = (\varepsilon - \boldsymbol{\sigma}\cdot\boldsymbol{p} - m\sigma^z - \Sigma_\varepsilon^{R(A)}(\boldsymbol{p}))^{-1}$, where $\Sigma_\varepsilon^{R(A)}$ is the self-energy. All diagrams of the response function can be organized in the way shown by Fig. 2(a), which is described by the Bethe-Salpeter equation:

$$\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}(\boldsymbol{q},\omega) = (\mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}_+))_{\alpha\gamma}(\mathcal{G}^A_{\varepsilon_-}(\boldsymbol{p}_-))_{\gamma'\alpha'}\Bigg(-\frac{\delta_{\boldsymbol{p}\boldsymbol{p}'}\delta_{\gamma\beta}\delta_{\beta'\gamma'}}{2\pi i} + \sum_{\boldsymbol{k}} U^{\boldsymbol{p}\boldsymbol{k}}_{\gamma\delta,\delta'\gamma'}(\boldsymbol{q},\omega)\phi^{\boldsymbol{k}\boldsymbol{p}'}_{\delta\beta,\beta'\delta'}(\boldsymbol{q},\omega)\Bigg). \quad (6)$$

Here the kernel U is a two-particle irreducible vertex function. As exemplified by Fig. 2(c), each diagram of U has ends joined by the disorder scattering line, and cannot be divided into disconnected parts through cutting the upper and the lower particle line simultaneously. Furthermore, by adapting the method of Ref. [36] it can be shown that U obeys the Ward identity:

$$\sum_{\boldsymbol{p}} (\delta\mathcal{G}_\varepsilon(\boldsymbol{p}))_{\gamma'\gamma} U^{\boldsymbol{p}\boldsymbol{k}}_{\gamma\delta,\delta'\gamma'} = (\delta\Sigma_\varepsilon(\boldsymbol{k}))_{\delta'\delta}, \quad (7)$$

where $\delta\mathcal{G}_\varepsilon(\boldsymbol{p}) := \mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}_+) - \mathcal{G}^A_\varepsilon(\boldsymbol{p}_-)$ and $\delta\Sigma_\varepsilon(\boldsymbol{p})$ is defined in the similar way. Equations (6) and (7) are rigorous, laying down a foundation for the analysis of the density response of massive Dirac fermions.

Multiplying both sides of Eq. (6) by the inverse of the matrix $\mathcal{G}^R$, we obtain

$$\big(\varepsilon_+ - \boldsymbol{\sigma}\cdot\boldsymbol{p}_+ - m\sigma^z - \Sigma^R_{\varepsilon_+}(\boldsymbol{p}_+)\big)_{\gamma\alpha}\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'} = \big(\mathcal{G}^A_{\varepsilon_-}(\boldsymbol{p}_-)\big)_{\gamma'\alpha'}\Bigg(-\frac{\delta_{\boldsymbol{p}\boldsymbol{p}'}\delta_{\gamma\beta}\delta_{\beta'\gamma'}}{2\pi i} + \sum_{\boldsymbol{k}} U^{\boldsymbol{p}\boldsymbol{k}}_{\gamma\delta,\delta'\gamma'}\phi^{\boldsymbol{k}\boldsymbol{p}'}_{\delta\beta,\beta'\delta'}\Bigg), \quad (8)$$

and similarly, we have

$$\big(\varepsilon_- - \boldsymbol{\sigma}\cdot\boldsymbol{p}_- - m\sigma^z - \Sigma^A_{\varepsilon_-}(\boldsymbol{p}_-)\big)_{\alpha'\gamma'}\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'} = \big(\mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}_+)\big)_{\alpha\gamma}\Bigg(-\frac{\delta_{\boldsymbol{p}\boldsymbol{p}'}\delta_{\gamma\beta}\delta_{\beta'\gamma'}}{2\pi i} + \sum_{\boldsymbol{k}} U^{\boldsymbol{p}\boldsymbol{k}}_{\gamma\delta,\delta'\gamma'}\phi^{\boldsymbol{k}\boldsymbol{p}'}_{\delta\beta,\beta'\delta'}\Bigg), \quad (9)$$

where the arguments $\boldsymbol{q}, \omega$ are suppressed in order to make the formulas compact. Let us set $\beta = \beta'$ in both equations, set $\gamma = \alpha'$ in the first equation and $\gamma' = \alpha$ in the second, and sum up the spin indices and the momenta. Subtracting the two equations thereby obtained and using Eq. (7), we obtain the macroscopic equation describing the particle number conservation:

$$-i\omega\phi_0(\boldsymbol{q},\omega) + i\boldsymbol{q}\cdot\boldsymbol{\phi}_j(\boldsymbol{q},\omega) = i\nu. \quad (10)$$

