Intrinsically gapless topological phases

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Topology in quantum matter is typically associated with gapped phases. For example, in symmetry protected topological (SPT) phases, the bulk energy gap localizes edge modes near the boundary. In this work we identify a new mechanism that leads to topological phases which are not only gapless but where the absence of a gap is essential. These "intrinsically gapless SPT phases" have no gapped counterpart and are hence also distinct from recently discovered examples of gapless SPT phases. The essential ingredient of these phases is that on-site symmetries act in an anomalous fashion at low energies. Intrinsically gapless SPT phases are found to display several unique properties including (i) protected edge modes that are impossible to realize in a gapped system with the same symmetries, (ii) string order parameters that are likewise forbidden in gapped phases, and (iii) constraints on the phase diagram obtained upon perturbing the phase. We verify predictions of the general theory in a specific realization protected by $\mathbb{Z}_4$ symmetry, the one dimensional Ising-Hubbard chain, using both numerical simulations and effective field theory. We also discuss extensions to higher dimensions and possible experimental realizations.

I. INTRODUCTION

Topology appears in various fascinating guises in many-body quantum physics. For instance, symmetry-protected topological (SPT) phases generalize the notion of topological insulators and superconductors [1] to include interactions and to other symmetry classes [2,3]. Such systems are well understood when there is a gapped symmetric bulk protecting the edge modes.

Previous work has also established the remarkable stability of gapped SPT physics upon closing the bulk gap [4–34]. Here we will instead be interested in a different question—can fundamentally *new* topological phenomena occur in gapless systems? In this paper, we show that gapless topological phases considerably expand our notion of what is possible: there exist gapless SPT phases which in a precise sense have no gapped counterpart. For concreteness, in this work, we will focus on such examples in 1+1D, although we briefly discuss how the physics can be generalized to higher dimensions.

These intrinsically gapless SPT phases share certain properties with gapped SPT phases. They have an on-site symmetry action and their topology can be diagnosed—at least in 1+1D—by studying the charges residing at the ends of string order parameters. Furthermore, the nontrivial string charges imply the existence of exponentially localized zero-energy edge modes. However, while in a gapped phase such charges are constrained to take certain values associated with projective representations of the symmetry group [35–38], in these gapless phases, there is no such restriction. In fact, we will see the charge assignments for intrinsically gapless SPT string order parameters are forbidden in gapped phases.

As an illustrative example, we show that charge doping an Ising chain restores spin flip symmetry in the bulk but leaves it broken on the boundary, leading to edge modes. This is similar to an SPT phase but differs in at least three respects. First, the bulk contains a Luttinger liquid of gapless charge fluctuations. Second, any small perturbation which opens a gap will restore antiferromagnetic order, so that the topological edge modes only occur in the gapless region. Third, the system has a long-range ordered string operator whose charge is forbidden in any gapped phase. Earlier studies of related gapless models observed such string order and its relation to edge degeneracy [4,9,16,17,25,39], but a general mechanism and the definite break from gapped SPT physics were lacking.

We propose such a general mechanism to understand these nontrivial string order parameters, the protected edge modes, and the curious nearby phase diagrams in terms of emergent anomalies. Anomalies yield powerful constraints on the ground states of systems. Indeed, they play a key role in determining the gapless/symmetry breaking behavior of our 1+1D model. However, anomalies are usually associated with the edge of a topological bulk, for example the 1+1D edge of a 2+1D nontrivial SPT which must exhibit either gapless behavior or broken symmetry [40–44]. How can anomalies be relevant for a genuine 1+1D system? There are two ways. The first is if the symmetry action cannot be characterized as "on-site." This arises, for example, in the spin 1/2 chain, where translation symmetry, which is clearly not an on-site symmetry, leads to an anomaly and there are associated Lieb-Schultz-Mattis theorems which constrain the phases of this system [45–50]. The Fermi-arc surface states [51] of Weyl semimetals can similarly be interpreted as "impossible" surface states arising from an anomalous 3+1 D bulk. However

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in that case as well, translation symmetry is invoked. The second way—explored in this work since our symmetries are on-site—is that of an *emergent* anomaly.

An emergent anomaly can occur whenever the (nonanomalous) microscopic symmetry is not faithfully represented on the gapless modes, due to fundamental charges being gapped [48]. In the doped Ising chain example that we shall discuss, electrons are gapped, so fermion parity symmetry acts trivially on all gapless modes. The effective low-energy symmetry is the quotient of the full symmetry group by the part that only acts on gapped modes. The emergent anomaly of the low-energy symmetry can be diagnosed from the charges of the string order parameters, which is nontrivial whenever these are inconsistent with any gapped SPT.

A fundamentally new feature of emergent anomalies arises when the microscopic symmetry is *on-site*, since one can then consider boundaries for which the symmetry is well-defined. Usually, since anomalies live on the edge of a higher-dimensional system, this is tantamount to considering the "boundary of a boundary," which goes beyond the scope of the conventional theory of anomalies. We argue that emergent anomalies of such on-site symmetries lead to nontrivial edge modes protected by symmetry and gaplessness.

The remainder of this paper is structured as follows. In Sec. II, we introduce the Ising-Hubbard chain which gives a simple realization of an intrinsically gapless SPT phase whose edge modes and unusual string order are protected by $\mathbb{Z}_4$ symmetry. The emergent anomaly of this topological phase is more readily apparent in the field theoretic perspective explored in Sec. III. The key ingredients are summarized and generalized in Sec. IV, applicable to any dimension and symmetry group G.

