

Lieb-Schultz-Mattis type theorems for Majorana models with discrete symmetriesÖmer M. Aksoy,¹ Apoorv Tiwari ^{1,2} and Christopher Mudry ^{1,3}¹*Condensed Matter Theory Group, Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland*²*Department of Physics, University of Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland*³*Institut de Physique, EPF Lausanne, Lausanne, CH-1015, Switzerland*

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We prove two Lieb-Schultz-Mattis type theorems that apply to any translationally invariant and local fermionic d -dimensional lattice Hamiltonian for which fermion-number conservation is broken down to the conservation of fermion parity. We show that when the internal symmetry group G_f is realized locally (in a repeat unit cell of the lattice) by a nontrivial projective representation, then the ground state cannot be simultaneously nondegenerate, symmetric (with respect to lattice translations and G_f), and gapped. We also show that when the repeat unit cell hosts an odd number of Majorana degrees of freedom and the cardinality of the lattice is even, then the ground state cannot be simultaneously nondegenerate, gapped, and translation symmetric.

DOI: [10.1103/PhysRevB.104.075146](https://doi.org/10.1103/PhysRevB.104.075146)**I. INTRODUCTION**

In 1961, Lieb, Schultz, and Mattis (LSM) proved a theorem on the low-lying excited states of the nearest-neighbor antiferromagnetic quantum spin-1/2 chain [1]. Accordingly, a quantum spin chain with an odd number of spin-1/2 degrees of freedom per repeat unit cell that is simultaneously translation and $SO(3)$ -spin-rotation symmetric cannot realize a gapped and symmetric ground state in the thermodynamic limit.

Since its original formulation, the LSM theorem has been generalized in a number of ways including extensions to higher dimensions, other global continuous symmetry groups, and different kinds of spatial symmetries [2–21]. For instance, it has been understood that the $SO(3)$ -spin-rotation symmetry is not an essential requirement for an LSM constraint. In fact, LSM-type theorems for $U(1)$ -number-conserving Hamiltonians have been established in arbitrary dimensions. These theorems state that systems with noninteger filling fraction ν , defined as the average number of particles per unit cell, cannot have a translationally invariant, nondegenerate, and short-range entangled ground state [2–6,8,9,16]. Similar constraints have also been worked out for number-conserving Hamiltonians that have nonsymmorphic or magnetic space group symmetries [10–12,15].

A number of LSM-type theorems pertaining to discrete internal symmetries combined with crystallographic symmetries have also been worked out [11,12,15,22–27]. In the context of spin chains with discrete symmetries, LSM-type theorems were proved by Ogata *et al.* in Refs. [26–28]. They found that a translationally invariant spin chain with half-integer spin at each site that possesses either time-reversal or $\mathbb{Z}_2 \times \mathbb{Z}_2$ -rotation symmetries (rotations by π around two axis, say x and z), cannot have a nondegenerate, gapped, and symmetric ground state. The proof of this statement is based on the fact that a nondegenerate gapped ground state of a

local Hamiltonian satisfies the so called *split property* [29,30]. LSM-type no-go theorems are then derived for states satisfying the split property by using operator algebra techniques and Gelfand–Naimark–Segal (GNS) construction. Alternatively, similar no-go constraints are obtained in Refs. [31,32] within the framework of matrix product states (MPS) [33–35]. These derivations are based on the fact that one can approximate [36] the nondegenerate gapped ground states of local Hamiltonians by injective MPS [37,38]. There are two hypothesis common to many of these LSM-type theorems with discrete symmetries. It is presumed that there exist local (on-site) degrees of freedom that span a local Hilbert space and realize a nontrivial projective representation of the global symmetry. It is also presumed that the global Hilbert space is obtained by postulating that the local degrees of freedom commute when separated in space. The resulting LSM-type theorems are applicable to bosonic systems. Generalizations to bosonic quantum systems in arbitrary dimension with crystallographic symmetries and general discrete Abelian symmetries have been proposed using the notion of lattice homotopy [24].

There exists similarities between Hamiltonians obeying LSM-type constraints and the boundary modes of a symmetry-protected-topological (SPT) insulator. An SPT insulator has a nondegenerate, symmetric, and gapped ground state when periodic boundary conditions are imposed. When open boundary conditions are imposed, the effective low-energy quantum Hamiltonian governing the dynamics of the boundary modes of an SPT insulator supports a ground state that is either (i) gapless, (ii) symmetry-broken, (iii) or topologically ordered if the boundary is no less than two dimensional [22,39–43]. From a low-energy perspective, both the effective boundary Hamiltonian of an SPT insulator and the bulk Hamiltonian satisfying an LSM-type constraint display quantum anomalies [17,44,45]. For the former case, the quantum anomaly is typically that for a global symmetry that acts locally on the boundary [46–50]. In the latter case, there is a

mixed quantum anomaly between a global symmetry that acts locally and a spatial symmetry such as translation. These parallels led to the formulation of so-called weak-SPT-LSM-type theorems [18,23,44,51,52]. In particular, Ref. [51] has conjectured an LSM-type constraint for d -dimensional fermionic lattice Hamiltonians with the help of a bulk-boundary correspondence. The fermionic d -dimensional lattice Hamiltonian is interpreted as the low-energy effective theory of a fermionic $d + 1$ -dimensional lattice Hamiltonian that is gapped but supports midgap boundary states such that (i) they can be localized at each site of the d -dimensional lattice (ii) where they span a local fermionic Fock space. The parent fermionic $(d + 1)$ -dimensional lattice Hamiltonian is an example of a weak-SPT fermionic insulator. LSM-like constraints on the ground states of the d -dimensional lattice Hamiltonian are inherited from the symmetries that protect the boundary states of the $(d + 1)$ -dimensional lattice Hamiltonian.

As compared to LSM-type theorems for bosonic and $U(1)$ -charge-conserving fermionic Hamiltonians, LSM-type theorems for fermionic Hamiltonians without any $U(1)$ -conserving symmetries are much less explored [49,51]. These LSM-type constraints would be relevant for any long-range superconducting order with fully broken $SU(2)$ -spin-rotation that coexists with some additional long-range order. Such fermionic Hamiltonians always admit a formulation in terms of Majorana degrees of freedom. To the best of our knowledge, there are no proofs of LSM-type constraints relevant to fermionic lattice Hamiltonians with translation symmetry and some discrete internal symmetry (such as time-reversal symmetry, say) for which fermion-number conservation is broken down to the conservation of fermion parity.

In the present work, we state and prove two LSM-type theorems. They apply to translationally invariant lattice Hamiltonians acting on a fermionic Fock space spanned by Majorana degrees of freedom. The lattice is embedded in d -dimensional Euclidean space. For Theorem 1, there also exists a global symmetry associated to a symmetry group G_f that can be realized locally, i.e., the number of Majorana degrees of freedom in a repeat unit cell of the lattice is even. We prove within the framework of fermionic MPS (FMPS) that whenever the Majorana degrees of freedom within a single repeat unit cell realize a nontrivial projective representation of G_f , then the lattice Hamiltonian cannot have a nondegenerate, gapped, and symmetric ground state that can be described by an even- or odd-parity injective FMPS (for $d > 1$ we must assume that G_f is Abelian and all its elements are represented by unitary operators), in agreement with the conjecture made by Cheng in Ref. [51] when the fermion number is conserved. For Theorem 2, it is only assumed that the repeat unit cell supports an odd number of Majorana modes, the cardinality of the lattice is even, and translation symmetry holds. It then follows that the ground state cannot be simultaneously nondegenerate, gapped, and translation symmetric.

The rest of the paper is organized as follows. We introduce the main results of the present work in the form of Theorems 1 and 2 in Sec. II. We present an overview of the internal symmetry group G_f and its projective representations in Sec. III. We introduce the framework for FMPS and present a FMPS-based proof of Theorem 1 in Sec. IV when the dimension d of space is one. Theorem 2 when $d = 1$ is then

proved by making use of Theorem 1. An independent proof of Theorem 2 is given for any dimensions d of space in Sec. V. A weaker version of Theorem 1 for any $d \geq 1$ is also provided in Sec. V. The latter proof is based on symmetry twisted boundary conditions on twisted lattices. Finally, we collect several examples in Sec. VI and conclude the main body of the paper with a summary in Sec. VII. We present details about group cohomology, FMPS construction and proof of Theorem 1 in Appendices A, B, and C, respectively.

II. MAIN RESULTS

The notion of a global fermionic symmetry group G_f with a local action plays a central role in this paper. What we have in mind with this terminology is any lattice model obtained by discrete translations of a repeat unit cell. The same even integer number $n = 2m$ of Majorana degrees of freedom (flavors), i.e., a local fermionic Fock space of dimension 2^m , is attached to any repeat unit cell. The fermion parity can then be defined for any repeat unit cell. Even though the fermion parity is generically not conserved locally, it must be conserved globally. Hence, the symmetry group G_f necessarily contains as a subgroup the cyclic group of two elements generated by the fermion parity. The action of any other element from the symmetry group G_f can be defined locally, i.e., its action is represented by the same polynomial on the algebra generated by the Majorana operators within the repeat unit cell for any repeat unit cell of the lattice. As with the fermion parity, this symmetry element need not be conserved locally but must be conserved globally. A translation along the basis that generates the lattice relates different repeat unit cells. Similarly, any crystalline symmetry lies outside of the symmetry group G_f .

The motivation for Theorem 1 is the following. Given is a local lattice Hamiltonian \hat{H} that is symmetric under the group $G_{\text{trsl}} \times G_f$, where G_{trsl} denotes the group of lattice translations. Assume that the ground state $|\Psi\rangle$ is symmetric under both G_{trsl} and G_f , i.e., the ground state can change by no more than a multiplicative phase factor (combined with complex conjugation when the symmetry is represented by an antiunitary operator). Assume that a gap separates the ground state from all excited states. Are there sufficient and necessary conditions for $|\Psi\rangle$ to be degenerate?

A sufficient condition for all energy eigenvalues to be degenerate is that at least one generator from G_{trsl} and one element g from G_f are represented globally by operators that commute projectively, i.e., when passing the translation operator from the left to the right of the operator representing globally the element g , a multiplicative phase factor that cannot be gauged to unity arises. Another sufficient condition for the case when G_f has an antiunitarily represented \mathbb{Z}_2 subgroup (such as time-reversal symmetry) is that the symmetry generator g is represented globally by an operator that squares to minus the identity (Kramers' theorem). However, none of these conditions are necessary for the ground state to be degenerate in the thermodynamic limit. Even if all operators representing G_{trsl} and all global operators representing G_f commute pairwise and if all global operators representing G_f square to unity, a gapped and symmetric ground state might still be degenerate in the thermodynamic limit.

To develop an intuition for this last claim, we consider a one-dimensional lattice with the topology of a ring and assume that G_f is an Abelian group such that any element $g \in G_f$ is represented globally by the unitary operator \widehat{U}_g . Let \widehat{H}_{pbc} be any local Hamiltonian that is invariant under the actions of the unitary operators \widehat{T}_1 representing a translation by one repeat unit cell and \widehat{U}_g . Assume that the spectrum of \widehat{H}_{pbc} shows a gap between its ground and excited states. We define the operator $\widehat{T}_{1,h}$ which translates all lattice repeat unit cells by one to the right and acts locally with the local unitary operator representing $h \in G_f$ on the last repeat unit cell of the chain. Suppose that there exists a Hamiltonian $\widehat{H}_{\text{twist}}$ that is constructed by deforming subextensively many local terms from \widehat{H}_{pbc} and such that $\widehat{H}_{\text{twist}}$ is invariant under the actions of both $\widehat{T}_{1,h}$ and \widehat{U}_g . As long as the spectral gaps at the end points $\lambda = 0$ and 1 of

$$\widehat{H}(\lambda) := \lambda \widehat{H}_{\text{pbc}} + (1 - \lambda) \widehat{H}_{\text{twist}} \quad (2.1)$$

does not close for $0 \leq \lambda \leq 1$, \widehat{H}_{pbc} and $\widehat{H}_{\text{twist}}$ must share the same ground-state degeneracy. Note that generically Hamiltonian $\widehat{H}(\lambda)$ is not invariant under the actions of \widehat{T}_1 or $\widehat{T}_{1,h}$ when $0 < \lambda < 1$. This scenario is plausible, for \widehat{H}_{pbc} and $\widehat{H}_{\text{twist}}$ differ by subextensively many terms in the thermodynamic limit. Now, we denote with $|\exp(iK_h), \exp(iU_g)\rangle$ a many-body ground state of $\widehat{H}_{\text{twist}}$ that is a simultaneous eigenstate of $\widehat{T}_{1,h}$ and \widehat{U}_g with the eigenvalues $\exp(iK_h)$ and $\exp(iU_g)$, respectively. The many-body state $\widehat{T}_{1,h}|\exp(iK_h), \exp(iU_g)\rangle$ is then also a ground state of $\widehat{H}_{\text{twist}}$. A sufficient condition for $\widehat{T}_{1,h}|\exp(iK_h), \exp(iU_g)\rangle$ to be orthogonal to $|\exp(iK_h), \exp(iU_g)\rangle$ is that the product $\widehat{U}_g \widehat{T}_{1,h}$ differs from the product $\widehat{T}_{1,h} \widehat{U}_g$ by a multiplicative phase $\exp(i\chi_{g,h}) \neq 1$ that cannot be gauged away. If so the ground state of $\widehat{H}_{\text{twist}}$ is necessarily degenerate. If the gap never closes as a function of λ in the interval $\lambda \in [0, 1]$ in Eq. (2.1), then \widehat{H}_{pbc} and $\widehat{H}_{\text{twist}}$ share the same ground-state degeneracy even though \widehat{T}_1 and \widehat{U}_g commute for all $g \in G_f$.

We are ready to state Theorems 1 and 2 for which G_{trsl} stands for the group of lattice translations while G_f stands for the global fermionic symmetry group with a local action (i.e., it is an internal symmetry group).

Theorem 1. Any one-dimensional lattice Hamiltonian that is local and admits the symmetry group $G_{\text{trsl}} \times G_f$ cannot have a nondegenerate, gapped, and $G_{\text{trsl}} \times G_f$ -symmetric ground state that can be described by an even- or odd-parity injective FMPS if G_f is realized by a nontrivial projective representation on the local Fock space.

Theorem 2. A local Majorana Hamiltonian with an odd number of Majorana degrees of freedom per repeat unit cell that is invariant under the symmetry group $G_{\text{trsl}} \times G_f$, cannot have a nondegenerate, gapped and translationally invariant ground state.

Comment 1. The thermodynamic limit is implicit in both theorems.

Comment 2. Theorem 1 is only predictive when G_f is realized by a nontrivial projective representation on the local Fock space. When G_f is a Lie group its projective representation

on the local Fock space can be trivial. If so, Theorem 1 is not predictive. However, one can use complementary arguments such as the adiabatic threading of a gauge flux to decide if the ground state is degenerate. It is when G_f is a finite group that the full power of Theorem 1 is unleashed.

Comment 3. Theorem 1 is proved within the FMPS framework in Sec. IV. A weaker form of Theorem 1 holds in any dimension if it is assumed that G_f is Abelian and can be realized locally using unitary operators. The weaker version of Theorem 1 that is valid in any dimension is proved using tilted and twisted boundary conditions in Sec. V.

Comment 4. Theorem 2 applies in any dimension of space without any restriction on the internal fermionic symmetry group G_f .

Comment 5. The direct product structure of the symmetry group $G_{\text{trsl}} \times G_f$ is crucial in Theorems 1 and 2, and their generalizations to higher dimensions. Indeed, it has been shown that when the total symmetry group does not have a direct product structure, such as is the case with magnetic translation symmetries, a symmetric, nondegenerate, gapped, and short-range entangled ground state is not ruled out when closed-boundary conditions are imposed [25,53,54]. However, such a short-range entangled ground state must then necessarily support gapless symmetry-protected boundary states when open boundary conditions are imposed.

III. PROJECTIVE REPRESENTATIONS OF SYMMETRIES OBEYED BY MAJORANAS

Theorems 1 and 2 relate the quantum dynamics obeyed by Majoranas to the symmetries they obey. The smallest symmetry group associated to Majoranas originates from the conservation of the parity (evenness or oddness) of the total number of fermions. This symmetry is associated to a cyclic group of order two that we shall denote with \mathbb{Z}_2^F . Other symmetries are possible, say time-reversal symmetry or spin rotation symmetry. All such additional symmetries define a second group G . The first question to be answered is how many different ways are there to marry into a group G_f the intrinsic symmetry group \mathbb{Z}_2^F of Majoranas with the model-dependent symmetry group G . This problem is known in group theory as the central extension of G by \mathbb{Z}_2^F . It delivers a family of distinct equivalence classes with each equivalence class $[\gamma]$ in one-to-one correspondence with the cohomology group $H^2(G, \mathbb{Z}_2^F)$. This result is motivated in Sec. III A.

Once a representative symmetry group G_f has been selected from $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$, its representation on the Fock space spanned by all the local quantum degrees of freedom, a set that includes Majorana operators, must be constructed. Hereto, there are many possibilities. Their enumeration amounts to classifying the inequivalent projective representations of the group G_f . All the inequivalent projective representations of any one of the groups G_f obtained in Sec. III A are in one-to-one correspondence with the cohomology group $H^2(G_f, \text{U}(1)_c)$. This result is motivated in Sec. III B. The computation of $H^2(G_f, \text{U}(1)_c)$ is done in Sec. III B and Sec. VI.

Given two projective representations, a third one can be obtained from a graded tensor product as is explained in

Sec. III C. We shall describe the stacking rules used to construct nontrivial projective representations.

A. Marrying the fermion parity with the symmetry group G

For quantum systems built out of an even number of local Majorana operators, it is always possible to express all Majorana operators as the real and imaginary parts of local fermionic creation or annihilation operators. The parity (evenness or oddness) of the total fermion number is always a constant of the motion. If \widehat{F} denotes the operator whose eigenvalues counts the total number of local fermions in the Fock space, then the parity operator $(-1)^{\widehat{F}}$ necessarily commutes with the Hamiltonian that dictates the quantum dynamics, even though \widehat{F} might not, as is the case in any mean-field treatment of superconductivity.

We denote the group of two elements e and p

$$\mathbb{Z}_2^F := \{e, p \mid ep = pe = p, \quad e = ee = pp\}, \quad (3.1)$$

whereby e is the identity element and we shall interpret the quantum representation of p as the fermion parity operator. It is because of this interpretation of the group element p that we attach the upper index F to the cyclic group \mathbb{Z}_2 . In addition to the symmetry group \mathbb{Z}_2^F , we assume the existence of a second symmetry group G with the composition law \cdot and the identity element id . We would like to construct a new symmetry group G_f out of the two groups G and \mathbb{Z}_2^F . Here, the symmetry group G_f inherits the ‘‘fermionic’’ label f from its center \mathbb{Z}_2^F . One possibility is to consider the Cartesian product

$$G \times \mathbb{Z}_2^F := \{(g, h) \mid g \in G, \quad h \in \mathbb{Z}_2^F\} \quad (3.2a)$$

with the composition rule

$$(g_1, h_1) \circ (g_2, h_2) := (g_1 \cdot g_2, h_1 h_2). \quad (3.2b)$$

The resulting group G_f is the direct product of G and \mathbb{Z}_2^F . However, the composition rule (3.2b) is not the only one compatible with the existence of a neutral element, inverse, and associativity. To see this, we assume first the existence of the map

$$\begin{aligned} \gamma : G \times G &\rightarrow \mathbb{Z}_2^F, \\ (g_1, g_2) &\mapsto \gamma(g_1, g_2), \end{aligned} \quad (3.3a)$$

whereby we impose the conditions

$$\gamma(\text{id}, g) = \gamma(g, \text{id}) = e, \quad \gamma(g^{-1}, g) = \gamma(g, g^{-1}), \quad (3.3b)$$

for all $g \in G$ and

$$\gamma(g_1, g_2) \gamma(g_1 \cdot g_2, g_3) = \gamma(g_1, g_2 \cdot g_3) \gamma(g_2, g_3), \quad (3.3c)$$

for all $g_1, g_2, g_3 \in G$. Second, we define G_f to be the set of all pairs (g, h) with $g \in G$ and $h \in \mathbb{Z}_2^F$ obeying the composition rule

$$\begin{aligned} \circ_\gamma : (G \times \mathbb{Z}_2^F) \times (G \times \mathbb{Z}_2^F) &\rightarrow G \times \mathbb{Z}_2^F, \\ ((g_1, h_1), (g_2, h_2)) &\mapsto (g_1, h_1) \circ_\gamma (g_2, h_2), \end{aligned} \quad (3.3d)$$

where

$$(g_1, h_1) \circ_\gamma (g_2, h_2) := (g_1 \cdot g_2, h_1 h_2 \gamma(g_1, g_2)). \quad (3.3e)$$

One verifies the following properties. First, the order within the composition $h_1 h_2 \gamma(g_1, g_2)$ is arbitrary since \mathbb{Z}_2^F is Abelian. Second, conditions (3.3b) and (3.3c) ensure that G_f is a group with the neutral element

$$(\text{id}, e), \quad (3.4a)$$

the inverse to (g, h) is

$$(g^{-1}, [\gamma(g, g^{-1})]^{-1} h^{-1}), \quad (3.4b)$$

and the center (those elements of the group that commute with all group elements) given by

$$(\text{id}, \mathbb{Z}_2^F), \quad (3.4c)$$

i.e., the group G_f is a central extension of G by \mathbb{Z}_2^F . Third, the map γ can be equivalent to a map γ' of the form (3.3a) in that they generate two isomorphic groups. This is true if there exists the one-to-one map

$$\begin{aligned} \tilde{\kappa} : G \times \mathbb{Z}_2^F &\rightarrow G \times \mathbb{Z}_2^F, \\ (g, h) &\mapsto (g, \kappa(g)h) \end{aligned} \quad (3.5a)$$

induced by the map

$$\begin{aligned} \kappa : G &\rightarrow \mathbb{Z}_2^F, \\ g &\mapsto \kappa(g), \end{aligned} \quad (3.5b)$$

such that the condition

$$\tilde{\kappa}((g_1, h_1) \circ_\gamma (g_2, h_2)) = \tilde{\kappa}((g_1, h_1)) \circ_{\gamma'} \tilde{\kappa}((g_2, h_2)) \quad (3.6)$$

holds for all $(g_1, h_1), (g_2, h_2) \in G \times \mathbb{Z}_2^F$. In other words, γ and γ' generate two isomorphic groups if the identity

$$\kappa(g_1 \cdot g_2) \cdot \gamma(g_1, g_2) = \kappa(g_1) \cdot \kappa(g_2) \cdot \gamma'(g_1, g_2) \quad (3.7)$$

holds for all $g_1, g_2 \in G$. This group isomorphism defines an equivalence relation. We say that the group G_f obtained by extending the group G with the group \mathbb{Z}_2^F through the map γ splits when a map (3.5b) exists such that

$$\kappa(g_1 \cdot g_2) \cdot \gamma(g_1, g_2) = \kappa(g_1) \cdot \kappa(g_2) \quad (3.8)$$

holds for all $g_1, g_2 \in G$, i.e., G_f splits when it is isomorphic to the direct product (3.2).

The task of classifying all the nonequivalent central extensions of G by \mathbb{Z}_2^F through γ is achieved by enumerating all the elements of the second cohomology group $H^2(G, \mathbb{Z}_2^F)$, see Appendix A. We define an index $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$ to represent such an equivalence class, whereby the index $[\gamma] = 0$ is assigned to the case when G_f splits.

B. Projective representations of the group G_f

We denote with Λ a d -dimensional lattice with $\mathbf{j} \in \mathbb{Z}^d$ labeling the repeat unit cells. We are going to attach to Λ a Fock space on which projective representations of the group G_f constructed in Sec. III A are realized. This will be done using four assumptions.

Assumption 1. We attach to each repeat unit cell $\mathbf{j} \in \Lambda$ the local Fock space $\mathcal{F}_{\mathbf{j}}$. This step requires that the number of Majorana degrees of freedom in each repeat unit cell is even.

It is then possible to define the local fermion number operator \hat{f}_j and the local fermion-parity operator

$$\hat{p}_j := (-1)^{\hat{f}_j}. \quad (3.9)$$

We assume that all local Fock spaces \mathcal{F}_j with $j \in \Lambda$ are “identical,” in particular they share the same dimensionality \mathcal{D} . This assumption is a prerequisite to imposing translation symmetry.

Assumption 2. Each repeat unit cell $j \in \Lambda$ is equipped with a representation $\hat{u}_j(g)$ of G_f through the conjugation

$$\hat{o}_j \mapsto \hat{u}_j(g) \hat{o}_j \hat{u}_j^\dagger(g), \quad [\hat{u}_j(g)]^{-1} = \hat{u}_j^\dagger(g), \quad (3.10a)$$

of any operator \hat{o}_j acting on the local Fock space \mathcal{F}_j . The representation (3.10a) of $g \in G_f$ can either be unitary or antiunitary. More precisely, let

$$\begin{aligned} \mathfrak{c} : G_f &\rightarrow \{1, -1\}, \\ g &\mapsto \mathfrak{c}(g), \end{aligned} \quad (3.10b)$$

be a homomorphism. We then have the decomposition

$$\hat{u}_j(g) := \begin{cases} \hat{v}_j(g), & \text{if } \mathfrak{c}(g) = +1, \\ \hat{v}_j(g) \mathbf{K}, & \text{if } \mathfrak{c}(g) = -1, \end{cases} \quad (3.10c)$$

where

$$\hat{v}_j^{-1}(g) = \hat{v}_j^\dagger(g), \quad \hat{p}_j \hat{v}_j(g) \hat{p}_j = (-1)^{\rho(g)} \hat{v}_j(g), \quad (3.10d)$$

is a unitary operator with the fermion parity $\rho(g) \in \{0, 1\} \equiv \mathbb{Z}_2$ acting linearly on \mathcal{F}_j and \mathbf{K} denotes complex conjugation on the local Fock space \mathcal{F}_j . Accordingly, the homomorphism $\mathfrak{c}(g)$ dictates if the representation of the element $g \in G_f$ is implemented through a unitary operator [$\mathfrak{c}(g) = 1$] or an antiunitary operator [$\mathfrak{c}(g) = -1$]. Finally, we always choose to represent locally the fermion parity $p \in \mathbb{Z}_2^F$ by the Hermitian operator \hat{p}_j ,

$$\hat{u}_j(p) := \hat{p}_j \equiv (-1)^{\hat{f}_j}. \quad (3.10e)$$

Assumption 3. For any two elements $g, h \in G_f$ [to simplify notation, $g \circ h \equiv gh$ for all $g, h \in G_f$], whereby $e = g g^{-1} = g^{-1} g$ denotes the neutral element and $g^{-1} \in G_f$ the inverse of $g \in G_f$, we postulate the projective representation

$$\hat{u}_j(e) = \hat{\mathbb{1}}_{\mathcal{D}}, \quad (3.11a)$$

$$\hat{u}_j(g) \hat{u}_j(h) = e^{i\phi(g,h)} \hat{u}_j(gh), \quad (3.11b)$$

$$[\hat{u}_j(g) \hat{u}_j(h)] \hat{u}_j(f) = \hat{u}_j(g) [\hat{u}_j(h) \hat{u}_j(f)], \quad (3.11c)$$

whereby the identity operator acting on \mathcal{F}_j is denoted $\hat{\mathbb{1}}_{\mathcal{D}}$ and the function

$$\begin{aligned} \phi : G_f \times G_f &\rightarrow [0, 2\pi), \\ (g, h) &\mapsto \phi(g, h), \end{aligned} \quad (3.12a)$$

must be compatible with the associativity in G_f , i.e.,

$$\phi(g, h) + \phi(gh, f) = \phi(g, hf) + \mathfrak{c}(g) \phi(h, f), \quad (3.12b)$$

for all $g, h, f \in G_f$.¹ The map ϕ taking values in $[0, 2\pi)$ and satisfying (3.12b) is an example of a 2-cocycle with the group action specified by the \mathbb{Z}_2 -valued homomorphism \mathfrak{c} . In the vicinity of the value 0, ϕ generates the Lie algebra $\mathfrak{u}(1)$. The associated Lie group is denoted $U(1)$. Given the neutral element $e \in G_f$, a normalized 2-cocycle obeys the additional constraint

$$\phi(e, g) = \phi(g, e) = 0 \quad (3.12c)$$

for all $g \in G_f$. Two 2-cocycles $\phi(g, h)$ and $\phi'(g, h)$ are said to be equivalent if they can be consistently related through a map

$$\begin{aligned} \xi : G_f &\rightarrow [0, 2\pi), \\ g &\mapsto \xi(g), \end{aligned} \quad (3.13)$$

as follows. The equivalence relation $\phi \sim \phi'$ holds if the transformation

$$\hat{u}(g) = e^{i\xi(g)} \hat{u}'(g), \quad (3.14a)$$

implies the relation

$$\phi(g, h) - \phi'(g, h) = \xi(g) + \mathfrak{c}(g) \xi(h) - \xi(gh), \quad (3.14b)$$

between the 2-cocycle $\phi(g, h)$ associated to the projective representation $\hat{u}(g)$ and the 2-cocycle $\phi'(g, h)$ associated to the projective representation $\hat{u}'(g)$.² In particular, \hat{u} is equivalent to an ordinary representation (a trivial projective representation) if $\phi'(g, h) = 0$ for all $g, h \in G_f$. Any $\phi \sim 0$ is called a coboundary. For any coboundary ϕ there must exist a ξ such that

$$\phi(g, h) = \xi(g) + \mathfrak{c}(g) \xi(h) - \xi(gh). \quad (3.15)$$

The space of equivalence classes of projective representations is obtained by taking the quotient of 2-cocycles (3.12) by coboundaries (3.15). The resulting set is the second cohomology group $H^2(G_f, U(1)_c)$, which has an additive group structure. Appendix A gives more details on group cohomology.

Assumption 4. We attach to Λ the global Fock space \mathcal{F}_Λ by taking the appropriate product over j of the local Fock spaces \mathcal{F}_j . This means that we impose some algebra on all local operators differing by their repeat unit cell labels.

Example 1. Any two local fermion number operators \hat{f}_j and $\hat{f}_{j'}$ must commute

$$[\hat{f}_j, \hat{f}_{j'}] = 0 \quad (3.16)$$

¹One recognizes that Eq. (3.12b) is a generalization of Eq. (3.3c) if one identifies the exponential of ϕ in Eq. (3.12b) with γ in Eq. (3.3c) [up to the homomorphism (3.10b)].

²One recognizes that Eq. (3.14b) is a generalization of Eq. (3.7) if one identifies the exponential of ξ in Eq. (3.14b) with κ in Eq. (3.7) [up to the homomorphism (3.10b)].

for any two distinct repeat unit cell $\mathbf{j} \neq \mathbf{j}' \in \Lambda$. The total fermion and fermion-parity numbers are

$$\hat{F}_\Lambda := \sum_{\mathbf{j} \in \Lambda} \hat{f}_{\mathbf{j}}, \quad \hat{P}_\Lambda := (-1)^{\hat{F}_\Lambda}, \quad (3.17)$$

respectively.

Example 2. Any two Majorana operators labeled by $\mathbf{j} \neq \mathbf{j}' \in \Lambda$ must anticommute.

Example 3. The algebra

$$\hat{u}_{\mathbf{j}}(g) \hat{u}_{\mathbf{j}'}(g') = (-1)^{\rho(g)\rho(g')} \hat{u}_{\mathbf{j}'}(g') \hat{u}_{\mathbf{j}}(g) \quad (3.18)$$

holds for any distinct $\mathbf{j} \neq \mathbf{j}' \in \Lambda$ and any $g, g' \in G_f$ because of Eq. (3.10d). We then define the operator

$$\hat{U}(g) := \begin{cases} \prod_{\mathbf{j} \in \Lambda} \hat{v}_{\mathbf{j}}(g), & \text{if } \mathfrak{c}(g) = +1, \\ \left[\prod_{\mathbf{j} \in \Lambda} \hat{v}_{\mathbf{j}}(g) \right] K, & \text{if } \mathfrak{c}(g) = -1, \end{cases} \quad (3.19)$$

that implements globally on the Fock space \mathcal{F}_Λ the operation corresponding to the group element $g \in G_f$.

Theorem 1 refers to nontrivial projective representations of the symmetry group G_f constructed in Sec. III A. We are going to characterize the projective representations constructed in Sec. III B by two indices $[(v, \rho)]$ and show that the second cohomology group $H^2(G_f, \text{U}(1)_c)$ is equivalent to the equivalence classes $[(v, \rho)]$ of a tuple (v, ρ) .

Theorem 1 presumes the existence of a local projective representation of the symmetry group G_f . This is only possible if the local Fock space $\mathcal{F}_{\mathbf{j}}$ defined in Sec. III B is spanned by an even number of Majorana operators. This hypothesis precludes a situation in which a fermion number operator is well defined globally but not locally, for example when the lattice Λ is made of an even number of repeat unit cells, but a repeat unit cell is assigned an odd number of Majorana operators. (This can happen upon changing the parameters governing the quantum dynamics as is illustrated in Fig. 1.) We introduce the index $\mu = 0, 1$ to distinguish both possibilities. The case $\mu = 0$ applies when the number of local Majorana operators at site $\mathbf{j} \in \Lambda$ is even, in which case the number of repeat unit cells in Λ is any positive integer. The case $\mu = 1$ applies when the number of local Majorana operators at site $\mathbf{j} \in \Lambda$ is odd, in which case the number of repeat unit cell in Λ must necessarily be an even positive integer. The triplet

$$[(v, \rho), \mu] := \begin{cases} [(v, \rho), 0], & \text{if } \mu = 0, \\ (0, 0, 1), & \text{if } \mu = 1, \end{cases} \quad (3.20)$$

of indices allows to treat Theorems 1 and 2 together, as we are going to explain.

A similar set of three indices also appears in the classification of one-dimensional fermionic SPT phases. This is not an accident, for fermionic SPT phases can also be classified in terms of the projective representations realized by the global symmetries after projection on the boundaries [55–59].

1. Indices $[(v, \rho)]$

Theorem 1 is only predictive once it is established that a projective representation of G_f is non trivial. We recall that (i) the group G_f is a central extension of the group G by \mathbb{Z}_2^F through the map γ defined in Eq. (3.3) and (ii)

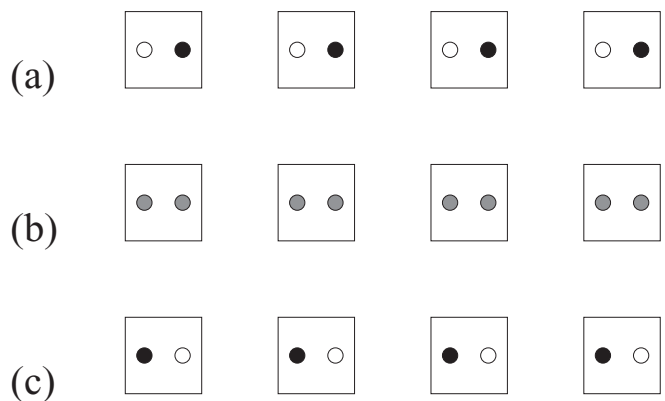


FIG. 1. The repeat unit cells of a lattice Λ are represented pictorially by squares. The lattice Λ is chosen for simplicity to be one dimensional. (a) The repeat unit cell is decorated with two circles, one empty, the other filled. If periodic boundary conditions are imposed, translations by one repeat unit cell are symmetries. The permutation of the empty and filled circle within all repeat unit cells is not a symmetry. (b) If the filling pattern is smoothly tuned (through an on-site potential whose magnitude is color coded, say) so that both circles in an repeat unit cell have the same filling, then the permutation of the left and right circles within all repeat unit cells is a symmetry. One may then choose a smaller repeat unit cell, a square centered about one circle only. (c) Image of (a) under the permutation of the empty and filled circle within all repeat unit cells.

only the equivalence classes $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$ deliver non-isomorphic groups. Choosing an element $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$ specifies G_f . In turn, a projective representation of G_f is specified by choosing an element ϕ from the equivalence class $[\phi] \in H^2(G_f, \text{U}(1)_c)$ where the 2-cocycle ϕ was defined in Eq. (3.12). By convention, a projective representation of G_f is trivial if $\phi \sim 0$, i.e.,

$$[\phi] \equiv 0 \quad (3.21)$$

is the trivial (i.e., neutral) element of $H^2(G_f, \text{U}(1)_c)$ equipped with the addition as group composition.