Here $i\nu$ is the source. ϕ_0 and $\boldsymbol{\phi}_j$ are the density and the current relaxation function, respectively, whose microscopic expressions are

$$\phi_0 = \sum_{\boldsymbol{p},\boldsymbol{p}'}\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta\alpha}, \quad \boldsymbol{\phi}_j = \sum_{\boldsymbol{p},\boldsymbol{p}'}\boldsymbol{\sigma}_{\alpha'\alpha}\phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta\alpha'}. \quad (11)$$

It should be emphasized that Eqs. (10) and (11) are exact, irrespective of the disorder strength, i.e., $\varepsilon\tau$. That they follow from Eqs. (8) and (9) is in spirit similar to that hydrodynamic equations follow from the Boltzmann kinetic equation. Should an additional relation between ϕ_0 and $\boldsymbol{\phi}_j$ exist, then the macroscopic equation (10) is closed.

Now we establish such a relation for $\varepsilon\tau \gg 1$. Note that the diagrams dominating $\mathcal{G}_\varepsilon^{R(A)}(\boldsymbol{p})$ have a rainbow-like structure (the self-consistent Bonn approximation). Their sum gives $\mathrm{Im}\Sigma_\varepsilon^{R(A)}(\boldsymbol{p}) = \mp\frac{1}{2\tau}$. (The real part is unimportant and ignored.) To calculate the microscopic expression of $\boldsymbol{\phi}_j$, we multiply Eqs. (8) and (9) by the matrix elements of $\boldsymbol{\sigma}$, sum up spin and momentum indices, and subtract the two equations obtained thereby. With the substitution of the following expansion

$$\sum_{\boldsymbol{p}'} \phi^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta\alpha'} = -\frac{1}{2\pi i\nu}(\delta\mathcal{G}_\varepsilon(\boldsymbol{p}))_{\alpha\alpha'}\phi_0 + \frac{1}{\pi\nu\tau}(\mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}_+)\boldsymbol{\sigma}\mathcal{G}^A_{\varepsilon_-}(\boldsymbol{p}_-))_{\alpha\alpha'}\cdot\boldsymbol{\phi}_j, \quad (12)$$

we obtain (the group velocity is ≈1 for $m/\varepsilon \ll 1$)

$$\boldsymbol{\phi}_j(\boldsymbol{q},\omega) = -i\boldsymbol{q}D(\omega)\phi_0(\boldsymbol{q},\omega),\ D(\omega) = \frac{1}{-i\omega+\gamma(\omega)}, \quad (13)$$

and the microscopic expression of $\gamma(\omega)$:

$$\gamma(\omega) = \frac{1}{\tau}\Bigg(1 - \frac{1}{\pi\nu\tau}\sum_{\boldsymbol{p},\boldsymbol{p}'}(\mathcal{G}^A_{\varepsilon_-}(\boldsymbol{p})\hat{\boldsymbol{q}}\cdot\boldsymbol{\sigma}\mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}))_{\alpha'\alpha} \times U^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}(\boldsymbol{q},\omega)(\mathcal{G}^R_{\varepsilon_+}(\boldsymbol{p}')\hat{\boldsymbol{q}}\cdot\boldsymbol{\sigma}\mathcal{G}^A_{\varepsilon_-}(\boldsymbol{p}'))_{\beta\beta'}\Bigg). \quad (14)$$

Equation (14) implies that τ is renormalized and the ω dependence implies a retarded effect. Substituting Eq. (13) into Eq. (10) gives

$$\phi_0(\boldsymbol{q},\omega) = \frac{i\nu}{-i\omega + D(\omega)\boldsymbol{q}^2}. \quad (15)$$

It shows that ϕ_0 has a diffusive pole. Physically, it implies that a number density fluctuation excited locally relaxes according to a diffusion-like equation. It differs from the normal diffusion equation in that the diffusion constant $D(\omega)$, as shown by its microscopic expression Eqs. (13) and (14), depends generally on ω. (In principle, it also depends on $\boldsymbol{q}$, but this plays no role for $\boldsymbol{q}\to 0$ in this work.) Such dependence accounts for the memory recovery developed in the course of propagation. Whenever $D(\omega)$ is independent of ω and the length scale L, the normal diffusion equation and thus Eq. (1) follows.