II. AN EXOTIC GAPLESS TOPOLOGICAL PHASE

Let us describe a simple model which illustrates this phenomenon, namely the doped Ising model, or equivalently, the Ising-Hubbard model. This describes a chain of spinful fermions $c_s^\dagger$ with Hamiltonian $H = H_{\text{Ising}} + H_{\text{Hub}}$, where

$$H_{\text{Ising}} = \sum_n \left(J_z S_n^z S_{n+1}^z + h_x S^x \right), \quad (1)$$

$$H_{\text{Hub}} = -t \sum_{j,s} (c_{j+1,s}^\dagger c_{j,s} + \text{H.c.}) + U \sum_j n_{j,\uparrow} n_{j,\downarrow} - \mu N \quad (2)$$

with $n_{j,s} = c_{j,s}^\dagger c_{j,s}$ and $S_j^\alpha = \frac{1}{2} c_{j,s}^\dagger \sigma_{s,s'}^\alpha c_{j,s'}$. The spin rotation is broken by the J_z and h_x terms to the π-rotation R_x around the x axis. Note that this is a $\mathbb{Z}_4$ symmetry since $R_x^2 = (-1)^F = P$. At half-filling, the Hubbard interaction U drives the model into a Mott phase (i.e., $\langle n_j \rangle = 1$), such that we obtain an effective spin-1/2 chain, which is either in an Ising phase (spontaneously breaking R_x down to its $\mathbb{Z}_2$ subgroup of fermion parity) or a symmetry-preserving paramagnet, depending on the value of h_x. However, if we dope the system using a chemical potential (i.e., $\langle n_j \rangle \neq 1$), a gapless Luttinger liquid appears, while spins remain gapped. The numerical phase diagram for $t = J_z = 1$ and $U = 5$ is shown in Fig. 1(a), which was obtained using the density matrix renormalization group (DMRG) [52,53].
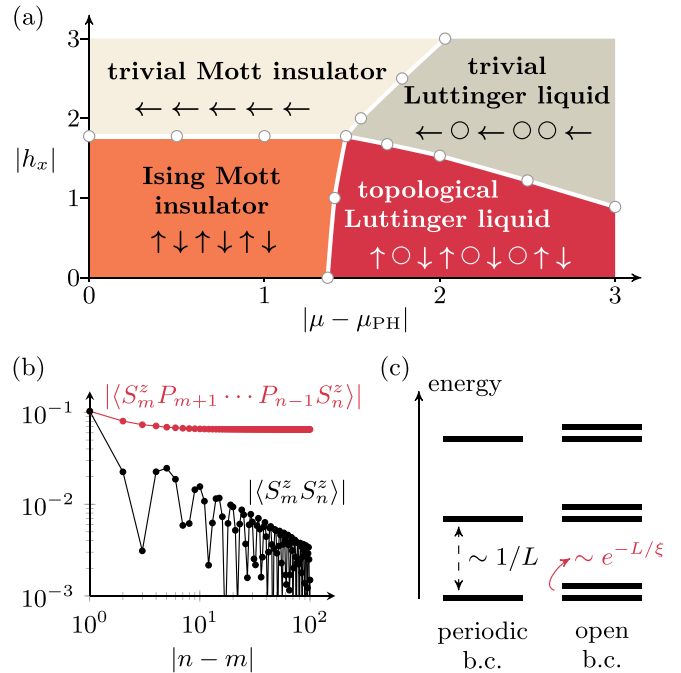

FIG. 1. Intrinsically gapless SPT phase in the Ising-Hubbard chain. (a) Phase diagram for the model in Eqs. (1) and (2) for $t = J_z = 1$ and $U = 5$. The chemical potential is relative to the particle-hole-symmetric value $\mu_{\text{PH}} = U/2$. The trivial and symmetry-breaking phases for small chemical potential are in the Mott limit. Eventually, these make way for a trivial and topological Luttinger liquid (LL), respectively. The topology of the latter is protected by $\mathbb{Z}_4$; if one additionally imposes translation symmetry, this is a stable gapless phase. (b) The topological order (or hidden symmetry breaking) in the topological LL can be detected by the string order parameter (plotted for $|\mu - \mu_{\text{PH}}| = 2$ and $h_x = 0$). (c) Whilst the topological LL has a unique ground state for periodic boundary conditions (b.c.), it has a twofold degeneracy for open b.c. whose exponentially small splitting is determined by the spin correlation length.

The key point is that doping the Ising antiferromagnet means that we lose long-range order in S^z since an arbitrary number of holes (or doublons) can appear between any two antiferromagnetically aligned spins. However, as sketched in Fig. 1(a), the resulting Luttinger liquid retains a hidden symmetry-breaking pattern where the "squeezed state" with the holes (or doublons) removed retains antiferromagnetic correlations [39,54]. This can be probed by measuring the S^z-correlation function, while keeping track of all the holes, which is done by inserting a string of fermionic parity: $\langle S_m^z (\prod_{m<k<n} P_k) S_n^z \rangle$. Figure 1(b) shows that this indeed has long-range order in the Luttinger liquid adjacent to the Ising phase, while the local Ising order parameter decays algebraically.

While this string order parameter was pointed out before [39,54,55], it was not appreciated how this implies that the ground state is topologically nontrivial with respect to $\mathbb{Z}_4$. As we now discuss, (i) this implies the presence of edge modes and (ii) a gapped SPT phase can never host this particular string order—hence this constitutes an intrinsically gapless topological state.

The R_x symmetry is fractionalized on this order parameter in the sense that its endpoints, which carry an S^z insertion, are charged. This is a hallmark feature of symmetry-protected topological (SPT) phases [36], harking back to the seminal work by den Nijs and Rommelse [56]. Usually, SPT phases are gapped, yet here we have long-range order in a string order parameter despite the system being gapless. Such gapless topological phases have been studied before [4,8,10–12,14–17,19,20,23,24,26–33]. However, what is entirely novel is that in our example, the topological phase is protected by $\mathbb{Z}_4$ alone; indeed, the charge of the above string operator[1] is well-defined as long as we preserve R_x. This is the first example that goes beyond the gapped classification [since $H^2(\mathbb{Z}_4, U(1)) = 0$],[2] giving an intrinsically gapless SPT phase. Indeed, this string order cannot have long-range order in a gapped phase.[3]

To avoid confusion, let us note that we call a system topological (with respect to a symmetry group G) if its topological phenomena can only be destroyed by (i) explicitly breaking the protecting symmetry or (ii) changing the bulk universality class [either by (iia) tuning off criticality, or (iib) tuning through a multicritical point]. In this sense, the above model is a (gapless) topological phase with respect to $\mathbb{Z}_4$. However, if one also wants the gaplessness itself to be a robust property, we might call it a *stable* gapless (topological) phase. For example, the above topological Luttinger liquid is stable if we preserve translation symmetry in addition to $\mathbb{Z}_4$; this can be inferred from measuring its Luttinger liquid parameter to be smaller than $1/2$, as discussed in Appendix A.