Deciding if a projective representation of G_f is non trivial amounts to calculating $H^2(G_f, \text{U}(1)_c)$. This task is facilitated by the fact that $H^2(G_f, \text{U}(1)_c)$ can be related to $H^2(G, \mathbb{Z}_2^F)$, $C^2(G, \text{U}(1))$, and $C^1(G, \mathbb{Z}_2)$ together with a set of appropriately defined cocycle and coboundary conditions, (see Appendix A for definitions of these sets and conditions) as was shown in the physics literature by Turzillo and You in Ref. [58]. When the equivalence class $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$ is trivial, $[\gamma] = 0$, the second cohomology group $H^2(G_f, \text{U}(1)_c)$ reduces to the direct product of the cohomology groups $H^2(G, \text{U}(1)_c)$ and $H^1(G, \mathbb{Z}_2)$ as we shall explain.

To understand this claim, we consider first the simpler case when the central extension G_f of the group G by \mathbb{Z}_2^F splits, i.e., when

$$[\gamma] = 0 \iff G_f = G \times \mathbb{Z}_2^F \quad (3.22)$$

according to Sec. III A. If so, we can apply the Kuenneth formula for group cohomology,

$$\begin{aligned} H^2(G_f, \text{U}(1)_c) &= H^2(G \times \mathbb{Z}_2^F, \text{U}(1)_c) \\ &= H^2(G, \text{U}(1)_c) \times H^1(G, \mathbb{Z}_2). \end{aligned} \quad (3.23)$$

Equation (3.23) states that the Abelian group $H^2(G_f, \text{U}(1)_c)$ is the direct product between the Abelian group $H^2(G, \text{U}(1)_c)$ and the Abelian group $H^1(G, \mathbb{Z}_2)$.

The cohomology group $H^2(G, \text{U}(1)_c)$ is obtained by restricting the domain of definition of the function ϕ in Eq. (3.12) to the domain of definition $G \times G$. We will reserve the letter ν to denote such a function. An element of the Abelian group $H^2(G, \text{U}(1)_c)$ with the addition as group composition is the equivalence class $[\nu]$ with the neutral element

$$[\nu] = 0. \quad (3.24)$$

The presence of the cohomology group $H^1(G, \mathbb{Z}_2)$ on the right-hand side of Eq. (3.23) can be understood as follows. Choose h in Eq. (3.11b) to be the fermion parity p (by the inclusion map). We then have

$$\hat{u}(g)\hat{u}(p) = e^{i\phi(g,p)}\hat{u}(gp), \quad \hat{u}(p)\hat{u}(g) = e^{i\phi(p,g)}\hat{u}(pg). \quad (3.25)$$

Because p belongs to the center of G_f , $gp = pg$ implies that

$$\hat{u}(g)\hat{u}(p) = e^{i[\phi(g,p) - \phi(p,g)]}\hat{u}(p)\hat{u}(g). \quad (3.26)$$

Because the eigenvalues of the fermion parity operator are 0 or 1,

$$\phi(g, p) - \phi(p, g) = n\pi \quad (3.27)$$

for some integer n , i.e., the projective representation of the parity operator $\hat{u}(p)$ either commutes (n even) or anticommutes (n odd) with the projective representation $\hat{u}(g)$ of any element of G . Hence, we may define the map

$$\begin{aligned} \rho : G &\rightarrow \mathbb{Z}_2, \\ g &\mapsto \rho(g) := \frac{\phi(g, p) - \phi(p, g)}{\pi} \bmod 2, \end{aligned} \quad (3.28)$$

whose equivalence classes $[\rho]$ under the gauge transformation induced by Eq. (3.14a) define $H^1(G, \mathbb{Z}_2)$. We recognize that the map (3.28) is the fermion parity $\rho(g) \in \{0, 1\} \equiv \mathbb{Z}_2$ defined in Eq. (3.10d). Even though the subgroup G of $G \times \mathbb{Z}_2^F$ commutes with the subgroup \mathbb{Z}_2^F , this need not be true under a projective representation. This possibility is captured by the presence of $H^1(G, \mathbb{Z}_2)$ on the right-hand side of Eq. (3.23). The neutral element of $H^1(G, \mathbb{Z}_2)$ equipped with the addition is

$$[\rho] = 0. \quad (3.29)$$

All told, when the central extension G_f of the group G by \mathbb{Z}_2^F splits, the Kuenneth formula (3.23) predicts that

$$[\phi] \equiv ([\nu], [\rho]) \quad (3.30a)$$

with the trivial projective representation defined by the condition

$$\mathbf{0} = [\phi] = ([\nu], [\rho]) = (0, 0). \quad (3.30b)$$

When the central extension G_f of the group G by \mathbb{Z}_2^F does not split, i.e., when

$$[\gamma] \neq 0 \quad (3.31)$$

according to Sec. III A, then the identification (3.30a) is no longer correct. Turzillo and You in Ref. [58] have shown in the context of SPT phases of matter that, the equivalence classes $[(\nu, \rho)]$ of the pair $(\nu, \rho) \in C^2(G, \text{U}(1)) \times C^1(G, \mathbb{Z}_2)$ that satisfy the conditions

$$(\delta\nu - \pi\rho \smile \gamma, \delta\rho) = (0, 0), \quad (3.32a)$$

are in one to one correspondence with the second cohomology classes $[\phi] \in H^2(G_f, \text{U}(1)_c)$. Here, the operation δ is defined in Eq. (A3), while \smile denotes the cup product defined in Eq. (A11). Two pairs (ν, ρ) and (ν', ρ') are equivalent to each other if there exists another pair (α, β) , whereby

$$\begin{aligned} \alpha : G &\rightarrow [0, 2\pi), \\ g &\mapsto \alpha(g), \end{aligned} \quad (3.32b)$$

while $\beta \in \mathbb{Z}_2$ such that

$$(\nu, \rho) = (\nu', \rho') + (\delta\alpha + \pi\beta \smile \gamma, \delta\beta). \quad (3.32c)$$

(See Appendix A2 for a detailed discussion.) Hence, the second cohomology class is identified with

$$[\phi] \equiv [(\nu, \rho)]. \quad (3.32d)$$

It can be seen from Eqs. (3.32a) and (3.32c) that if $[\gamma] = 0$ then $\delta\nu = 0$, which is the defining condition for ν to be a 2-cocycle. We may then identify the gauge equivalent classes $[\nu]$ with the elements of the Abelian group $H^2(G, \text{U}(1)_c)$ and the identification (3.32d) reduces to the one in Eq. (3.30a). However, when $[\gamma] \neq 0$, the function $\nu : G \times G \rightarrow \text{U}(1)$ is called a 2-cochain and belongs to the set $C^2(G, \text{U}(1))$. In practice, Eq. (3.32) ties the 2-cochain ν to the 2- and 1-cochains γ and ρ that belong to the sets $C^2(G, \mathbb{Z}_2^F)$ and $C^1(G, \mathbb{Z}_2)$, respectively. From now on, we shall use the notation (3.32d) to trade in general the second cohomology class $[\phi]$ with the equivalence class $[(\nu, \rho)]$ and reserve the notation $([\nu], [\rho])$ of Eq. (3.30a) for the case when the underlying fermionic symmetry group G_f splits (i.e., $[\gamma] = 0$).

C. Stacking rules

In this section, we review the so-called stacking rules, according to which the indices $[(\nu, \rho)]$ and μ classifying an LSM constraint can be ‘‘added.’’

Given two Fock spaces $\mathcal{F}_j^{(1)}$ and $\mathcal{F}_j^{(2)}$, let $\hat{u}_j^{(1)}$ and $\hat{u}_j^{(2)}$ be two local projective representations of G_f with indices $([(\nu_1, \rho_1)], 0)$ and $([(\nu_2, \rho_2)], 0)$, respectively. Stacking $\mathcal{F}_j^{(1)}$ and $\mathcal{F}_j^{(2)}$ refers to taking the graded tensor product³

$$\mathcal{F}_j^{(1 \otimes_g 2)} := \mathcal{F}_j^{(1)} \otimes_g \mathcal{F}_j^{(2)}. \quad (3.33)$$

³To account for the fermionic statistics, instead of the standard tensor product one must use a \mathbb{Z}_2 graded one. Fermionic Fock spaces then carry the structure of a \mathbb{Z}_2 graded vector space, also called a supervector space. See Appendix B for more details on this construction.

The local representation of G_f over $\mathcal{F}_j^{(1 \otimes_{\mathfrak{g}} 2)}$ is

$$\hat{u}_j^{(1 \otimes_{\mathfrak{g}} 2)}(g) := \begin{cases} \hat{v}_j^{(1)}(g) \hat{v}_j^{(2)}(g), & \text{if } c(g) = +1, \\ \hat{v}_j^{(1)}(g) \hat{v}_j^{(2)}(g) \mathbf{K}, & \text{if } c(g) = -1. \end{cases} \quad (3.34a)$$

This representation satisfies the composition rule

$$\hat{u}_j^{(1 \otimes_{\mathfrak{g}} 2)}(g) \hat{u}_j^{(1 \otimes_{\mathfrak{g}} 2)}(h) = e^{i\phi(g,h)} \hat{u}_j^{(1 \otimes_{\mathfrak{g}} 2)}(gh), \quad (3.34b)$$

where the phase $\phi(g, h) \in [0, 2\pi)$ is

$$\phi(g, h) = \phi_1(g, h) + \phi_2(g, h) + \pi \rho_1(h) \rho_2(g). \quad (3.34c)$$

The first two terms arise due to the composition rule of the two representations $\hat{u}_j^{(1)}$ and $\hat{u}_j^{(2)}$. The last term encodes the fact that representation (1) of element h and representation (2) of element g anticommute if both of them have odd fermion parity. Equation (3.34c) can be decomposed into the stacking relations

$$([\langle v, \rho \rangle], 0) = ([\langle v_1 + v_2 + \pi \rho_1 \smile \rho_2, \rho_1 + \rho_2 \rangle], 0). \quad (3.35)$$

Here, $\rho_1 \smile \rho_2$ is the cup product of the two 1-cochains ρ_1 and ρ_2 (see Appendix A for the definition of the cup product). Turzillo and You in Ref. [58] have shown that a similar relationship holds in the context of SPT phases of matter. The graded tensor product also allows to stack two projective representations $(\mathbf{0}, \mathbf{0}, \mu_1)$ and $(\mathbf{0}, \mathbf{0}, \mu_2)$ with $\mu_1 = 1 \pmod 2$ and $\mu_2 = 1 \pmod 2$. The result is a projective representation of the form $([\langle v, \rho \rangle], 0)$ since it is then possible to define a local Fock space.

IV. MAJORANA LSM THEOREM IN ONE DIMENSION

In this section, we sketch the proofs for Theorems 1 and 2 in 1D space using the machinery of fermionic matrix product states (FMPS). We relegate some intermediate steps and technical details to Appendix C. We begin with a definition of FMPS in Sec. IV A (further background can be found in Appendix B and Refs. [56–58,60]).

The strategy that we follow is to prove that so-called injective FMPS are only compatible with a trivial projective representation of the symmetry group G_f discussed in Sec. III. The main steps of the proof for Theorems 1 and 2, the first main results of this paper, are provided in Sec. IV B and IV C, respectively. We close by discussing parallels with the SPT phases in Sec. IV D.

A. Fermionic matrix product states

Consider a one-dimensional lattice $\Lambda \cong \mathbb{Z}_N$. At the repeat unit cell $j = 1, \dots, N$, the local fermion number operator is denoted \hat{f}_j and the local Fock space of dimension \mathcal{D}_j is denoted $\mathcal{F}_j \cong \mathbb{C}^{\mathcal{D}_j}$. We define with

$$|\psi_{\sigma_j}\rangle, \quad \sigma_j = 1, \dots, \mathcal{D}_j, \quad (4.1a)$$

an orthonormal basis of \mathcal{F}_j such that

$$(-1)^{\hat{f}_j} |\psi_{\sigma_j}\rangle = (-1)^{|\sigma_j|} |\psi_{\sigma_j}\rangle. \quad (4.1b)$$

The fermion parity eigenvalue of the basis element $|\psi_{\sigma_j}\rangle$ is thus denoted $(-1)^{|\sigma_j|}$ with $|\sigma_j| \equiv 0, 1$. The local Fock space \mathcal{F}_j admits the direct sum decomposition

$$\mathcal{F}_j = \mathcal{F}_j^{(0)} \oplus \mathcal{F}_j^{(1)}, \quad (4.2a)$$

where, given $p = 0, 1$,

$$\mathcal{F}_j^{(p)} := \text{span}\{|\psi_{\sigma_j}\rangle, \sigma_j = 1, \dots, \mathcal{D}_j \mid |\sigma_j| = p\}. \quad (4.2b)$$

One verifies that $\dim \mathcal{F}_j^{(0)} = \dim \mathcal{F}_j^{(1)} = \mathcal{D}_j/2$. To construct the Fock space \mathcal{F}_Λ for the lattice Λ , we demand that the direct sum (4.2) also holds for \mathcal{F}_Λ . This is achieved with the help of the \mathbb{Z}_2 tensor product $\otimes_{\mathfrak{g}}$. This tensor product preserves the \mathbb{Z}_2 -grading structure. We define the reordering rule

$$|\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_{j'}}\rangle \equiv (-1)^{|\sigma_j| |\sigma_{j'}|} |\psi_{\sigma_{j'}}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \quad (4.3)$$

on any two basis elements $|\psi_{\sigma_j}\rangle$ and $|\psi_{\sigma_{j'}}\rangle$ of \mathcal{F}_j and $\mathcal{F}_{j'}$ for any two distinct sites $j \in \Lambda$ and $j' \in \Lambda$, respectively. The rule (4.3) guarantees that states are antisymmetric under the exchange of an odd number of fermions on site j with an odd number of fermions on site j' while symmetric otherwise. We then define the fermionic Fock space \mathcal{F}_Λ for the lattice Λ to be

$$\mathcal{F}_\Lambda := \text{span} \left\{ |\Psi_\sigma\rangle \mid |\Psi_\sigma\rangle \equiv \bigotimes_{j=1}^N |\psi_{\sigma_j}\rangle, \right. \\ \left. \sigma \equiv (\sigma_1, \dots, \sigma_N) \in \{1, \dots, \mathcal{D}_1\} \times \dots \times \{1, \dots, \mathcal{D}_N\} \right\}. \quad (4.4)$$

As the parity $|\sigma_j|$ of the state $|\psi_{\sigma_j}\rangle$ can be generalized to the parity $|\sigma|$ of the state $|\Psi_\sigma\rangle$ through the action of the global fermion number operator

$$\hat{F}_\Lambda := \sum_{j=1}^N \hat{f}_j, \quad |\sigma| \equiv \sum_{j=1}^N |\sigma_j| \pmod 2, \quad (4.5)$$

the Fock space (4.4) inherits the direct sum decomposition (4.2a),

$$\mathcal{F}_\Lambda = \mathcal{F}_\Lambda^{(0)} \oplus \mathcal{F}_\Lambda^{(1)}. \quad (4.6)$$

Any state $|\Psi\rangle \in \mathcal{F}_\Lambda$ has the expansion

$$|\Psi\rangle = \sum_{\sigma} c_{\sigma} |\Psi_{\sigma}\rangle \quad (4.7a)$$

with the expansion coefficient $c_{\sigma} \in \mathbb{C}$. Such a state is homogeneous if it belongs to either $\mathcal{F}_\Lambda^{(0)}$ or $\mathcal{F}_\Lambda^{(1)}$, in which case it has a definite parity $|\Psi| \equiv 0, 1$. From now on, we assume that all local Fock spaces are pairwise isomorphic, i.e.,

$$\mathcal{D}_j = \mathcal{D}, \quad \mathcal{F}_j \cong \mathcal{F}_{j'} \quad 1 \leq j < j' \leq N. \quad (4.8)$$

This assumption is needed to impose translation symmetry below. We describe the construction of two families of states that lie in $\mathcal{F}_\Lambda^{(0)}$ and $\mathcal{F}_\Lambda^{(1)}$, respectively. To this end, we choose the positive integer M , denote with $\mathbb{1}_M$ the unit $M \times M$ matrix and define the following pair of $2M \times 2M$ matrices

$$P := \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & \mathbb{1}_M \\ -\mathbb{1}_M & 0 \end{pmatrix}. \quad (4.9)$$

The 2×2 grading that is displayed is needed to represent the \mathbb{Z}_2 grading in Eq. (4.6) as will soon become apparent. The anticommuting matrices P and Y belong to the set $\text{Mat}(2M, \mathbb{C})$ of all $2M \times 2M$ matrices. This set is a $4M^2$ -dimensional vector space over the complex numbers.⁴ For any $\sigma_j = 1, \dots, \mathcal{D}$ with $j \in \Lambda$, we choose the matrices

$$B_{\sigma_j}, C_{\sigma_j}, D_{\sigma_j}, E_{\sigma_j}, G_{\sigma_j} \in \text{Mat}(M, \mathbb{C}) \quad (4.10a)$$

with the help of which we define the matrices

$$A_{\sigma_j}^{(0)} := \begin{cases} \begin{pmatrix} B_{\sigma_j} & 0 \\ 0 & C_{\sigma_j} \end{pmatrix}, & \text{if } |\sigma_j| = 0, \\ \begin{pmatrix} 0 & D_{\sigma_j} \\ E_{\sigma_j} & 0 \end{pmatrix}, & \text{if } |\sigma_j| = 1, \end{cases} \quad (4.10b)$$

and

$$A_{\sigma_j}^{(1)} := \begin{cases} \begin{pmatrix} G_{\sigma_j} & 0 \\ 0 & G_{\sigma_j} \end{pmatrix}, & \text{if } |\sigma_j| = 0, \\ \begin{pmatrix} 0 & G_{\sigma_j} \\ -G_{\sigma_j} & 0 \end{pmatrix}, & \text{if } |\sigma_j| = 1, \end{cases} \quad (4.10c)$$

from $\text{Mat}(2M, \mathbb{C})$. Observe that Eq. (4.10c) is a special case of Eq. (4.10b). For any $\sigma_j = 1, \dots, \mathcal{D}$ with $j \in \Lambda$, the matrix P commutes (anticommutes) with $A_{\sigma_j}^{(p)}$ when $|\sigma_j| = 0$ ($|\sigma_j| = 1$),

$$P A_{\sigma_j}^{(p)} = (-1)^{|\sigma_j|} A_{\sigma_j}^{(p)} P \quad (4.11)$$

for both $p = 0, 1$. In contrast, the matrix Y commutes with $A_{\sigma_j}^{(1)}$

$$Y A_{\sigma_j}^{(1)} = A_{\sigma_j}^{(1)} Y \quad (4.12)$$

for all $\sigma_j = 1, \dots, \mathcal{D}$ with $j \in \Lambda$.

We are ready to define the FMPS. We define states with either periodic boundary conditions (PBC) or antiperiodic boundary conditions (APBC) denoted by the parameter $b = 0$ or 1, respectively. They are

$$|\{A_{\sigma_j}^{(0)}\}; b\rangle := \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} \cdots A_{\sigma_N}^{(0)}] |\Psi_{\sigma}\rangle \quad (4.13a)$$

and

$$|\{A_{\sigma_j}^{(1)}\}; b\rangle := \sum_{\sigma} \text{tr}[P^b Y A_{\sigma_1}^{(1)} \cdots A_{\sigma_N}^{(1)}] |\Psi_{\sigma}\rangle \quad (4.13b)$$

for any choice of the matrices (4.10b) and (4.10c), respectively, and with the basis (4.4) of the Fock space \mathcal{F}_{Λ} . The following properties follow from the cyclicity of the trace and from the fact that Y is traceless.

Property 1. The FMPS $|\{A_{\sigma_j}^{(p)}\}; b\rangle$ is homogeneous and belongs to $\mathcal{F}_{\Lambda}^{(p)}$ for $p = 0, 1$. This claim is a consequence of the

identities

$$\sum_{j=1}^N |\sigma_j| = 1 \pmod{2} \Rightarrow \text{tr}(P^b P A_{\sigma_1}^{(0)} \cdots A_{\sigma_N}^{(0)}) = 0, \quad (4.14a)$$

$$\sum_{j=1}^N |\sigma_j| = 0 \pmod{2} \Rightarrow \text{tr}(P^b Y A_{\sigma_1}^{(1)} \cdots A_{\sigma_N}^{(1)}) = 0. \quad (4.14b)$$

Property 2. The FMPS $|\{A_{\sigma_j}^{(p)}\}; b\rangle$ changes by a multiplicative phase under a translation by one repeat unit cell. Indeed, one verifies that

$$\hat{T}_b |\{A_{\sigma_j}^{(p)}\}; b\rangle = e^{i\pi(2k+b)/N} |\{A_{\sigma_j}^{(p)}\}; b\rangle, \quad (4.15)$$

where \hat{T}_b is the generator of translation by one repeat unit cell with boundary conditions $b = 0, 1$ and $k \in \mathbb{Z}$.

Property 3. The FMPS (4.13a) and (4.13b) are not uniquely specified by the choices $\{A_{\sigma_j}^{(p)}\}$ for $p = 0, 1$, respectively. For example, the similarity transformation

$$A_{\sigma_j}^{(0)} \mapsto U A_{\sigma_j}^{(0)} U^{-1}, \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N, \quad (4.16)$$

with U any matrix that commutes with P leaves the trace unchanged. Another example occurs if there exists a nonvanishing matrix $Q = Q^2 \in \text{Mat}(2M, \mathbb{C})$ such that

$$Q A_{\sigma_j}^{(0)} = Q A_{\sigma_j}^{(0)} Q, \quad \sigma_j = 1, \dots, \mathcal{D}. \quad (4.17)$$

Indeed, one verifies that Eq. (4.17) implies the identity

$$\text{tr}[P^{b+1} A_{\sigma_1}^{(0)} \cdots A_{\sigma_N}^{(0)}] = \text{tr}[P^{b+1} \tilde{A}_{\sigma_1}^{(0)} \cdots \tilde{A}_{\sigma_N}^{(0)}] \quad (4.18a)$$

with $\tilde{A}_{\sigma_j}^{(0)}$ the matrix

$$\tilde{A}_{\sigma_j}^{(0)} := Q A_{\sigma_j}^{(0)} Q + (\mathbb{1}_M - Q) A_{\sigma_j}^{(0)} (\mathbb{1}_M - Q). \quad (4.18b)$$

While conditions (4.17) imply that all matrices $A_1^{(0)}, \dots, A_{\mathcal{D}}^{(0)}$ are reducible, conditions (4.18b) imply that all matrices $\tilde{A}_1^{(0)}, \dots, \tilde{A}_{\mathcal{D}}^{(0)}$ are decomposable into the same block diagonal form. A necessary and sufficient condition on the \mathcal{D} matrices $A_1^{(0)}, \dots, A_{\mathcal{D}}^{(0)}$ to prevent that Eq. (4.17) holds for some $Q \in \text{Mat}(2M, \mathbb{C})$ is to demand that there exists an integer $1 \leq \ell^* \leq N$ such that the vector space spanned by the \mathcal{D}^{ℓ^*} matrix products

$$A_{\sigma_1}^{(0)} \cdots A_{\sigma_{\ell^*}}^{(0)}, \quad \sigma_1, \dots, \sigma_{\ell^*} = 1, \dots, \mathcal{D}, \quad (4.19a)$$

is $\text{Mat}(2M, \mathbb{C})$.⁵ More precisely, for any $A \in \text{Mat}(2M, \mathbb{C})$, it is possible to find \mathcal{D}^{ℓ^*} coefficients $a_{\sigma_1, \dots, \sigma_{\ell^*}}^{(0)} \in \mathbb{C}$ such that⁶

$$A = \sum_{\sigma_1, \dots, \sigma_{\ell^*}=1}^{\mathcal{D}} a_{\sigma_1, \dots, \sigma_{\ell^*}}^{(0)} A_{\sigma_1}^{(0)} \cdots A_{\sigma_{\ell^*}}^{(0)}. \quad (4.19b)$$

In order to restrict the redundancy in the choice of the matrices (4.10) that enter the FMPS (4.13), we make the following definitions.

⁵If $\mathcal{D} = 2$, $M = 1$, $A_1^{(0)}$ is i times the second Pauli matrix, and $A_2^{(0)}$ is the third Pauli matrix, then $\ell^* = 4$.

⁶The basis (4.19a) is in general overcomplete owing to the condition $\mathcal{D}^{\ell^*} \geq 4M^2$.

⁴The set $\text{Mat}(2M, \mathbb{C})$ of all $2M \times 2M$ matrices is a $8M^2$ -dimensional vector space over the real numbers.

Definition 1. The even-parity FMPS (4.13a) is *injective* if there exists an integer $\ell^* \geq 1$ such that the \mathcal{D}^{ℓ^*} products $A_{\sigma_1}^{(0)} \cdots A_{\sigma_{\ell^*}}^{(0)}$ of $2M \times 2M$ matrices span $\text{Mat}(2M, \mathbb{C})$.

Definition 2. The odd-parity FMPS (4.13b) is *injective* if there exists an integer $\ell^* \geq 1$ such that the \mathcal{D}^{ℓ^*} products $G_{\sigma_1} \cdots G_{\sigma_{\ell^*}}$ of $M \times M$ matrices span $\text{Mat}(M, \mathbb{C})$.

The need to distinguish the definitions of injectivity for even- and odd-parity FMPS stems from the fact that for an odd-parity FMPS the matrix Y commutes with $A_1^{(1)}, \dots, A_{\mathcal{D}}^{(1)}$. In other words, Y is in the center of the algebra closed by products of $A_1^{(1)}, \dots, A_{\mathcal{D}}^{(1)}$. Injectivity requires this center to be generated by $\mathbb{1}_{2M}$ and Y , i.e., the algebra closed by products of matrices $A_1^{(1)}, \dots, A_{\mathcal{D}}^{(1)}$ is a \mathbb{Z}_2 -graded simple algebra. For the center to be generated by no more than $\mathbb{1}_{2M}$ and Y , the product of $G_1, \dots, G_{\mathcal{D}}$ must close a simple algebra of $M \times M$ matrices, which is precisely the Definition 2. The following properties of FMPS are essential to the proofs of Theorems 1 and 2.

Property 4. Let $\ell \geq \ell^*$. The \mathcal{D}^{ℓ} products $A_{\sigma_1}^{(0)} \cdots A_{\sigma_{\ell}}^{(0)}$ of $2M \times 2M$ matrices span $\text{Mat}(2M, \mathbb{C})$ for any injective even-parity FMPS. The \mathcal{D}^{ℓ} products $G_{\sigma_1} \cdots G_{\sigma_{\ell}}$ of $M \times M$ matrices span $\text{Mat}(M, \mathbb{C})$ for any $\ell \geq \ell^*$ injective odd-parity FMPS.

Property 5. If two sets of matrices $\{A_{\sigma_j}^{(p)}\}$ and $\{\tilde{A}_{\sigma_j}^{(p)}\}$ generate the same injective FMPS, there then exists an invertible matrix U and a phase $\varphi_U \in [0, 2\pi)$ such that [56]

$$\tilde{A}_{\sigma_j}^{(p)} = e^{i\varphi_U} U A_{\sigma_j}^{(p)} U^{-1}, \quad (4.20a)$$

for any $\sigma_j = 1, \dots, \mathcal{D}$, and

$$P = \pm U P U^{-1}, \quad (4.20b)$$

for $p = 0$, while

$$P = U P U^{-1}, \quad Y = \pm U Y U^{-1}, \quad (4.20c)$$

for $p = 1$. Here, the phase φ_U is needed to compensate for the possibility that the matrix U anticommutes with P or Y . We also observe that the index σ_j that labels the local fermion number is preserved under the conjugation by U . The transformation (4.20) that leaves an injective FMPS invariant is called a gauge transformation.

Property 6. Definitions 1 and 2 ensure that the two-point correlation function of any pair of local operators taken in an injective FMPS decays exponentially fast with their separation. This provides an additional motivation to study them as they can be used to describe nondegenerate and gapped ground states [56].

With the formalism introduced in Secs. III and IV A, we restate Theorem 1 as follows: If an even- or odd-parity injective FMPS is translation-invariant and symmetric under a projective representation of the symmetry group G_f defined in Sec. III, then the projective representation of G_f must have a trivial second group cohomology class $[\phi] = 0$. We recover Theorem 1 by negating this statement.

B. Proof of Theorem 1

Our strategy is inspired by the study of injective bosonic MPS assumed to be $G_{\text{trsl}} \times G$ -invariant made by Tasaki in Ref. [32]. For the fermionic case, we shall distinguish the cases of even- and odd-parity FMPS, as each case demands

distinct conditions for injectivity. For the case of even-parity injective FMPS, we shall establish the following identity between any matrix $A \in \text{Mat}(2M, \mathbb{C})$ and a given norm preserving $W \in \text{Mat}(2M, \mathbb{C})$ that is induced by a projective representation of the symmetry group G_f . There exists a phase $\delta \in [0, 2\pi)$ and a nonvanishing positive integer ℓ^* such that

$$A = e^{i\ell^* \delta} W^{-1} A W, \quad (4.21a)$$

holds for all $\ell = \ell^*, \ell^* + 1, \ell^* + 2, \dots$ and all $A \in \text{Mat}(2M, \mathbb{C})$. This is only possible if

$$\delta = 0, \quad (4.21b)$$

which obviously holds when A is the identity matrix $\mathbb{1}_{2M}$. For the case of odd FMPS, we shall establish the same identity as (4.21) for any matrix $A \in \text{Mat}(2M, \mathbb{C})$ that commutes with matrix Y , i.e., Y is in the center of the algebra spanned by such matrices A . Theorem 1 will follow from the interpretation of the condition $\delta = 0$ as the projective representation of G_f defined in Sec. III to have trivial second group cohomology class.

1. Case of even-parity injective FMPS

We start from the even-parity injective FMPS

$$|\{A_{\sigma_j}^{(0)}\}; b\rangle := \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} \cdots A_{\sigma_N}^{(0)}] |\Psi_{\sigma}\rangle. \quad (4.22)$$

Let g be an element from G_f be represented by the operator $\widehat{U}(g)$ as defined in Sec. III B.

On the one hand, we have the identity

$$\begin{aligned} \widehat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle &= \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} \cdots A_{\sigma_N}^{(0)}] \widehat{U}(g) |\Psi_{\sigma}\rangle \\ &\equiv \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)}(g) \cdots A_{\sigma_N}^{(0)}(g)] |\Psi_{\sigma}\rangle, \end{aligned} \quad (4.23a)$$

where

$$A_{\sigma_j}^{(0)}(g) := \sum_{\sigma'_j=1}^{\mathcal{D}} [U(g)]_{\sigma_j \sigma'_j} K_g[A_{\sigma'_j}^{(0)}], \quad (4.23b)$$

$$[U(g)]_{\sigma_j \sigma'_j} := \langle \psi_{\sigma_j} | \hat{v}_j | \psi_{\sigma'_j} \rangle, \quad (4.23c)$$

$$K_g[A_{\sigma_j}^{(0)}] := \begin{cases} A_{\sigma_j}^{(0)}, & \text{if } \mathfrak{c}(g) = 0, \\ \mathbb{K} A_{\sigma_j}^{(0)} \mathbb{K}, & \text{if } \mathfrak{c}(g) = 1. \end{cases} \quad (4.23d)$$

(Complex conjugation is denoted with \mathbb{K} .) On the other hand, we have the identity

$$\widehat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle = e^{i\eta(g;b)} |\{A_{\sigma_j}^{(0)}\}; b\rangle = | \{ e^{i\eta(g;b)/N} A_{\sigma_j}^{(0)} \}; b \rangle \quad (4.24)$$

for some phase $\eta(g;b) \in [0, 2\pi)$ if we assume that $\widehat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle$ is an eigenstate of the norm-preserving operator $\widehat{U}(g)$, as it should be if G_f is a symmetry. By the assumption of injectivity, the matrices $A_{\sigma_j}^{(0)}(g)$ and $e^{i\eta(g;b)/N} A_{\sigma_j}^{(0)}$ are related by a similarity transformation (4.20), i.e., there exists an invertible matrix $U(g)$ and a phase $\varphi_{U(g)}^{(b)} \in [0, 2\pi)$ such that

$$e^{i\eta(g;b)/N} A_{\sigma_j}^{(0)} = e^{i\varphi_{U(g)}^{(b)}} U(g) A_{\sigma_j}^{(0)}(g) U^{-1}(g) \quad (4.25)$$

for any σ_j . We massage Eq. (4.25) into

$$e^{i\theta(g;b)} U^\dagger(g) A_{\sigma_j}^{(0)} U(g) = \sum_{\sigma'_j=1}^{\mathcal{D}} [U(g)]_{\sigma_j \sigma'_j} K_g [A_{\sigma'_j}^{(0)}], \quad (4.26a)$$

where we have introduced the phase

$$\theta(g; b) := \frac{\eta(g; b)}{N} - \phi_{U(g)}^{(b)}. \quad (4.26b)$$

Consider a second element $h \in G_f$ besides from $g \in G_f$. We can use the relation (4.26a) with g replaced by the composition gh . We can also iterate the relation (4.26a) by evaluating the composition $\tilde{U}(g) \tilde{U}(h) |A_{\sigma_j}^{(0)}; b\rangle$. After some algebra (Appendix C 1), one finds that (i) the phase

$$\delta(g, h; b) := c(g) \theta(h; b) + \theta(g; b) - \phi(g, h) - \theta(gh; b) \quad (4.27a)$$

that relates the normalized 2-cocycle defined in Eqs. (3.11) and (3.12) to the phase (4.26a), (ii) the map represented by

$$V(g) := \begin{cases} U(g), & \text{if } c(g) = 0, \\ U(g) K, & \text{if } c(g) = 1, \end{cases} \quad (4.27b)$$

and the \mathcal{D} matrices $A_{\sigma_j}^{(0)}$, are related by

$$e^{i\delta(g,h;b)} A_{\sigma_j}^{(0)} W(g, h) = W(g, h) A_{\sigma_j}^{(0)} \quad (4.27c)$$

for any $\sigma_j = 1, \dots, \mathcal{D}$, where

$$W(g, h) := V(g) V(h) V^{-1}(gh). \quad (4.27d)$$

We are going to make use of the injectivity of the FMPS a second time after massaging Eq. (4.27c) into

$$A_{\sigma_j}^{(0)} = e^{i\delta(g,h;b)} W^{-1}(g, h) A_{\sigma_j}^{(0)} W(g, h) \quad (4.28)$$

for any $\sigma_j = 1, \dots, \mathcal{D}$. For any integer $\ell = 1, 2, \dots$, iteration of Eq. (4.28) gives

$$\prod_{j=1}^{\ell} A_{\sigma_j}^{(0)} = e^{i\ell \delta(g,h;b)} W^{-1}(g, h) \left[\prod_{j=1}^{\ell} A_{\sigma_j}^{(0)} \right] W(g, h). \quad (4.29)$$

When $\ell \geq \ell^*$, injectivity of the FMPS implies that any matrix $A \in \text{Mat}(2M, \mathbb{C})$ can be written as a linear superposition of all the possible monomials $\prod_{j=1}^{\ell} A_{\sigma_j}^{(0)}$ of order ℓ , each of which obeys Eq. (4.29) [recall Eq. (4.19b)]. Hence, we arrive at the identity

$$A = e^{i\ell \delta(g,h;b)} W^{-1}(g, h) A W(g, h), \quad \forall \ell \geq \ell^*, \quad (4.30)$$

for any $A \in \text{Mat}(2M, \mathbb{C})$, which implies, in turn, that $W(g, h)$ belongs to the center of the algebra spanned by monomials $\prod_{j=1}^{\ell} A_{\sigma_j}^{(0)}$. For even-parity FMPS, this center is one-dimensional as it is generated by the unit matrix $\mathbb{1}_{2M}$. In particular, we can choose $A = \mathbb{1}_{2M}$ for which

$$\mathbb{1}_{2M} = e^{i\ell \delta(g,h;b)} \mathbb{1}_{2M}, \quad (4.31a)$$

which implies that

$$\delta(g, h; b) = 0, \quad (4.31b)$$

and, therefore, $[\phi] = 0$ [recall Eq. (3.15)].