As a simple application of the general theory, we ignore the second term in Eq. (14), obtaining $\gamma = \frac{1}{\tau}$. So $D(\omega) = \tau \equiv D_0$ for $\omega \ll \gamma$. This result can also be derived by summing up all the ladder diagrams of ϕ_0, and the sum is thus called "diffuson". Alternatively, it can be obtained by generalizing the Boltzmann kinetic theory developed for spinless disordered Hamiltonians [11,12].

However, this result cannot be extended to arbitrarily small ω or equivalently arbitrarily large L, for which we need to consider diagrams beyond the first order in U_0 and, in particular, those giving rise to singular contributions to U. Let us sum up the maximally crossing diagrams shown in Fig. 2(c), obtaining: $U^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'} = \frac{\pi\nu U_0^2}{-i\omega+\tau_m^{-1}+D_0(\boldsymbol{p}+\boldsymbol{p}')^2}\Psi^0_{\alpha\beta'}(\Psi^0)^*_{\beta\alpha'} \equiv C^{\boldsymbol{p}+\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}(\omega)$. In the presence of the $\hat{T}$ symmetry, $m = 0$ and τ_m^{-1} vanishes. So C has a diffusive pole: It is singular at $\boldsymbol{p} \approx -\boldsymbol{p}'$, i.e., $\boldsymbol{Q} \equiv \boldsymbol{p}+\boldsymbol{p}' \approx 0$, and is called "cooperon". When the symmetry is broken, τ_m is finite, which was observed in Ref. [32] and may be regarded as the lifetime of the cooperon. Here Ψ^0 is a projector. $\Psi^0_{\alpha\beta'}$ implies that the spins with indices α, β' form a singlet pair, and so does $(\Psi^0)^*_{\beta\alpha'}$. In principle, there are triplet contributions to U; however, for $\varepsilon/m \gg 1$ they do not display any singularities and can be ignored. With the substitution of U into Eq. (14), we obtain the leading quantum correction to D_0, denoted as δD_1:

$$\frac{\delta D_1}{D_0} = \frac{1}{\pi\nu}\int_{Q<\frac{1}{\tau}}\frac{d\boldsymbol{Q}}{(2\pi)^2}\frac{1}{-i\omega+\tau_m^{-1}+D_0\boldsymbol{Q}^2}. \quad (16)$$

It holds for $|\frac{\delta D_1}{D_0}| \ll 1$, i.e., $\tau\max(\omega,\tau_m^{-1}) \gg e^{-4\pi^2\nu D_0}$. Provided $\omega\tau_m \ll 1$ and $L \gg \sqrt{D_0\tau_m}$, δD_1 is independent of ω, L and reduces to the first line of Eq. (3), giving rise to the weak quantum diffusion. Due to $\nu D_0 = \frac{\varepsilon\tau}{2\pi}$ the inequality $\frac{\tau}{\tau_m} \gg e^{-4\pi^2\nu D_0}$ gives the condition in the introduction that defines the regime I. Equation (16) differs from the well-known weak antilocalization [30] in the appearance of τ_m^{-1}.

For smaller τ_m^{-1}, namely, smaller ω or larger L, we need to go beyond the perturbative cooperon contributions. To perform such a nonperturbative analysis we note that, similar to the spinless case [7], the most singular contributions to U have the diagrammatical structure as shown in Fig. 2(b). There, two singlet cooperons cross an arbitrary diagram (e.g. an infinite series of cooperons) denoted as Z. This gives an expression for the dominant Bethe-Salpeter kernel:

$$U^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'} \overset{\text{dominant}}{\longrightarrow} C^{\boldsymbol{p}+\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}(\omega) + C^{\boldsymbol{p}+\boldsymbol{p}'}_{\alpha\tau,\beta'\rho'}(\omega)Z^{\boldsymbol{p}+\boldsymbol{p}'}_{\tau\rho,\rho'\tau'}(\omega)C^{\boldsymbol{p}+\boldsymbol{p}'}_{\rho\beta,\tau'\alpha'}(\omega). \quad (17)$$

For $m = 0$ it has been shown [27] that $\Psi^0_{\beta'\alpha}U^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'}\Psi^0_{\alpha'\beta} = i\pi U_0^2\phi_0(\boldsymbol{p}+\boldsymbol{p}',\omega)$ with ϕ_0 given by Eq. (15). Intuitively, this identity reflects a reciprocal relation resulting from the $\hat{T}$ symmetry, i.e., when the lower particle line in Fig. 2(b) is rotated so that it goes in the same direction as the upper particle line, the diagrams representing the right-hand side of Eq. (17) are converted into those representing $U_0^2\phi_0(\boldsymbol{p}+\boldsymbol{p}',\omega)$. For $m \ll \varepsilon$ the $\hat{T}$ symmetry is broken only weakly. As a result, the reciprocal relation remains valid, except that similar to the difference between the diffuson and the cooperon, the symmetry breaking term τ_m^{-1} is added to the diffusive pole carried by dominant U. Taking this into account, we obtain