The topology of this gapless phase appears in the form of edge modes. Unlike in the Ising phase, long-range order in this string operator does not lead to a bulk symmetry breaking degeneracy. Instead, it leads to degeneracy on an interval with open boundaries. To see this, note that the string will still have long-range order even as its endpoints approach the boundaries. Applying the global symmetry transformation P, the correlator then becomes a correlation function for a pair of *local* charged operators localized at each boundary. By locality, this means that each boundary has a spontaneous expectation value for $\langle S_n^z \rangle \neq 0$. Summarizing this schematically: $0 \neq \langle S_1^z P_2 \cdots P_{N-1} S_N^z \rangle = \pm \langle P_1 S_1^z P_N S_N^z \rangle$ implies $\langle P_1 S_1^z \rangle \neq 0$ by clustering. This leads to an exponentially split ground state degeneracy, with correlation length set by the spin gap. The energy splitting of the bulk spectrum is much larger at $\sim 1/L$, so this degeneracy can be detected in the finite size spectrum as sketched in Fig. 1(c).

The special properties of this phase can be described in terms of an *emergent anomaly*. Indeed, since fermions are gapped, the parity subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_4$ only acts on gapped degrees of freedom, such that R_x acts as a $\mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2$ symmetry on the low-energy theory. In fact, we will see that its action is incompatible with an on-site microscopic $\mathbb{Z}_2$ symmetry, which is the essence of the anomaly. In our model R_x is

on-site, but the loophole is that $R_x^2 = (-1)^F$, so the anomaly is only an emergent property of the low-energy degrees of freedom.

The anomaly is illustrated by R_x string correlators of the form $\langle \mathcal{O}_m (\prod_{m<k<n} R_x) \mathcal{O}_n \rangle$, where $\mathcal{O}$ is a local operator. This correlation function tends to zero either algebraically or exponentially quickly. As we will see, it turns out that we are in the algebraic case if and only if $\mathcal{O}$ has odd fermion parity. Since such operators have a nontrivial charge under R_x^2, from the point of view of the gapless degrees of freedom, these R_x strings, associated with an effective $\mathbb{Z}_2$ symmetry, have fractional charge. This fractional charge is a hallmark of the anomaly, as we describe in more detail in Appendix B. We will derive the anomaly from the low-energy field theory in Sec. III and argue that emergent anomalies are always associated with edge modes in Sec. IV.

III. EFFECTIVE FIELD THEORY

In this section, we present a field theory of the intrinsically gapless SPT phase above. To focus on the essential features, we present a stripped-down version, where we start from free spinful fermions ($U = J_z = h_x = 0$ above) and consider a single perturbation that drives us into one of two topologically distinct Luttinger liquids, with one of the two having protected edge modes and an emergent anomaly. The free spinful fermion thus plays the role of a phase transition where the fermion becomes gapless and the emergent anomaly jumps.

We represent the fermion in Abelian bosonization as a pair of 2π-periodic compact boson fields ($\varphi_\uparrow, \theta_\uparrow$), ($\varphi_\downarrow, \theta_\downarrow$) satisfying $[\partial_x \varphi_s(y), \theta_{s'}(x)] = 2\pi i \delta_{ss'} \delta(x - y)$, in which the fermions may be expressed as $\psi_{s,\pm}^\dagger = U_s e^{\pm i\varphi_s/2 + i\theta_s}$. Here $\pm$ denote the left and right-movers and $U_{1,2}$ the Klein factors, necessary to make these two operators anticommute [58]. In this theory, $\phi_s \mapsto \phi_s + 2\pi$, $\theta_s \mapsto \theta_s + \pi$ are gauge symmetries for each spin species, see Appendix F. Our $\mathbb{Z}_4$ symmetry of interest acts as $\psi_{s,\pm}^\dagger \mapsto i \psi_{-s,\pm}^\dagger$ from which we infer the action on the compact boson fields:

$$ R_x : \begin{cases} \varphi_s \mapsto \varphi_{-s} \\ \theta_\uparrow \mapsto \theta_\downarrow + \pi/2 \\ \theta_\downarrow \mapsto \theta_\uparrow - \pi/2 \end{cases} \qquad \begin{cases} U_\uparrow \mapsto U_\downarrow \\ U_\downarrow \mapsto -U_\uparrow \end{cases}, \qquad (3) $$

i.e., rotation about the x axis exchanges the opposite spin fermions and they acquire a phase so that this satisfies $R_x^2 = (-1)^F$.

The operator which tunes us into the two topologically distinct Luttinger liquids is $\mathcal{O}_{zz} = \cos(\varphi_\uparrow - \varphi_\downarrow)$. This pins[4] the spin field $\Phi_1 = \varphi_\uparrow - \varphi_\downarrow$, and hence all states of odd fermion parity are gapped. The remaining low-energy degrees of freedom can be described as a Luttinger liquid of spinless Cooper pairs

$$ \psi_{\uparrow,+}^\dagger \psi_{\downarrow,-}^\dagger \sim \exp\left(i(\theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2) \right). \qquad (4) $$

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[1]We automatically have P since it is an unbreakable fermionic symmetry. Alternatively, $R_x^2 = P$—hence the same physics occurs in purely bosonic system with $\mathbb{Z}_4$ symmetry, see Appendix E.

[2]Strictly speaking, for fermions these phases are classified by spin cobordism. The group is still zero, however [57].

[3]See Appendix B 1 for a direct proof.

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[4]This operator is marginal at the free-fermion point, but it can be made relevant by tuning exactly marginal parameters. See Appendix A or D for more details.

We can express this in terms of the conjugate compact boson fields[5] $\Phi_2 = \varphi_\downarrow$ and $\Theta_2 = \theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2$. See Appendix E for more details.