2. Case of odd-parity injective FMPS

The odd-parity FMPS differs from the even-parity FMPS in that the \mathcal{D}^ℓ products $A_{\sigma_1}^{(1)} \cdots A_{\sigma_\ell}^{(1)}$ for any $\ell \geq \ell^*$ span a subalgebra of $\text{Mat}(2M, \mathbb{C})$ with the center spanned by $\mathbb{1}_{2M}$ and Y . This difference is of no consequence until reaching the odd-parity counterpart to Eq. (4.25). However, for the odd-parity counterpart to Eq. (4.25) multiplication of $U(g)$ from the left by any element from the center generated by $\mathbb{1}_{2M}$ and Y ,

$$[a(g) \mathbb{1}_{2M} + b(g) Y] U(g), \quad |a(g)|^2 + |b(g)|^2 = 1, \quad (4.32)$$

leaves Eq. (4.25) unchanged. To fix this subtlety, we replace $U(g)$ in Eq. (4.25) by $U^{(0)}(g)$ which is defined by

$$U(g) := [a(g) \mathbb{1}_{2M} + b(g) Y] U^{(0)}(g), \quad (4.33a)$$

$$P U^{(0)}(g) P = U^{(0)}(g). \quad (4.33b)$$

With this change in mind, all the steps leading to Eq. (4.27) for the even-parity case can be repeated for the odd-parity case. The analog to the even-parity coboundary condition (4.31b) then follows, thereby completing the proof of Theorem 1.

C. Proof of Theorem 2

Theorem 1 presumes the existence of a local fermionic Fock space, i.e., of an even number of Majorana degrees of freedom per repeat unit cell. This hypothesis precludes translation invariant lattice Hamiltonians with odd number of Majorana operators per repeat unit cell such as

$$\hat{H}_K := \sum_{j=1}^{2M} i \hat{\gamma}_j \hat{\gamma}_{j+1}. \quad (4.34a)$$

Here, the Hermitian operators $\{\hat{\gamma}_j = \hat{\gamma}_j^\dagger\}$ obey the Majorana algebra

$$\{\hat{\gamma}_j, \hat{\gamma}_{j'}\} = 2\delta_{jj'}, \quad j, j' = 1, \dots, 2M, \quad (4.34b)$$

and the total number $2M$ of repeat unit cell is an even integer. Hamiltonian \hat{H}_K realizes the critical point between the two topologically distinct phases of the Kitaev chain. In the continuum limit, it describes a helical pair of Majorana fields and has a gapless spectrum.

Motivated by this example, we now prove a separate LSM constraint on Majorana lattice models with an odd Majorana flavors per repeat unit site. We use Theorem 1 for the proof.

Let $n \geq 0$ be an integer and

$$\hat{\chi}_j := (\hat{\chi}_{j,1}, \hat{\chi}_{j,2}, \dots, \hat{\chi}_{j,2m+1})^\top \quad (4.35)$$

be the spinor made of $2m+1$ Majorana operators. Let the Hamiltonian \hat{H} be local and translationally invariant. We write

$$\hat{H} \equiv \sum_{j=1}^{2M} \hat{h}(\hat{\chi}_{j-q}, \dots, \hat{\chi}_j, \dots, \hat{\chi}_{j+q}), \quad (4.36)$$

where \hat{h} is a Hermitian polynomial of $2q$ Majorana spinors $\{\hat{\chi}_{j-q}, \dots, \hat{\chi}_{j+q}\}$ with q a positive integer. The finiteness of q renders \hat{H} local. Hamiltonian (4.36) is defined over $2M$ sites, since an even number of Majorana operators are needed to have a well-defined Fock space. We assume that \hat{H} has

a nondegenerate gapped ground state $|\Psi_0\rangle$. We are going to deliver a contradiction by making use of Theorem 1, thereby proving Theorem 2.

Define the Hamiltonian,

$$\widehat{H}' := \sum_{j=1}^{2M} \sum_{\alpha=1}^2 \widehat{h}(\widehat{\chi}_{j-q}^{(\alpha)}, \dots, \widehat{\chi}_{j+q}^{(\alpha)}), \quad (4.37a)$$

which is the sum of two copies of Hamiltonian (4.36). The repeat unit cell labeled by $j = 1, \dots, 2M$ now contains two Majorana spinors labeled by $\alpha = 1, 2$. Hamiltonian (4.37) thus acts on a Fock space which is locally spanned by an even number of Majorana flavors. At each site $j = 1, \dots, 2M$, one can define a local fermionic Fock space. Since there is no coupling between the two copies $\alpha = 1, 2$ of Majorana spinors, \widehat{H}' inherits from \widehat{H} the nondegenerate gapped ground state

$$|\Psi_0'\rangle := |\Psi_0\rangle \otimes_g |\Psi_0\rangle. \quad (4.37b)$$

Since at each site j , there is no term coupling the two copies $\widehat{\chi}_j^{(1)}$ and $\widehat{\chi}_j^{(2)}$, \widehat{H}' is invariant under any local permutation

$$\begin{pmatrix} \widehat{\chi}_j^{(1)} \\ \widehat{\chi}_j^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} \widehat{\chi}_j^{(2)} \\ \widehat{\chi}_j^{(1)} \end{pmatrix}. \quad (4.38a)$$

The local representation of the fermion parity operator is

$$\widehat{P}_j := \prod_{l=1}^{2n+1} [i \widehat{\chi}_{j,l}^{(1)} \widehat{\chi}_{j,l}^{(2)}]. \quad (4.38b)$$

Under the transformation (4.38a), the local fermion parity operator \widehat{P}_j acquires the phase $(-1)^{2n+1} = -1$. Therefore the symmetry transformation (4.38a) anticommutes with \widehat{P}_j . This anticommutation relation implies a nontrivial second group cohomology class $[\phi] \neq 0$ of G_f , independent of the group of onsite symmetries of Hamiltonian (4.36).⁷ Therefore by Theorem 1 Hamiltonian \widehat{H}' cannot have a nondegenerate gapped ground state. This is in contradiction with the initial assumption that Hamiltonian (4.36) has the nondegenerate gapped ground state $|\Psi_0\rangle$.

We observe that dimensionality d of space played no role in the proof of Theorem 2 until Theorem 1 was used. Hence, Theorem 2 holds for any d if Theorem 1 holds for any d .

One can interpret Theorem 2 as the inability to write down an injective FMPS for the ground state of translationally invariant Hamiltonians with an odd number of Majorana flavors per repeat unit cell. This is because one cannot define the matrices A_{σ_j} as there is no well-defined Fock space at site j to begin with.

D. LSM constraints and classification of 1D fermionic SPTs

In order to prove Theorem 1, we have shown that $\delta(g, h; b)$ defined in Eq. (4.27a) vanishes. Consequently, $W(g, h; b)$ defined in Eq. (4.27d) must be proportional to the unit matrix

$\mathbb{1}_{2M}$. If so, the similarity transformations $V(g; b)$, $V(h; b)$, and $V^{-1}(g; h; b)$ that enter $W(g, h; b)$ must also realize a projective representation of G_f . This observation allows us to draw a bridge to the classification of 1D fermionic SPT phases.

It is known that group cohomology classes corresponding to representations of G_f induced by similarity transformations $V(g)$ classify bosonic [22,39,61] and fermionic [55–59,62] SPT phases. Similarly, for a given symmetry group G_f and in 1D, fermionic SPT phases are classified by a triplet of indices $([(v, \rho)], \mu)$. Indices $[(v, \rho)]$ are related to $[\phi] \in H^2(G_f, U(1)_c)$. The component v of the indices $[(v, \rho)]$ encodes the information about the projective representations of the symmetry group G . The component ρ of the indices $[(v, \rho)]$ encodes the algebra between the representations of a group element $g \in G$ and the fermion parity $p \in \mathbb{Z}_2^F$. Finally, the index $\mu \in \{0, 1\}$ characterizes the total fermion parity of the SPT ground state, or equivalently the parity of the total number of boundary Majorana modes.

Although the same cohomology group $H^2(G_f, U(1)_c)$ appears in the classification of 1D LSM-type constraints and 1D SPT phases, they have a different origin. For the 1D LSM-type constraints, $H^2(G_f, U(1)_c)$ arises when classifying the projective representations of G_f on a local Fock space. For the 1D SPT phases, $H^2(G_f, U(1)_c)$ arises when classifying the boundary projective representations of global symmetries.

V. MAJORANA LSM THEOREMS IN HIGHER DIMENSIONS

In this section, we extend Theorem 1 to any dimension d of space when the symmetry group G_f is Abelian and all elements $g \in G_f$ are represented by unitary operators. Our method is inspired by the one used recently in Ref. [63] for quantum spin Hamiltonians.

Consider a d -dimensional lattice Λ with periodic boundary conditions in each linearly independent direction $\hat{\mu} = \hat{1}, \dots, \hat{d}$ such that Λ realizes a d -torus. Let each repeat unit cell be labeled as \mathbf{j} and host a local fermionic Fock space \mathcal{F}_j that is generated by a Majorana spinor $\widehat{\chi}_j$ with $2n$ components $\widehat{\chi}_{j,l}$, $l = 1, \dots, 2n$. The fermionic Fock space attached to the lattice Λ is denoted by \mathcal{F}_Λ . We impose the global symmetry corresponding to the central extension G_f of G by \mathbb{Z}_2^F as defined in Sec. III A, whereby G_f is assumed to be Abelian. We also impose translation symmetry. If the d -dimensional lattice Λ has $N_{\hat{\mu}}$ repeat unit cell in the $\hat{\mu}$ direction and thus the cardinality

$$|\Lambda| \equiv \prod_{\hat{\mu}=\hat{1}}^d N_{\hat{\mu}}, \quad (5.1)$$

the translation group is

$$G_{\text{trsl}} \equiv \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \dots \times \mathbb{Z}_{N_d}. \quad (5.2)$$

By assumption, the combined symmetry group is the Cartesian product group

$$G_{\text{total}} \equiv G_{\text{trsl}} \times G_f. \quad (5.3)$$

⁷This anticommutation relation implies nontrivial index $[\rho]$ which is defined in Sec. III B 1.

The representation of the translation group (5.2) is generated by the unitary operator $\widehat{T}_{\hat{\mu}}$ whose action on the Majorana spinors is

$$\widehat{T}_{\hat{\mu}} \hat{\chi}_j \widehat{T}_{\hat{\mu}}^{-1} = \hat{\chi}_{j+e_{\hat{\mu}}}, \quad \widehat{T}_{\hat{\mu}}^{-1} = \widehat{T}_{\hat{\mu}}^{\dagger}, \quad (5.4a)$$

along the $\hat{\mu}$ direction ($e_{\hat{\mu}}$ is a basis-vector along the $\hat{\mu}$ direction). Imposing periodic boundary conditions implies

$$(\widehat{T}_{\hat{\mu}})^{N_{\hat{\mu}}+1} = \widehat{T}_{\hat{\mu}}. \quad (5.4b)$$

The representation $\widehat{U}(g)$ of $g \in G_f$ is defined in Sec. III B. Any translationally and G_f -invariant local Hamiltonian acting on \mathcal{F}_{Λ} can be written in the form

$$\widehat{H}_{\text{pbc}} := \sum_{\hat{\mu}=\hat{i}}^d \sum_{n_{\hat{\mu}}=1}^{N_{\hat{\mu}}} (\widehat{T}_{\hat{\mu}})^{n_{\hat{\mu}}} \hat{h}_j (\widehat{T}_{\hat{\mu}}^{\dagger})^{n_{\hat{\mu}}}, \quad (5.5a)$$

where \hat{h}_j is a local Hermitian operator centered at an arbitrary repeat unit cell j . More precisely, it is a finite-order polynomial in the Majorana operators centered at j that is also invariant under all the nonspatial symmetries, i.e.,

$$\hat{h}_j = \widehat{U}(g) \hat{h}_j \widehat{U}^{-1}(g) = (\hat{h}_j)^{\dagger} \quad (5.5b)$$

for any $g \in G_f$. Instead of extracting spectral properties of Hamiltonian \widehat{H}_{pbc} directly, we shall do so with the family of Hamiltonians indexed by $g \in G_f$ and given by

$$\widehat{H}_{\text{twis}}^{\text{ilt}}(g) := \sum_{a=1}^{|\Lambda|} (\widehat{T}_1(g))^a \hat{h}_1^{\text{ilt}} (\widehat{T}_1^{-1}(g))^a, \quad (5.6)$$

where \hat{h}_1^{ilt} is a G_f -symmetric and local Hermitian operator and $\widehat{T}_1(g)$ is the “ g -twisted translation operator” to be defined shortly. We shall derive LSM-like constraints for $\widehat{H}_{\text{twis}}^{\text{ilt}}(g)$ and then explain why those LSM-like constraints also apply to \widehat{H}_{pbc} . To this end, we will explain what is meant by the upper index “ilt” for tilted and the lower index “twis” for twisted and how $\widehat{H}_{\text{twis}}^{\text{ilt}}(g)$ and \widehat{H}_{pbc} differ.

A. Case of a $d = 1$ -dimensional lattice

As a warm up, we first consider the one-dimensional case, i.e., $\Lambda \cong \mathbb{Z}_N$. We impose two assumptions in addition to those previously assumed. These are that every element in G_f is unitarily represented (*Assumption 5*) and that G_f is an Abelian group (*Assumption 6*). These two assumptions were superfluous when proving Theorem 1 using injective FMPS in Sec. IV. This drawback is compensated by the possibility to extend the proof that follows to any dimension d of space.

Twisted boundary conditions are implemented by defining the symmetry twisted translation operator

$$\widehat{T}_1(g) := \hat{v}_1(g) \widehat{T}_1 \quad (5.7a)$$

through its action

$$\widehat{T}_1(g) \hat{\chi}_j \widehat{T}_1^{-1}(g) = \begin{cases} (-1)^{\rho(g)} \hat{\chi}_{j+1}, & \text{if } j \neq N, \\ \hat{v}_1(g) \hat{\chi}_1 \hat{v}_1^{-1}(g), & \text{if } j = N, \end{cases} \quad (5.7b)$$

for $j = 1, \dots, N$, where $\rho(g) \in \{0, 1\} \equiv \mathbb{Z}_2$ is defined in Eq. (3.10d) [see also Eq. (3.28)]. We then consider any Hamiltonian of the form (5.6) where the operator \hat{h}_1^{ilt} in Eq. (5.6) is

nothing but the operator \hat{h}_j in Eq. (5.5a) with Λ restricted to a one-dimensional lattice. Such a twisted boundary condition is equivalent to coupling the Majorana operators to a background Abelian gauge field with a holonomy $g \in G_f$ around the spatial cycle. The effect of turning on such a background field is that it delivers the operator algebra (see Appendix D)

$$[\widehat{T}_1(g)]^N = \widehat{U}(g), \quad g \in G_f \quad (5.8a)$$

and

$$\widehat{U}(h)^{-1} \widehat{T}_1(g) \widehat{U}(h) = e^{i\chi(g,h)} \widehat{T}_1(g), \quad h \in G_f, \quad (5.8b)$$

where

$$\chi(g, h) := \phi(h, g) - \phi(g, h) + (N-1)\pi \rho(h)[\rho(g) + 1]. \quad (5.8c)$$

The same algebra with $\rho(g) \equiv 0$ for all $g \in G_f$ was obtained by Yao and Oshikawa in Refs. [63,64]. The phase $\chi(g, h)$ is vanishing if and only if the second cohomology class $[\phi]$ is trivial (see Appendix D). As explained in Sec. III B, we can trade the index $[\phi]$ with the indices $[(\nu, \rho)]$.

If $\chi(g, h) \bmod 2\pi$ is nonvanishing, one-dimensional representations of (5.8) are not allowed. The ground state of any Hamiltonian of the form (5.6) is either degenerate or spontaneously breaks the symmetry in the thermodynamic limit. We have rederived the Theorem 1 for the Abelian group G_f that is represented unitarily when twisted boundary conditions apply.

If we assume that the choice of boundary conditions cannot change the ground-state degeneracy when all excited states are separated from the ground states by an energy gap, then the Theorem 1 applies to all boundary conditions compatible with translation symmetry that are imposed on the one-dimensional chain Λ and, in particular, to Hamiltonians of the form (5.5) with Λ restricted to a one-dimensional lattice that obey periodic boundary conditions. A necessary condition for this assumption to hold is that all correlation functions between local operators decay sufficiently fast, a condition known to be an attribute of any Hamiltonian with gapped ground states [65].

We emphasize that, in rederiving Theorem 1, we have taken (i) the group G_f to be Abelian and (ii) representation $\hat{u}_j(g)$ to be unitary for all $g \in G_f$. There exist several challenges in relaxing both of these assumptions. When the group is taken to be non-Abelian, one cannot consistently define a twisted Hamiltonian (5.6) that is invariant under both global symmetry transformations $\widehat{U}(h)$ and symmetry twisted translation operators $\widehat{T}_1(g)$ without imposing stricter constraints on local operators \hat{h}_1^{ilt} than Eq. (5.6). The challenges with imposing antiunitary twisted boundary conditions with the group element $g \in G_f$ are the following. First, complex conjugation is applied on all the states in the Fock space \mathcal{F}_{Λ} . This means that Hamiltonian (5.5) can differ from Hamiltonian (5.6) through an extensive number of terms when $\mathfrak{c}(g) = -1$, in which case it is not obvious to us how to safely tie some spectral properties of Hamiltonians (5.6) and (5.5). Second, not all representations of the group G_f are either even or odd under complex conjugation, in which case conjugation of $\widehat{T}_1(g)$ by $\widehat{U}(h)^{-1}$ need not result anymore in a mere phase factor multiplying $\widehat{T}_1(g)$ when $\mathfrak{c}(g) = -1$. In view of this difficulty with

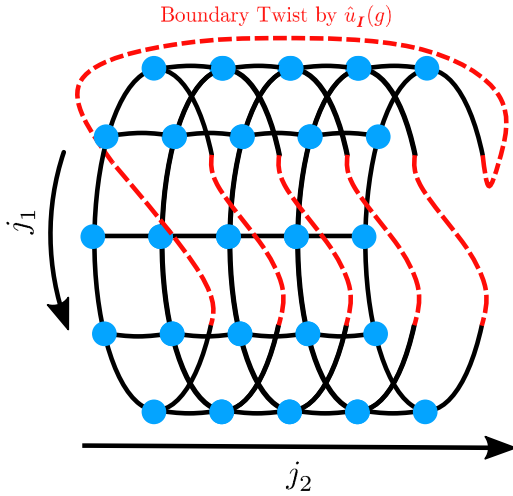


FIG. 2. Example of a path that visits all the sites of a two-dimensional lattice that decorates the surface of a torus.

interpreting antiunitary twisted boundary conditions, we observe that the FMPS construction of LSM-type constraints is more general than the one using twisted boundary conditions.

B. Case of a $d > 1$ -dimensional lattice

We now assume that Λ is a $d > 1$ -dimensional lattice. We would like to generalize the twisted boundary conditions (5.7) obeyed by the Majorana operators to arbitrary spatial dimensions. There is no unique way for doing so. In what follows, we construct a group of translations $G_{\text{trsl}}^{\text{tilt}}$ that is cyclic. This is achieved by imposing *tilted* or *sheared* boundary conditions. After constructing $G_{\text{trsl}}^{\text{tilt}}$, we twist the boundary conditions in a particular way using the local representations of elements of the on-site (internal) symmetry group G_f . The operators representing translations on the lattice with tilted and twisted boundary conditions may not commute with the operators representing elements of G_f , even though all elements of $G_{\text{trsl}}^{\text{tilt}}$ commute with all elements of G_f by assumption (5.3). When this is so, the representation of $G_{\text{total}}^{\text{tilt}} = G_{\text{trsl}}^{\text{tilt}} \times G_f$ is necessarily larger than one dimensional, in which case the ground states are either degenerate or the symmetry group $G_{\text{total}}^{\text{tilt}} = G_{\text{trsl}}^{\text{tilt}} \times G_f$ is spontaneously broken.

Our strategy is to construct the counterpart of Eqs. (5.7) and (5.8). To this end, we are going to trade the translation symmetry group (5.2), which is a polycyclic group when $d > 1$, for the cyclic group

$$G_{\text{trsl}}^{\text{tilt}} \equiv \mathbb{Z}_{N_1 \cdots N_d} \quad (5.9)$$

and define the combined symmetry group

$$G_{\text{total}}^{\text{tilt}} \equiv G_{\text{trsl}}^{\text{tilt}} \times G_f. \quad (5.10)$$

The intuition underlying the construction of the tilted translation symmetry group $G_{\text{trsl}}^{\text{tilt}}$ is provided by Fig. 2. As a set, the elements of $G_{\text{trsl}}^{\text{tilt}}$ can be labeled by the elements of G_{trsl} , namely

$$G_{\text{trsl}}^{\text{tilt}} := \left\{ ((t_1)^{n_1}, \dots, (t_d)^{n_d}) \mid n_{\hat{\mu}} = 1, \dots, N_{\hat{\mu}}, \quad \hat{\mu} = \hat{1}, \dots, \hat{d} \right\}. \quad (5.11)$$

However, as a group we would like to label the elements of $G_{\text{trsl}}^{\text{tilt}}$ as those of the cyclic group with $|\Lambda|$ elements, i.e.,

$$G_{\text{trsl}}^{\text{tilt}} := \{t^n \mid n = 1, \dots, |\Lambda|\}. \quad (5.12)$$

This is achieved by carefully choosing the group composition for the elements (5.11), i.e., by iterating $d - 1$ central extensions.

Step 1. We consider \mathbb{Z}_{N_1} generated by t_1 and extend it by \mathbb{Z}_{N_2} generated by t_2 through the map

$$\Theta_1 : \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_1} \rightarrow \mathbb{Z}_{N_2}, \quad (5.13a)$$

$$\Theta_1((t_1)^a, (t_1)^b) := (t_2)^{\frac{1}{N_1}(a+b-[a+b]_{N_1})},$$

for any $a, b = 1, \dots, N_1$, to obtain $\mathbb{Z}_{N_1 N_2}$, the group of translations on the tilted lattice restricted to \mathbb{R}^2 . Here, the notation $[a+b]_n$ is used to denote addition modulo n . This group extension can be summarized by the short exact sequence

$$1 \rightarrow \mathbb{Z}_{N_2} \rightarrow \mathbb{Z}_{N_1 N_2} \rightarrow \mathbb{Z}_{N_1} \rightarrow 1 \quad (5.13b)$$

and is labeled by the extension classes

$$[\Theta_1] \in H^2(\mathbb{Z}_{N_1}, \mathbb{Z}_{N_2}). \quad (5.13c)$$

Using this extension class and the standard expression for group composition in an extended group, we may identify t_1 as the generator of $\mathbb{Z}_{N_1 N_2}$.

Step 2. We consider $\mathbb{Z}_{N_1 N_2}$ generated by t_1 and extend it by \mathbb{Z}_{N_3} generated by t_3 through the map

$$\Theta_2 : \mathbb{Z}_{N_1 N_2} \times \mathbb{Z}_{N_1 N_2} \rightarrow \mathbb{Z}_{N_3}, \quad (5.14a)$$

$$\Theta_2((t_1)^a, (t_1)^b) := (t_3)^{\frac{1}{N_1 N_2}(a+b-[a+b]_{N_1 N_2})},$$

for any $a, b = 1, \dots, N_1 N_2$, to obtain $\mathbb{Z}_{N_1 N_2 N_3}$ the group of translations on the tilted lattice restricted to \mathbb{R}^3 . This group extension can be summarized by the short exact sequence

$$1 \rightarrow \mathbb{Z}_{N_3} \rightarrow \mathbb{Z}_{N_1 N_2 N_3} \rightarrow \mathbb{Z}_{N_1 N_2} \rightarrow 1 \quad (5.14b)$$

and is labeled by the extension classes

$$[\Theta_2] \in H^2(\mathbb{Z}_{N_1 N_2}, \mathbb{Z}_{N_3}). \quad (5.14c)$$

Using this extension class and the standard expression for group composition in an extended group, we may identify t_1 as the generator of $\mathbb{Z}_{N_1 N_2 N_3}$.

Step $d - 1$. We consider $\mathbb{Z}_{N_1 \cdots N_{d-1}}$ generated by t_1 and extend it by \mathbb{Z}_{N_d} generated by t_d through the map

$$\Theta_{d-1} : \mathbb{Z}_{N_1 \cdots N_{d-1}} \times \mathbb{Z}_{N_1 \cdots N_{d-1}} \rightarrow \mathbb{Z}_{N_d}, \quad (5.15a)$$

$$\Theta_{d-1}((t_1)^a, (t_1)^b) := (t_d)^{\frac{1}{N_1 \cdots N_{d-1}}(a+b-[a+b]_{N_1 \cdots N_{d-1}})},$$

for any $a, b = 1, \dots, N_1 \cdots N_{d-1}$, to obtain $\mathbb{Z}_{N_1 \cdots N_d}$ the group of translations on the tilted lattice Λ . This group extension can be summarized by the short exact sequence

$$1 \rightarrow \mathbb{Z}_{N_d} \rightarrow \mathbb{Z}_{N_1 \cdots N_d} \rightarrow \mathbb{Z}_{N_1 \cdots N_{d-1}} \rightarrow 1 \quad (5.15b)$$

and is labeled by the extension classes

$$[\Theta_{d-1}] \in H^2(\mathbb{Z}_{N_1 \cdots N_{d-1}}, \mathbb{Z}_{N_d}). \quad (5.15c)$$

Using this extension class and the standard expression for group composition in an extended group, we may identify $t_{\hat{1}}$ as the generator of $\mathbb{Z}_{N_1 \dots N_d}$.

At the quantum level, we represent the cyclic group (5.9) by replacing Eq. (5.4) with

$$\widehat{T}_{\hat{\mu}} \hat{\chi}_{\mathbf{j}} \widehat{T}_{\hat{\mu}}^{-1} = \hat{\chi}_{t_{\hat{\mu}}(\mathbf{j})}, \quad (5.16a)$$

where $t_{\hat{\mu}}(\mathbf{j})$ is the action of the cyclic group $G_{\text{trsl}}^{\text{tilt}}$ on the repeat unit cell $\mathbf{j} \in \Lambda$. The cyclicity of $G_{\text{trsl}}^{\text{tilt}} \equiv \mathbb{Z}_{N_1 \dots N_d} \equiv \mathbb{Z}_{|\Lambda|}$ is enforced by

$$(\widehat{T}_{\hat{\mu}})^{N_{\hat{\mu}}} = \begin{cases} \widehat{T}_{\hat{\mu}+1}, & \text{if } \hat{\mu} = \hat{1}, \dots, \hat{d} - \hat{1}, \\ \mathbb{1}_{\Lambda}, & \text{if } \hat{\mu} = \hat{d}. \end{cases} \quad (5.16b)$$

With this convention, $(\widehat{T}_{\hat{1}})^a$ with $a = 1, \dots, |\Lambda|$ represents all the elements $(t_{\hat{1}})^a$ with $a = 1, \dots, |\Lambda|$ of $G_{\text{trsl}}^{\text{tilt}}$. Equation (5.5) is replaced by

$$\widehat{H}^{\text{tilt}} := \sum_{a=1}^{|\Lambda|} (\widehat{T}_{\hat{1}})^a \hat{h}_1^{\text{tilt}} (\widehat{T}_{\hat{1}}^{-1})^a, \quad (5.17a)$$

whereby

$$\hat{h}_1^{\text{tilt}} = \widehat{U}(g) \hat{h}_1^{\text{tilt}} \widehat{U}^{-1}(g) = (\hat{h}_1^{\text{tilt}})^{\dagger} \quad (5.17b)$$

holds for any $g \in G_f$. The locality of the polynomial \hat{h}_1^{tilt} is no longer manifest when comparing the integers that now label the local Majorana operators in \hat{h}_1^{tilt} . The locality of \hat{h}_1^{tilt} is inherited from the fact that $\hat{h}_{\mathbf{j}}$ is local while \hat{h}_1^{tilt} is nothing but a mere rewriting of $\hat{h}_{\mathbf{j}}$ in the cyclic representation of $\mathbf{j} \in \Lambda$. Hamiltonian (5.17) differs from Hamiltonian (5.5) by a subextensive number of terms of order $|\Lambda|/N_{\hat{1}}$. The same number of terms would distinguish Hamiltonian (5.5) from the Hamiltonian $\widehat{H}_{\text{twis}}(g)$ obtained by replacing the periodic boundary conditions (5.4b) by twisted a one, i.e., by multiplying the right-hand side of Eq. (5.4b) with $\widehat{U}(g)$ for $g \neq e$ and $\hat{\mu} = \hat{1}$.

Because of the cyclicity of $G_{\text{trsl}}^{\text{tilt}} \equiv \mathbb{Z}_{N_1 \dots N_d} \equiv \mathbb{Z}_{|\Lambda|}$ and of its quantum representation, we can adapt the definition (5.7) for the twisted translation operator when $d = 1$ to that when $d > 1$. We define for any $g \in G_f$ with $c(g) = +1$ the generator of twisted translation

$$\widehat{T}_{\hat{1}}(g) := \hat{u}_{\mathbf{I}}(g) \widehat{T}_{\hat{1}}, \quad \hat{u}_{\mathbf{I}}^{-1}(g) = \hat{u}_{\mathbf{I}}^{\dagger}(g), \quad (5.18a)$$

through its action

$$\widehat{T}_{\hat{1}}(g) \hat{\chi}_{\mathbf{j}} \widehat{T}_{\hat{1}}^{-1}(g) = \begin{cases} (-1)^{\rho(g)} \hat{\chi}_{t_{\hat{1}}(\mathbf{j})}, & \text{if } \mathbf{j} \neq \mathbf{N}, \\ \hat{u}_{\mathbf{I}}(g) \hat{\chi}_{\mathbf{I}} \hat{u}_{\mathbf{I}}^{-1}(g), & \text{if } \mathbf{j} = \mathbf{N}, \end{cases} \quad (5.18b)$$

on any Majorana operator labeled by $\mathbf{j} \in \Lambda$. Here, $\mathbf{I} \equiv (1, \dots, 1) \in \Lambda$, $\mathbf{N} = (N_1, \dots, N_d) \in \Lambda$, and $\mathbf{j} = (n_1, \dots, n_d)$ with $n_{\hat{\mu}} = 1, \dots, N_{\hat{\mu}}$. One verifies that these twisted translation operators satisfy the twisted operator algebra

$$\widehat{U}(h)^{-1} \widehat{T}_{\hat{1}}(g) \widehat{U}(h) = e^{i\chi(g,h)} \widehat{T}_{\hat{1}}(g), \quad (5.19a)$$

where

$$\chi(g, h) := \phi(g, h) - \phi(h, g) + (|\Lambda| - 1)\pi \rho(h)[\rho(g) + 1], \quad (5.19b)$$

which is nothing but the algebra (5.8) with the identification $N \rightarrow |\Lambda| \equiv N_1 \dots N_d$. Finally, we define the family of Hamiltonians (5.6) that obey twisted boundary conditions. The proof of Theorem 1 when $d > 1$ for Hamiltonians of the form (5.6) is the same as that when $d = 1$. Because the family of Hamiltonians (5.6) only differ from the family of Hamiltonians (5.5) obeying periodic boundary conditions by a subextensive number of terms, the LSM-like conditions characterizing the existence of nondegenerate gapped ground states valid for Hamiltonians of the form (5.6) are conjectured to be also valid for Hamiltonians of the form (5.5).

C. Theorem 2 in $d > 1$ dimensions

We have extended Theorem 1 to any spatial dimension d . As discussed at the end of Sec. IV C, if Theorem 1 holds for any d , then so does Theorem 2. It is nevertheless instructive to provide an alternative proof of Theorem 2 for any spatial dimension d without relying on Theorem 1.

We consider a d -dimensional lattice Λ such that at each repeat unit cell labeled by $\mathbf{j} \in \Lambda$, there exists a Majorana spinor $\hat{\chi}_{\mathbf{j}}$ with $2n + 1$ components $\hat{\chi}_{\mathbf{j},l}$, $l = 1, \dots, 2n + 1$. To have a well-defined total Fock space on lattice Λ , we set the total number of sites $|\Lambda|$ in the lattice to be even. On lattice Λ , we impose the tilted translation symmetry group $G_{\text{trsl}}^{\text{tilt}}$ defined in Eq. (5.12). Let $\widehat{T}_{\hat{1}}$ be the representation of the generator of the cyclic group $G_{\text{trsl}}^{\text{tilt}}$ with the action (5.16) on the Majorana spinors $\hat{\chi}_{\mathbf{j}}$.

In terms of the Majorana spinors $\hat{\chi}_{\mathbf{j}}$, the total fermion parity operator \widehat{P} has the representation

$$\widehat{P} := i^{|\Lambda|/2} \prod_{\mathbf{j} \in \Lambda} \prod_{l=1}^{2n+1} \hat{\chi}_{\mathbf{j},l}. \quad (5.20)$$

Conjugation of the fermion parity operator \widehat{P} by the tilted translation operator $\widehat{T}_{\hat{1}}$ delivers

$$\widehat{T}_{\hat{1}} \widehat{P} \widehat{T}_{\hat{1}}^{-1} = (-1)^{|\Lambda|-1} \widehat{P} = -\widehat{P}, \quad (5.21)$$

where we arrived at the last equality by noting that $|\Lambda|$ is an even integer. The factor $(-1)^{|\Lambda|-1}$ arises since each spinor $\hat{\chi}_{\mathbf{j}}$ consists of an odd number of Majorana operators. The nontrivial algebra (5.21) implies that the ground state of any Hamiltonian that commutes with \widehat{P} , the generators of the tilted translation group, and the generators of G_f is either degenerate or spontaneously breaks translation or G_f symmetry. We have proven Theorem 2. We note that the algebra (5.21) was shown in Ref. [49] for a one dimensional Majorana chain and interpreted as the existence of Witten's quantum-mechanical supersymmetry [66].