$$U^{\boldsymbol{p}\boldsymbol{p}'}_{\alpha\beta,\beta'\alpha'} \overset{\text{dominant}}{\longrightarrow} \frac{\pi\nu U_0^2}{-i\omega+\tau_m^{-1}+D(\omega)(\boldsymbol{p}+\boldsymbol{p}')^2}\Psi^0_{\alpha\beta'}(\Psi^0)^*_{\beta\alpha'}. \quad (18)$$

Substituting it into Eqs. (13) and (14) gives

$$\frac{D_0}{D(\omega)} = 1 - \frac{1}{\pi\nu}\int_{Q<\frac{1}{\tau}}\frac{d\boldsymbol{Q}}{(2\pi)^2}\frac{1}{-i\omega+\tau_m^{-1}+D(\omega)\boldsymbol{Q}^2}. \quad (19)$$

This result differs crucially from the self-consistent equation, that describes localization when the breaking of time-reversal symmetry is weak, in the sign of the second term [6]. It cannot be obtained by the nonperturbative field theory for massive Dirac fermions [35] where all cooperon contributions vanish. For low frequencies $\omega\tau_m \ll 1$ one may ignore the ω term. So

$D(\omega)$ is independent of ω, but depends on the length scale L in general. Solving the equation numerically we obtain Fig. 1. We see that as L increases the low-frequency $D(\omega)$ increases from D_0, and levels off at a value, which gives the quantum diffusion constant D. To find an analytic form of the latter, we note that for $L \gg \sqrt{D\tau_m}$ Eq. (19) reduces to Eq. (2). From Eq. (2) we reproduce for the regime I the first line of Eq. (3); this was obtained before from Eq. (16), which is perturbative and corresponds to replacing $D(\omega)$ on the right-hand side of Eq. (19) by D_0. Most importantly, from Eq. (2) we see that $D \gg D_0$ in the regime II; in this case solving Eq. (2) up to the logarithmic accuracy, we obtain the second line of Eq. (3).

IV. INTERPLAY WITH LOCALIZATION

Consider a path representing a quantum amplitude, which moves diffusively at length scale of $\sqrt{D\tau_m}$. Suppose that during this diffusive motion the path self-intersects twice with two loops formed. Then, another quantum amplitude can pass the two loops in different order, and pass each loop along the same direction as the former amplitude. These two paths have the same phase and thus constructively interfere with each other. They give an interference correction to D (not D_0), denoted as δD_2, which is immune to even strong $\hat{T}$-symmetry breaking and cannot be described by the theory developed above. It can be calculated by the field-theoretical approach [17], which is $\frac{\delta D_2}{D} = -\frac{1}{2\pi^2(\nu D)^2} \ln \frac{L}{\sqrt{D\tau_m}}$, a unitary-class weak localization correction to D. δD_2 and D are comparable for $L \sim \sqrt{D\tau_m} e^{2\pi^2(\nu D)^2}$, where the quantum diffusion crosses over to the unitary-class localization. Our findings thus show neither the scaling law for various sympletic class systems [27–29,31] nor that for unitary class systems [31] applies to the present system. It is even not clear to us whether some generalization of the well-known single parameter scaling theory of Anderson localization [37] may exist, since the present system is in the crossover from the sympletic to the unitary class.

V. IMPLICATIONS FOR TOPOLOGICAL INSULATORS

Our findings imply an exotic quantum transport phenomenon in topological insulators. Consider a 3D topological insulator on the substrate of a ferromagnet. The surface electronic states of the topological insulator are described by the 2D massive Dirac equation, with the mass term arising from the Zeeman splitting. Then, by the Einstein relation, the second line of Eq. (3) implies that at zero temperature, as the sample size increases, the surface electron conductance increases from the Drude value and levels off at a value larger than the Drude conductance, as shown in Fig. 1. Finally, we note that the exchange interaction between electrons can give rise to an Altshuler-Aronov type correction [10], which might not be negligible in real experiments on topological insulators [38]. We leave the interplay between such kind of interaction corrections and presently found quantum diffusion for future studies.

ACKNOWLEDGMENTS

Support by Hong Kong Research Grant Council Projects 16307114, 16300818, and by National Natural Science Foundation China Projects 11925507, 12047503, is acknowledged. We wish to thank Yayu Wang for helpful discussions.

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