To determine the symmetry action on these fields, we use (3) and replace Φ_1 by its vev wherever it appears. We see that R_x acts as a $\mathbb{Z}_2$ symmetry in the effective low-energy theory:

$$R_x : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle, \\ \Theta_2 \mapsto \Theta_2 + \langle \Phi_1 \rangle, \end{cases} \tag{5}$$

where $\langle \Phi_1 \rangle = 0$ or π depending on the sign of the $\cos \Phi_1$ perturbation. If we take the sign to be positive, then $\langle \Phi_1 \rangle = \pi$ and in this case the action of R_x matches the anomalous action at the boundary of the CZX/Levin-Gu $\mathbb{Z}_2$-SPT phase (compare to Eq. (53) of Ref. [41] and also [59]). This field theory thus describes an intrinsically gapless SPT phase which is equivalent to the one identified in Sec. II. Indeed, it describes the same CFT, and R_x acts as the unique anomalous $\mathbb{Z}_2$ symmetry. If we perturb with the negative sign of $\mathcal{O}_{zz}$ on the other hand, we find a trivial Luttinger liquid phase with $\langle \Phi_1 \rangle = 0$ and trivial R_x action. In the lattice model in Sec. II, the trivial phase is obtained by driving a different parameter and we cannot capture this alternative transition in this field theory. We have also confirmed the above prediction in a lattice model that closely realizes this field theory; see Appendix A.

The string order for fermion parity in this theory may be derived from the above two-component Luttinger liquid as follows. By the canonical commutation relations, we see the generator of fermion parity $\theta_{1,2} \mapsto \theta_{1,2} + \pi$ is given by $\exp(i \int dx (\partial_x \varphi_\uparrow/2 + \partial_x \varphi_\downarrow/2))$. To obtain the string operator, we first truncate the integral so it goes from $-\infty$ to x, which reduces it to a boundary term. In the new variables, it becomes $\exp(i\Phi_1(x)/2 + i\Phi_2(x))$. Note that in either $c = 1$ phase, where Φ_1 is gapped and has a v.e.v., this operator is mutually local with the low-energy operators. Indeed fermion parity is a gapped symmetry. However, because of the $e^{i\Phi_2(x)}$, its correlation function has algebraic decay, so we must take the endpoint operator $\mathcal{O}(x) = e^{-i\Phi_2(x)}$ to cancel this factor and obtain a string with long-range order. This end point operator is charged under R_x in the topological phase [cf. (5)], just as we observed in Sec. II.

We can see the edge modes by studying a spatial interface from the topological to trivial Luttinger liquid where we tune the coefficient of the $\mathcal{O}_{zz} = \cos \Phi_1$ perturbation from a positive to a negative value, adapting an argument from Ref. [16]. There is an edge mode associated with the path Φ_1 takes from 0 to π across the interface. Any continuous path that minimizes the energy across this interface comes with a degenerate partner by exchanging $R_x : \Phi_1 \to -\Phi_1$. One way to see this is to observe that because Φ_1 is pinned to its vev far from the interface, $\langle S^z \rangle = \frac{1}{2} \langle \int_{-\infty}^{\infty} \frac{d\Phi_1}{2\pi} \rangle = \pm \frac{1}{4}$, from which we see that the edge mode is a spin-$1/2$ qubit. Together with the gapless charges in the bulk, this implies that at the boundary, fermions become gapless, as we anticipated in Sec. II.

This field theory has appeared in Refs. [16,17,25] although the protecting symmetry group there was $U(1) \rtimes T$ instead of

our $\mathbb{Z}_4$. In the former case, the gapped $\mathbb{Z}_2$ subgroup of $U(1)$ has a string order whose charge is incompatible with any $U(1)$-symmetric gapped SPT. However, if one explicitly breaks the $U(1)$ symmetry down to its $\mathbb{Z}_2$ subgroup, one obtains a gapped $\mathbb{Z}_4^T$ SPT phase. In this sense, the $\mathbb{Z}_4$ example in this work gives a conceptually cleaner instance of an intrinsically gapless SPT phase, as there is not even any subgroup that protects a gapped SPT phase. See also Table I and Appendix E. To the best of our knowledge, the emergent anomaly viewpoint which is central to our discussion of these models, has not appeared in earlier works.

IV. GENERAL FRAMEWORK

To recap, the key ingredients of an intrinsically gapless phase of the sort explored above is (i) a system with an on-site symmetry (e.g., R_x), (ii) a gapless phase where part of the symmetry [e.g., $(-1)^F$] acts only on gapped degrees of freedom and (iii) the action of the remaining symmetry on the gapless degrees of freedom is anomalous, i.e., an emergent anomaly. In 1+1D, one can detect the emergent anomaly by measuring the charges of string order parameters—note that unbroken symmetries that act purely on gapped degrees of freedom always admit a string order parameter (see Appendix B). The nontrivial charges of the latter also imply edge modes, which seems analogous to an SPT phase, but the charges themselves are incompatible with any gapped phase.

Let us now describe a general picture of this phenomenon, which applies to a general symmetry group and any dimension. Let G be the microscopic on-site symmetry, G_{gap} be the (normal) subgroup of G which acts trivially on the gapless degrees of freedom, $G_{\text{low}} = G/G_{\text{gap}}$ be the effective symmetry of the low-energy theory, with $\pi : G \to G_{\text{low}}$ being the quotient map.

We consider the partition function $Z(X, A)$ on a spacetime X coupled to background G gauge field A.[6] If the gapped sector is trivial, meaning that it preserves all symmetries and it is not topologically ordered, at leading order it contributes at most a phase to this partition function, so we may write

$$Z(X, A) = Z_{\text{low}}(X, A_{\text{low}}) e^{2\pi i \int_X \alpha(A)} + \cdots, \tag{6}$$

where $Z_{\text{low}}(X, A_{\text{low}})$ is the partition function of the gapless degrees of freedom coupled to background G_{low} gauge field derived from A by $A_{\text{low}} = \pi(A)$, $\alpha(A)$ is a topological term obtained after integrating out the gapped degrees of freedom, and $\cdots$ are terms exponentially small in the gap. For the partition functions we are interested in, since the system is gapless, $Z_{\text{low}}(X, A)$ is order one (see also Appendix C 7), so we discard these extra terms.

In the case that there is no emergent anomaly, $Z_{\text{low}}(X, A_{\text{low}})$ and $\alpha(A)$ are both gauge invariant, and $\alpha(A)$ describes a G-SPT phase in the gapped sector. When there is an emergent

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[5] We are grateful to Max Metlitski for a discussion about these variables.

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[6] Our discrete gauge fields can be interpreted as simplicial cochains, where d is the simplicial differential, and products are cup products. Thus they generalize the familiar $U(1)$ gauge potentials with the extra provision that for, e.g., $G = \mathbb{Z}_N$ they are valued in integers mod N, so that the bosonic Chern-Simons term $\frac{1}{2\pi} A \wedge dA$ becomes $\frac{2\pi}{N^2} A \cup dA$. See Appendix G for more details.