VI. EXAMPLES

All our results apply to any central extension G_f of the group G by the group \mathbb{Z}_2^F associated to the fermion parity. Establishing LSM-type conditions requires (i) constructing a projective representation of G_f and (ii) verifying which one of the group cohomology classes $[\phi] \in H^2(G_f, \text{U}(1)_c)$

is realized by this projective representation. We have shown how the group cohomology classes $[\phi] \in H^2(G_f, U(1)_c)$ are associated to the indices $([(\nu, \rho)], \mu)$ with $\nu \in C^2(G, U(1))$, $\rho \in C^1(G, \mathbb{Z}_2)$, and $\mu = 0, 1$ the evenness or oddness of the local number of Majorana degrees of freedom (flavors). It is impossible to proceed any further without choosing the group G .

We shall choose the central extension G_f of the group G by the group \mathbb{Z}_2^F to be the split Abelian group $G_f = G \times \mathbb{Z}_2^F$ with $G = \mathbb{Z}_2^T$, the split Abelian group $G_f = G \times \mathbb{Z}_2^F$ with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and the nonsplit Abelian group $G_f = \mathbb{Z}_4^{FT}$ with $G = \mathbb{Z}_2^T$. Their group cohomology is reviewed in Appendix A. For the first two cases (split groups), we denote the corresponding indices by $([\nu], [\rho], \mu)$ as explained in Sec. III B 1.

Given any one of these groups, we shall define a global fermionic Fock space $\mathcal{F}_\Lambda = \mathcal{F}_0 \oplus \mathcal{F}_1$ and construct a projective representation that realizes the indices $([(\nu, \rho)], \mu)$ labeling $H^2(G_f, U(1)_c)$. The global fermionic Fock space $\mathcal{F}_\Lambda = \mathcal{F}_0 \oplus \mathcal{F}_1$ is here always constructed from $n|\Lambda|$ Hermitian operators

$$\hat{\chi}_{j,a} = \hat{\chi}_{j,a}^\dagger, \quad \mathbf{j} \in \Lambda, \quad a = 1, \dots, n, \quad (6.1a)$$

obeying the Majorana (Clifford) algebra

$$\{\hat{\chi}_{j,a}, \hat{\chi}_{j',a'}\} = 2\delta_{j,j'}\delta_{a,a'}, \quad \mathbf{j}, \mathbf{j}' \in \Lambda, \quad a, a' = 1, \dots, n. \quad (6.1b)$$

The index μ takes the value 0 when $n = 2m$ is an even integer, in which case the cardinality $|\Lambda|$ of the lattice Λ is any positive integer and we may always define the fermionic creation and annihilation operators

$$\hat{c}_{j,2b-1}^\dagger := \frac{\hat{\chi}_{j,2b-1} - i\hat{\chi}_{j,2b}}{2}, \quad \hat{c}_{j,2b-1} := \frac{\hat{\chi}_{j,2b-1} + i\hat{\chi}_{j,2b}}{2}, \quad (6.2a)$$

with $\mathbf{j} \in \Lambda$ and $b = 1, \dots, m$. The local and global fermionic Fock space \mathcal{F}_j and \mathcal{F}_Λ are then

$$\mathcal{F}_j := \text{span} \left\{ \prod_{b=1}^m (\hat{c}_{j,2b-1}^\dagger)^{n_{j,2b-1}} |0\rangle \mid n_{j,2b-1} = 0, 1, \quad \hat{c}_{j,2b-1} |0\rangle = 0 \right\} \quad (6.2b)$$

and

$$\mathcal{F}_\Lambda := \text{span} \left\{ \prod_{\mathbf{j} \in \Lambda} \prod_{b=1}^m (\hat{c}_{j,2b-1}^\dagger)^{n_{j,2b-1}} |0\rangle \mid n_{j,2b-1} = 0, 1, \quad \hat{c}_{j,2b-1} |0\rangle = 0 \right\}, \quad (6.2c)$$

respectively. In order to define the operation of complex conjugation \mathbf{K} on both the local and global Fock space, we define

$$\mathbf{K}(z\hat{c}_{j,2b-1}^\dagger + w\hat{c}_{j,2b-1})\mathbf{K} := z^*\hat{c}_{j,2b-1}^\dagger + w^*\hat{c}_{j,2b-1} \quad (6.3a)$$

for any pair of complex number $z, w \in \mathbb{C}$ (for any $\mathbf{j}, \mathbf{j}' \in \Lambda$ and $b, b' = 1, \dots, m$) and

$$\mathbf{K}|0\rangle \equiv |0\rangle. \quad (6.3b)$$

This implies the transformation law

$$\mathbf{K}\hat{\chi}_{j,2b-1}\mathbf{K} = +\hat{\chi}_{j,2b-1}, \quad \mathbf{K}\hat{\chi}_{j,2b}\mathbf{K} = -\hat{\chi}_{j,2b}, \quad (6.3c)$$

for any $\mathbf{j} \in \Lambda$ and $b = 1, \dots, m$.

The index μ takes the value 1 when $n = 2m + 1$ is an odd integer, in which case the cardinality $|\Lambda|$ of the lattice Λ must be an even positive integer. For notational simplicity, we shall assume that Λ is bipartite, that is the disjoint union of two interpenetrating sublattices Λ_A and Λ_B such that all nearest neighbors of the sites in Λ_A belong to Λ_B and vice versa. We impose on Λ the topology of a torus. We define the fermionic creation and annihilation operators

$$\hat{c}_{j,a}^\dagger := \frac{\hat{\chi}_{j,a} - i\hat{\chi}_{(j+\mu),a}}{2}, \quad \hat{c}_{j,a} := \frac{\hat{\chi}_{j,a} + i\hat{\chi}_{(j+\mu),a}}{2}, \quad (6.4a)$$

where $\mathbf{j} \in \Lambda_A$, $a = 1, \dots, 2m + 1$, and μ is any fixed basis vectors spanning Λ . The global fermionic Fock space \mathcal{F}_Λ is then

$$\mathcal{F}_\Lambda := \text{span} \left\{ \prod_{\mathbf{j} \in \Lambda_A} \prod_{a=1}^{2m+1} (\hat{c}_{j,a}^\dagger)^{n_{j,a}} |0\rangle \mid n_{j,a} = 0, 1, \quad \hat{c}_{j,a} |0\rangle = 0 \right\}. \quad (6.4b)$$

In order to define the operation of complex conjugation \mathbf{K} on the global Fock space, we define

$$\mathbf{K}(z\hat{c}_{j,a}^\dagger + w\hat{c}_{j',a'})\mathbf{K} := z^*\hat{c}_{j,a}^\dagger + w^*\hat{c}_{j',a'} \quad (6.5a)$$

for any pair of complex number $z, w \in \mathbb{C}$ (for any $\mathbf{j}, \mathbf{j}' \in \Lambda_A$ and $a, a' = 1, \dots, 2m + 1$) and

$$\mathbf{K}|0\rangle \equiv |0\rangle. \quad (6.5b)$$

This implies the transformation law

$$\mathbf{K}\hat{\chi}_{j,a}\mathbf{K} = +\hat{\chi}_{j,a}, \quad \mathbf{K}\hat{\chi}_{(j+\mu),a}\mathbf{K} = -\hat{\chi}_{(j+\mu),a}, \quad (6.5c)$$

for any $\mathbf{j} \in \Lambda_A$ and $a = 1, \dots, 2m + 1$.

Both for $\mu = 0, 1$, we shall assume a quantum dynamics governed by Hamiltonians of the form (5.5) where \hat{h}_j in Eq. (5.5b) is a finite-order polynomial in the Majorana operators. The order of each monomial entering this polynomial is necessarily even for \mathbb{Z}_2^F to be a symmetry group. The Hamiltonian is noninteracting if the order of \hat{h}_j is two, interacting otherwise. The finiteness of the order guarantees locality. We also introduce the notion of range of \hat{h}_j which is the largest separation between the lattice indices of the Majorana operators present in \hat{h}_j . If the range vanishes, then Hamiltonian (5.5) is the sum over $|\Lambda|$ commuting Hermitian operators, in which case the spectrum of Hamiltonian (5.5) is obtained by diagonalizing \hat{h}_j .

For the split Abelian group $G_f = G \times \mathbb{Z}_2^F$ with $G = \mathbb{Z}_2^T$, we find that only the projective representations of the group algebra that belong to the trivial group cohomology class can be realized by noninteracting fermions. Any projective representation of the group algebra that belong to a nontrivial group

cohomology class prohibits bilinear terms in the fermions in any G_f -symmetric Hamiltonian of the form (5.5). Such intrinsically interacting Hamiltonians are quantum perturbations of classical Ising-type Hamiltonians.

For the split Abelian group $G_f = G \times \mathbb{Z}_2^F$ with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the nonsplit group $G_f = \mathbb{Z}_4^{FT}$ with $G = \mathbb{Z}_2^T$, we find that any projective representation of the group algebra that belongs to a nontrivial group cohomology class implies that any G_f -symmetric Hamiltonian of the form (5.5) is necessarily gapless when quadratic in the fermions. For any one-dimensional lattice Hamiltonian, Theorem 1 then predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

A. One-dimensional space with the symmetry group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ for an even number of local Majorana flavors

The lattice is $\Lambda = \{1, \dots, N\}$ with $N = 2M$ an even integer and the global fermionic Fock space \mathcal{F}_Λ is of dimension 2^{nM} with n the number of local Majorana flavors. By choosing the cardinality $|\Lambda| = 2M$ to be even, we make sure that the lattice is bipartite. This allows to treat the two values of the index $\mu := n \bmod 2$ in parallel. The symmetry group $G_f := \mathbb{Z}_2^T \times \mathbb{Z}_2^F$ is a split group. The group $G := \mathbb{Z}_2^T = \{e, t\}$ corresponds to reversal of time.

The local antiunitary representation $\hat{u}_j(t)$ of reversal of time generates a projective representation of the group \mathbb{Z}_2^T . The local unitary representation $\hat{u}_j(p)$ of the fermion parity p generates a projective representation of the group \mathbb{Z}_2^F . According to Appendix A 3, all cohomologically distinct projective representations of $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ are determined by the independent indices $[\nu] = 0, 1$ and $[\rho] = 0, 1$ through the relations⁸

$$\hat{u}_j(t) \hat{u}_j(t) = (-1)^{[\nu]} \hat{u}_j(e), \quad (6.6a)$$

$$\hat{u}_j(t) \hat{u}_j(p) = (-1)^{[\rho]} \hat{u}_j(p) \hat{u}_j(t), \quad (6.6b)$$

This gives the four distinct group cohomology classes

$$([\nu], [\rho], 0) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\}. \quad (6.6c)$$

All but the group cohomology class $([\nu], [\rho], 0) = (1, 0, 0)$ can be realized using $n = 2$ local Majorana flavors. The group cohomology class $([\nu], [\rho], 0) = (1, 0, 0)$ requires at least $n = 4$ local Majorana flavors for it to be realized. We will start with the nontrivial projective representation in the group cohomology class $([\nu], [\rho], 0) = (1, 1, 0)$ that we shall represent using two local Majorana flavors. We will then construct successively the projective representations in the group cohomology classes $([\nu], [\rho], 0) = (1, 0, 0)$, $([\nu], [\rho], 0) = (0, 1, 0)$, and $([\nu], [\rho], 0) = (0, 0, 0)$ by using the graded tensor product, i.e., by considering 4, 6, and 8 flavors of local Majoranas, respectively. This will allow us to verify explicitly the stacking rules of Sec. III C according to which Eq. (3.35)

⁸We have chosen the convention of always representing the generator p of \mathbb{Z}_2^F by a Hermitian operator according to Eq. (3.10e).

simplifies to the rule

$$([\nu], [\rho], 0) = ([\nu_1] + [\nu_2] + [\rho_1][\rho_2], [\rho_1] + [\rho_2], 0) \quad (6.7)$$

when $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^F$. The indices (6.6c) will thus be shown to form the cyclic group \mathbb{Z}_4 with respect to the stacking rule (6.7).

1. Group cohomology class $([\nu], [\rho], \mu) = (1, 1, 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 2$ is generated by the doublet of Majorana operators

$$\hat{\chi}_j \equiv \begin{pmatrix} \hat{\chi}_{j,1} \\ \hat{\chi}_{j,2} \end{pmatrix}, \quad j = 1, \dots, 2M. \quad (6.8)$$

One verifies that

$$\hat{u}_j(t) := -i\hat{\chi}_{j,2} \mathbf{K}, \quad (6.9a)$$

$$\hat{u}_j(p) := i\hat{\chi}_{j,1} \hat{\chi}_{j,2}, \quad (6.9b)$$

realizes the projective algebra (6.6) with

$$[\nu] = 1, \quad [\rho] = 1. \quad (6.9c)$$

One verifies that the Majorana doublet (6.8) is odd under conjugation by both $\hat{u}_j(t)$ and $\hat{u}_j(p)$. Time-reversal symmetry forbids any Hermitian quadratic form for the doublet (6.8).

The only Hamiltonian of the form (5.5) that is of quartic order and of range $r = 1$ is

$$\hat{H}_{\text{pbc}} = \lambda \sum_{j=1}^{2M} \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j+1,1} \hat{\chi}_{j+1,2}, \quad \lambda \in \mathbb{R}. \quad (6.10)$$

This Hamiltonian is nothing but the sum

$$\hat{H}_{\text{pbc}} = \sum_{j=1}^{2M} \hat{h}_j \quad (6.11a)$$

over commuting operators

$$\hat{h}_j := -4\lambda(\hat{n}_j - \frac{1}{2})(\hat{n}_{j+1} - \frac{1}{2}) \quad (6.11b)$$

when expressed in terms of the fermion-number operator

$$\begin{aligned} \hat{n}_j &= \hat{c}_j^\dagger \hat{c}_j, \quad \hat{n}_j \hat{c}_{j'}^\dagger |0\rangle = \delta_{j,j'} n_j \hat{c}_{j'}^\dagger |0\rangle, \\ n_j &= 0, 1, \quad j, j' \in \Lambda. \end{aligned} \quad (6.11c)$$

It is thus diagonalized in the fermion-number basis

$$\mathcal{F}_\Lambda = \text{span}\{|n_1, \dots, n_{2M}\rangle\} \quad (6.12a)$$

in which it is represented by the classical Ising Hamiltonian

$$\hat{H}_{\text{pbc}}^{(1)} := -\lambda \sum_{j=1}^{2M} \sigma_j \sigma_{j+1}, \quad \sigma_j := 2n_j - 1. \quad (6.12b)$$

The subspace \mathcal{F}_{gs} in the global Fock space $\mathcal{F}_\Lambda = \mathcal{F}_0 \oplus \mathcal{F}_1$ that is spanned by the linearly independent ground states is twofold degenerate. It is spanned by either the ferromagnetic states

$$\mathcal{F}_{\text{gs}} = \text{span}\{|0, 0, \dots, 0, 0\rangle, |1, 1, \dots, 1, 1\rangle\} \subset \mathcal{F}_0 \quad (6.13a)$$

when $\lambda > 0$ or the antiferromagnetic states

$$\mathcal{F}_{\text{gs}} = \text{span}\{|1, 0, \dots, 1, 0\rangle, |0, 1, \dots, 0, 1\rangle\} \subset \mathcal{F}_{M \bmod 2} \quad (6.13b)$$

when $\lambda < 0$. Because we made sure that the lattice is bipartite ($|\Lambda| = N = 2M$), \mathcal{F}_{gs} is homogeneous.⁹ All ground states are separated from all excited states by the gap $|2\lambda|$. The action of “time reversal” in the fermion representation is that of a “particle-hole” transformation under which $\hat{n}_j \mapsto (1 - \hat{n}_j)$ and $\sigma_j \mapsto -\sigma_j$. The action of parity is trivial (the identity) in the fermion representation. Reversal of time is broken spontaneously at zero temperature and in the thermodynamic limit in the sense that applying either a uniform or staggered magnetic field that couples to the Ising spins through a Zeeman coupling, taking the thermodynamic limit, and switching off the Zeeman coupling selects one of the two degenerate ground states when $\lambda > 0$ and $\lambda < 0$, respectively.

The complexity of the Hamiltonian of the form (5.5) that is of quartic order and of range $r = 2$ increases dramatically. For any cluster made of the three repeat unit cells $j - 1$, j , and $j + 1$, one can construct three groups of five monomials. The first group is made of

$$\widehat{X}_{j;12|12|00} := \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j,2}, \quad (6.14a)$$

$$\widehat{X}_{j;12|10|10} := \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j+1,1}, \quad (6.14b)$$

$$\widehat{X}_{j;12|10|02} := \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j+1,2}, \quad (6.14c)$$

$$\widehat{X}_{j;12|02|10} := \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,2} \hat{\chi}_{j+1,1}, \quad (6.14d)$$

$$\widehat{X}_{j;12|02|02} := \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,2} \hat{\chi}_{j+1,2}. \quad (6.14e)$$

The second group is made of $\widehat{X}_{j;12|00|12}$, $\widehat{X}_{j;10|10|12}$, $\widehat{X}_{j;10|02|12}$, $\widehat{X}_{j;02|10|12}$, and $\widehat{X}_{j;02|02|12}$. The third group is made of $\widehat{X}_{j;00|12|12}$, $\widehat{X}_{j;10|12|10}$, $\widehat{X}_{j;10|12|02}$, $\widehat{X}_{j;02|12|10}$, and $\widehat{X}_{j;02|12|02}$. Because of translation invariance, only one of the two monomials $\widehat{X}_{j;00|12|12}$ and $\widehat{X}_{j;12|12|00}$ needs to be accounted for. We are left with the generic cluster Hamiltonian

$$\begin{aligned} \hat{h}_j &= \lambda_1 \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j,2} \\ &+ \lambda_2 \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j+1,1} \hat{\chi}_{j+1,2} \\ &+ \lambda_3 \hat{\chi}_{j-1,1} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j+1,1} \\ &\quad \dots \\ &+ \lambda_{14} \hat{\chi}_{j-1,2} \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j+1,2} \end{aligned} \quad (6.15)$$

with fourteen real-valued couplings. If we set the twelve couplings $\lambda_3 = \dots = \lambda_{14} = 0$ to zero, we obtain the classical Ising model

$$\widehat{H}_{\text{pbcc}}^{\text{I}} := - \sum_{j=1}^{2M} (\lambda_1 \sigma_j \sigma_{j+1} + \lambda_2 \sigma_j \sigma_{j+2}) \quad (6.16)$$

in the same fermion-number basis of the Fock space that delivered the classical Ising Hamiltonian (6.12b). Any perturbation with one of the couplings $\lambda_3, \dots, \lambda_{14}$ breaks the

fermion-number conservation down to the conservation of the fermion-parity number. Any gapped phase in the 14-dimensional coupling space is predicted by Theorem 1 to be either degenerate or break spontaneously time-reversal or translation symmetry. This prediction can be verified explicitly for the classical Ising model (6.16) with nearest- and next-nearest-neighbor interactions.

2. Group cohomology class $([v], [\rho], \mu) = (1, 0, 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 4$ is generated by the quartet of Majorana operators¹⁰

$$\hat{\chi}_j \equiv (\hat{\chi}_{j,1} \quad \dots \quad \hat{\chi}_{j,4})^T, \quad j = 1, \dots, 2M. \quad (6.17)$$

One verifies that

$$\hat{u}_j(t) := -i \hat{\chi}_{j,2} \hat{\chi}_{j,4} \mathbf{K}, \quad (6.18a)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4}, \quad (6.18b)$$

realizes the projective algebra (6.6) with

$$[v] = 1, \quad [\rho] = 0. \quad (6.18c)$$

One verifies that the Majorana quartet (6.17) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. Time-reversal symmetry forbids any Hermitian quadratic form for the quartet (6.17).

The only Hamiltonian of the form (5.5) that is of quartic order and of range $r = 0$ is

$$\widehat{H}_{\text{pbcc}} = \lambda \sum_{j=1}^{2M} \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4}, \quad \lambda \in \mathbb{R}. \quad (6.19)$$

This Hamiltonian is nothing but the sum

$$\widehat{H}_{\text{pbcc}} = \sum_{j=1}^{2M} \hat{h}_j \quad (6.20a)$$

over commuting operators

$$\hat{h}_j := -4\lambda (\hat{n}_{j,1} - \frac{1}{2}) (\hat{n}_{j,3} - \frac{1}{2}) \quad (6.20b)$$

when expressed in terms of the fermion-number operator

$$\begin{aligned} \hat{n}_{j,2b-1} &:= \hat{c}_{j,2b-1}^\dagger \hat{c}_{j,2b-1}, \\ \hat{n}_{j,2b-1} \hat{c}_{j',2b'-1}^\dagger |0\rangle &= \delta_{j,j'} \delta_{b,b'} n_{j',2b'-1} \hat{c}_{j',2b'-1}^\dagger |0\rangle, \\ n_{j,2b-1} &= 0, 1, \quad j, j' \in \Lambda, \quad b, b' = 1, 2. \end{aligned} \quad (6.20c)$$

It is thus diagonalized in the fermion-number basis

$$\mathcal{F}_\Lambda = \text{span} \left\{ \left| \begin{pmatrix} n_{1,1} \\ n_{1,3} \end{pmatrix}, \dots, \begin{pmatrix} n_{2M,1} \\ n_{2M,3} \end{pmatrix} \right\rangle \right\} \quad (6.21a)$$

in which it is represented by two Ising chains (labeled 1 and 3) coupled through their rungs only (but not along the chains),

$$\begin{aligned} \widehat{H}_{\text{pbcc}}^{(\text{Irrung})} &:= -\lambda \sum_{j=1}^{2M} \sigma_{j,1} \sigma_{j,3}, \\ \sigma_{j,2b-1} &:= 2n_{j,2b-1} - 1, \quad b = 1, 2. \end{aligned} \quad (6.21b)$$

¹⁰It is not possible to represent the group cohomology class $([v], [\rho], \mu) = (1, 0, 0)$ with a doublet of Majorana operators.

⁹This is not so when $|\Lambda| = N = 2M + 1$ is odd.

The subspace \mathcal{F}_{gs} in the global Fock space $\mathcal{F}_\Lambda = \mathcal{F}_0 \oplus \mathcal{F}_1$ that is spanned by the linearly independent ground states is 2^{2M} -fold degenerate. It is spanned by either

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle, \dots, \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.22a)$$

when $\lambda > 0$ or the antiferromagnetic states

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \dots, \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.22b)$$

when $\lambda < 0$. All ground states are separated from all excited states by the gap $|2\lambda|$. The action of “time reversal” in the fermion representation is that of a “particle-hole” transformation under which $\hat{n}_{j,2b-1} \mapsto (1 - \hat{n}_{j,2b-1})$ and $\sigma_{j,2b-1} \mapsto -\sigma_{j,2b-1}$. The action of parity is trivial (the identity) in the fermion representation.

The complexity of the Hamiltonian of the form (5.5) that is of quartic order and of range $r = 1$ increases dramatically. For any cluster made of the two repeat unit cells j and $j + 1$, the generic cluster Hamiltonian \hat{h}_j that is summed over in Hamiltonian (5.5) is the sum over 70 (choose 4 out of 8) monomials of the form

$$\hat{\chi}_{j,a} \hat{\chi}_{j,b} \hat{\chi}_{j+1,c} \hat{\chi}_{j+1,d}, \quad 1 \leq a < b \leq 4, \quad 1 \leq c < d \leq 4, \quad (6.23)$$

each one weighted by a real-valued coupling. Of these 70 couplings, 69 are independent by translation symmetry. Three of the 69 monomials are compatible with fermion-number conservation. All other monomials break the fermion-number conservation down to the conservation of the fermion-parity number. If these 66 couplings are set to zero and the remaining three couplings are set to λ , we obtain the classical Ising ladder

$$\hat{H}_{\text{pbc}}^{(\text{ladder})} := -\lambda \sum_{j=1}^{2M} \left(\sigma_{j,1} \sigma_{j,3} + \sum_{b=1,2} \sigma_{j,2b-1} \sigma_{j,2b-1} \right) \quad (6.24)$$

in the same fermion-number basis of the Fock space that delivered Hamiltonian (6.21b). Any gapped phase in the 66-dimensional coupling space is predicted by Theorem 1 to be either degenerate or break spontaneously time-reversal or translation symmetry. This prediction can be verified explicitly for the classical Ising ladder (6.24) for which the subspace \mathcal{F}_{gs} in the global Fock space $\mathcal{F}_\Lambda = \mathcal{F}_0 \oplus \mathcal{F}_1$ that is spanned by the linearly independent ground states is twofold degenerate. It is spanned by either

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.25a)$$

when $\lambda > 0$ or the antiferromagnetic states

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.25b)$$

when $\lambda < 0$. Reversal of time is broken spontaneously at zero temperature and in the thermodynamic limit in the sense that applying either a uniform or staggered (within the repeat unit

cell) magnetic field that couples to the Ising spins through a Zeeman coupling, taking the thermodynamic limit, and switching off the Zeeman coupling selects one of the two degenerate ground states when $\lambda > 0$ and $\lambda < 0$, respectively.

3. Group cohomology class $([v], [\rho], \mu) = (0, 1, 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 8$ is generated by the sextet of Majorana operators

$$\hat{\chi}_j \equiv (\hat{\chi}_{j,1} \quad \cdots \quad \hat{\chi}_{j,6})^\top, \quad j = 1, \dots, 2M. \quad (6.26)$$

One verifies that

$$\hat{u}_j(t) := -i \hat{\chi}_{j,2} \hat{\chi}_{j,4} \hat{\chi}_{j,6} \mathbf{K}, \quad (6.27a)$$

$$\hat{u}_j(p) := i \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4} \hat{\chi}_{j,5} \hat{\chi}_{j,6}, \quad (6.27b)$$

realizes the projective algebra (6.6) with

$$[v] = 0, \quad [\rho] = 1. \quad (6.27c)$$

One verifies that the Majorana sextet (6.26) is odd under conjugation by both $\hat{u}_j(t)$ and $\hat{u}_j(p)$. Time-reversal symmetry forbids any Hermitian quadratic form for the quartet (6.26).

The generic Hamiltonian of the form (5.5) that is of quartic order and of range $r = 0$ is

$$\hat{H}_{\text{pbc}} := \sum_{j=1}^{2M} \sum_{1 \leq a_1 < a_2 < a_3 < a_4 \leq 6} \lambda_{a_1 a_2 a_3 a_4} \prod_{i=1}^4 \hat{\chi}_{j,a_i} \quad (6.28)$$

with $\lambda_{a_1 a_2 a_3 a_4}$ a real-valued coupling. Each monomial

$$\hat{X}_{j|a_1 a_2 a_3 a_4} := \prod_{i=1}^4 \hat{\chi}_{j,a_i}, \quad 1 \leq a_1 < a_2 < a_3 < a_4 \leq 6, \quad (6.29)$$

is a Hermitian operator with the eigenvalues ± 1 as it squares to the identity. Any two such monomials on the right-hand side of Eq. (6.29) either commute or anticommute. In the fermion-number basis (6.2), the three monomials

$$\hat{X}_{j|1234} := -4(\hat{n}_{j,1} - \frac{1}{2})(\hat{n}_{j,3} - \frac{1}{2}), \quad (6.30a)$$

$$\hat{X}_{j|1256} := -4(\hat{n}_{j,1} - \frac{1}{2})(\hat{n}_{j,5} - \frac{1}{2}), \quad (6.30b)$$

$$\hat{X}_{j|3456} := -4(\hat{n}_{j,3} - \frac{1}{2})(\hat{n}_{j,5} - \frac{1}{2}), \quad (6.30c)$$

are the only ones that are compatible with conservation of the fermion-number. All remaining 12 monomials break the conservation of the fermion-number in the basis (6.2) down to that of the fermion-parity number. Any gapped phase in the 15-dimensional coupling space is predicted by Theorem 1 to be either degenerate or break spontaneously time-reversal or translation symmetry. This prediction can be verified explicitly by summing with the same weight the three monomials defined in Eq. (6.30). One obtains three Ising chains coupled through their rungs only. The ground state is then 2^{2M} -fold degenerate. This macroscopic degeneracy becomes twofold by increasing the range from $r = 0$ to $r = 1$ and considering the three Ising chains (labeled 1, 3, 5) with the Hamiltonian

[in the fermion-number basis (6.2)]

$$\begin{aligned} \widehat{H}_{\text{pbc}}^{(13\text{chains})} := & -\lambda \sum_{b=1}^3 \sum_{j=1}^{2M} \sigma_{j,2b-1} \sigma_{j+1,2b-1} \\ & -\lambda \sum_{j=1}^{2M} (\sigma_{j,1} \sigma_{j,3} + \sigma_{j,3} \sigma_{j,5}) \end{aligned} \quad (6.31)$$

with $\lambda \in \mathbb{R}$, say. The subspace \mathcal{F}_{gs} in the global Fock space $\mathcal{F}_{\Lambda} = \mathcal{F}_0 \oplus \mathcal{F}_1$ that is spanned by the linearly independent ground states is either

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.32a)$$

when $\lambda > 0$ or the antiferromagnetic states

$$\mathcal{F}_{\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_0 \quad (6.32b)$$

when $\lambda < 0$. Reversal of time is broken spontaneously at zero temperature and in the thermodynamic limit in the sense that applying either a uniform or staggered (within the repeat unit cell) magnetic field that couples to the Ising spins through a Zeeman coupling, taking the thermodynamic limit, and switching off the Zeeman coupling selects one of the two degenerate ground states when $\lambda > 0$ and $\lambda < 0$, respectively.

4. Group cohomology class $([v], [\rho], \mu) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators

$$\hat{\chi}_j \equiv (\hat{\chi}_{j,1} \ \cdots \ \hat{\chi}_{j,8})^T, \quad j = 1, \dots, 2M. \quad (6.33)$$

One verifies that

$$\hat{u}_j(t) := -i \hat{\chi}_{j,2} \hat{\chi}_{j,4} \hat{\chi}_{j,6} \hat{\chi}_{j,8} \mathbf{K}, \quad (6.34a)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4} \hat{\chi}_{j,5} \hat{\chi}_{j,6} \hat{\chi}_{j,7} \hat{\chi}_{j,8}, \quad (6.34b)$$

realizes the projective algebra (6.6) with

$$[v] = 0, \quad [\rho] = 0. \quad (6.34c)$$

One verifies that the Majorana octuplet (6.33) is even under conjugation by $\hat{u}_j(t)$ and odd under conjugation by $\hat{u}_j(p)$. Time-reversal symmetry forbids any Hermitian quadratic form for the quartet (6.33).

Theorem 1 is inoperative. It is possible to find examples of both nondegenerate and degenerate gapped Hamiltonians that are translation invariant and G_f invariant.

To prove this claim, it suffices to consider a generic Hamiltonian of the form (5.5) that is of quartic order and of range $r = 0$. It is given by

$$\widehat{H}_{\text{pbc}} := \sum_{j=1}^{2M} \sum_{1 \leq a_1 < a_2 < a_3 < a_4 \leq 8} \lambda_{a_1 a_2 a_3 a_4} \prod_{i=1}^4 \hat{\chi}_{j,a_i} \quad (6.35)$$

with $\lambda_{a_1 a_2 a_3 a_4}$ a real-valued coupling. Each monomial

$$\widehat{X}_{j|a_1 a_2 a_3 a_4} := \prod_{i=1}^4 \hat{\chi}_{j,a_i}, \quad 1 \leq a_1 < a_2 < a_3 < a_4 \leq 8, \quad (6.36)$$

is a Hermitian operator with the eigenvalues ± 1 as it squares to the identity. Its two degenerate eigenspaces are therefore eight-dimensional. Any two monomials of the form (6.36) either commute or anticommute. There are 70 (choose 4 out of 8) such monomials. The six monomials

$$\widehat{X}_{j|1234} := -4(\hat{n}_{j,1} - \frac{1}{2})(\hat{n}_{j,3} - \frac{1}{2}), \quad (6.37a)$$

$$\widehat{X}_{j|1256} := -4(\hat{n}_{j,1} - \frac{1}{2})(\hat{n}_{j,5} - \frac{1}{2}), \quad (6.37b)$$

$$\widehat{X}_{j|1278} := -4(\hat{n}_{j,1} - \frac{1}{2})(\hat{n}_{j,7} - \frac{1}{2}), \quad (6.37c)$$

$$\widehat{X}_{j|3456} := -4(\hat{n}_{j,3} - \frac{1}{2})(\hat{n}_{j,5} - \frac{1}{2}), \quad (6.37d)$$

$$\widehat{X}_{j|3478} := -4(\hat{n}_{j,3} - \frac{1}{2})(\hat{n}_{j,7} - \frac{1}{2}), \quad (6.37e)$$

$$\widehat{X}_{j|5678} := -4(\hat{n}_{j,5} - \frac{1}{2})(\hat{n}_{j,7} - \frac{1}{2}), \quad (6.37f)$$

are the only ones that are compatible with conservation of the fermion-number in the fermion-number basis (6.2). All remaining 64 monomials break the conservation of this fermion-number down to conservation of the fermion-parity number. Among these, the 16 monomials generated by expanding

$$(\hat{\chi}_{j,1} + \hat{\chi}_{j,2})(\hat{\chi}_{j,3} + \hat{\chi}_{j,4})(\hat{\chi}_{j,5} + \hat{\chi}_{j,6})(\hat{\chi}_{j,7} + \hat{\chi}_{j,8}) \quad (6.38)$$

are special because they form the basis to represent all 16 terms of the form

$$\begin{aligned} \widehat{A}_{j|0000} &:= \hat{c}_{j,1}^\dagger \hat{c}_{j,3}^\dagger \hat{c}_{j,5}^\dagger \hat{c}_{j,7}^\dagger, \\ \widehat{A}_{j|0001} &:= \hat{c}_{j,1}^\dagger \hat{c}_{j,3}^\dagger \hat{c}_{j,5}^\dagger \hat{c}_{j,7}, \dots, \widehat{A}_{j|1000} := \hat{c}_{j,1} \hat{c}_{j,3}^\dagger \hat{c}_{j,5}^\dagger \hat{c}_{j,7}^\dagger, \\ &\vdots \\ \widehat{A}_{j|1110} &:= \hat{c}_{j,1} \hat{c}_{j,3} \hat{c}_{j,5} \hat{c}_{j,7}^\dagger, \dots, \widehat{A}_{j|0111} := \hat{c}_{j,1}^\dagger \hat{c}_{j,3} \hat{c}_{j,5} \hat{c}_{j,7}, \\ \widehat{A}_{j|1111} &:= \hat{c}_{j,1} \hat{c}_{j,3} \hat{c}_{j,5} \hat{c}_{j,7}, \end{aligned} \quad (6.39)$$

in the fermion-number basis (6.2). In the 70-dimensional coupling space of Hamiltonian (6.36), there is room to find gapped Hamiltonians with either a degenerate or a nondegenerate ground state.