TABLE I. Emergent anomalies in intrinsically gapless SPT phases (protected by G in d spatial dimensions). These examples can occur in fermionic systems where all odd-parity states are gapped (or alternatively in a bosonic system with $G_{\text{gap}} = \mathbb{Z}_2$). The details of these calculations can be found in Appendix C 2. The case $G = \mathbb{Z}_4$ was discussed in Secs. II and III above. Meanwhile, $G = U(1) \times \mathbb{Z}_2$ captures the phenomena of the gapless Haldane phase in Ref. [4] or the charge-conserving topological superconductors in Refs. [8,11,14,15,24,26] and $G = \text{Pin}^-(2)$, generated by U(1) charge and $T^2 = (-1)^F$, that of Refs. [16,17,25]. We describe the field theory for these examples in Appendix E. Unlike $G = \mathbb{Z}_4$, the robustness of the edge modes in these models can be understood by considering a subgroup of G, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4^T$, respectively. In the latter case, the 1st Stiefel-Whitney class w_1 (of the tangent bundle of X) appears in both ω and α, which plays the role of a time-reversal gauge field [60,61]. The anomaly of the deconfined quantum critical point (DQCP) [48,62–64] can also be cured by embedding the system into a fermionic Hilbert space with rotation and charge conservation [i.e., $\text{Spin}^c(3)$ or even $\text{Spin}(5)$ symmetry] realized as on-site symmetries [65–67]. In this case, there are several possible solutions to the anomaly vanishing equation, indicated by $\cdots$ in α, which essentially contain $\text{Spin}^c(3)$ Chern-Simons terms. In the anomaly, w_4 is obtained by restriction of the fourth Stiefel-Whitney class by the block-diagonal embedding $SO(3) \times U(1) \subset SO(5)$.

d	G	G_{low}	Comment	ω	α	SPT$_d$ subgroup
1	$\mathbb{Z}_4$	$\mathbb{Z}_2$	Levin-Gu/CZX anomaly	$\frac{1}{4} A_{\text{low}} dA_{\text{low}}$	$\frac{1}{2} A_{\text{gap}} A_{\text{low}}$	none
1	$U(1) \times \mathbb{Z}_2$	$U(1) \times \mathbb{Z}_2$	bosonic QSH anomaly	$\frac{1}{4\pi} A_{\text{low}}^{\mathbb{Z}_2} dA_{\text{low}}^{U(1)}$	$\frac{1}{2} A_{\text{gap}} A_{\text{low}}^{\mathbb{Z}_2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
1	$\text{Pin}^-(2)$	$U(1) \rtimes \mathbb{Z}_2^T$	bosonic TI anomaly	$\frac{1}{4\pi} w_1 dA_{\text{low}}^{U(1)}$	$\frac{1}{2} w_1 A_{\text{gap}}$	$\mathbb{Z}_4^T$
2	$\text{Spin}^c(3)$	$SO(3) \times U(1)$	Neel-VBS DQCP	$\frac{1}{2} w_4(A_{\text{low}})$	$\frac{1}{4\pi} A_{\text{gap}} dA_{\text{low}}^{U(1)} + \cdots$	$U(1)$ or $SU(2)$

anomaly, on the other hand, $Z_{\text{low}}(X, A_{\text{low}})$ and $\alpha(A)$ are not separately gauge invariant, and instead transform in such a way that only their combination $Z(X, A)$ is gauge invariant. We cannot interpret α as an SPT class in this case. Instead, invoking the bulk-boundary correspondence, we can express the emergent anomaly in terms of a topological term $\omega(A_{\text{low}})$ for a G_{low} SPT in one higher dimension [42,68]. This means that for $\partial X = 0$, $Z_{\text{low}}(X, A_{\text{low}}) \exp(2\pi i \int_{X \times \mathbb{R}_{\geq 0}} \omega(A_{\text{low}}))$ is gauge invariant. By standard arguments, gauge invariance of (6) on closed spacetime manifolds is then equivalent to the anomaly vanishing equation $d\alpha = \omega$ (see Appendix G).

We see that for this equation to be solvable, $\alpha(A)$ has to depend on the G_{gap} part of A, since otherwise ω would describe a trivial G_{low} SPT. This means that when we perform a G_{gap} gauge transformation, $\alpha(A)$ will shift by an exact but nonzero form, meaning $\alpha(A + dg) = \alpha(A) + d\lambda(A, g)$ for some λ, while $Z(X, A_{\text{low}})$ will remain unchanged. When we study the partition function on a spacetime X with boundary, this will lead to a boundary term $e^{2\pi i \int_{\partial X} \lambda(A, g)}$, indicating that there must be an extra boundary contribution which makes the combination (6) gauge invariant again. This extra boundary contribution must come from some kind of edge mode.

Let us illustrate the above for our $G = \mathbb{Z}_4$ example, where $G_{\text{gap}} = \mathbb{Z}_2$ and $G_{\text{low}} = \mathbb{Z}_2$. The $\mathbb{Z}_2$ anomaly has the Chern-Simons form $\omega(A_{\text{low}}) = \frac{1}{4} A_{\text{low}} dA_{\text{low}}$ [68]. We can write the $\mathbb{Z}_4$-valued A as a combination of $\mathbb{Z}_2$ gauge fields A_{low} and A_{gap} according to $A = 2A_{\text{gap}} + A_{\text{low}}$, with the extension to $\mathbb{Z}_4$ encoded into the equation $dA = 0 \mod 4 \Leftrightarrow 2dA_{\text{gap}} = dA_{\text{low}} \mod 4$. This equation says that a 2π flux of A_{low} equals a π flux of A_{gap}, which captures the relation $R_x^2 = (-1)^F$. We find the only topological term which satisfies the anomaly vanishing equation is $\alpha(A) = \frac{1}{2} A_{\text{gap}} A_{\text{low}}$. Under a gauge transformation $A_{\text{gap}} \mapsto A_{\text{gap}} + dg$, there is a boundary term $\frac{1}{2} g A_{\text{low}}$, as claimed. This implies the existence of an edge mode to restore gauge invariance.

This argument can be related to the 1d argument for edge modes based on string operators. Indeed, one may observe that $\alpha(A) = \frac{1}{2} A_{\text{gap}} A_{\text{low}}$ in the example above means that the fermion parity string has an endpoint operator charged under

R_x, as we have observed in Secs. II and III. This is because the fermion parity string corresponds to the ground state in the parity twisted sector, which has $\int_{\text{space}} A_{\text{gap}} = 1 \mod 2$. The topological term α thus contributes 1 to the total A_{low} charge. Likewise we find that every state in the R_x-twisted sector of the low-energy theory has odd fermion parity, as we claimed in Sec. II. See Appendix C for more details and examples, summarized in Table I.

This approach also tells us that *up to stacking with a G-SPT*, the partition function of the gapless SPT is classified by the low-energy theory and its emergent anomaly (we do not know if two theories with the same $Z(X, A)$ can be deformed into one another, although it is a standard assumption in gapped classifications [60,69]). Indeed, in general there may be different α that solve the anomaly vanishing equation. If α and α' are two solutions, then $d(\alpha - \alpha') = 0$ so $\alpha - \alpha'$ describes a G-SPT which can be interpreted as relating the two topological phases by stacking.[7]

Another viewpoint on our work emerges by thinking about "failed" 2+1D bosonic SPT phases, i.e., those that can be trivialized by embedding them into a bigger Hilbert space—a natural instance being the addition of fermions [70–77]—meaning that the edge theory is no longer absolutely protected. The present work shows that there is still a well-defined emergent anomaly at the edge, and moreover it can therefore be realized in its own dimension as an intrinsically gapless topological phase.

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[7]Therefore, if there is one solution to the anomaly vanishing equation, then there are as many as there are G-SPTs in the same dimension. It may be that not all of these describe distinct phases however, since often the gapless system can "absorb" an SPT involving only the gapless symmetries [33]. For instance, if $G_{\text{gap}} = 1$, so there is no anomaly and all symmetries are gapless, it can happen that even if $\alpha \neq 0$, there are no protected edge modes. See Appendix C 7. This illustrates the importance of using the G_{gap} gauge transformations in the argument above.

Thus far, we have studied various examples of intrinsically gapless topological phases, but how do we systematically construct them? This question is answered in Appendix C 4 where we discuss the construction of intrinsically gapless SPT on the lattice, beginning from an anomalous theory (i.e., a gapless edge of an SPT lattice model) and a solution to the anomaly vanishing equation, i.e. a prescription to augment the degrees of freedom to "cure" the anomaly. Moreover, adapting the results of Refs. [78–80] (who studied the problem of constructing gapped symmetric boundaries of SPTs and encountered the same equation $d\alpha = \omega$), for any theory with a bosonic global G_{low} anomaly, there is some G_{gap} which realizes it as an intrinsically gapless phase where the anomaly is completely emergent.

V. OUTLOOK

Anomalies are a key nonperturbative phenomenon in high-energy and condensed matter physics [44,81–84]. However, they can be difficult to realize, either living on the boundary of a higher-dimensional system or requiring a noninternal symmetry action. In this work, we have shown how they can arise from *on-site* symmetries in the *same* dimension and how—in one dimension—they can be diagnosed using unusual string order. This paves the way for experimental realization of anomalies. In fact, most of the machinery is already in place. Doped spin-1/2 Heisenberg chains [85] have been realized and their string order has been measured— although the latter decayed algebraically due to spin-rotation symmetry. If one can engineer an Ising anisotropy, then the lattice model in Sec. II is realized, which should endow the string order parameter with long-range order. As discussed, the latter implies edge modes with $S^z = \pm\frac{1}{4}$, whose fractional value should be measurable by considering statistical ensembles.

The framework introduced in this work provides guidance to constructing interesting new models, not just in 1+1D but also in higher dimensions. A paradigmatic example of an anomaly in 2+1D is at a deconfined quantum critical point, which has been proposed to describe a transition between a Heisenberg antiferromagnet and a valence bond solid [62]. Here, SO(3) spin rotation and an effective U(1) symmetry arise, despite the anomaly, by utilizing spatial symmetries to implement (a discrete subgroup of) the latter symmetry. Alternatively, the U(1) can be identified with an internal symmetry related to charge conservation. In this case the anomaly of the deconfined critical point can be lifted by embedding it into a fermionic Hilbert space [65–67]; see Table I. Our theory predicts that with on-site SO(3) and U(1) symmetries, implemented at the expense of including gapped fermions, an exotic 1+1D edge theory will appear at the gapless deconfined critical point. This direction demands further study.

Finally, although we have highlighted one mechanism for intrinsically gapless topological phases, we do not know if it is the only mechanism. In particular, because the emergent anomaly relies on there being a gapped symmetry, one can ask if there are also intrinsically gapless SPTs for which the whole symmetry group is gapless, or in models with no gapped sector at all.

APPENDIX A: MORE NUMERICAL RESULTS

1. The Ising-Hubbard chain

a. Confirming criticality and topology

To confirm that we indeed have a Luttinger liquid, we extract the central charge $c = 1$ using entanglement scaling [86,87], plotted in Fig. 2(a). We have already numerically confirmed that the system does not spontaneously break the Ising symmetry; see the algebraically decaying spin correlations in Fig. 1(b). However, with open boundaries, the edge-to-edge spin correlation function has long-range order [Fig. 2(b)], indicating that the boundaries spontaneously break R_x symmetry. The resulting degeneracy has an exponentially small finite-size splitting for finite systems, shown by the red dots in Fig. 2(c). This is significantly smaller than the bulk $\sim 1/L$ energy scale. The scaling of the latter [blue dots in Fig. 2(c)] is significantly affected by Friedel oscillations.

b. Lattice-continuum correspondence and Luttinger liquid parameter

Here we build a correspondence between the lattice model of Sec. II and the field theory discussed in Sec. III. This will also allow us to numerically extract the Luttinger liquid parameter of the topological phase, which in turn carries information about the stability of the phase.