On the one hand, the local Hamiltonian

$$\begin{aligned} \hat{h}_j^{(14\text{rungs})} &:= \lambda(\widehat{X}_{j|1234} + \widehat{X}_{j|3456} + \widehat{X}_{j|5678}) \\ &= -\lambda(\sigma_{j,1} \sigma_{j,3} + \sigma_{j,3} \sigma_{j,5} + \sigma_{j,5} \sigma_{j,7}) \end{aligned} \quad (6.40)$$

in the fermion-number basis (6.2) is none but the classical nearest-neighbor Ising Hamiltonian for a rung of four Ising spins ordered from bottom to top as 1, 3, 5, 7. As such, the subspace $\mathcal{F}_{j\text{gs}}$ in the local Fock space $\mathcal{F}_j = \mathcal{F}_{j0} \oplus \mathcal{F}_{j1}$ that is spanned by the linearly independent ground states of $\hat{h}_j^{(14\text{rungs})}$ is either

$$\mathcal{F}_{j\text{gs}} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_{j0} \quad (6.41a)$$

when $\lambda > 0$ or

$$\mathcal{F}_{jgs} = \text{span} \left\{ \left| \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left| \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \right\} \subset \mathcal{F}_{j0} \quad (6.41b)$$

when $\lambda < 0$ in the fermion-number basis (6.2).

On the other hand, the local Hamiltonian

$$\hat{h}_j := \lambda(\hat{X}_{j|1234} + \hat{X}_{j|3456} + \hat{X}_{j|5678} + \hat{X}_{j|1357}) \quad (6.42)$$

has a nondegenerate ground state. This is so because the monomial $\hat{X}_{j|1357}$ is the sum over all 16 operators (6.39) with equal weight. Hence, its action on either the basis (6.41a) or the basis (6.41b) is to exchange the two basis states, thereby lifting their degeneracies.

The counterpart to this mechanism to lift the twofold degeneracy of a 2-rung Ising Hamiltonian is not available in Sec. (VI A 3) because of fermion-parity conservation (to exchange the ferromagnetic ground states, one would need the odd-parity perturbation $\hat{c}_{j,1}^\dagger \hat{c}_{j,3}^\dagger \hat{c}_{j,5}^\dagger + \text{H.c.}$). The same is true in Sec. (VI A 2). Lifting the twofold degeneracy of a 1-rung Ising Hamiltonian with the help of the perturbation $\hat{c}_{j,1}^\dagger \hat{c}_{j,3}^\dagger + \text{H.c.}$ is not possible because time-reversal symmetry prohibits any local quadratic term.

By identifying the set

$$([\nu], [\rho], 0) \in \{(1, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 0)\} \quad (6.43a)$$

with the four distinct group cohomology classes (6.6) and by defining a group operation using the stacking rules (6.7), we have justified the identification

$$\begin{aligned} g &\rightarrow (1, 1, 0), & g^2 &\rightarrow (1, 0, 0), \\ g^3 &\rightarrow (0, 1, 0), & g^4 &\rightarrow (0, 0, 0), \end{aligned} \quad (6.43b)$$

where g is the generator of the cyclic group

$$\mathbb{Z}_4 := \{g, g^2, g^3, g^4 \equiv e\}. \quad (6.43c)$$

The fact that the stacking rules (6.7) obeyed by the indices $([\rho], [\nu], \mu = 0)$ realize the cyclic group \mathbb{Z}_4 is reminiscent of the fact that the topological index (the integer number of Majorana boundary zero modes) of noninteracting fermions belonging to the symmetry class BDI in one-dimensional space is to be replaced by one belonging to the cyclic group \mathbb{Z}_8 if interactions compatible with the symmetry class BDI are allowed [55,67]. The difference between the cyclic group \mathbb{Z}_4 for LSM-type constraints and the cyclic group \mathbb{Z}_8 in Ref. [55] arises because we must set $\mu = 0$ when defining locally the Fock space, whereas there is no such constraint at the boundary of an SPT phase.

B. One-dimensional space with the symmetry group

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ for an even number of local Majorana flavors

The lattice is $\Lambda = \{1, \dots, N\}$ with $N = 2M$ an even integer and the global fermionic Fock space \mathcal{F}_Λ is of dimension 2^{nM} with n the number of local Majorana flavors. By choosing the cardinality $|\Lambda| = 2M$ to be even, we make sure that the lattice is bipartite. This allows to treat the two values of the index $\mu := n \bmod 2$ in parallel. The symmetry group $G_f :=$

$G \times \mathbb{Z}_2^F$ is a split group. As usual, \mathbb{Z}_2^F is generated by p . We choose $G := \mathbb{Z}_2 \times \mathbb{Z}_2$. The Abelian group G has hence two generators g_1 and g_2 that commute pairwise, while each of them squares to the identity. We shall only consider the case when the local number of Majorana flavors $n = 2m$ is an even positive integer. The index μ then takes the value $\mu = 0$.

Any local projective representation of G_f can be labeled by the pair of indices $[\nu] \in H^2(G, \text{U}(1)_c)$ and $[\rho] = ([\rho]_1, [\rho]_2)$ with $[\rho]_1, [\rho]_2 \in H^1(G, \mathbb{Z}_2)$ through the relations¹¹

$$\hat{u}(g_1) \hat{u}(g_2) = (-1)^{|\nu|} \hat{u}(g_2) \hat{u}(g_1), \quad [\nu] = 0, 1, \quad (6.44a)$$

$$\hat{u}(g_i) \hat{u}(p) = (-1)^{|\rho|_i} \hat{u}(p) \hat{u}(g_i), \quad [\rho]_i = 0, 1. \quad (6.44b)$$

This gives the eight distinct group cohomology classes

$$\begin{aligned} ([\nu], [\rho], 0) = \{ &(0, (0, 0), 0), (1, (0, 0), 0), \\ &(0, (0, 1), 0), (1, (0, 1), 0), \\ &(0, (1, 0), 0), (1, (1, 0), 0), \\ &(0, (1, 1), 0), (1, (1, 1), 0)\}. \end{aligned} \quad (6.45)$$

Here, the group cohomology class $(0, (0, 0), 0)$ is interpreted as the trivial representation. Theorem 1 is predictive for any of the remaining seven group cohomology classes. It is shown in Appendix A 4 that these eight distinct group cohomology classes form the (stacking) group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, whereby the group composition is defined by the stacking rule (A39e). This (stacking) group is generated by the three group cohomology classes $(1, (1, 0), 0)$, $(1, (0, 1), 0)$, and $(1, (0, 0), 0)$, as we are going to verify explicitly.

1. Group cohomology class $([\nu], [\rho], 0) = (1, (1, 0), 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 4$ is generated by the quartet of Majorana operators¹²

$$\hat{\chi}_j \equiv (\hat{\chi}_{j,1} \quad \cdots \quad \hat{\chi}_{j,4})^\top, \quad j = 1, \dots, 2M. \quad (6.46)$$

One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1}, \quad (6.47a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1} \hat{\chi}_{j,3}, \quad (6.47b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4}, \quad (6.47c)$$

realizes the projective representation (6.44) with $([\nu], [\rho], \mu) = (1, (1, 0), 0)$.

An example of a translation- and G_f -invariant Hamiltonian of the form (5.5), order 2, and range $r = 1$ is

$$\begin{aligned} \hat{H}_{\text{pbc}} := &\sum_{j=1}^{2M} \sum_{a=1}^4 \lambda_a i \hat{\chi}_{j,a} \hat{\chi}_{j+1,a} \\ &+ \sum_{j=1}^{2M} \lambda_{2,4} i \hat{\chi}_{j,2} \hat{\chi}_{j,4}, \quad \lambda_a, \lambda_{2,4} \in \mathbb{R}. \end{aligned} \quad (6.48)$$

¹¹We have chosen the convention of always representing the generator p of \mathbb{Z}_2^F by a Hermitian operator according to Eq. (3.10e).

¹²It is not possible to represent the cohomology class $([\nu], [\rho], \mu) = (1, (1, 0), 0)$ with a doublet of Majorana operators.

This Hamiltonian does not conserve the fermion-number in the fermion-number basis (6.2). It can be thought of as four Kitaev chains each of which has an effective index $\mu = 1$. When $\lambda_{24} = 0$, all Kitaev chains decouple and are fine-tuned to their quantum critical point (4.34) between their symmetry-protected and topologically inequivalent gapped phases. Kitaev chains 2 and 4 are coupled by the on-site term $i\hat{\chi}_{j,2}\hat{\chi}_{j,4}$. The on-site term $i\hat{\chi}_{j,2}\hat{\chi}_{j,4}$ gaps chains 2 and 4. The low-energy sector of the theory is that of two decoupled quantum critical Kitaev chains labeled 1 and 3. The quadratic term $i\hat{\chi}_{j,1}\hat{\chi}_{j,3}$ that would gap chains 1 and 3, thereby delivering a nondegenerate gapped ground state, is odd under conjugation by $\hat{u}_j(g_1)$ and thus forbidden by symmetry. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $(1, (1, 0), 0)$ representation are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators

$$\hat{\chi}_j \equiv (\hat{\chi}_{j,1} \cdots \hat{\chi}_{j,8})^T, \quad j = 1, \dots, 2M. \quad (6.49)$$

One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1} \hat{\chi}_{j,5}, \quad (6.50a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1} \hat{\chi}_{j,3} \hat{\chi}_{j,5} \hat{\chi}_{j,7}, \quad (6.50b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4} \hat{\chi}_{j,5} \hat{\chi}_{j,6} \hat{\chi}_{j,7} \hat{\chi}_{j,8}, \quad (6.50c)$$

realizes the projective representation (6.44) with $([v], [\rho], \mu) = (0, (0, 0), 0)$, i.e., the trivial projective representation. In this trivial representation, any flavor $a = 1, \dots, 8$ has an image $a' = (a + 4) \bmod 8$ such that $\hat{\chi}_{j,a}$ and $\hat{\chi}_{j,a'}$ transform identically under G_f . All on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+4}$ with $a = 1, \dots, 4$ are then G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form (5.5) can be lifted by including the four on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+4}$ with $a = 1, \dots, 4$, i.e., Theorem 1 is not predictive.

2. Group cohomology class $([v], [\rho], \mu) = (1, (0, 1), 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 4$ is generated by the quartet of Majorana operators (6.46).¹³ One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1} \hat{\chi}_{j,3}, \quad (6.51a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1}, \quad (6.51b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4}, \quad (6.51c)$$

realizes the projective representation (6.44) with $([v], [\rho], \mu) = (1, (0, 1), 0)$.

Equation (6.51) differs from Eq. (6.47) by interchanging g_1 and g_2 . This difference does not affect the reasoning leading

to the conclusion that the gapless Hamiltonian (6.48) is the most general translation-invariant, G_f -invariant, order 2, and range $r = 1$ Hamiltonian of the form (5.5) whose hopping is diagonal with respect to the flavor index. This difference also implies that stacking two copies of the $(1, (0, 1), 0)$ representation (6.51) delivers the trivial projective representation $(0, (0, 0), 0)$ encoded by Eqs. (6.49) and (6.50), for which Theorem 1 is not predictive anymore.

3. Group cohomology class $([v], [\rho], \mu) = (1, (0, 0), 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 4$ is generated by the quartet of Majorana operators (6.46).¹⁴ One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,2} \hat{\chi}_{j,3}, \quad (6.52a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1} \hat{\chi}_{j,3}, \quad (6.52b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4}, \quad (6.52c)$$

realizes the projective representation (6.44) with $([v], [\rho], \mu) = (1, (0, 0), 0)$.

The most general translation- and G_f -invariant Hamiltonian of the form (5.5) of order 2, range $r = 1$, and whose hopping is diagonal with respect to the flavor index is then

$$\hat{H}_{\text{pbc}} = \sum_{j=1}^N \sum_{a=1}^4 \lambda_a i\hat{\chi}_{j,a} \hat{\chi}_{j+1,a}, \quad \lambda_a \in \mathbb{R}, \quad (6.53)$$

i.e., four decoupled Kitaev chains that are fine-tuned to their quantum critical point (4.34) between their symmetry-protected and topologically inequivalent gapped phases. No on-site quadratic term is allowed by the symmetries. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $(1, (0, 0), 0)$ representation are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators (6.49) with the projective representation

$$\hat{u}_j(g_1) := \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,6} \hat{\chi}_{j,7}, \quad (6.54a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1} \hat{\chi}_{j,3} \hat{\chi}_{j,5} \hat{\chi}_{j,7}, \quad (6.54b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4} \hat{\chi}_{j,5} \hat{\chi}_{j,6} \hat{\chi}_{j,7} \hat{\chi}_{j,8}, \quad (6.54c)$$

that realizes the group cohomology class $([v], [\rho], \mu) = (0, (0, 0), 0)$, i.e., the trivial group cohomology class. In this trivial representation, any flavor $a = 1, \dots, 8$ has an image $a' = (a + 4) \bmod 8$ such that $\hat{\chi}_{j,a}$ and $\hat{\chi}_{j,a'}$ transform identically under G_f . All on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+4}$ with $a = 1, \dots, 4$ are then G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form

¹³It is not possible to represent the cohomology class $([v], [\rho], \mu) = (1, (0, 1), 0)$ with a doublet of Majorana operators.

¹⁴It is not possible to represent the cohomology class $([v], [\rho], \mu) = (1, (0, 0), 0)$ with a doublet of Majorana operators.

(5.5) can be lifted by including the 4 on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+4}$ with $a = 1, \dots, 4$, i.e., Theorem 1 is not predictive.

4. Group cohomology class $([v], [\rho], \mu) = (1, (1, 1), 0)$

When representations (6.47) and (6.51) are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators (6.49). One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1}\hat{\chi}_{j,5}\hat{\chi}_{j,7}, \quad (6.55a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1}\hat{\chi}_{j,3}\hat{\chi}_{j,5}, \quad (6.55b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1}\hat{\chi}_{j,2}\hat{\chi}_{j,3}\hat{\chi}_{j,4}\hat{\chi}_{j,5}\hat{\chi}_{j,6}\hat{\chi}_{j,7}\hat{\chi}_{j,8}, \quad (6.55c)$$

realizes the projective representation (6.44) with $([v], [\rho], \mu) = (1, (1, 1), 0)$.

An example of a translation- and G_f -invariant Hamiltonian of the form (5.5), order 2, and range $r = 1$ is

$$\begin{aligned} \hat{H}_{\text{pbc}} := \sum_{j=1}^N \left[\sum_{a=1}^8 \lambda_a i\hat{\chi}_{j,a}\hat{\chi}_{j+1,a} + \lambda_{1,5} i\hat{\chi}_{j,1}\hat{\chi}_{j,5} \right. \\ \left. + \lambda_{2,4} i\hat{\chi}_{j,2}\hat{\chi}_{j,4} + \lambda_{6,8} i\hat{\chi}_{j,6}\hat{\chi}_{j,8} \right] \end{aligned} \quad (6.56)$$

with $\lambda_a, \lambda_{a,b} \in \mathbb{R}$ for $a, b = 1, \dots, 8$. By construction, this Hamiltonian is gapless since the quantum critical Kitaev chains 3 and 7 are decoupled from all other Kitaev chains. This gapless phase is stable to any on-site quadratic perturbation since the only on-site quadratic terms $i\hat{\chi}_{j,3}\hat{\chi}_{j,a}$ and $i\hat{\chi}_{j,7}\hat{\chi}_{j,a}$ that could gap the quantum critical Kitaev chains 3 and 7 are odd under G for any $a \neq 3, 7$. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $(1, (1, 1), 0)$ representation are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 256$ is generated by 16 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1}\hat{\chi}_{j,5}\hat{\chi}_{j,7}\hat{\chi}_{j,9}\hat{\chi}_{j,13}\hat{\chi}_{j,15}, \quad (6.57a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1}\hat{\chi}_{j,3}\hat{\chi}_{j,5}\hat{\chi}_{j,9}\hat{\chi}_{j,11}\hat{\chi}_{j,13}, \quad (6.57b)$$

$$\hat{u}_j(p) := \prod_{a=1}^{16} \hat{\chi}_{j,a}, \quad (6.57c)$$

realizes the group cohomology class $([v], [\rho], \mu) = (0, (0, 0), 0)$, i.e., the trivial group cohomology class. There exists a bijective map $a \mapsto a' := (a + 8) \bmod 16$ such that all on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a'}$ with $a = 1, \dots, 8$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form (5.5) can be lifted by including the 8 on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+8}$ with $a = 1, \dots, 8$, i.e., Theorem 1 is not predictive.

5. Group cohomology class $([v], [\rho], \mu) = (0, (1, 0), 0)$

When representations (6.52) and (6.47) are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is generated by the octuplet of Majorana operators (6.49). One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,2}\hat{\chi}_{j,3}\hat{\chi}_{j,5}, \quad (6.58a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1}\hat{\chi}_{j,3}\hat{\chi}_{j,5}\hat{\chi}_{j,7}, \quad (6.58b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1}\hat{\chi}_{j,2}\hat{\chi}_{j,3}\hat{\chi}_{j,4}\hat{\chi}_{j,5}\hat{\chi}_{j,6}\hat{\chi}_{j,7}\hat{\chi}_{j,8}, \quad (6.58c)$$

realizes the projective representation (6.44) with $([v], [\rho], \mu) = (0, (1, 0), 0)$.

An example of a translation- and G_f -invariant Hamiltonian of the form (5.5), order 2, and range $r = 1$ is

$$\begin{aligned} \hat{H}_{\text{pbc}} := \sum_{j=1}^N \left[\sum_{a=1}^8 \lambda_a i\hat{\chi}_{j,a}\hat{\chi}_{j+1,a} + \lambda_{1,7}\hat{\chi}_{j,1}\hat{\chi}_{j,7} \right. \\ \left. + \lambda_{4,8}\hat{\chi}_{j,4}\hat{\chi}_{j,8} + \lambda_{3,5}\hat{\chi}_{j,3}\hat{\chi}_{j,5} \right] \end{aligned} \quad (6.59)$$

with $\lambda_a, \lambda_{a,b} \in \mathbb{R}$ for $a, b = 1, \dots, 8$. By construction, this Hamiltonian is gapless since the quantum critical Kitaev chains 2 and 6 are decoupled from all other Kitaev chains. This gapless phase is stable to any on-site and quadratic perturbation since the only on-site quadratic terms $i\hat{\chi}_{j,2}\hat{\chi}_{j,a}$ and $i\hat{\chi}_{j,6}\hat{\chi}_{j,a}$ that could gap the quantum critical Kitaev chains 2 and 6 are odd under G for any $a \neq 2, 6$. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $(0, (1, 0), 0)$ representation are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 256$ is generated by 16 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,2}\hat{\chi}_{j,3}\hat{\chi}_{j,5}\hat{\chi}_{j,10}\hat{\chi}_{j,11}\hat{\chi}_{j,13}, \quad (6.60a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,1}\hat{\chi}_{j,3}\hat{\chi}_{j,5}\hat{\chi}_{j,7}\hat{\chi}_{j,9}\hat{\chi}_{j,11}\hat{\chi}_{j,13}\hat{\chi}_{j,15}, \quad (6.60b)$$

$$\hat{u}_j(p) := \prod_{a=1}^{16} \hat{\chi}_{j,a}, \quad (6.60c)$$

realizes the group cohomology class $([v], [\rho], \mu) = (0, (0, 0), 0)$, i.e., the trivial group cohomology class. There exists a bijective map $a \mapsto a' := (a + 16) \bmod 16$ such that all on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a'}$ with $a = 1, \dots, 8$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form (5.5) can be lifted by including the eight on-site terms $i\hat{\chi}_{j,a}\hat{\chi}_{j,a+8}$ with $a = 1, \dots, 8$, i.e., Theorem 1 is not predictive.

6. Group cohomology class $([v], [\rho], \mu) = (0, (0, 1), 0)$

When representations (6.52) and (6.51) are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 16$ is gener-

ated by the octuplet of Majorana operators (6.49). One verifies that

$$\hat{u}_j(g_1) := \hat{\chi}_{j,1} \hat{\chi}_{j,3} \hat{\chi}_{j,5} \hat{\chi}_{j,7}, \quad (6.61a)$$

$$\hat{u}_j(g_2) := \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,5}, \quad (6.61b)$$

$$\hat{u}_j(p) := \hat{\chi}_{j,1} \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,4} \hat{\chi}_{j,5} \hat{\chi}_{j,6} \hat{\chi}_{j,7} \hat{\chi}_{j,8}, \quad (6.61c)$$

realizes the projective representation (6.44) with $([\nu], [\rho], \mu) = (0, (0, 1), 0)$.

Equation (6.61) differs from Eq. (6.58) by interchanging g_1 and g_2 . This difference does not affect the reasoning leading to the conclusion that the gapless Hamiltonian (6.59) is the most general translation-invariant, G_f -invariant, order 2, and range $r = 1$ Hamiltonian of the form (5.5) whose hopping is diagonal with respect to the flavor index. This difference also implies that stacking two copies of the $(1, (0, 1), 0)$ representation (6.61) delivers the trivial projective representation $(0, (0, 0), 0)$ encoded by Eqs. (6.60), for which Theorem 1 is not predictive anymore.

7. Group cohomology class $([\nu], [\rho], \mu) = (0, (1, 1), 0)$

When representations (6.52) and (6.55) are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 64$ is generated by 12 Majorana operators. One verifies that

$$\hat{u}_j(g_1) := i \hat{\chi}_{j,2} \hat{\chi}_{j,3} \hat{\chi}_{j,5} \hat{\chi}_{j,9} \hat{\chi}_{j,11}, \quad (6.62a)$$

$$\hat{u}_j(g_2) := i \hat{\chi}_{j,1} \hat{\chi}_{j,3} \hat{\chi}_{j,5} \hat{\chi}_{j,7} \hat{\chi}_{j,9}, \quad (6.62b)$$

$$\hat{u}_j(p) := \prod_{a=1}^{12} \hat{\chi}_{j,a}, \quad (6.62c)$$

realizes the projective representation (6.44) with $([\nu], [\rho], \mu) = (0, (1, 1), 0)$.

An example of a translation- and G_f -invariant Hamiltonian of the form (5.5), order 2, and range $r = 1$ is

$$\begin{aligned} \hat{H}_{\text{pbc}} := & \sum_{j=1}^N \left[\sum_{a=1}^{12} \lambda_a i \hat{\chi}_{j,a} \hat{\chi}_{j+1,a} + \lambda_{1,7} i \hat{\chi}_{j,1} \hat{\chi}_{j,7} \right. \\ & + \lambda_{3,5} i \hat{\chi}_{j,3} \hat{\chi}_{j,5} + \lambda_{2,11} i \hat{\chi}_{j,2} \hat{\chi}_{j,11} \\ & \left. + \lambda_{4,6} i \hat{\chi}_{j,4} \hat{\chi}_{j,6} + \lambda_{8,10} i \hat{\chi}_{j,8} \hat{\chi}_{j,10} \right] \end{aligned} \quad (6.63)$$

with $\lambda_a, \lambda_{a,b} \in \mathbb{R}$ for $a, b = 1, \dots, 12$. By construction, this Hamiltonian is gapless since the quantum critical Kitaev chains 9 and 12 are decoupled from all other Kitaev chains. This gapless phase is stable to any quadratic on-site perturbation since the only on-site quadratic terms $i \hat{\chi}_{j,9} \hat{\chi}_{j,a}$ and $i \hat{\chi}_{j,12} \hat{\chi}_{j,a}$ that could gap the quantum critical Kitaev chains 9 and 12 are odd under G for any $a \neq 9, 12$. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of this $(0, (1, 1), 0)$ representation are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 2^{12}$ is generated by 24 Majorana operators. The \mathbb{Z}_2 -graded

tensor product of the projective representation (6.62) with itself realizes the group cohomology class $([\nu], [\rho], \mu) = (0, (0, 0), 0)$, i.e., the trivial group cohomology class. There exists a bijective map $a \mapsto a' := (a + 12) \bmod 24$ such that all on-site terms $i \hat{\chi}_{j,a} \hat{\chi}_{j,a'}$ with $a = 1, \dots, 12$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form (5.5) can be lifted by including the 12 on-site terms $i \hat{\chi}_{j,a} \hat{\chi}_{j,a+12}$ with $a = 1, \dots, 12$, i.e., Theorem 1 is not predictive.

C. One-dimensional space with the symmetry group \mathbb{Z}_4^{FT} for an even number of local Majorana flavors

The lattice is $\Lambda = \{1, \dots, N\}$ with $N = 2M$ an even integer and the global fermionic Fock space \mathcal{F}_Λ is of dimension 2^{nM} with n the number of local Majorana flavors. By choosing the cardinality $|\Lambda| = 2M$ to be even, we make sure that the lattice is bipartite. This allows to treat the two values of the index $\mu := n \bmod 2$ in parallel. We shall only consider the case when $n = 2m$ with m a positive integer and $\mu = 0$. The symmetry group $G_f := \mathbb{Z}_4^{FT} := \{t, t^2, t^3, t^4\}$ is the nontrivial central extension of $G \equiv \mathbb{Z}_2^T = \{t, t^2\}$ by $\mathbb{Z}_2^F \equiv \{p, p^2\}$, where the identification $t^2 = p$ is made. The upper index T for the cyclic group $G \equiv \mathbb{Z}_2^T \equiv \{e, t\}$ refers to the interpretation of t as reversal of time (see Appendix A 5). As usual, p denotes fermion parity. The symmetry group G_f is thus generated by reversal of time t , whereby reversal of time squares to the fermion parity p .

The local antiunitary representation $\hat{u}_j(t)$ of reversal of time generates a projective representation of the group \mathbb{Z}_4^{FT} . According to Appendix A 5, all cohomologically distinct projective representations of \mathbb{Z}_4^{FT} are determined by the indices $[(\nu, \rho)]$, with $(\nu, \rho) \in C^2(G, \text{U}(1)) \times C^1(G, \mathbb{Z}_2)$, through the relations¹⁵

$$[(\nu, \rho)] = (\rho(t), \rho(t)), \quad (6.64a)$$

$$\hat{u}_j(t) \hat{u}_j(p) = (-1)^{\rho(t)} \hat{u}_j(p) \hat{u}_j(t),$$

where

$$\hat{u}_j^2(t) = e^{i\phi(t,t)} \hat{u}_j(p) \quad (6.64b)$$

and ϕ is the 2-cocycle defined in Eq. (3.11). This gives two distinct group cohomology classes

$$((\nu, \rho), 0) \in \{(0, 0, 0), (1, 1, 0)\}. \quad (6.64c)$$

We will start with the nontrivial projective representation in the cohomology class $([\nu], [\rho], 0) = (1, 1, 0)$ that we shall represent using two local Majorana flavors. We will then construct a projective representation in the group cohomology classes $([\nu], [\rho], 0) = (0, 0, 0)$ by using the graded tensor product, i.e., by considering 4 local Majorana flavors. This will allow us to verify explicitly the stacking rules of Sec. III C according to which Eq. (3.35) simplifies to the rule (see Appendix A 5)

$$\begin{aligned} ((\nu, \rho), 0) &= (\rho(t), \rho(t), 0), \\ \rho(t) &= \rho_1(t) + \rho_2(t) \bmod 2, \end{aligned} \quad (6.65)$$

¹⁵We have chosen the convention of always representing the generator p of \mathbb{Z}_2^F by a Hermitian operator according to Eq. (3.10e).

when $G_f = \mathbb{Z}_4^{FT}$. The indices defined in Eqs. (6.64) thus form the cyclic group \mathbb{Z}_2 with respect to the stacking rule (6.65).

1. Group cohomology classes $([(v, \rho)], \mu) = (1, 1, 0)$
and $([(v, \rho)], \mu) = (0, 0, 0)$

The local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 2$ is generated by the doublet of Majorana operators

$$\hat{\chi}_j \equiv \begin{pmatrix} \hat{\chi}_{j,1} \\ \hat{\chi}_{j,2} \end{pmatrix}, \quad j = 1, \dots, 2M. \quad (6.66)$$

One verifies that

$$\hat{u}_j(t) := \frac{1}{\sqrt{2}}(\hat{\chi}_{j,1} - \hat{\chi}_{j,2})\mathbf{K}, \quad (6.67a)$$

$$\hat{u}_j(p) := i\hat{\chi}_{j,1} \hat{\chi}_{j,2}, \quad (6.67b)$$

realizes the projective representation (6.64) with $([(v, \rho)], \mu) = (1, 0, 0)$. With the help of

$$[\hat{u}_j(t)]^{-1} = \frac{1}{\sqrt{2}}(\hat{\chi}_{j,1} + \hat{\chi}_{j,2})\mathbf{K}, \quad (6.68)$$

one also verifies that

$$\hat{u}_j(t) \begin{pmatrix} \hat{\chi}_{j,1} \\ \hat{\chi}_{j,2} \end{pmatrix} [\hat{u}_j(t)]^{-1} = \begin{pmatrix} -\hat{\chi}_{j,2} \\ +\hat{\chi}_{j,1} \end{pmatrix}. \quad (6.69)$$

It follows from Eq. (6.69) that the only on-site Hermitian quadratic form $i\hat{\chi}_{j,1} \hat{\chi}_{j,2}$ is odd under reversal of time. Consequently,

$$\hat{H}_{\text{pbc}} := \sum_{j=1}^{2M} \lambda (i\hat{\chi}_{j,1} \hat{\chi}_{j+1,1} - i\hat{\chi}_{j,2} \hat{\chi}_{j+1,2}) \quad (6.70)$$

with $\lambda \in \mathbb{R}$ is the most general translation- and G_f -invariant Hamiltonian of the form (5.5) of order 2, range $r = 1$, and whose hopping is diagonal with respect to the flavor index. This Hamiltonian describes two Kitaev chains that have been fine-tuned to their quantum critical point (4.34) between their symmetry-protected and topologically inequivalent gapped phases. The stability of this gapless phase to on-site quadratic perturbations can thus be thought of as a consequence of Theorem 1. Theorem 1 also predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum must break spontaneously at least one of the symmetries responsible for the noninteracting spectrum being gapless.

When two copies of the projective representation (6.67) are stacked, the local fermionic Fock space \mathcal{F}_j of dimension $\mathcal{D} = 4$ is generated by four Majorana operators. The \mathbb{Z}_2 -graded tensor product of the projective representation (6.67) with itself realizes the group cohomology class $([(v, \rho)], \mu) = (0, 0, 0)$, i.e., the trivial group cohomology class. There exists a bijective map $a \mapsto a' := (a + 2) \bmod 4$ such that all on-site terms $i\hat{\chi}_{j,a} \hat{\chi}_{j,a'} - i\hat{\chi}_{j,a+1} \hat{\chi}_{j,(a+1)'}$ with $a = 1, 2$ can be shown to be G_f symmetric. The ground-state degeneracy of any translation- and G_f -invariant Hamiltonian of the form (5.5) can be lifted by increasing the strength of $i\hat{\chi}_{j,a} \hat{\chi}_{j,a'} - i\hat{\chi}_{j,a+1} \hat{\chi}_{j,(a+1)'}$ with $a = 1, 2$, i.e., Theorem 1 is not predictive.

D. One-dimensional space with the symmetry group \mathbb{Z}_2^F for an odd number of local Majorana flavors

The lattice is $\Lambda = \{1, \dots, N\}$ with $N = 2M$ an even integer and the global fermionic Fock space \mathcal{F}_Λ is of dimension $2^{(2m+1)M}$ with $(2m + 1)$ the number of local Majorana flavors, i.e., $\mu = 1$. By choosing the cardinality $|\Lambda| = 2M$ to be even, we make sure that the lattice is bipartite. The symmetry group is $G_f := \{p, p^2\} \equiv \mathbb{Z}_2^F$, where p denotes fermion parity. If we reinterpret $G_f = G \times \mathbb{Z}_2^F$ with $G = \{e\}$ the group with one element, we deduce that the indices $[v]$ and $[\rho]$ are trivial, i.e., $[v] = [\rho] = 0$. The index associated with this group is then $(0,0,1)$, for which we illustrate how translation symmetry prevents a nondegenerate gapped ground state in agreement with Theorem 2.

We define the $2^{(2m+1)M}$ -dimensional global Fock space \mathcal{F}_Λ using the $2(2m + 1)M$ Majorana operators obeying the algebra

$$\hat{\chi}_{j,a}^\dagger = \chi_{j,a}, \quad \hat{\chi}_{j,a}^2 = 1, \quad \{\chi_{j,a}, \chi_{j',a'}\} = 2\delta_{j,j'} \delta_{a,a'}, \quad (6.71)$$

for $j, j' = 1, \dots, 2M, a, a' = 1, \dots, 2m + 1$.

For simplicity, we first choose $2m + 1 = 1$ local Majorana flavors. Hamiltonian

$$\hat{H}_{\text{pbc}} := \lambda \sum_{j=1}^{2M} i \hat{\chi}_j \hat{\chi}_{j+1}, \quad \lambda \in \mathbb{R}, \quad (6.72)$$

is the most general translation- and G_f -invariant Hamiltonian of the form (5.5) of range $r = 1$. It is gapless as it describes one Kitaev chain on M sites that has been fine-tuned to its quantum critical point (4.34) between its symmetry-protected and topologically inequivalent gapped phases.

If we now consider $2m + 1 > 1$ local Majorana flavors arranged into the $(2m + 1)$ -multiplet $\hat{\chi}_j$, the most general translation- and G_f -invariant Hamiltonian of the form (5.5) of order 2 and range $r = 1$ is

$$\hat{H}_{\text{pbc}} := \sum_{j=1}^{2M} i \hat{\chi}_j^\top \mathbf{M} \hat{\chi}_{j+1} \quad (6.73)$$

where the $(2m + 1) \times (2m + 1)$ -dimensional matrix \mathbf{M} is real-valued and antisymmetric. As \mathbf{M} has necessarily a zero eigenvalue, the spectrum of \hat{H}_{pbc} is gapless.

Theorem 2 predicts that any G_f -symmetric interaction of the form (5.5) that opens a spectral gap in the noninteracting spectrum of Hamiltonian (6.73) must break spontaneously the symmetries responsible for the noninteracting spectrum being gapless.

After stacking two copies of the projective representation $(0,0,1)$, it is possible to define a local fermionic Fock space with the gapless Hamiltonian

$$\hat{H}_{\text{pbc}} := \sum_{j=1}^{2M} \sum_{a=1}^2 i \hat{\chi}_{j,a}^\top \mathbf{M}_a \hat{\chi}_{j+1,a}, \quad \mathbf{M}_a = -\mathbf{M}_a^\top \in \text{Mat}(2m + 1, \mathbb{R}). \quad (6.74)$$

The on-site mass term $i\hat{\chi}_{j,1}^\top \hat{\chi}_{j,2}$ is compatible with parity conservation. When added to the right-hand side so as to preserve translation symmetry, it selects a nondegenerate gapped ground state. There is no contradiction with Theorem 2 since

the the projective representation under stacking rules is the trivial one (0,0,0).