Let us first consider the Ising order parameter S^z_j. We have already seen in Fig. 1(b) that this decays algebraically. To determine which CFT operator it generates, we write (suppressing momentum-dependent prefactors):

$$S^z_j \sim \partial(\varphi_\uparrow - \varphi_\downarrow) + (e^{i\varphi_\uparrow} - e^{i\varphi_\downarrow}) + \cdots$$
$$= \partial\Phi_1 + e^{i\Phi_2}(e^{i\Phi_1} - 1) + \cdots. \tag{A1}$$

Hence, in the trivial phase, where $\Phi_1 = 0$, we obtain that $S^z_j \sim 0$. Indeed, this must happen at all orders, since S^z_j is odd under R_x, whereas in Sec. III, we have seen that the low-energy fields of the trivial Luttinger liquid are all even under R_x. We have numerically confirmed that in the trivial phase, $\langle S^z_i S^z_j \rangle$ decays exponentially fast.

In the topological phase, we have $\Phi_1 = \pi$, such that $S^z \sim \cos\Phi_2$. We can thus read off the effective Luttinger liquid parameter K_{eff} from $\langle S^z_i S^z_j \rangle \sim 1/|i - j|^{2K_{\text{eff}}}$. Since $\Phi_2 = \varphi_\downarrow$, this will carry momentum $\langle n \rangle \pi$, where $\langle n \rangle$ is the particle filling. Fixing $t = J_z = 1$, $\mu = \frac{1}{2}$ and $U = 5$ (and $h_x = 0$)—i.e., the
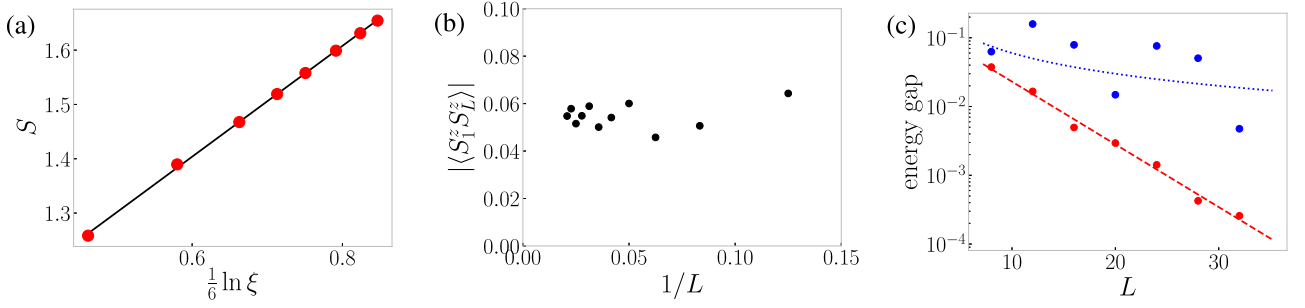
FIG. 2. The Ising-Hubbard chain for $\mu = 0$, $h_x = 0.1$, $J_z = 1 = t$, and $U = 20$. (a) The entanglement scaling indicates a conformal field theory with central charge $c_{\text{fit}} \approx 1.02$. (b) The boundaries are spontaneously magnetized, confirmed by the long-range order in the boundary-boundary correlator. (c) The finite-size splitting is exponentially small in system size (red). The blue dots are the gap to the next excited state; the blue dotted line is $\sim 1/L$ as a guide to the eye.

same parameters as in Fig. 1(b)—we measure $\langle n \rangle \approx 0.7935$. Dividing out this oscillatory phase factor, we obtain a straight line in the log-log plot of Fig. 3, from which we extract $K_{\text{eff}} \approx 0.326$.

We are thus in a strongly repulsive regime. In particular, all symmetry-allowed U(1)-breaking terms [i.e., $\cos(2n\Theta_2)$ and $\sin(2n\Theta_2)$ since R_x shifts Θ by π; see Eq. (5)] are RG-irrelevant. The relevant symmetry-allowed operators are $\cos(2\Phi_2)$ and $\sin(2\Phi_2)$, with dimensions $4K_{\text{eff}}$. Since these carry incommensurate momentum, the gapless phase is stable if we preserve translation symmetry—in addition to the R_x symmetry. For a discussion of the nearby phases generated by the above operators, see Appendix D.

To obtain a lattice operator that generates $\cos(\Theta_2)$, it is natural to consider Eq. (4), which we reproduce here for convenience:

$$\psi^\dagger_{\uparrow,+}\psi^\dagger_{\downarrow,-} \sim \exp\left(i(\theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2)\right) = e^{i\Theta_2}. \quad (A2)$$

One could wonder whether the lattice operator $c^\dagger_{\uparrow,i}c^\dagger_{\downarrow,i}$ generates this field. However, it cannot: this lattice operator is manifestly R_x-symmetric, whereas $e^{i\Theta_2}$ is odd under R_x (at least in the topological phase). Indeed, instead one finds that it has contributions of the sort

$$\psi^\dagger_{\uparrow,+}\psi^\dagger_{\downarrow,+} \sim e^{i(\frac{\varphi_\uparrow + \varphi_\downarrow}{2} + \theta_\uparrow + \theta_\downarrow)} \sim e^{i\Phi_1}e^{i(\Theta_2 + \Phi_2)}. \quad (A3)$$
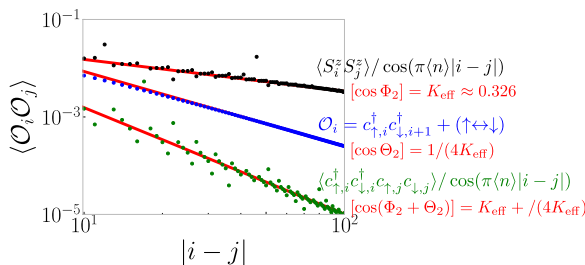

FIG. 3. Lattice operators in the Ising-Hubbard chain and the corresponding low-energy CFT operators that they generate in the topological Luttinger liquid. We use one of these to extract the Luttinger liquid parameter K_{eff}; the other two then give strong consistency checks. We have multiplied some of these correlation functions by an overall constant prefactor to shift the curves up for easier presentation.

In conclusion, $c^\dagger_{\uparrow,i}c^\dagger_{\downarrow,i} + \text{H.c.} \sim \cos(\Theta_2 + \Phi_2)$, which has scaling dimension $K_{\text{eff}} + \frac{1}{4K_{\text{eff}}}$. This prediction is confirmed in Fig. 3.

To generate $\cos\Theta_2$, we must thus consider a lattice operator that is odd under R_x. A simple tweak of the above is $c^\dagger_{\uparrow,i}c^\dagger_{\downarrow,i+1} + (\uparrow\leftrightarrow\downarrow)$. We indeed confirm that this has dimension $1/(4K_{\text{eff}})$, shown in Fig. 3. Note that this operator has no momentum, in contrast to the Φ_2 fields.