As was suggested just after Eq. (6.72), we can always do the reinterpretation

$$\widehat{H}_{\text{pbc}} := \lambda \sum_{l=1}^M i(\widehat{\chi}_{o,l} \widehat{\chi}_{e,l} + \widehat{\chi}_{o,l} \widehat{\chi}_{e,l+1}), \quad (6.75a)$$

$$\widehat{\chi}_{o,l} := \widehat{\chi}_{2l-1}, \quad \widehat{\chi}_{e,l} := \widehat{\chi}_{2l}, \quad (6.75b)$$

according to which the enlarged repeat unit cell is labeled by the odd sites Λ_A of Λ and there are two flavors (even and odd) of Majorana per enlarged repeat unit cell. It is then tempting to ask if one could use Theorem 1 for some group G to understand the spectrum of Eq. (6.75), in which case the need for Theorem 2 would be superfluous at best or contradictory at worst. However, there is no conflict between Theorem 2 and Theorem 1, as Theorem 1 cannot be applied to understand the spectrum of Eq. (6.75). To see this, we observe that the translation by one original repeat unit cell that is presumed by Theorem 2 is represented in Eq. (6.75) by

$$\widehat{\chi}_{o,l} \mapsto \widehat{\chi}_{e,l}, \quad \widehat{\chi}_{e,l} \mapsto \widehat{\chi}_{o,l+1}, \quad l = 1, \dots, M. \quad (6.76)$$

As this transformation is not internal to the enlarged repeat unit cell, it cannot be interpreted as the group G of on-site symmetries needed to establish Theorem 1.

VII. SUMMARY

In this work, we have obtained two Lieb-Schultz-Mattis type no-go constraints that forbid the existence of a nondegenerate gapped ground state for translationally invariant local lattice Hamiltonians of the form (5.5) in any dimension. Theorem 1 is proved within the FMPS framework and presumes that the repeat unit cell hosts a finite even number of Majorana degrees of freedom that, in turn, realize a nontrivial projective representation of a global symmetry group G_f . We have extended Theorem 1 to any dimension $d > 1$ by using tilted and twisted boundary conditions, albeit with some restrictions on the global symmetry group G_f . Theorem 2 presumes that the repeat unit cell of any d -dimensional lattice hosts a finite odd number of Majorana degrees of freedom (of course the lattice must then host an even number of sites). Such Lieb-Schultz-Mattis-type theorems provide nonperturbative constraints on the nature of the ground state which are expected to have applications in fermionic models with broken U(1) number-conservation symmetry, but with additional global symmetries G_f present. Notably, such LSM-type constraints dictate the conditions under which a Fermi liquid can be unstable to a low-temperature phase in which superconducting long-range order coexists with the long-range order associated to the spontaneous breaking of some symmetry group G_f .

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APPENDIX A: GROUP COHOMOLOGY

We review some basic concepts in group cohomology in Appendix A 1. We then calculate the relevant group cohomologies for the projective representations of the group G_f considered in Appendix A 3–A 5.

1. Some definitions

Given two groups G and M , an n -cochain is the map

$$\begin{aligned} \phi : G^n &\rightarrow M, \\ (g_1, g_2, \dots, g_n) &\mapsto \phi(g_1, g_2, \dots, g_n), \end{aligned} \quad (A1)$$

that maps an n -tuple (g_1, g_2, \dots, g_n) to an element $\phi(g_1, g_2, \dots, g_n) \in M$. The set of all n -cochains from G^n to M is denoted by $C^n(G, M)$. We define an M -valued 0-cochain to be an element of the group M itself, i.e., $C^0(G, M) = M$. Henceforth, we will denote the group composition rule in G by \cdot and the group composition rule in M additively by $+$ ($-$ denoting the inverse element).

Given the group homomorphism $\mathfrak{c} : G \rightarrow \{-1, 1\}$, for any $g \in G$, we define the group action

$$\begin{aligned} \mathfrak{C}_g : M &\rightarrow M, \\ m &\mapsto \mathfrak{c}(g)m. \end{aligned} \quad (A2)$$

The homomorphism \mathfrak{c} indicates whether and element $g \in G$ is represented unitarily [$\mathfrak{c}(g) = +1$] or antiunitarily [$\mathfrak{c}(g) = -1$]. We define the map $\delta_{\mathfrak{c}}^n$

$$\begin{aligned} \delta_{\mathfrak{c}}^n : C^n(G, M) &\rightarrow C^{n+1}(G, M), \\ \phi &\mapsto (\delta_{\mathfrak{c}}^n \phi), \end{aligned} \quad (A3a)$$

from n -cochains to $(n+1)$ -cochains such that

$$\begin{aligned} (\delta_{\mathfrak{c}}^n \phi)(g_1, \dots, g_{n+1}) &:= \mathfrak{C}_{g_1}(\phi(g_2, \dots, g_n, g_{n+1})) \\ &+ \sum_{i=1}^n (-1)^i \phi(g_1, \dots, g_i \cdot g_{i+1}, \dots, g_{n+1}) \\ &- (-1)^n \phi(g_1, \dots, g_n). \end{aligned} \quad (A3b)$$

The map $\delta_{\mathfrak{c}}^n$ is called a *coboundary operator*.

Example $n = 2$. The coboundary operator $\delta_{\mathfrak{c}}^2$ is defined by

$$\begin{aligned} (\delta_{\mathfrak{c}}^2 \phi)(g_1, g_2, g_3) &= \mathfrak{C}_{g_1}(\phi(g_2, g_3)) + (-1)^1 \phi(g_1 \cdot g_2, g_3) \\ &+ (-1)^2 \phi(g_1, g_2 \cdot g_3) - (-1)^2 \phi(g_1, g_2) \\ &= \mathfrak{c}(g_1) \phi(g_2, g_3) - \phi(g_1 \cdot g_2, g_3) \\ &+ \phi(g_1, g_2 \cdot g_3) - \phi(g_1, g_2). \end{aligned} \quad (A4)$$

We observe that

$$\begin{aligned} (\delta_{\mathfrak{c}}^2 \phi)(g_1, g_2, g_3) = 0 &\iff \phi(g_1, g_2) + \phi(g_1 \cdot g_2, g_3) \\ &= \phi(g_1, g_2 \cdot g_3) + \mathfrak{c}(g_1) \phi(g_2, g_3) \end{aligned} \quad (A5)$$

is nothing but the 2-cocycle condition (3.12b) obeyed by ϕ .

Example $n = 1$. The coboundary operator $\delta_{\mathfrak{c}}^1$ is defined by

$$\begin{aligned} (\delta_{\mathfrak{c}}^1 \phi)(g_1, g_2) &= \mathfrak{C}_{g_1}(\phi(g_2)) + (-1)^1 \phi(g_1 \cdot g_2) - (-1)^1 \phi(g_1) \\ &= \mathfrak{c}(g_1) \phi(g_2) - \phi(g_1 \cdot g_2) + \phi(g_1). \end{aligned} \quad (A6)$$

One verifies the important identity

$$\Phi(g_1, g_2) := (\delta_c^1 \phi)(g_1, g_2) \Rightarrow (\delta_c^2 \Phi)(g_1, g_2, g_3) = 0. \quad (\text{A7})$$

We observe that Eq. (3.15) implies that ϕ is the image of ξ under the coboundary operator δ_c^1 . Using the coboundary operator, we define two sets

$$Z^n(G, M_c) := \ker(\delta_c^n) = \{\phi \in C^n(G, M) \mid \delta_c^n \phi = 0\}, \quad (\text{A8a})$$

and

$$\begin{aligned} B^n(G, M_c) &:= \text{im}(\delta_c^{n-1}) \\ &= \{\phi \in C^n(G, M) \mid \phi = \delta_c^{n-1} \phi', \phi' \in C^{n-1}(G, M)\}. \end{aligned} \quad (\text{A8b})$$

The cochains in $Z^n(G, M_c)$ are called *n-cocycles*. The cochains in $B^n(G, M_c)$ are called *n-coboundaries*. The action of the boundary operator on the elements of the group M is sensitive to the homomorphism \mathfrak{c} . For this reason, we label M by \mathfrak{c} in $Z^n(G, M_c)$ and $B^n(G, M_c)$. The importance of the coboundaries is that the identity (A7) generalizes to

$$\phi = \delta_c^{n-1} \phi' \Rightarrow \delta_c^n \phi = 0. \quad (\text{A9})$$

The n th cohomology group is defined as the quotient of the n -cocycles by the n -coboundaries, i.e.,

$$H^n(G, M_c) := Z^n(G, M_c) / B^n(G, M_c). \quad (\text{A10})$$

From now on, we omit the labels n and \mathfrak{c} in δ_c^n for convenience. It should be understood that the map δ acting on a cochain ϕ maps it to a cochain of one higher degree. The n th cohomology group $H^n(G, M_c)$ is an additive Abelian group. We denote its elements by $[\phi] \in H^n(G, M_c)$, i.e., the equivalence class of the n -cocycle ϕ .

Finally, we define the following operation on the cochains. Given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$, we produce the cochain $(\phi \cup \theta) \in C^{n+m}(G, N \times M)$ through

$$\begin{aligned} (\phi \cup \theta)(g_1, \dots, g_n, g_{n+1}, \dots, g_m) \\ := (\phi(g_1, \dots, g_n), \mathfrak{C}_{g_1, g_2, \dots, g_n}(\theta(g_{n+1}, \dots, g_{n+m}))). \end{aligned} \quad (\text{A11a})$$

If we compose operation (A.11) with the pairing map $f : N \times M \rightarrow M'$ where M' is an Abelian group, we obtain the cup product

$$\begin{aligned} (\phi \smile \theta)(g_1, \dots, g_n, g_{n+1}, \dots, g_m) \\ := f\left(\left(\phi(g_1, \dots, g_n), \mathfrak{C}_{g_1, g_2, \dots, g_n}(\theta(g_{n+1}, \dots, g_{n+m}))\right)\right). \end{aligned} \quad (\text{A11b})$$

Hence, $(\phi \smile \theta) \in C^{n+m}(G, M')$. For our purposes, both N and M are subsets of the integer numbers, $M' = \mathbb{Z}_2$, while the pairing map f is

$$\begin{aligned} f\left(\left(\phi(g_1, \dots, g_n), \mathfrak{C}_{g_1, g_2, \dots, g_n}(\theta(g_{n+1}, \dots, g_{n+m}))\right)\right) \\ := \phi(g_1, \dots, g_n) \mathfrak{C}_{g_1, g_2, \dots, g_n}(\theta(g_{n+1}, \dots, g_{n+m})) \bmod 2 \end{aligned} \quad (\text{A12})$$

where multiplication of cochains ϕ and θ is treated as multiplication of integers numbers modulo 2. For instance, for the cup product of a 1-cochain $\alpha \in C^1(G, \mathbb{Z}_2)$ and a 2-cochain $\beta \in C^2(G, \mathbb{Z}_2)$, we write

$$\begin{aligned} (\alpha \smile \beta)(g_1, g_2, g_3) &= \alpha(g_1) \mathfrak{C}_{g_1}(\beta(g_2, g_3)) \\ &= \alpha(g_1) \beta(g_2, g_3), \end{aligned} \quad (\text{A13})$$

where the cup product takes values in $\mathbb{Z}_2 = \{0, 1\}$ and multiplication of α and β is the multiplication of integers. In reaching the last equality, we have used the fact that the 2-cochain $\beta(g_2, g_3)$ takes values in \mathbb{Z}_2 for which $\mathfrak{C}_{g_1}(\beta(g_2, g_3)) = \beta(g_2, g_3)$ for any g_1 . The cup product defined in Eq. (A11b) satisfies

$$\delta(\phi \smile \theta) = (\delta\phi \smile \theta) + (-1)^n (\phi \smile \delta\theta), \quad (\text{A14})$$

given two cochains $\phi \in C^n(G, N)$ and $\theta \in C^m(G, M)$. Hence, the cup product of two cocycles is again a cocycle as the right-hand side of Eq. (A14) vanishes. Having introduced the basics of group cohomology, next we shall compute the cohomology groups $[\phi] \in H^2(G_f, \text{U}(1)_c)$ for some specific finite Abelian groups G encountered in Sec. VI and whose projective representations are defined by Eqs. (3.11) and (3.12).

2. Classification of projective representations of G_f

It was described in Sec. III A, how a global symmetry group G_f for a fermionic quantum system naturally contains the fermion-number parity symmetry group \mathbb{Z}_2^F in its center, i.e., it is a central extension of a group G by \mathbb{Z}_2^F . Such group extension are classified by prescribing an element $[\gamma] \in H^2(G, \mathbb{Z}_2^F)$, such that we may think of G_f as the set of tuples $(g, h) \in G \times \mathbb{Z}_2^F$ with composition rule as in Eq. (3.3e). From this perspective, there is an implicit choice of trivialization $\tau : G_f \rightarrow \mathbb{Z}_2^F$ and projection $b : G_f \rightarrow G$ such that

$$\tau((g, h)) = h, \quad b((g, h)) = g. \quad (\text{A15})$$

Importantly, τ is related to the extension class γ that defines the group extension via the relation

$$b^* \gamma = \delta \tau, \quad (\text{A16})$$

where $b^* \gamma \in H^2(G_f, \mathbb{Z}_2^F)$ is the pullback of γ via b .

As explained in Sec. III B 1, we shall trade the 2-cocycle $\phi(g, h) \in \mathbb{Z}^2(G, \text{U}(1)_c)$ with the tuple $(\nu, \rho) \in C^2(G, \text{U}(1)) \times C^1(G, \mathbb{Z}_2)$ that satisfy certain cocycle and coboundary conditions. To this end, it is convenient to define the modified 2-coboundary operator

$$\mathcal{D}_\nu^2(\nu, \rho) := (\delta\nu - \pi \rho \smile \gamma, \delta\rho), \quad (\text{A17})$$

acting on a tuple of cochains $(\nu, \rho) \in C^2(G, \text{U}(1)) \times C^1(G, \mathbb{Z}_2)$ together with the modified 1-coboundary operator

$$\mathcal{D}_\gamma^1(\alpha, \beta) := (\delta\alpha + \pi \beta \smile \gamma, \delta\beta) \quad (\text{A18})$$

acting on a tuple of cochains $(\alpha, \beta) \in C^1(G, \text{U}(1)) \times C^0(G, \mathbb{Z}_2)$. Being a 0-cochain β does not take any arguments and takes values in \mathbb{Z}_2 , i.e., $\beta \in \mathbb{Z}_2$. Note that for the 0-cochain β , the coboundary operator (A3b) acts as

$$(\delta_c^0 \beta)(g) = \mathfrak{C}_g(\beta) - \beta, \quad (\text{A19})$$

which in fact vanishes for any $g \in G$ since β takes values in \mathbb{Z}_2 and $\mathfrak{C}_g(\beta) = \beta$. Using Eq. (A14) and the fact that γ is a cocycle, i.e., $\delta\gamma = 0$, one verifies that

$$\mathcal{D}_\gamma^2 \mathcal{D}_\gamma^1(\alpha, \beta) = (0, 0) \quad (\text{A20})$$

for any tuple $(\alpha, \beta) \in C^1(G, U(1)) \times C^0(G, \mathbb{Z}_2)$.

It was proved in Ref. [58] that one may assign to any 2-cocycle $[\phi] \in H^2(G_f, U(1)_c)$ an equivalence class $[(v, \rho)]$ of those tuples $(v, \rho) \in C^2(G, U(1)) \times C^1(G, \mathbb{Z}_2)$ that satisfy the cocycle condition under the modified 2-coboundary operator (A17) given by

$$\mathcal{D}_\gamma^2(v, \rho) = (\delta v - \pi \rho \smile \gamma, \delta \rho) = (0, 0). \quad (\text{A21})$$

Indeed, two tuples (v, ρ) and (v', ρ') that satisfy Eq. (3.6) are said to be equivalent if there exists a tuple $(\alpha, \beta) \in C^1(G, U(1)) \times C^0(G, \mathbb{Z}_2)$ such that

$$(v, \rho) = (v', \rho') + \mathcal{D}_\gamma^1(\alpha, \beta) = (v' + \delta\alpha + \pi\beta \smile \gamma, \delta\beta). \quad (\text{A22})$$

In other words, using this equivalence relation we define an equivalence class $[(v, \rho)]$ of the tuple (v, ρ) as an element of the set

$$[(v, \rho)] \in \frac{\ker(\mathcal{D}_\gamma^2)}{\text{im}(\mathcal{D}_\gamma^1)}. \quad (\text{A23})$$

The proof of the one-to-one correspondence between $[\phi]$ and $[(v, \rho)]$ then follows in three steps which we sketch out below. We refer the reader to Ref. [58] for more details.

(1) First, given a cocycle $\phi \in Z^2(G_f, U(1)_c)$, one can define $\rho \in Z^1(G, \mathbb{Z}_2)$ via Eq. (3.28). The fact that ρ is a cocycle follows from that fact that ϕ is a cocycle.

(2) Next, one can always find a representative ϕ in every cohomology class $[\phi] \in H^2(G_f, U(1)_c)$ that satisfies the relation $\phi = v + \pi\rho \smile \tau$.

(3) Finally, the fact that $\delta\phi = 0$ implies that $\delta v = \pi\rho \smile \gamma$.

We note that when the $[\gamma] = 0$, i.e., the group G_f splits as $G_f = G \times \mathbb{Z}_2^F$, the modified coboundary operators (A17) and (A18) reduce to the coboundary operator (A3b) with $n = 2$ and $n = 1$, respectively. If so the cochains v and ρ are both cocycles, i.e., $(v, \rho) \in Z^2(G, U(1)_c) \times Z^1(G, \mathbb{Z}_2)$. The equivalence classes $[(v, \rho)]$ of the tuple (v, ρ) is then equal to the equivalence cohomology classes of each of its components, i.e.,

$$[(v, \rho)] = ([v], [\rho]) \in H^2(G, U(1)_c) \times H^1(G, \mathbb{Z}_2). \quad (\text{A24})$$

We use the notation $([v], [\rho])$ for the two indices whenever the group G_f splits ($[\gamma] = 0$). The notation $[(v, \rho)]$ applies whenever the group G_f does not split ($[\gamma] \neq 0$).

3. The split group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$

The group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$, where the upper index T for the cyclic group $\mathbb{Z}_2^T \equiv \{e, t\}$ refers to the interpretation of t as time, is a split group. Since the group splits ($[\gamma] = 0$) upon using Eq. (A21) one finds that $[\phi] \in H^2(G_f, U(1)_c)$ separates into the pair of independent indices $[v] \in H^2(\mathbb{Z}_2^T, U(1)_c)$ and

$[\rho] \in H^1(\mathbb{Z}_2^T, \mathbb{Z}_2)$. Since $[v] = 0, 1$ and $[\rho] = 0, 1$,

$$\begin{aligned} H^2(\mathbb{Z}_2^T \times \mathbb{Z}_2^F, U(1)_c) \\ = \{([v], [\rho]) \mid [v] = 0, 1, \quad [\rho] = 0, 1\}. \end{aligned} \quad (\text{A25})$$

Below we describe how to “measure” these indices as well as the product or monoidal structure that these indices satisfy.

Claim 1: $[v] = 0, 1$.

Proof. Any cochain v belonging to the equivalence class $[v]$ is defined by the substitution $G = \mathbb{Z}_2^T$ in Eq. (3.11) and must satisfy the cocycle and coboundary conditions in (3.12b) and (3.14), respectively. If one chooses $g = h = f = t$ in Eq. (3.12b), where $t \in \mathbb{Z}_2^T$ is the generator of time-reversal which is represented antiunitarily $[c(t) = -1]$. One finds

$$v(t, t) + v(e, t) = v(t, e) - v(t, t) \text{ mod } 2\pi \Rightarrow v(t, t) = 0, \pi. \quad (\text{A26})$$

Equation (A26) is nothing but the statement that the representation of time reversal should square to either the identity or minus the identity. These two possibilities are not connected by a coboundary. Hence, they correspond to different second cohomology classes. To see this, assume they were connected by a coboundary, i.e., they satisfy the equivalence condition (3.14). On the one hand, choosing $g = t$ and $h = t$ in Eq. (3.14) implies that

$$v(t, t) - v'(t, t) = \varphi(t) - \varphi(t) - \varphi(e) = -\varphi(e) \Rightarrow \varphi(e) = \pi \quad (\text{A27})$$

if $v'(t, t) = \pi$ and $v(t, t) = 0$. However, on the other hand, choosing $g = t$ and $h = e$ in Eq. (3.14) implies that

$$\begin{aligned} v(t, e) - v'(t, e) &= \varphi(t) - \varphi(e) - \varphi(t) \\ &= -\varphi(e) \Rightarrow \varphi(e) = 0, \end{aligned} \quad (\text{A28})$$

since $v(g, e) = v(e, g) = 0$ for all g . Equations (A27) and (A28) contradict each other. This contradiction implies that one cannot consistently define a gauge transformation φ that interpolates between v such that $v(t, t) = \pi$ to v' such that $v(t, t) = 0$. We denote the cases $v(t, t) = \pi, 0$ with the equivalence classes $[v] = 1, 0$, respectively. ■

Claim 2: $[\rho] = 0, 1$.

Proof. For the second index $[\rho] \in H^1(\mathbb{Z}_2^T, \mathbb{Z}_2)$, two 1-cochains ρ and ρ' are equivalent if and only if they are 1-cocycles that differ by a coboundary of a 0-cochain. But, by definition, a \mathbb{Z}_2 -valued 0-cochain has a vanishing coboundary. Hence, the coset $H^1(\mathbb{Z}_2^T, \mathbb{Z}_2)$ is just the set of all distinct 1-cocycles. By definition, a 1-cocycle ρ must obey [recall Eq. (A6)]

$$\rho(g) + c(g)\rho(h) - \rho(gh) = 0. \quad (\text{A29a})$$

Choosing $g = t$ and $h = p$ delivers

$$\rho(t) = \rho(p) + \rho(t \cdot p) = \rho(t \cdot p), \quad (\text{A29b})$$

where we used the fact that $\rho(p) = 0$ by the definition (3.28). The value of $\rho(t)$ in \mathbb{Z}_2 indicates whether the projective representation of reversal of time commutes or anticommutes with the projective representation of the fermion parity operator p . Equation (A29b) states that fermion parity of the quantum representation of t is equal to the fermion parity of the quantum

representation of $t \cdot p$. We assign the indices $[\rho] = 0, 1$ to the values $\rho(t) = 0, 1$, respectively. ■

Given two local projective representations \hat{u}_1 and \hat{u}_2 of the group $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^F$ acting on the Fock spaces \mathcal{F}_1 and \mathcal{F}_2 , respectively, we now derive the indices associated with the local projective representation \hat{u} acting on the graded tensor product $\mathcal{F} = \mathcal{F}_1 \otimes_{\mathfrak{g}} \mathcal{F}_2$ of the Fock spaces \mathcal{F}_1 and \mathcal{F}_2 . The definition (3.34a) implies

$$\hat{u}(t) := \hat{v}_1(t) \hat{v}_2(t) \mathbf{K}, \quad \hat{u}(p) := \hat{v}_1(p) \hat{v}_2(p), \quad (\text{A30})$$

for the representations of elements $t \in \mathbb{Z}_2^T$ and $p \in \mathbb{Z}_2^F$. In turn, using the relation (3.34c) and the definition (3.28), we find (for $g, h \in G_f$)

$$\phi(g, h) = \phi_1(g, h) + \phi_2(g, h) + \pi \rho_1(h) \rho_2(g) \bmod 2\pi, \quad (\text{A31a})$$

$$\nu(t, t) = \nu_1(t, t) + \nu_2(t, t) + \pi \rho_1(t) \rho_2(t) \bmod 2\pi, \quad (\text{A31b})$$

$$\begin{aligned} \rho(t) &= \frac{\phi(t, p) - \phi(p, t)}{\pi} \bmod 2 \\ &= \frac{1}{\pi} [\phi_1(t, p) + \phi_2(t, p) + \pi \rho_1(p) \rho_2(t) - \phi_1(p, t) \\ &\quad - \phi_2(p, t) - \pi \rho_1(t) \rho_2(p)] \bmod 2 \\ &= \frac{\phi_1(t, p) - \phi_1(p, t)}{\pi} + \frac{\phi_2(t, p) - \phi_2(p, t)}{\pi} \bmod 2 \\ &= \rho_1(t) + \rho_2(t) \bmod 2, \end{aligned} \quad (\text{A31c})$$

for the 2-cocycle ν and 1-cocycle ρ associated with the representation \hat{u} . In reaching the last line of Eq. (A31c), we have used the fact that by definition $\rho_1(p) = \rho_2(p) = 0$. Here, ν_1 and ν_2 are the 2-cocycles, and ρ_1 and ρ_2 are the 1-cocycles associated with the representations \hat{u}_1 and \hat{u}_2 , respectively. Assignments of indices $[\nu]$ and $[\rho]$ to the local projective representations of the group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ (as shown above) and the identity (A31) imply that the indices of the tensor product representation are related to the indices of the constituent representations via

$$\begin{aligned} [\nu] &= [\nu_1] + [\nu_2] + [\rho_1][\rho_2] \bmod 2, \\ [\rho] &= [\rho_1] + [\rho_2] \bmod 2, \end{aligned} \quad (\text{A32a})$$

We note that for the group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ the cup product $[\pi \rho_1 \smile \rho_2]$ in Eq. (3.35) simplifies to

$$[\pi \rho_1 \smile \rho_2] \equiv [\rho_1][\rho_2], \quad (\text{A32b})$$

One thus finds that different local projective representations of the group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ form the cyclic group \mathbb{Z}_4 under the stacking rule (A32a).

4. The split group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$

As in Appendix A3, the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ is a split group. We denote the two generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by g_1 and g_2 , both of which are represented by unitary operators. Because of the Cartesian products, $[\phi] \in H^2(G_f, \text{U}(1)_c)$ separates into the pair of independent indices $[\nu] \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1)_c)$ and $[\rho] \in H^1(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ according to Eq. (3.30). Since

the group representation is unitary (and hence linear as opposed to antilinear), there is no negative sign that appears on the right-hand side of the equality in Eq. (A26). It is not possible to constrain the possible values of $\nu(g_1, g_1)$ or $\nu(g_2, g_2)$ as was done in Eq. (A26). Cocycle conditions that are akin to Eq. (A26) are trivially satisfied. On the other hand, examining the algebra between projective representations of g_1 and g_2 provides useful information. Since each symmetry transformation must have a fixed fermion parity, the projective representations of g_1 and g_2 must either commute or anticommute with each other, i.e.,

$$\nu(g_1, g_2) - \nu(g_2, g_1) = 0, \pi. \quad (\text{A33})$$

These two possible values constitute the two inequivalent cohomology classes for the index $[\nu]$. To show this, we consider two cochains ν and ν' that are connected by the coboundary condition (3.14). One finds

$$\nu(g_1, g_2) - \nu'(g_1, g_2) = \varphi(g_1) + \varphi(g_2) - \varphi(g_1 \cdot g_2), \quad (\text{A34a})$$

$$\nu(g_2, g_1) - \nu'(g_2, g_1) = \varphi(g_2) + \varphi(g_1) - \varphi(g_2 \cdot g_1). \quad (\text{A34b})$$

Because $G \equiv \mathbb{Z}_2 \times \mathbb{Z}_2$ is Abelian, this pair of equations implies that

$$\nu(g_1, g_2) - \nu(g_2, g_1) = \nu'(g_1, g_2) - \nu'(g_2, g_1). \quad (\text{A34c})$$

Therefore the projective representations of $g_1 \in G$ and $g_2 \in G$ that either commute pairwise or anticommute pairwise must belong to distinct second cohomology classes. We assign the values $[\nu] = 0, 1$ to $\nu(g_1, g_2) - \nu(g_2, g_1) = 0, \pi$, respectively.

The index $[\rho]$ characterizes whether the representations of $g_1 \in G$ and $g_2 \in G$ commute or anticommute with the fermion parity. Note that the parity of the element $g_1 \cdot g_2$ is determined by the parities of g_1 and g_2 . Therefore $[\rho]$ retains the $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure. We assign a pair of indices

$$[\rho] = ([\rho]_1, [\rho]_2), \quad [\rho]_1 = 0, 1, \quad [\rho]_2 = 0, 1, \quad (\text{A35})$$

to indicate the parities of the projective representations of $g_1 \in G$ and $g_2 \in G$, respectively. We may then write

$$\begin{aligned} H^2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F, \text{U}(1)_c) &= \{([\nu], [\rho]) \mid [\nu] = 0, 1, \\ &[\rho] = ([\rho]_1, [\rho]_2), \quad [\rho]_1, [\rho]_2 = 0, 1\}. \end{aligned} \quad (\text{A36})$$

Given two local projective representations \hat{u}_1 and \hat{u}_2 of the group $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ acting on two Fock spaces \mathcal{F}_1 and \mathcal{F}_2 , respectively, we shall derive the indices associated with the local projective representation \hat{u} acting on the graded tensor product $\mathcal{F} = \mathcal{F}_1 \otimes_{\mathfrak{g}} \mathcal{F}_2$ of Fock spaces \mathcal{F}_1 and \mathcal{F}_2 . The definition (3.34a) implies

$$\begin{aligned} \hat{u}(g_1) &= \hat{v}_1(g_1) \hat{v}_2(g_1), \quad \hat{u}(g_2) = \hat{v}_1(g_2) \hat{v}_2(g_2), \\ \hat{u}(p) &= \hat{v}_1(p) \hat{v}_2(p), \end{aligned} \quad (\text{A37})$$

for the representations of elements $g_1, g_2 \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $p \in \mathbb{Z}_2^F$. In turn, using the relation (3.34c) and the definition

(3.28), we find

$$\begin{aligned} \nu(g_1, g_2) - \nu(g_2, g_1) &= \nu_1(g_1, g_2) + \nu_2(g_1, g_2) + \pi \rho_1(g_2) \rho_2(g_1) - \nu_1(g_2, g_1) - \nu_2(g_2, g_1) - \pi \rho_1(g_1) \rho_2(g_2) \bmod 2\pi \\ &= \nu_1(g_1, g_2) - \nu_1(g_2, g_1) + \nu_2(g_1, g_2) - \nu_2(g_2, g_1) + \pi[\rho_1(g_2) \rho_2(g_1) - \rho_1(g_1) \rho_2(g_2)] \bmod 2\pi, \end{aligned} \quad (\text{A38a})$$

$$\begin{aligned} \rho(g_i) &= \frac{\phi(g_i, p) - \phi(p, g_i)}{\pi} \bmod 2 \\ &= \frac{1}{\pi} [\phi_1(g_i, p) + \phi_2(g_i, p) + \pi \rho_1(p) \rho_2(g_i) - \phi_1(p, g_i) - \phi_2(p, g_i) - \pi \rho_1(g_i) \rho_2(p)] \bmod 2 \\ &= \frac{\phi_1(g_i, p) - \phi_1(p, g_i)}{\pi} + \frac{\phi_2(g_i, p) - \phi_2(p, g_i)}{\pi} \bmod 2 \quad [\text{where we made use of } \rho_1(p) = \rho_2(p) = 0] \\ &= \rho_1(g_i) + \rho_2(g_i) \bmod 2, \quad i = 1, 2, \end{aligned} \quad (\text{A38b})$$

for the 2-cocycle ν and 1-cocycle ρ associated with the representation \hat{u} . Here, ν_1 and ν_2 are the 2-cocycles, and ρ_1 and ρ_2 are the 1-cocycles associated with the representations \hat{u}_1 and \hat{u}_2 , respectively. Assignments of indices $[\nu]$ and $[\rho]$ to the local projective representations of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ (as shown above) and the identity (A38) imply that the indices of the tensor product representation are related to the indices of the constituent representations via

$$[\nu] = [\nu_1] + [\nu_2] + [\rho_1]_2 [\rho_2]_1 - [\rho_1]_1 [\rho_2]_2 \bmod 2, \quad (\text{A39a})$$

$$[\rho]_i = [\rho_1]_i + [\rho_2]_i \bmod 2, \quad i = 1, 2, \quad (\text{A39b})$$

where we have denoted by $[\rho_1], [\rho_2] \in H^1(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ the first cohomology classes associated to the local projective representations \hat{u}_1 and \hat{u}_2 , and we used the notation

$$[\rho_1] \equiv ([\rho_1]_1, [\rho_1]_2), \quad [\rho_2] \equiv ([\rho_2]_1, [\rho_2]_2). \quad (\text{A39c})$$

We note that for the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ the cup product $[\pi \rho_1 \smile \rho_2]$ in Eq. (3.35) simplifies to

$$[\pi \rho_1 \smile \rho_2] \equiv [\rho_1]_2 [\rho_2]_1 - [\rho_1]_1 [\rho_2]_2, \quad (\text{A39d})$$

and the stacking rule (3.35) becomes

$$([\nu], [\rho], 0) \equiv \left([\nu], ([\rho]_1, [\rho]_2), 0 \right) = \left([\nu_1] + [\nu_2] + [\rho_1]_2 [\rho_2]_1 - [\rho_1]_1 [\rho_2]_2, ([\rho_1]_1 + [\rho_2]_1, [\rho_1]_2 + [\rho_2]_2), 0 \right). \quad (\text{A39e})$$

One thus finds that different local projective representations of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^F$ form the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ under the stacking rule (A39e).

5. The nonsplit group \mathbb{Z}_4^{FT}

The group \mathbb{Z}_4^{FT} is the nontrivial central extension of $G \equiv \mathbb{Z}_2^T$ by $\mathbb{Z}_2^F \equiv \{p, p^2\}$, where the upper index T for the cyclic group $\mathbb{Z}_2^T \equiv \{t, t^2\}$ refers to the interpretation of t as reversal of time. This central extension of time reversal by fermion parity is specified by the map γ obeying the nonsplit condition (3.31) because of $\gamma(t, t) = p$ (which implies the group composition rule $t \cdot t = p$). If so, ν is not a cocycle but a cochain with nonvanishing coboundary according to Eq. (3.32) and (A21). On the other hand, ρ is a 1-cocycle. We thus observe that two tuples (ν, ρ) and (ν', ρ') are not in the same equivalence class if $\rho(t) \neq \rho'(t)$. Since $\rho(t)$ can take two values, 0 or 1, there exist at least two distinct equivalence classes of the tuple (ν, ρ) , labeled by $\rho(t)$. Given this value of $\rho(t)$, we shall construct the distinct equivalence classes of (ν, ρ) corresponding to different values of $\nu \in C^2(G, \text{U}(1))$. Choosing $g = h = f = t$ in Eq. (3.32) delivers

$$\nu(t, e) - \nu(t, t) - \nu(t, t) - \nu(e, t) = \pi \rho(t) \gamma(t, t) \bmod 2\pi. \quad (\text{A40})$$

With the help of Eq. (3.12c) and with the choice of the convention $\gamma(t, t) = p \equiv 1$ for the nonsplit group \mathbb{Z}_4^{FT} , we find the pair of solutions to Eq. (A40) given by

$$\nu(t, t) = -\frac{\pi}{2} \rho(t), \quad \nu(t, t) = -\frac{\pi}{2} \rho(t) + \pi. \quad (\text{A41})$$

The multiplicative factor π appears on the right-hand sides since ν takes values in $\text{U}(1)$ and is thus defined modulo 2π . Now, the two solutions (A41) are equivalent under the equivalence relation (A22) as can be seen by choosing $\alpha = 0$ and $\beta = p \equiv 1$ in Eq. (A22). Indeed, the term $\pi\beta \smile \gamma = \pi$ then cancels the factor π between the two solutions (A41). Thus, for each value of $\rho(t) = 0, 1$, there exists a single distinct equivalence class $[(\nu, \rho)]$. From now on, we choose the solution $(-\pi \rho(t)/2, \rho(t))$ as the representative of the equivalence class $[(\nu, \rho)]$. We assign $[(\nu, \rho)] = (1, 1)$ to the case $(\nu(t, t), \rho(t)) = (-\pi/2, 1)$ and $[(\nu, \rho)] = (0, 0)$ to the case $(\nu(t, t), \rho(t)) = (0, 0)$ and write

$$H^2(\mathbb{Z}_4^{FT}, \text{U}(1)_c) = \{[(\nu, \rho)] \mid [(\nu, \rho)] = (0, 0), (1, 1)\}. \quad (\text{A42})$$

Given two local projective representations \hat{u}_1 and \hat{u}_2 of the group $G_f = \mathbb{Z}_4^{FT}$ acting on two Fock spaces \mathcal{F}_1 and \mathcal{F}_2 , respectively, we shall derive the indices associated with the local projective representation \hat{u} acting on the graded tensor product $\mathcal{F} = \mathcal{F}_1 \otimes_{\mathfrak{g}} \mathcal{F}_2$ of Fock spaces \mathcal{F}_1 and \mathcal{F}_2 . The definition

(3.34a) implies

$$\hat{u}(t) = \hat{v}_1(t) \hat{v}_2(t) \mathbf{K}, \quad \hat{u}(p) = \hat{v}_1(p) \hat{v}_2(p), \quad (\text{A43})$$

for the representations of elements $t, p \in \mathbb{Z}_4^{FT}$. In turn, using the relation (3.34c) and the definition (3.28), we find that if v_1 and v_2 are 2-cochains, and ρ_1 and ρ_2 are two 1-cocycles associated with the representations \hat{u}_1 and \hat{u}_2 , respectively, then the 2-cochain v and 1-cocycle ρ associated with the representation \hat{u} are given by

$$v(t, t) = v_1(t, t) + v_2(t, t) + \pi \rho_1(t) \rho_2(t) \bmod 2\pi, \quad (\text{A44a})$$

$$\rho(t) = \rho_1(t) + \rho_2(t) \bmod 2, \quad (\text{A44b})$$

respectively. Although this pair of relations are identical to their counterparts in Eq. (A31), the nonsplit nature of the group \mathbb{Z}_4^{FT} carries over to the stacked projective representation. Indeed, inserting twice (A41) on the right-hand side of (A44a) gives

$$v(t, t) = -\frac{\pi}{2} \rho_1(t) - \frac{\pi}{2} \rho_2(t) + \pi \rho_1(t) \rho_2(t) \bmod 2\pi, \quad (\text{A45a})$$

$$\rho(t) = \rho_1(t) + \rho_2(t) \bmod 2. \quad (\text{A45b})$$

There are four cases to consider. When $(\rho_1(t), \rho_2(t)) = (0, 0)$, $\rho(t) = 0$ and $v(t, t) = 0$. When $(\rho_1(t), \rho_2(t)) = (0, 1)$, $\rho(t) = 1$ and $v(t, t) = -\pi/2$. When $(\rho_1(t), \rho_2(t)) = (1, 0)$, $\rho(t) = 1$ and $v(t, t) = -\pi/2$. When $(\rho_1(t), \rho_2(t)) = (1, 1)$, $\rho(t) = 0$ and $v(t, t) = 0$. Hence, $v(t, t)$ and $\rho(t)$ on the left-hand sides of Eqs. (A44a) and (A44b), respectively, obey Eq. (A41). The stacked representation can then be labeled by the indices $[(\rho, \rho)]$ where $\rho = \rho_1 + \rho_2 \bmod 2$. At last, the stacking rule (3.35) takes the form

$$[(v, \rho), 0] = ([(\rho_1 + \rho_2 \bmod 2, \rho_1 + \rho_2 \bmod 2)], 0). \quad (\text{A46})$$

One thus finds that different local projective representations of the group \mathbb{Z}_4^{FT} form the cyclic group \mathbb{Z}_2 under the stacking rule (A46).

APPENDIX B: CONSTRUCTION OF FERMIONIC MATRIX PRODUCT STATES (FMPS)

We review the construction of fermionic matrix product states (FMPS). We refer the reader to Refs. [56,57] and references therein on the topic of matrix product states (MPS). As in bosonic matrix product states (BMPS), FMPS can be expressed as a contraction of objects belonging to a graded tensor product of vector spaces. The need for *graded tensor product of vector spaces* stems from the underlying fermionic algebra.

1. \mathbb{Z}_2 -graded vector spaces and their \mathbb{Z}_2 -graded tensor products

Any fermionic Fock space \mathcal{F} can be seen, in the basis that diagonalizes the total fermionic number operator, to be the direct sum over a subspace \mathcal{F}_0 with even total fermionic number and a subspace \mathcal{F}_1 with odd total fermionic number. This property endows fermionic Fock space with a natural \mathbb{Z}_2 grading.

A \mathbb{Z}_2 -graded vector space V admits the direct sum decomposition

$$V = V_0 \oplus V_1. \quad (\text{B1})$$

We shall identify the subscripts 0 and 1 as the elements of the additive group \mathbb{Z}_2 . We say that V_0 (V_1) has parity 0 (1). Any vector space is \mathbb{Z}_2 -graded since the choice $V_0 = V$ and $V_1 = \emptyset$ is always possible. Any subspace of V_0 shares its parity 0. Any subspace of V_1 shares its parity 1. A vector $|v\rangle \in V$ is called *homogeneous* if it entirely resides in either one of the subspaces V_0 and V_1 . The parity $|v|$ of the homogeneous state $|v\rangle$ is either 0 if $|v\rangle \in V_0$ or 1 if $|v\rangle \in V_1$. These observation on the \mathbb{Z}_2 -grading of a vector space V only become useful when one demands that any operation acting on V preserves the \mathbb{Z}_2 grading.

For example, certain operations need to be defined carefully between two \mathbb{Z}_2 -graded vector space V and W that preserve their \mathbb{Z}_2 structure. One such operation is the \mathbb{Z}_2 -graded tensor product. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two \mathbb{Z}_2 -graded vector spaces. We define their graded tensor product as the map

$$\otimes_{\mathfrak{g}} : V \times W \rightarrow V \otimes W, \quad (\text{B2a})$$

such that

$$V_i \otimes_{\mathfrak{g}} W_j \subseteq (V \otimes W)_{(i+j) \bmod 2}, \quad i, j = 0, 1. \quad (\text{B2b})$$

By design, the operation $\otimes_{\mathfrak{g}}$ carries the \mathbb{Z}_2 -grading of V and W to their \mathbb{Z}_2 -graded tensor product. In particular, for any homogeneous vectors $|v\rangle \in V$ with parity $|v| = 0, 1$ and $|w\rangle \in W$ with parity $|w| = 0, 1$, the graded tensor product $|v\rangle \otimes_{\mathfrak{g}} |w\rangle$ of two homogeneous vectors has the parity

$$||v\rangle \otimes_{\mathfrak{g}} |w\rangle| := (|v| + |w|) \bmod 2. \quad (\text{B2c})$$

The connection between the \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$ and fermionic Fock spaces $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$, is established through the identifications $\mathcal{F}_0 \rightarrow V_0$ and $\mathcal{F}_1 \rightarrow V_1$. However, a fermionic Fock space has more structure than a mere \mathbb{Z}_2 -graded vector space. Wave functions in a fermionic Fock space are fully antisymmetric under the permutation of two fermions. This requirement can be implemented as follows on a \mathbb{Z}_2 -graded vector space. The exchange of two fermions can be represented by the isomorphism

$$R : V \otimes_{\mathfrak{g}} W \rightarrow W \otimes_{\mathfrak{g}} V, \quad (\text{B3a})$$

by which the graded tensor product of the homogeneous vectors $|v\rangle \in V$ and $|w\rangle \in W$ obeys

$$|v\rangle \otimes_{\mathfrak{g}} |w\rangle \mapsto (-1)^{|v||w|} |w\rangle \otimes_{\mathfrak{g}} |v\rangle. \quad (\text{B3b})$$

The map R is called the *reordering* operation. It is invertible with itself as inverse since R^2 is the identity map.

For every \mathbb{Z}_2 -graded vector space V , we define the dual \mathbb{Z}_2 -graded vector space V^* . We denote an element of the dual \mathbb{Z}_2 -graded vector space V^* by $\langle v|$, the dual to the vector $|v\rangle \in V$. The dual \mathbb{Z}_2 -graded vector space V^* inherits a \mathbb{Z}_2 grading from assigning the parity $|v|$ to the vector $\langle v| \in V^*$ if $|v\rangle \in V$ is homogeneous with parity $|v|$. The *contraction* \mathcal{C} is the map

$$\mathcal{C} : V^* \otimes_{\mathfrak{g}} V \rightarrow \mathbb{C}, \quad (\text{B4a})$$

$$\langle \psi| \otimes_{\mathfrak{g}} |\phi\rangle \mapsto \langle \psi|\phi\rangle,$$

where $\langle \psi | \phi \rangle$ denotes the scalar product between the pair $|\psi\rangle, |\phi\rangle \in V$. Hence,

$$\mathcal{C}(\langle i | \otimes_{\mathfrak{g}} | j \rangle) = \delta_{ij} \quad (\text{B4b})$$

holds for any pair of orthonormal and homogeneous basis vectors $|i\rangle, |j\rangle \in V$. The *contraction* \mathcal{C}^* is the map $\mathcal{C}^* : V \otimes_{\mathfrak{g}} V^* \rightarrow \mathbb{C}$ defined by its action

$$\begin{aligned} \mathcal{C}^*(|i\rangle \otimes_{\mathfrak{g}} \langle j|) &:= \mathcal{C}(R(|i\rangle \otimes_{\mathfrak{g}} \langle j|)) = \mathcal{C}((-1)^{|i||j|} \langle j | \otimes_{\mathfrak{g}} |i\rangle) \\ &= (-1)^{|i||j|} \langle j | i \rangle = (-1)^{|i||j|} \delta_{ij} \end{aligned} \quad (\text{B4c})$$

for any pair of orthonormal basis vectors $|i\rangle, |j\rangle \in V$. It is common practice to use the same symbol \mathcal{C} for both \mathcal{C} and \mathcal{C}^* . Any linear operator

$$M : V \rightarrow V \quad (\text{B5a})$$

can be represented in the orthonormal and homogeneous basis $\{|i\rangle\}$ of V by the matrix

$$M_{ij} = (-1)^{|i||j|} M_{ji} \quad (\text{B5b})$$

through the expansion

$$M := \sum_{i,j} M_{ij} |i\rangle \otimes_{\mathfrak{g}} \langle j| \in V \otimes_{\mathfrak{g}} V^*. \quad (\text{B5c})$$

The linear operator M has a well defined parity if and only if each term $|i\rangle \otimes_{\mathfrak{g}} \langle j|$ in the summation has the same parity, in which case

$$|M| := (|i| + |j|) \bmod 2. \quad (\text{B5d})$$

More generally, if we define

$$T := \sum_{i_1, \dots, i_n} T_{i_1, \dots, i_n} |i_1\rangle \otimes_{\mathfrak{g}} \dots \otimes_{\mathfrak{g}} |i_n\rangle \in V_{i_1} \otimes_{\mathfrak{g}} \dots \otimes_{\mathfrak{g}} V_{i_n} \quad (\text{B6a})$$

we can assign the parity

$$|T| := (|i_1| + \dots + |i_n|) \bmod 2 \quad (\text{B6b})$$

when all $|i_1\rangle \otimes_{\mathfrak{g}} \dots \otimes_{\mathfrak{g}} |i_n\rangle$ share the same parity.

2. Fermionic matrix product state (FMPS)

We attach to each integer $j = 1, \dots, N$ three \mathbb{Z}_2 -graded vector spaces

$$\begin{aligned} V_j &:= \text{span}\{|\alpha\rangle \mid \alpha = 1, \dots, \mathcal{D}_{v,j}\}, \\ \mathcal{F}_j &:= \text{span}\{|\psi_{\sigma}\rangle \mid \sigma = 1, \dots, \mathcal{D}_j\}, \\ V_j^* &:= \text{span}\{|\beta\rangle \mid \beta = 1, \dots, \mathcal{D}_{v,j}\}. \end{aligned} \quad (\text{B7a})$$

The basis states $|\alpha\rangle$ and $|\beta\rangle$ of the dual pair V_j and V_j^* of \mathbb{Z}_2 -graded vector spaces are virtual (auxiliary) states. They are denoted by rounded kets and bras and are introduced for convenience. Each auxiliary basis state has a well defined parity by assumption. The basis states $\{|\psi_{\sigma}\rangle\}$ span the physical fermionic Fock space \mathcal{F}_j . Each physical basis state $|\psi_{\sigma}\rangle$ has a well defined parity by assumption, as follows from working in the fermion-number basis of \mathcal{F}_j say. The auxiliary \mathbb{Z}_2 -graded vector space V_j has the dimension $\mathcal{D}_{v,j}$.

The physical fermionic Fock space \mathcal{F}_j has dimension \mathcal{D}_j . A fermionic matrix product state (FMPS) takes the form

$$|\Psi\rangle := \mathcal{C}_v(Q(b)Y \mathbf{A}[1] \otimes_{\mathfrak{g}} \mathbf{A}[2] \otimes_{\mathfrak{g}} \dots \otimes_{\mathfrak{g}} \mathbf{A}[N]) \quad (\text{B7b})$$

and has a well defined parity provided the objects $Q(b)$, Y , $\mathbf{A}[1]$, $\mathbf{A}[2]$, \dots , $\mathbf{A}[N]$ are defined as follows. For any $j = 1, \dots, N$, element $\mathbf{A}[j] \in V_j \otimes_{\mathfrak{g}} \mathcal{F}_j \otimes_{\mathfrak{g}} V_{j+1}^*$ is defined by

$$\mathbf{A}[j] := \sum_{\alpha_j=1}^{\mathcal{D}_{v,j}} \sum_{\sigma_j=1}^{\mathcal{D}_j} \sum_{\beta_{j+1}=1}^{\mathcal{D}_{v,j+1}} (A_{\sigma_j})_{\alpha_j \beta_j} |\alpha_j\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} \langle \beta_j| \quad (\text{B7c})$$

once the matrices A_{σ_j} , labeled as they are by the basis elements of the local Fock space \mathcal{F}_j and with the matrix elements $(A_{\sigma_j})_{\alpha_j \beta_j}$, have been chosen. The contraction \mathcal{C}_v labeled by the lower index v is understood to be over all virtual indices belonging to the dual pair (V_j^*, V_j) of auxiliary \mathbb{Z}_2 -graded vector spaces, thereby producing the tensor proportional to

$$T_{\alpha_1 \dots \alpha_N | \beta_1 \dots \beta_N} := \delta_{\beta_1 \alpha_2} \delta_{\beta_2 \alpha_3} \dots \delta_{\beta_{N-1} \alpha_N} \delta_{\beta_N \alpha_1} \quad (\text{B7d})$$

if $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$ and $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ were chosen to be the identity

$$Q(b) \equiv Y \equiv \sum_{\alpha} |\alpha\rangle \otimes_{\mathfrak{g}} \langle \alpha|. \quad (\text{B7e})$$

The integer $b = 0, 1$ labels the boundary conditions selected by $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$. The element $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ is needed to fix the fermion parity of $|\Psi\rangle$. More precisely, we demand that the parity (B5d) of $Q(b) \in V_1 \otimes_{\mathfrak{g}} V_1^*$ and the parity (D6b) of $\mathbf{A}[j] \in V_j \otimes_{\mathfrak{g}} \mathcal{F}_j \otimes_{\mathfrak{g}} V_{j+1}^*$ are *both* even, while the parity (B5d) of $Y \in V_1 \otimes_{\mathfrak{g}} V_1^*$ is *either* even *or* odd. Consequently, the parity of $|\Psi\rangle$ is determined by the parity of Y since

$$|\Psi| = \left(|Q(b)| + |Y| + \sum_{j=1}^N |\mathbf{A}[j]| \right) \bmod 2 = |Y|. \quad (\text{B7f})$$

A prerequisite to imposing translation symmetry on any FMPS is that all dimensions $\mathcal{D}_{v,j}$ and \mathcal{D}_j are independent of $j = 1, \dots, N$. Hence, we assume from now on that

$$\mathcal{D}_{v,j} \equiv \mathcal{D}_v, \quad \mathcal{D}_j \equiv \mathcal{D}, \quad j = 1, \dots, N. \quad (\text{B7g})$$

3. Even-parity fermionic matrix product state (FMPS)

The FMPS

$$|\Psi\rangle_0^b := \mathcal{C}_v(Q(b)Y \mathbf{A}[1] \otimes_{\mathfrak{g}} \dots \otimes_{\mathfrak{g}} \mathbf{A}[N]) \quad (\text{B8a})$$

is an even-parity FMPS obeying periodic ($b = 0$) or antiperiodic ($b = 1$) boundary conditions if, for any $j = 1, \dots, N$,

$$\mathbf{A}[j] := \sum_{\alpha_j=1}^{\mathcal{D}_v} \sum_{\sigma_j=1}^{\mathcal{D}} \sum_{\beta_j=1}^{\mathcal{D}_v} (A_{\sigma_j}^{(0)})_{\alpha_j \beta_j} |\alpha_j\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} \langle \beta_j|, \quad (\text{B8b})$$

$$|\mathbf{A}[j]| = (|\alpha_j| + |\sigma_j| + |\beta_j|) \bmod 2 = 0, \quad (\text{B8c})$$

$$Y := \sum_{\alpha=1}^{\mathcal{D}_v} |\alpha\rangle \otimes_{\mathfrak{g}} \langle \alpha|, \quad (\text{B8d})$$

$$Q(b=0) := \sum_{\alpha=1}^{\mathcal{D}_v} |\alpha\rangle \otimes_{\mathfrak{g}} |\alpha\rangle, \quad (\text{B8e})$$

$$Q(b=1) := \sum_{\alpha=1}^{\mathcal{D}_v} (-1)^{|\alpha|} |\alpha\rangle \otimes_{\mathfrak{g}} |\alpha\rangle. \quad (\text{B8f})$$

By construction, both $Q(b)$ and Y are of even parity. Moreover, $(|\alpha_j| + |\sigma_j| + |\beta_j|) \bmod 2 = 1$ implies that $(A_{\sigma_j}^{(0)})_{\alpha_j \beta_j} = 0$.

We are going to give an alternative representation of this even-parity FMPS under the assumption that the virtual dimension \mathcal{D}_v obeys the partition $\mathcal{D}_v = M_e + M_o$ where $M_e \equiv M$ and $M_o \equiv M$ are the numbers of even- and odd-parity virtual basis vectors, respectively. Parity evenness of $A[j]$ implies that the $\mathcal{D}_v \times \mathcal{D}_v$ dimensional matrices $A_{\sigma_j}^{(0)}$ with the matrix elements $(A_{\sigma_j}^{(0)})_{\alpha_j \beta_j}$ is either block diagonal

$$A_{\sigma_j}^{(0)} = \begin{pmatrix} B_{\sigma_j} & 0 \\ 0 & C_{\sigma_j} \end{pmatrix}, \quad \text{if } |\sigma_j| = 0, \quad (\text{B9a})$$

when the physical state is of even parity [as follows from Eq. (B8c)] or block off diagonal

$$A_{\sigma_j}^{(0)} = \begin{pmatrix} 0 & D_{\sigma_j} \\ F_{\sigma_j} & 0 \end{pmatrix}, \quad \text{if } |\sigma_j| = 1, \quad (\text{B9b})$$

when the physical state is of odd parity [as follows from Eq. (B8c)]. All the blocks are here $M \times M$ -dimensional. Parity evenness of $Q(b)$ with matrix elements $(Q(b))_{\alpha_j \beta_j}$ and Y with matrix elements $Y_{\alpha_j \beta_j}$ implies that

$$Y = Q(b=0) = \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & \mathbb{1}_M \end{pmatrix},$$

$$Q(b=1) = \begin{pmatrix} \mathbb{1}_M & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix} =: P. \quad (\text{B9c})$$

Hereby, we introduced the parity matrix P that satisfies

$$P A_{\sigma_j}^{(0)} P = (-1)^{|\sigma_j|} A_{\sigma_j}^{(0)}. \quad (\text{B9d})$$

Inserting these explicit representations of $Q(b)$ and Y in Eq. (B8a) delivers

$$|\Psi\rangle_0^b \equiv |\{A_{\sigma_j}^{(0)}\}; b\rangle := \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}] |\Psi_{\sigma}\rangle, \quad (\text{B10a})$$

where we used the shorthand notation $|\Psi_{\sigma}\rangle := |\psi_{\sigma_1}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_2}\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |\psi_{\sigma_N}\rangle$. The appearance of the matrix P when $b=0$ is counterintuitive. It is needed to eliminate from the sum over all physical basis states $\{|\Psi_{\sigma}\rangle\}$ those physical basis states of odd parity. The state $|\{A_{\sigma_j}^{(0)}\}; b\rangle$ has even parity since

$$\left(\sum_{j=1}^N |\sigma_j| \right) \bmod 2 = \left[\sum_{j=1}^N (|\alpha_j| + |\beta_j|) \right] \bmod 2$$

$$= \left(\sum_{j=1}^N 2|\alpha_j| \right) \bmod 2 = 0, \quad (\text{B10b})$$

where we used condition (B8c) to establish the first equality and the condition $|\beta_j| = |\alpha_{j+1}|$ that is imposed by the contractions of virtual indices to establish the second equality.

4. Odd-parity fermionic matrix product state (FMPS)

The FMPS

$$|\Psi\rangle_1^b := \mathcal{C}_v(Q(b)Y A[1] \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} A[N]), \quad (\text{B11a})$$

is an odd-parity FMPS obeying periodic ($b=0$) or antiperiodic ($b=1$) boundary conditions if, for any $j=1, \dots, N$,

$$A[j] := \sum_{\alpha_j=1}^{\mathcal{D}_v} \sum_{\sigma_j=1}^{\mathcal{D}} \sum_{\beta_j=1}^{\mathcal{D}_v} (A_{\sigma_j}^{(1)})_{\alpha_j \beta_j} |\alpha_j\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_j}\rangle \otimes_{\mathfrak{g}} |\beta_j\rangle, \quad (\text{B11b})$$

$$|A[j]| = (|\sigma_j| + |\alpha_j| + |\beta_j|) \bmod 2 = 0, \quad (\text{B11c})$$

$$Y := \sum_{\alpha, \beta=1}^{\mathcal{D}_v} Y_{\alpha \beta} |\alpha\rangle \otimes_{\mathfrak{g}} |\beta\rangle,$$

$$Y_{\alpha \beta} = 0 \text{ if } (|\alpha| + |\beta|) \bmod 2 = 0, \quad (\text{B11d})$$

$$Q(b=0) := \sum_{\alpha=1}^{\mathcal{D}_v} |\alpha\rangle \otimes_{\mathfrak{g}} |\alpha\rangle, \quad (\text{B11e})$$

$$Q(b=1) := \sum_{\alpha=1}^{\mathcal{D}_v} (-1)^{|\alpha|} |\alpha\rangle \otimes_{\mathfrak{g}} |\alpha\rangle. \quad (\text{B11f})$$

By construction, $Q(b)$ is of even parity while Y is of odd parity. Moreover, $(|\alpha_j| + |\sigma_j| + |\beta_j|) \bmod 2 = 1$ implies that $(A_{\sigma_j}^{(0)})_{\alpha_j \beta_j} = 0$.

We note that the only difference between definitions (B8) and (B11) is the choice for Y . In the former case its parity is even, in the latter case its parity is odd. Analogously to the even FMPS case, we define $2M \times 2M$ dimensional matrices $A_{\sigma_j}^{(1)}$ and Y with the matrix elements $(A_{\sigma_j}^{(1)})_{\alpha_j \beta_j}$ and $Y_{\alpha_j \beta_j}$. The parity $|Y| = 1$ implies that

$$Y = \begin{pmatrix} 0 & Y_1 \\ Y_2 & 0 \end{pmatrix}, \quad (\text{B12a})$$

where Y_1 and Y_2 are $M \times M$ and dimensional matrices, respectively. Imposing translation symmetry requires that

$$Y A_{\sigma_j}^{(1)} = A_{\sigma_j}^{(1)} Y. \quad (\text{B12b})$$

We choose

$$Y := \begin{pmatrix} 0 & \mathbb{1}_M \\ -\mathbb{1}_M & 0 \end{pmatrix}, \quad P Y P = -Y, \quad (\text{B12c})$$

which implies

$$A_{\sigma_j}^{(1)} = \begin{pmatrix} G_{\sigma_j} & 0 \\ 0 & G_{\sigma_j} \end{pmatrix}, \quad \text{if } |\sigma_j| = 0, \quad (\text{B12d})$$

$$A_{\sigma_j}^{(1)} = \begin{pmatrix} 0 & G_{\sigma_j} \\ -G_{\sigma_j} & 0 \end{pmatrix}, \quad \text{if } |\sigma_j| = 1, \quad (\text{B12e})$$

where G_{σ_j} are $M \times M$ dimensional matrices. Inserting these explicit representations of $Q(b)$ and Y in Eq. (B11a)

delivers

$$\begin{aligned} |\Psi\rangle_1^b &\equiv |\{A_{\sigma_j}^{(1)}\}; b\rangle := \sum_{\sigma} \text{tr}[P^b Y A_{\sigma_1}^{(1)} A_{\sigma_2}^{(1)} \cdots A_{\sigma_N}^{(1)}] |\Psi_{\sigma}\rangle, \\ |\Psi_{\sigma}\rangle &:= |\psi_{\sigma_1}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_2}\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |\psi_{\sigma_N}\rangle. \end{aligned} \quad (\text{B13a})$$

The state $|\{A_{\sigma_j}^{(1)}\}; b\rangle$ has odd parity since

$$\left(\sum_{j=1}^N |\sigma_j| \right) \bmod 2 = \left[\sum_{j=1}^N (|\alpha_j| + |\beta_j|) \right] \bmod 2 = \left(|\alpha| + |\beta| + \sum_{j=2}^{N-1} 2|\alpha_j| \right) \bmod 2 = 1, \quad (\text{B13b})$$

where we used condition (B11c) to establish the first equality. For the second equality we used the conditions $|\alpha| = |\beta_N|$ and $|\beta| = |\alpha_1|$ where $|\alpha|, |\beta|$ are the parities of the virtual indices corresponding to matrix elements $Y_{\alpha\beta}$, and $|\beta_j| = |\alpha_{j+1}|$ for $j = 2, \dots, N-1$ that is imposed by the contractions of virtual indices to establish the second equality.

APPENDIX C: PROOF OF THEOREM 1

We will prove Theorem 1 for one dimensional systems within the FMPS framework. Our proof follows closely that for the bosonic case¹⁶ introduced in Ref. [32]. We will show that a parity-even or parity-odd injective FMPS necessarily requires the local projective representation \hat{u}_j of the symmetry group G_f to have trivial second cohomology class $[\phi] \in H^2(G_f, \text{U}(1)_c)$. In other words, when this cohomology class is nontrivial there is no compatible injective FMPS with even or odd parity. The general forms (B10a) and (B13a) as well as the injectivity conditions (1) and (2) are distinct for even and odd parity FMPS. The proofs for the even- and the odd-parity cases are thus treated successively. For conciseness, we are going to suppress the symbol $\otimes_{\mathfrak{g}}$ when working with the orthonormal and homogeneous basis

$$\{|\Psi_{\sigma}\rangle \equiv |\psi_{\sigma_1}\rangle \otimes_{\mathfrak{g}} |\psi_{\sigma_2}\rangle \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} |\psi_{\sigma_N}\rangle\} \quad (\text{C1a})$$

of the Fock space

$$\mathcal{F}_{\Lambda} \equiv \mathcal{F}_1 \otimes_{\mathfrak{g}} \mathcal{F}_2 \otimes_{\mathfrak{g}} \cdots \otimes_{\mathfrak{g}} \mathcal{F}_N. \quad (\text{C1b})$$

1. Even-parity FMPS

Let [see Eq. (B9)]

$$|\{A_{\sigma_j}^{(0)}\}; b\rangle \equiv \sum_{\sigma} \text{tr}(P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}) |\psi_{\sigma_1}\rangle |\psi_{\sigma_2}\rangle \cdots |\psi_{\sigma_N}\rangle \quad (\text{C2})$$

be a translation-invariant, G_f -symmetric, even-parity, and injective FMPS obeying periodic boundary conditions when $b = 0$ or antiperiodic boundary conditions when $b = 1$. For any $g \in G_f$, the global representation $\hat{U}(g)$ of g is defined in Eq. (3.19). Hence, for any $g \in G_f$, there exists a phase $\eta(g; b) \in [0, 2\pi)$ such that

$$\hat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle = e^{i\eta(g;b)} |\{A_{\sigma_j}^{(0)}\}; b\rangle. \quad (\text{C3})$$

The action of the transformation $\hat{U}(g)$ on the right-hand side of Eq. (C2) gives

$$\begin{aligned} \hat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle &= \sum_{\sigma} \{ \mathbf{K}_g \text{tr}(P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}) \} \hat{v}_1(g) |\psi_{\sigma_1}\rangle \hat{v}_2(g) |\psi_{\sigma_2}\rangle \cdots \hat{v}_N(g) |\psi_{\sigma_N}\rangle \\ &= \sum_{\sigma} \left\{ \sum_{\sigma'} \text{tr}[P^{b+1} \mathbf{K}_g (A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)})] \prod_{j=1}^N \langle \psi_{\sigma_j} | \hat{v}_j(g) | \psi_{\sigma'_j} \rangle \right\} |\psi_{\sigma_1}\rangle |\psi_{\sigma_2}\rangle \cdots |\psi_{\sigma_N}\rangle \end{aligned} \quad (\text{C4})$$

after using N times the resolution of the identity, one for each local Fock space \mathcal{F}_j . The right-hand side can be written more elegantly with the definition of the g -dependent $2M \times 2M$ matrix

$$A_{\sigma_j}^{(0)}(g) := \sum_{\sigma'_j} \langle \psi_{\sigma_j} | \hat{v}_j(g) | \psi_{\sigma'_j} \rangle \mathbf{K}_g[A_{\sigma'_j}^{(0)}] \equiv \sum_{\sigma'_j} [\mathcal{U}(g)]_{\sigma_j \sigma'_j} \mathbf{K}_g[A_{\sigma'_j}^{(0)}], \quad \sigma_j = 1, \dots, D, \quad j = 1, \dots, N, \quad (\text{C5a})$$

¹⁶Bosonic matrix products states presume that the local Fock space \mathcal{F}_j has no more than the trivial \mathbb{Z}_2 grading, i.e., $\mathcal{F}_j \equiv \mathcal{F}_{j0} \oplus \mathcal{F}_{j1}$ with $\mathcal{F}_{j0} \equiv \mathcal{F}_j$ and $\mathcal{F}_{j1} \equiv \emptyset$.

where the $\mathcal{D} \times \mathcal{D}$ matrix $\mathcal{U}(g)$, whose matrix elements are the complex-valued coefficients weighting the sum over the $2M \times 2M$ matrices $\mathbb{K}_g[A_{\sigma'_j}^{(0)}]$, acts on the local Fock space \mathcal{F}_j and we have defined

$$\mathbb{K}_g[A_{\sigma'_j}^{(0)}] := \begin{cases} A_{\sigma'_j}^{(0)}, & \text{if } \mathfrak{c}(g) = 0, \\ \mathbb{K} A_{\sigma'_j}^{(0)} \mathbb{K}, & \text{if } \mathfrak{c}(g) = 1. \end{cases} \quad (\text{C5b})$$

As usual, \mathbb{K} denotes complex conjugation. Equation (C4) becomes

$$\widehat{U}(g) |\{A_{\sigma_j}^{(0)}\}; b\rangle = \sum_{\sigma} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)}(g) A_{\sigma_2}^{(0)}(g) \cdots A_{\sigma_N}^{(0)}(g)] |\psi_{\sigma_1}\rangle |\psi_{\sigma_2}\rangle \cdots |\psi_{\sigma_N}\rangle, \quad (\text{C5c})$$

which is nothing but the FMPS (C2) with $A_{\sigma_j}^{(0)}$ substituted for $A_{\sigma'_j}^{(0)}(g)$. Equating the right-hand sides of Eqs. (4.5) and (4.9) implies

$$\text{tr}[P^{b+1} A_{\sigma_1}^{(0)}(g) A_{\sigma_2}^{(0)}(g) \cdots A_{\sigma_N}^{(0)}(g)] = e^{i\eta(g;b)} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}]. \quad (\text{C6a})$$

This equation is satisfied by the ansatz

$$A_{\sigma_j}^{(0)}(g) = e^{i\theta(g)} U^{-1}(g) A_{\sigma_j}^{(0)} U(g), \quad P U(g) P = (-1)^{\kappa(g)} U(g), \quad \theta(g) := \frac{1}{N} [\eta(g;b) - \pi(b+1)\kappa(g)], \quad (\text{C6b})$$

where $\kappa(g) = 0, 1$ dictates if the $2M \times 2M$ unitary matrix $U(g)$ commutes or anticommutes with the $2M \times 2M$ parity matrix P defined in Eq. (B9c), since

$$\begin{aligned} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)}(g) A_{\sigma_2}^{(0)}(g) \cdots A_{\sigma_N}^{(0)}(g)] &= e^{i\theta(g)N} \text{tr}[P^{b+1} U^{-1}(g) A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)} U(g)] \\ &\stackrel{\text{cyclicity of the trace}}{=} e^{i\theta(g)N} \text{tr}[U(g) P^{b+1} U^{-1}(g) A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}] \\ &\stackrel{\text{Eq. (C6b)}}{=} e^{i\theta(g)N} (-1)^{(b+1)\kappa(g)} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}] \\ &= e^{i\theta(g)N + i\pi(b+1)\kappa(g)} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}] \\ &\equiv e^{i\eta(g;b)} \text{tr}[P^{b+1} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \cdots A_{\sigma_N}^{(0)}]. \end{aligned} \quad (\text{C6c})$$

The existence of the $2M \times 2M$ invertible matrix $U(g)$ is guaranteed because of the injectivity of the FMPS. In an injective even-parity FMPS, the matrices $A_{\sigma_1}^{(0)}, \dots, A_{\sigma_\ell}^{(0)}$ span the simple algebra of all $2M \times 2M$ matrices for any $\ell > \ell^*$ for some nonvanishing integer ℓ^* . Hence, provided N is sufficiently large, the family of matrices $\{A_{\sigma_1}^{(0)}(g), \dots, A_{\sigma_N}^{(0)}(g)\}$ is related to the family of matrices $\{e^{i\eta(g;b)/N} A_{\sigma_1}^{(0)}, \dots, e^{i\eta(g;b)/N} A_{\sigma_N}^{(0)}\}$ that give the *same* FMPS (C2) by the similarity transformation [see Eqs. (4.20) and (4.25)]

$$A_{\sigma_j}^{(0)}(g) = e^{i\varphi_{U(g)}^{(b)}} U^{-1}(g) [e^{i\eta(g;b)/N} A_{\sigma_j}^{(0)}] U(g), \quad (\text{C7a})$$

for some phase $\varphi_{U(g)}^{(b)} = [0, 2\pi)$ and some invertible $2M \times 2M$ matrix $U(g)$ that must also obey

$$U(p) = P, \quad P U(g) P = (-1)^{\kappa(g)} U(g). \quad (\text{C7b})$$

Here, the map $\kappa : G_f \rightarrow \{0, 1\}$ specifies the algebra between the similarity transformation $U(g)$ corresponding to element $g \in G_f$ and the fermion parity P . The effect of the factor $(-1)^{\kappa(g)}$ is nothing but the phase

$$\varphi_{U(g)}^{(b)} = -\frac{1}{N} \pi(b+1)\kappa(g), \quad (\text{C8})$$

as follows from Eq. (C6b).

Equating the right-hand sides of Eqs. (C7a) and (C5a) implies

$$e^{i\theta(g)} U^{-1}(g) A_{\sigma_j}^{(0)} U(g) = \sum_{\sigma'_j} [\mathcal{U}(g)]_{\sigma_j \sigma'_j} \mathbb{K}_g[A_{\sigma'_j}^{(0)}], \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N. \quad (\text{C9})$$

We would like to isolate $K_g[A_{\sigma_j}^{(0)}]$ on the right-hand side. To this end, we do the manipulations

$$\begin{aligned}
e^{i\theta(g)} \sum_{\sigma_j} [\mathcal{U}^\dagger(g)]_{\sigma_j''\sigma_j} U^{-1}(g) A_{\sigma_j}^{(0)} U(g) &= \sum_{\sigma_j} \sum_{\sigma_j'} [\mathcal{U}^\dagger(g)]_{\sigma_j''\sigma_j} [\mathcal{U}(g)]_{\sigma_j\sigma_j'} K_g[A_{\sigma_j'}^{(0)}] = \sum_{\sigma_j} \sum_{\sigma_j'} \langle \psi_{\sigma_j''} | \hat{v}_j^\dagger(g) | \psi_{\sigma_j} \rangle \langle \psi_{\sigma_j} | \hat{v}_j(g) | \psi_{\sigma_j'} \rangle K_g[A_{\sigma_j'}^{(0)}] \\
&= \sum_{\sigma_j'} \langle \psi_{\sigma_j''} | \hat{v}_j^\dagger(g) \hat{v}_j(g) | \psi_{\sigma_j'} \rangle K_g[A_{\sigma_j'}^{(0)}] = \sum_{\sigma_j'} \langle \psi_{\sigma_j''} | \psi_{\sigma_j'} \rangle K_g[A_{\sigma_j'}^{(0)}] = \sum_{\sigma_j'} \delta_{\sigma_j'',\sigma_j'} K_g[A_{\sigma_j'}^{(0)}] \\
&= K_g[A_{\sigma_j''}^{(0)}], \quad \sigma_j'' = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N.
\end{aligned} \tag{C10}$$

By applying K_g to both sides of this equation, we obtain the self-consistency condition

$$\begin{aligned}
A_{\sigma_j}^{(0)} &= K_g \left[e^{i\theta(g)} \sum_{\sigma_j'} [\mathcal{U}^\dagger(g)]_{\sigma_j\sigma_j'} U^{-1}(g) A_{\sigma_j'}^{(0)} U(g) \right] \\
&= e^{i\mathfrak{c}(g)\theta(g)} \sum_{\sigma_j'} \langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle V^{-1}(g) A_{\sigma_j'}^{(0)} V(g), \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N,
\end{aligned} \tag{C11a}$$

with $\hat{u}_j(g)$ defined in Eq. (3.19), $V(g) = U(g)$ if $\mathfrak{c}(g) = 0$ and $V(g) = U(g)K$ if $\mathfrak{c}(g) = 1$. We use the notation $(\hat{u}_j^\dagger(g) | \psi_{\sigma_j'})$ to indicate that the operator $\hat{u}_j^\dagger(g)$ acts on the right, an important fact to keep track of when $\hat{u}_j^\dagger(g)$ is an antiunitary operator. Had we chosen the elements $h \in G_f$ and $gh \in G_f$, Eq. (C11a) would give the self-consistency conditions

$$A_{\sigma_j}^{(0)} = e^{i\mathfrak{c}(h)\theta(h)} \sum_{\sigma_j'} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle V^{-1}(h) A_{\sigma_j}^{(0)} V(h), \quad \sigma_j'' = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N, \tag{C11b}$$

and

$$A_{\sigma_j}^{(0)} = e^{i\mathfrak{c}(gh)\theta(gh)} \sum_{\sigma_j'} \langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(gh) | \psi_{\sigma_j'}) \rangle V^{-1}(gh) A_{\sigma_j'}^{(0)} V(gh), \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N, \tag{C11c}$$

respectively.

Inserting the self-consistency condition (C11a) into the self-consistency condition (C11b) gives

$$\begin{aligned}
A_{\sigma_j}^{(0)} &= e^{i\mathfrak{c}(h)\theta(h)} \sum_{\sigma_j} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle V^{-1}(h) \left(e^{i\mathfrak{c}(g)\theta(g)} \sum_{\sigma_j'} \langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle V^{-1}(g) A_{\sigma_j'}^{(0)} V(g) \right) V(h) \\
&= e^{i\mathfrak{c}(h)\theta(h)+i\mathfrak{c}(h)\mathfrak{c}(g)\theta(g)} \sum_{\sigma_j, \sigma_j'} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle K_h[\langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle] V^{-1}(h) V^{-1}(g) A_{\sigma_j'}^{(0)} V(g) V(h) \\
&= e^{i\mathfrak{c}(h)\theta(h)+i\mathfrak{c}(h)\mathfrak{c}(g)\theta(g)} \sum_{\sigma_j'} \langle \psi_{\sigma_j'} | \hat{u}_j^\dagger(h) (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle V^{-1}(h) V^{-1}(g) A_{\sigma_j'}^{(0)} V(g) V(h) \\
&= e^{i\mathfrak{c}(h)\theta(h)+i\mathfrak{c}(h)\mathfrak{c}(g)\theta(g)-i\mathfrak{c}(gh)\theta(gh)} \sum_{\sigma_j'} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(gh) | \psi_{\sigma_j'}) \rangle V^{-1}(h) V^{-1}(g) A_{\sigma_j'}^{(0)} V(g) V(h).
\end{aligned} \tag{C12}$$

In reaching the penultimate and last equalities, we used two identities. First,

$$\sum_{\sigma_j} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle K_h[\langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle] = \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) \hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle \tag{C13}$$

is obviously true when $\mathfrak{c}(h) = 1$ since $\sum_{\sigma_j} |\psi_{\sigma_j}\rangle \langle \psi_{\sigma_j}|$ is the resolution of the identity on \mathcal{F}_j . When $\mathfrak{c}(h) = -1$, $\hat{u}_j^\dagger(h)$ is antiunitary so that

$$\begin{aligned}
&\sum_{\sigma_j} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle K_h[\langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle] \\
&= \sum_{\sigma_j} \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) | \psi_{\sigma_j}) \rangle [\langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle]^* = \sum_{\sigma_j} [(\langle \psi_{\sigma_j'} | \hat{u}_j(h) | \psi_{\sigma_j} \rangle)]^* [(\langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle)]^* \\
&= \left[\sum_{\sigma_j} (\langle \psi_{\sigma_j'} | \hat{u}_j(h) | \psi_{\sigma_j} \rangle) \langle \psi_{\sigma_j} | (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle \right]^* = [(\langle \psi_{\sigma_j'} | \hat{u}_j(h) | \psi_{\sigma_j} \rangle) (\hat{u}_j^\dagger(g) | \psi_{\sigma_j'})]^* = \langle \psi_{\sigma_j'} | (\hat{u}_j^\dagger(h) \hat{u}_j^\dagger(g) | \psi_{\sigma_j'}) \rangle.
\end{aligned} \tag{C14}$$

Second, we used the projective representation (3.11) to obtain

$$\hat{u}_j^\dagger(h) \hat{u}_j^\dagger(g) = [\hat{u}_j(g) \hat{u}_j(h)]^\dagger = [e^{+i\phi(g,h)} \hat{u}_j(g,h)]^\dagger = \hat{u}_j^\dagger(g,h) e^{-i\phi(g,h)} = e^{-i\mathfrak{c}(g,h)\phi(g,h)} \hat{u}_j^\dagger(g,h). \quad (\text{C15})$$

Equating the right-hand sides of Eqs. (C12) and (C11c) gives the condition

$$e^{i\mathfrak{c}(h)\theta(h)+i\mathfrak{c}(h)\mathfrak{c}(g)\theta(g)-i\mathfrak{c}(gh)\phi(g,h)} V^{-1}(h) V^{-1}(g) A_{\sigma_j}^{(0)} V(g) V(h) = e^{i\mathfrak{c}(gh)\theta(gh)} V^{-1}(gh) A_{\sigma_j}^{(0)} V(gh). \quad (\text{C16})$$

Upon using the fact that \mathfrak{c} is a homomorphism so that $\mathfrak{c}(gh) = \mathfrak{c}(g)\mathfrak{c}(h)$ holds, we arrive at

$$W^{-1}(g,h) A_{\sigma_j}^{(0)} W(g,h) = e^{-i\delta(g,h;b)} A_{\sigma_j}^{(0)}, \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N, \quad (\text{C17a})$$

where

$$W(g,h) := V(g) V(h) V^{-1}(gh), \quad \delta(g,h;b) := \mathfrak{c}(g)\theta(h) + \theta(g) - \phi(g,h) - \theta(gh). \quad (\text{C17b})$$

A fortiori

$$W^{-1}(g,h) A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)} W(g,h) = e^{-i\ell \delta(g,h;b)} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)} \quad (\text{C18})$$

holds for any positive integer ℓ .

Injectivity of a FMPS implies that for some integer $\ell^* > 1$ and any $\ell \geq \ell^*$ all the products of the form $A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)}$ span the space of all $2M \times 2M$ matrices. Therefore Eq. (C18) combined with injectivity implies that the $2M \times 2M$ matrix $W(g,h)$ is an element from the center of the algebra defined by the vector space of all $2M \times 2M$ matrices, i.e., $\{\mathbb{1}_{2M}\}$. Condition (C18) thus simplifies to

$$A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)} = e^{-i\ell \delta(g,h;b)} A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)} \quad (\text{C19})$$

for any $\ell \geq \ell^*$. Choosing a linear combination of $A_{\sigma_1}^{(0)} A_{\sigma_2}^{(0)} \dots A_{\sigma_\ell}^{(0)}$ equating the identity matrix $\mathbb{1}_{2M}$, delivers the constraint

$$\ell \delta(g,h;b) = 0, \quad \forall \ell > \ell^* \Rightarrow \delta(g,h;b) = 0. \quad (\text{C20a})$$

Inserting the value of $\delta(g,h;b)$ given in Eq. (C17) implies the final constraint

$$\phi(g,h) = \mathfrak{c}(g)\theta(h) + \theta(g) - \theta(gh). \quad (\text{C20b})$$

This is the coboundary condition (3.14) when $\phi' = 0$. In other words, the local representation \hat{u}_j is equivalent to the trivial projective representation.

2. Odd-parity FMPS

Let [see Eq. (B12)]

$$|\{A_{\sigma_j}^{(1)}\}; b\rangle = \sum_{\sigma} \text{tr}[P^b Y A_{\sigma_1}^{(1)} A_{\sigma_2}^{(1)} \dots A_{\sigma_N}^{(1)}] |\psi_{\sigma_1}\rangle |\psi_{\sigma_2}\rangle \dots |\psi_{\sigma_N}\rangle \quad (\text{C21})$$

be a translation-invariant, G_f -symmetric, odd-parity (each matrix $A_{\sigma_j}^{(1)}$ commutes with the matrix Y), and injective FMPS obeying periodic boundary conditions when $b = 0$ or antiperiodic boundary conditions when $b = 1$. For any $g \in G_f$, the global representation $\widehat{U}(g)$ of g is defined in Eq. (3.19). Hence, for any $g \in G_f$, there exists a phase $\eta(g;b) \in [0, 2\pi)$ such that

$$\widehat{U}(g) |\{A_{\sigma_j}^{(1)}\}; b\rangle = e^{i\eta(g;b)} |\{A_{\sigma_j}^{(1)}\}; b\rangle. \quad (\text{C22})$$

The counterpart to Eq. (C5) is

$$\widehat{U}(g) |\{A_{\sigma_j}^{(1)}\}; b\rangle = \sum_{\sigma} \text{tr}[P^b Y A_{\sigma_1}^{(1)}(g) A_{\sigma_2}^{(1)}(g) \dots A_{\sigma_N}^{(1)}(g)] |\psi_{\sigma_1}\rangle |\psi_{\sigma_2}\rangle \dots |\psi_{\sigma_N}\rangle, \quad (\text{C23a})$$

$$A_{\sigma_j}^{(1)}(g) := \sum_{\sigma'_j} \langle \psi_{\sigma_j} | \hat{v}_j(g) | \psi_{\sigma'_j} \rangle \mathbf{K}_g[A_{\sigma'_j}^{(1)}] = \sum_{\sigma'_j} \mathcal{U}(g)_{\sigma_j, \sigma'_j} A_{\sigma'_j}^{(1)}, \quad (\text{C23b})$$

$$\mathbf{K}_g[A_{\sigma_j}^{(1)}] := \begin{cases} A_{\sigma_j}^{(1)}, & \text{if } \mathfrak{c}(g) = 0, \\ \mathbf{K} A_{\sigma_j}^{(1)} \mathbf{K}, & \text{if } \mathfrak{c}(g) = 1. \end{cases} \quad (\text{C23c})$$

Odd-parity injective FMPS differ from the even ones in one crucial way. There exists a positive integer $\ell^* \geq 1$ such that for any $\ell \geq \ell^*$ the products of the form $A_{\sigma_1}^{(1)} A_{\sigma_2}^{(1)} \dots A_{\sigma_\ell}^{(1)}$ span the \mathbb{Z}_2 -graded algebra of $2M \times 2M$ matrices with the center $\{\mathbb{1}_{2M}, Y\}$. Consequently, there exists a $2M \times 2M$ invertible matrix $U(g)$ and a phase $\theta(g) \in [0, 2\pi)$ such that [recall Eq. (4.20)]

$$U(g) = P U(g) P, \quad U(g) = (-1)^{\zeta(g)} Y U(g) Y, \quad \zeta(g) = 0, 1, \quad (\text{C24a})$$

with $\zeta : G_f \rightarrow \{-1, +1\}$ a group homomorphism and

$$A_{\sigma_j}^{(1)}(g) = e^{i\theta(g)} U^{-1}(g) A_{\sigma_j}^{(1)} U(g) \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N. \quad (\text{C24b})$$

The same steps that lead to Eq. (C6) then give

$$\text{tr}[P^b Y A_{\sigma_1}^{(1)}(g) A_{\sigma_2}^{(1)}(g) \cdots A_{\sigma_N}^{(1)}(g)] = e^{in(g;b)} \text{tr}[P^b Y A_{\sigma_1}^{(1)} A_{\sigma_2}^{(1)} \cdots A_{\sigma_N}^{(1)}] \quad (\text{C25a})$$

with the solution

$$A_{\sigma_j}^{(1)}(g) = e^{i\theta(g)} U^{-1}(g) A_{\sigma_j}^{(1)} U(g), \quad Y U(g) Y = (-1)^{\zeta(g)} U(g), \quad \theta(g) := \frac{1}{N} [\eta(g; b) - \pi \zeta(g)]. \quad (\text{C25b})$$

All the steps leading to Eq. (C17) deliver

$$W^{-1}(g, h) A_{\sigma_j}^{(1)} W(g, h) = e^{-i\delta(g, h; b)} A_{\sigma_j}^{(1)}, \quad \sigma_j = 1, \dots, \mathcal{D}, \quad j = 1, \dots, N, \quad (\text{C26a})$$

where

$$W(g, h) := V(g) V(h) V^{-1}(gh), \quad \delta(g, h; b) := \mathfrak{c}(g) \theta(h) + \theta(g) - \phi(g, h) - \theta(gh), \quad (\text{C26b})$$

and $V(g) = U(g)$ if $\mathfrak{c}(g) = 0$ and $V(g) = U(g) \mathbf{K}$ if $\mathfrak{c}(g) = 1$. Because $U(g)$ commutes with P so does $W(g, h)$. Because all possible products of the form $A_{\sigma_1}^{(1)} A_{\sigma_2}^{(1)} \cdots A_{\sigma_r}^{(1)}$ span the \mathbb{Z}_2 -graded algebra of $2M \times 2M$ matrices with the center $\{\mathbb{1}_{2M}, Y\}$, $W(g, h)$ is, up to a phase factor, proportional to $\mathbb{1}_{2M}$. The counterpart to the even-parity coboundary condition (C20) then follows, thereby completing the proof of Theorem 1 for the parity-odd FMPS.

APPENDIX D: PROOF OF THEOREM 1 WITH TWISTED BOUNDARY CONDITIONS FOR ANY ABELIAN GROUP G_f WHOSE PROJECTIVE REPRESENTATIONS ARE ALL UNITARY

The lattice is $\Lambda = \{1, \dots, N\} \equiv \mathbb{Z}_N$ with N an integer. The global fermionic Fock space \mathcal{F}_Λ is of dimension 2^{mN} with $n = 2m$ an even number of local Majorana flavors. The local \mathcal{F}_j and global \mathcal{F}_Λ Fock spaces are generated by the Hermitian Majorana operators $\hat{\chi}_{j,a}$ obeying the Clifford algebra

$$\{\hat{\chi}_{j,a}, \hat{\chi}_{j',a'}\} = 2 \delta_{j,j'} \delta_{a,a'}, \quad j, j' = 1, \dots, N, \quad a, a' = 1, \dots, n = 2m. \quad (\text{D1})$$

The local and global fermion parity operators are

$$\hat{p}_j := \prod_{a=1}^m i \hat{\chi}_{j,2a-1} \hat{\chi}_{j,2a}, \quad \hat{P}_\Lambda := \prod_{j=1}^N \hat{p}_j, \quad (\text{D2})$$

respectively. Any polynomial \hat{h}_j in the Majorana operators that is of finite order, of finite range r (the integer r is the maximum separation between the space labels of the Majorana operators entering \hat{h}_j) of even parity ($\hat{P}_\Lambda \hat{h}_j \hat{P}_\Lambda = \hat{h}_j$), and Hermitian ($\hat{h}_j^\dagger = \hat{h}_j$) is a local Hamiltonian. We define the unitary operator \hat{T}_1 by its action

$$\hat{T}_1 \hat{\chi}_{j,a} \hat{T}_1^{-1} = \begin{cases} \hat{\chi}_{j+1,a}, & \text{if } j = 1, \dots, N-1 \text{ and } a = 1, \dots, n = 2m, \\ \hat{\chi}_{1,a}, & \text{if } j = N \text{ and } a = 1, \dots, n = 2m. \end{cases} \quad (\text{D3})$$

It follows that

$$\hat{T}_1^N = \hat{\mathbb{1}}_{2^{mN}}, \quad (\text{D4})$$

i.e., \hat{T}_1 is a unitary representation of the generator of the cyclic group \mathbb{Z}_N . For any Abelian central extension G_f of G by \mathbb{Z}_2^F and for any $g \in G_f$, we assume the projective representation (3.11) with

$$\hat{u}_j(g) = \hat{v}_j(g) \quad [\text{as } \mathfrak{c}(g) = +1 \text{ always hold by hypothesis}] \quad (\text{D5})$$

a polynomial in $\hat{\chi}_{j,a}$ with $a = 1, \dots, n = 2m$. We make the identifications

$$\begin{aligned} \hat{v}_j(e) &\equiv \hat{\mathbb{1}}_{2m}, & \hat{v}_j(p) &\equiv \hat{p}_j, & \hat{p}_j \hat{v}_j(g) \hat{p}_j &= (-1)^{\rho(g)} \hat{v}_j(g), & j &= 1, \dots, N, \\ \hat{U}(e) &\equiv \hat{\mathbb{1}}_{2^{mN}}, & \hat{U}(p) &\equiv \hat{P}_\Lambda, & \hat{U}(g) &:= \prod_{j=1}^N \hat{v}_j(g), & \forall g &\in G_f. \end{aligned} \quad (\text{D6})$$

We assume that the Hamiltonian \hat{h}_j is G_f -invariant (symmetric), i.e.,

$$\hat{h}_j = \hat{U}(g) \hat{h}_j \hat{U}^{-1}(g), \quad \forall g \in G_f. \quad (\text{D7})$$

By construction, the Hamiltonian defined by [recall Eq. (5.5)]

$$\widehat{H}_{\text{pbc}} := \sum_{n=1}^N (\widehat{T}_1)^n \widehat{h}_j (\widehat{T}_1^\dagger)^n, \quad \widehat{U}(g) \widehat{h}_j \widehat{U}^{-1}(g), \quad \forall g \in G_f, \quad (\text{D8a})$$

is translation-invariant (symmetric),

$$\widehat{T}_1 \widehat{H}_{\text{pbc}} \widehat{T}_1^{-1} = \widehat{H}_{\text{pbc}}, \quad (\text{D8b})$$

and G_f -invariant (symmetric),

$$\widehat{U}(g) \widehat{H}_{\text{pbc}} \widehat{U}^{-1}(g) = \widehat{H}_{\text{pbc}}, \quad \forall g \in G_f. \quad (\text{D8c})$$

We define the family of twisted translation operators

$$\widehat{T}_1(g) := \widehat{v}_1(g) \widehat{T}_1, \quad g \in G_f, \quad \mathfrak{c}(g) = +1. \quad (\text{D9})$$

Their action on the Majorana spinor

$$\widehat{\chi}_j := (\widehat{\chi}_{j,1} \quad \cdots \quad \widehat{\chi}_{j,n})^\top \quad (\text{D10a})$$

differ from that in Eq. (4.5),

$$\widehat{T}_1(g) \widehat{\chi}_j \widehat{T}_1^{-1}(g) = \begin{cases} (-1)^{\rho(g)} \widehat{\chi}_{j+1}, & \text{if } j \neq N, \\ \widehat{v}_1(g) \widehat{\chi}_1 \widehat{v}_1^{-1}(g), & \text{if } j = N. \end{cases} \quad (\text{D10b})$$

We have the identity

$$\begin{aligned} [\widehat{T}_1(g)]^N &= [\widehat{v}_1(g) \widehat{T}_1][\widehat{v}_1(g) \widehat{T}_1] \cdots [\widehat{v}_1(g) \widehat{T}_1][\widehat{v}_1(g) \widehat{T}_1] \\ &= \widehat{v}_1(g) [\widehat{T}_1 \widehat{v}_1(g) \widehat{T}_1] \cdots \widehat{v}_1(g) [\widehat{T}_1 \widehat{v}_1(g) \widehat{T}_1] \\ &= \widehat{v}_1(g) [\widehat{T}_1 \widehat{v}_1(g) \widehat{T}_1] \cdots \widehat{v}_1(g) [\widehat{T}_1 \widehat{v}_1(g) \widehat{T}_1^{-1}] \widehat{T}_1^2 \\ &\stackrel{\text{Eq. (D3)}}{=} \widehat{v}_1(g) [\widehat{T}_1 \widehat{v}_1(g) \widehat{T}_1] \cdots \widehat{v}_1(g) \widehat{v}_2(g) \widehat{T}_1^2 = \widehat{U}(g) \widehat{T}_1^N \\ &\stackrel{\text{Eq. (D4)}}{=} \widehat{U}(g). \end{aligned} \quad (\text{D11})$$

Finally, we define the family of twisted Hamiltonians

$$\widehat{H}_{\text{twis}}^{\text{tilt}}(g) := \sum_{j=1}^N [\widehat{T}_1(g)]^j \widehat{h}_1^{\text{tilt}} [\widehat{T}_1^{-1}(g)]^j, \quad \widehat{h}_1^{\text{tilt}} = \widehat{U}(h) \widehat{h}_1^{\text{tilt}} \widehat{U}^{-1}(h), \quad \forall h \in G_f. \quad (\text{D12})$$

By design,

$$\begin{aligned} \widehat{T}_1(g) \widehat{H}_{\text{twis}}^{\text{tilt}}(g) \widehat{T}_1^{-1}(g) &= \sum_{j=1}^{N-1} [\widehat{T}_1(g)]^{j+1} \widehat{h}_1^{\text{tilt}} [\widehat{T}_1^{-1}(g)]^{j+1} + [\widehat{T}_1(g)]^{N+1} \widehat{h}_1^{\text{tilt}} [\widehat{T}_1^{-1}(g)]^{N+1} \\ &\stackrel{\text{Eq. (D11)}}{=} \sum_{j=1}^{N-1} [\widehat{T}_1(g)]^{j+1} \widehat{h}_1^{\text{tilt}} [\widehat{T}_1^{-1}(g)]^{j+1} + \widehat{T}_1(g) [\widehat{U}(g) \widehat{h}_1^{\text{tilt}} \widehat{U}^{-1}(g)] \widehat{T}_1^{-1}(g) \\ &\stackrel{G_f \text{ symmetry}}{=} \sum_{j=1}^{N-1} [\widehat{T}_1(g)]^{j+1} \widehat{h}_1^{\text{tilt}} [\widehat{T}_1^{-1}(g)]^{j+1} + \widehat{T}_1(g) \widehat{h}_1^{\text{tilt}} \widehat{T}_1^{-1}(g) = \widehat{H}_{\text{twis}}^{\text{tilt}}(g). \end{aligned} \quad (\text{D13})$$

We are going to derive the important identity

$$\widehat{U}(h) \widehat{T}_1(g) \widehat{U}^{-1}(h) = e^{i\chi(g,h)} \widehat{T}_1(g), \quad \forall g, h \in G_f, \quad (\text{D14a})$$

with the phase

$$\chi(g, h) := \phi(h, g) - \phi(g, h) + \pi \rho(h) [\rho(g) + 1] (N - 1), \quad \forall g, h \in G_f. \quad (\text{D14b})$$

We shall then specify the conditions under which the algebra defined by Eqs. (D13) and (D14) guarantees that the spectrum of the twisted Hamiltonian is degenerate.

Proof. We begin with the proof of Eq. (D14). We choose two elements $g, h \in G_f$ with the local representations $\widehat{v}_1(g)$ and $\widehat{v}_1(h)$, respectively, both of which are unitary.

Step 1. We observe that

$$\widehat{U}(h) \hat{v}_1(g) = \hat{v}_1(h) \hat{v}_2(h) \cdots \hat{v}_N(h) \hat{v}_1(g). \quad (\text{D15})$$

We can then interchange the local operator $\hat{v}_j(h)$ and $\hat{v}_{j'}(g)$ pairwise at the cost of the fermionic phase $(-1)^{\rho(h)\rho(g)}$ for any $j, j' = 1, \dots, N$. This is done $(N-1)$ times

$$\widehat{U}(h) \hat{v}_1(g) = (-1)^{\rho(g)\rho(h)(N-1)} \hat{v}_1(h) \hat{v}_1(g) \hat{v}_2(h) \cdots \hat{v}_N(h). \quad (\text{D16})$$

We conclude with

$$\widehat{U}(h) \hat{v}_1(g) = (-1)^{\rho(g)\rho(h)(N-1)} \hat{v}_1(h) \hat{v}_1(g) \hat{v}_2(h) \cdots \hat{v}_N(h). \quad (\text{D17})$$

Step 2. We begin with

$$\begin{aligned} \widehat{T}_1 \widehat{U}^{-1}(h) &= \widehat{T}_1 \hat{v}_N^{-1}(h) \hat{v}_{N-1}^{-1}(h) \cdots \hat{v}_1^{-1}(h) = [\widehat{T}_1 \hat{v}_N^{-1}(h) \widehat{T}_1^{-1}] [\widehat{T}_1 \hat{v}_{N-1}^{-1}(h) \widehat{T}_1^{-1}] \cdots [\widehat{T}_1 \hat{v}_1^{-1}(h) \widehat{T}_1^{-1}] \widehat{T}_1 \\ &\stackrel{\text{Eq. (5.4a)}}{=} \hat{v}_1^{-1}(h) \hat{v}_N^{-1}(h) \cdots \hat{v}_2^{-1}(h) \widehat{T}_1. \end{aligned} \quad (\text{D18})$$

Hence,

$$\widehat{T}_1 \widehat{U}^{-1}(h) = (-1)^{\rho(h)(N-1)} \hat{v}_N^{-1}(h) \hat{v}_{N-1}^{-1}(h) \cdots \hat{v}_1^{-1}(h) \widehat{T}_1, \quad (\text{D19})$$

where we have reordered the factors $\hat{v}_j^{-1}(h)$ and, in doing so, obtained the coefficient $(-1)^{\rho(h)(N-1)}$ that encodes the fermionic algebra.

Step 3. We combine Eqs. (D17) and (D19) into

$$\begin{aligned} \widehat{U}(h) \widehat{T}_1(g) \widehat{U}^{-1}(h) &= (-1)^{\rho(h)[\rho(g)+1](N-1)} \hat{v}_1(h) \hat{v}_1(g) \hat{v}_2(h) \cdots \hat{v}_N(h) \hat{v}_N^{-1}(h) \cdots \hat{v}_1^{-1}(h) \widehat{T}_1 \\ &= (-1)^{\rho(h)[\rho(g)+1](N-1)} \hat{v}_1(h) \hat{v}_1(g) \hat{v}_1^{-1}(h) \widehat{T}_1. \end{aligned} \quad (\text{D20})$$

Step 4. We need to massage $\hat{v}_1(h) \hat{v}_1(g) \hat{v}_1^{-1}(h)$. To this end, we use the fact that the group G_f is Abelian to obtain

$$\begin{aligned} \hat{v}_1(h) \hat{v}_1(g) \hat{v}_1^{-1}(h) &= e^{i\phi(h,g)} \hat{v}_1(hg) \hat{v}_1^{-1}(h) = e^{i\phi(h,g)} \hat{v}_1(g) \hat{v}_1^{-1}(h) \\ &= e^{i\phi(h,g)} [e^{-i\phi(g,h)} \hat{v}_1(g) \hat{v}_1(h)] \hat{v}_1^{-1}(h) = e^{i\phi(h,g)-i\phi(g,h)} \hat{v}_1(g). \end{aligned} \quad (\text{D21})$$

Insertion into the right-hand side of Eq. (D20) delivers the result

$$\widehat{U}(h) \widehat{T}_1(g) \widehat{U}^{-1}(h) = (-1)^{\rho(h)[\rho(g)+1](N-1)} e^{i\phi(h,g)-i\phi(g,h)} \hat{v}_1(g) \widehat{T}_1 \equiv e^{i\chi(g,h)} \widehat{T}_1(g), \quad (\text{D22a})$$

with the definition

$$\chi(g, h) = \phi(h, g) - \phi(g, h) + \pi \rho(h) [\rho(g) + 1] (N - 1). \quad (\text{D22b})$$

Step 5. It is instructive to derive the transformation law of the phase (D22b) under the global U(1) gauge transformation generated by

$$\hat{v}_j(g) := e^{i\xi(g)} \hat{v}'_j(g), \quad j = 1, \dots, N, \quad \forall g \in G_f. \quad (\text{D23})$$

Under this transformation,

$$\phi'(g, h) = \phi(g, h) - \xi(g) - \xi(h) + \xi(g), \quad \forall g, h \in G_f, \quad (\text{D24})$$

is the phase entering the projective algebra obeyed by the operators $\{\hat{v}'_j(g) \mid g \in G_f\}$ according to Eq. (3.14b). Hence, if we define

$$\chi'(g, h) := \phi'(h, g) - \phi'(g, h) + \pi \rho'(h) [\rho'(g) + 1] (N - 1), \quad \forall g, h \in G_f, \quad (\text{D25})$$

we then have the relation

$$\begin{aligned} \chi(g, h) &= \phi(h, g) - \phi(g, h) + \pi \rho(h) [\rho(g) + 1] (N - 1) \\ &= \chi'(g, h) + \underline{\underline{\xi(h)}} + \underline{\underline{\xi(g)}} - \underline{\underline{\xi(hg)}} - \underline{\underline{\xi(g)}} - \underline{\underline{\xi(h)}} + \underline{\underline{\xi(g)}} \\ &= \chi'(g, h), \quad \forall g, h \in G_f. \end{aligned} \quad (\text{D26})$$

Hence, $\chi(g, h)$ is gauge invariant under the U(1) gauge transformation (D23). The pair of cocycles ϕ' and ϕ are equivalent if and only if they have the same second cohomology class

$[\phi] = [\phi'] \in H^2(G_f, \text{U}(1)_c)$, i.e., if and only if they are related by the U(1) gauge transformation (D24). The gauge invariance of χ implies that it is independent of the choice

made of ϕ within the equivalence class $[\phi] \in H^2(G_f, U(1)_c)$. For example, $\chi(g, h) = 0$ holds for all $g, h \in G_f$ for any ϕ belonging to the trivial second cohomology class $[\phi] = 0$ since the function $\phi = 0$ belongs to $[\phi] = 0$. Hence, the trivial second cohomology class $[\phi] = 0$ implies that $\chi(g, h) = 0$. Conversely, if we assume that $\chi(g, h) = 0$, we find that

$$\phi(g, h) = \phi(h, g) + \pi \rho(h)[\rho(g) + 1](N - 1) \quad (\text{D27a})$$

for any $g, h \in G_f$. We distinguish two cases. First, if the number of sites N is odd, then the last term is an integer multiple of 2π . As such it can be dropped and we find that

$$\phi(g, h) = \phi(h, g) \quad (\text{D27b})$$

for any $g, h \in G_f$. For an Abelian group G_f (that is unitarily represented), Eq. (D27b) implies that the representations of any two elements $g, h \in G_f$ commute pairwise and therefore $[\phi] = 0$. In other words, both $\phi(g, h)$ and $\phi(h, g)$ can be made to vanish by an appropriate gauge transformation. Second, if the number of sites N is even, choosing h to be the fermion

parity $h = p$ in Eq. (D27b) implies

$$\pi \rho(g) = \phi(g, p) - \phi(p, g) = \pi \rho(p)[\rho(g) + 1](N - 1) = 0, \quad (\text{D27c})$$

where we used the definition (3.28) and the fact that $\rho(p) = 0$ on the right-hand side of the second equality. Since $\rho(g) = 0$ for any $g \in G_f$, we can again drop the last term on the right-hand side of Eq. (D27a). We arrive at Eq. (D27b). Thus we have proven that $[\phi] = 0$ if and only if $\chi(g, h) = 0$ for any $g, h \in G_f$. As a corollary, there exists a pair $g, h \in G_f$ for which $\chi(g, h)$ is nonvanishing if and only if $[\phi] \neq 0$.

Step 6. The twisted Hamiltonian $\widehat{H}_{\text{twis}}^{\text{tilt}}(g)$ is constructed so as to commute with the generator $\widehat{T}_i(g)$ of twisted translations and with the representation $\widehat{U}(h)$ of any group element $h \in G_f$, whereby passing $\widehat{U}(h)$ from the left through $\widehat{T}_i(g)$ produces the phase $\exp(i\chi(g, h))$. If it is possible to find a pair (g, h) such that $\chi(g, h)$ is not 0 modulo 2π , then the spectrum of $\widehat{H}_{\text{twis}}^{\text{tilt}}(g)$ must be degenerate. Indeed, any simultaneous eigenstate $|E(g), \exp(iK(g))\rangle$ of $\widehat{H}_{\text{twis}}^{\text{tilt}}(g)$ and $\widehat{T}_i(g)$ must be orthogonal to the state $\widehat{U}(h)|E(g), \exp(iK(g))\rangle$, which is also an eigenstate of $\widehat{H}_{\text{twis}}^{\text{tilt}}(g)$ with the energy $E(g)$ but has the eigenvalue $\exp(i[K(g) + \chi(g, h)]) \neq \exp(iK(g))$ with respect to $\widehat{T}_i(g)$.

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Correction: The seventh paragraph of Sec. II Main Results contained a superfluous condition and has been corrected. A duplication of a term in text appearing after Eq. (4.21a) has been fixed. A minor typographical error in the subscript of the last operator of Eq. (5.18b) has been fixed.