2. Coupled Luttinger liquids

Instead of the Ising-Hubbard chain in Eqs. (1) and (2), we can also consider two decoupled interacting Luttinger liquids (for each spin species) which are coupled by the Ising term:

$$H = H_\uparrow + H_\downarrow + J_z\sum_n S^z_j S^z_{j+1}$$

where

$$H_s = -t\sum_n(c^\dagger_{j+1,s}c_{j,s} + \text{H.c.}) + V\sum_n n_{j,s}n_{j+1,s}, \quad (A4)$$

with $S^z_j = \frac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})$. This setup is closest to the field theory discussed in Sec. III. In particular, for $V = J_z = 0$, we have our spinful free-fermion starting point. The Ising term contains $\cos(\varphi_\uparrow - \varphi_\downarrow)$, however, at the free-fermion point this has dimension $K_\uparrow + K_\downarrow = 2$, i.e., it is is marginal. Therefore we include the Luttinger interaction which tunes the R_x-symmetric $(\partial\varphi_\uparrow)^2 + (\partial\varphi_\downarrow)^2$, such that for repulsive $V > 0$, the Luttinger liquid parameter $K_s < 1$, making $J_z\cos(\varphi_\uparrow - \varphi_\downarrow)$ relevant. Hence, for such a fixed value of $V > 0$, the field theory in Sec. III predicts that depending on the sign of J_z, we get a topological or trivial Luttinger liquid with central charge $c = 1$, separated by a $c = 2$ phase transition at $J_z = 0$.

This prediction is straightforwardly verified. Note that for $J_z = 0$, the model is integrable, and using the exact solution we can calculate the resulting Luttinger liquid parameter [88]. We fix $V = 10$, for which $K_\uparrow = K_\downarrow \approx 0.5266$, such that the Ising coupling is relevant with dimension ≈ 1.0532. As before, we numerically diagnosed criticality by observing $c = 1$ from entanglement scaling; the topological and trivial Luttinger liquids were distinguished by measuring long-range order of the topological string-order ($\cdots P_{j-2}P_{j-1}S^z_j$) or the trivial string-order ($\cdots P_{j-2}P_{j-1}P_j$) parameters. We find that the topological and trivial Luttinger liquids persist over a

wide range of parameter space: the phases are observed for $0 < |J_z| < 4$ (and slightly beyond), with $J_z > 0$ ($J_z < 0$) being topolgical (trivial), consistent with the field theory.

APPENDIX B: ANOMALIES AND STRING ORDER PARAMETERS

G anomalies of 1d systems are in correspondence with 2d G-SPTs via the bulk-boundary correspondence [42,68]. The latter have been classified by the topological terms which are generated when the SPT is coupled to a background gauge field [89]. These topological terms in turn determine the anomalous behavior of string order parameters (also known as symmetry fluxes) in the anomalous 1d theory. In this Appendix we describe these anomalous behaviors and how one can use the topological terms to further learn about our gapless phases with emergent anomaly.

1. String operators in gapped phases

Before diving into the anomalous case, let us discuss string operators in 1d gapped phases [56,90–92]. We consider on-site unitary symmetries, meaning

$$U = \prod_x U_x, \tag{B1}$$

where the product is over sites x and U_x is a unitary operator acting only on the local Hilbert space of the site and satisfying the group law. If we have a gapped symmetric ground state $|0\rangle$, it is known that if we apply the symmetry to the ground state in a finite region, it is equivalent applying exponentially localized operators near the ends, meaning

$$\prod_{j \leqslant x \leqslant k} U_x |0\rangle = U_{j,L} U_{k,R} |0\rangle, \tag{B2}$$

where $U_{j,L}$ and $U_{k,R}$ are unitary operators with support exponentially localized near j and k respectively. This is known as symmetry fractionalization [93–95].

By conjugation with U, we can assume U_L and U_R have some fixed charge q_L, q_R under U. Let us now show this charge is trivial. We consider the product of two semi-infinite strings (although the same argument can be carried out for finite strings, just with more bookkeeping):

$$\left(\prod_{x \leqslant n} U_x^\dagger\right)\left(\prod_{y \leqslant m} U_y\right)|0\rangle \tag{B3}$$

with $m \gg n$. We have

$$\left(\prod_{x \leqslant n} U_x^\dagger\right)\left(\prod_{y \leqslant m} U_y\right)|0\rangle = \prod_{n \leqslant y \leqslant m} U_y |0\rangle = U_{n,L} U_{m,R} |0\rangle \tag{B4}$$

as well as

$$\left(\prod_{x \leqslant n} U_x^\dagger\right)\left(\prod_{y \leqslant m} U_y\right)|0\rangle = \left(\prod_{x \leqslant n} U_x^\dagger\right) U_{m,R} |0\rangle$$
$$= U_{n,R}^\dagger U_{m,R} |0\rangle, \tag{B5}$$

so $U_L |0\rangle = U_R^\dagger |0\rangle$. We also have

$$\left(\prod_{x \leqslant n} U_x^\dagger\right)\left(\prod_{y \leqslant m} U_y\right)|0\rangle = \left(\prod_{y \leqslant m} U_y\right)\left(\prod_{x \leqslant n} U_x^\dagger\right)|0\rangle$$
$$= \left(\prod_{y \leqslant m} U_y\right) U_{n,L} |0\rangle$$
$$= e^{iq_L} U_{n,L} \left(\prod_{y \leqslant m} U_y\right)|0\rangle$$
$$= e^{iq_L} U_{n,L} U_{m,R} |0\rangle. \tag{B6}$$

This proves the claim.

Rearranging (B2), we find

$$\langle 0| U_{j,L} \left(\prod_{j \leqslant x \leqslant k} U_x\right) U_{k,R} |0\rangle = 1, \tag{B7}$$

so symmetry fractionalization implies that the string operator $(\prod_{x \leqslant k} U_x) U_{k,R}$ has long-range order. Although U_R cannot be charged under U, if there are other global symmetries, then it can be charged, which signals a nontrivial SPT phase. Note that this charge is unique unless the other global symmetries are broken, since otherwise with $U_{R,x} U_{R,x'}'$ two differently charged end operators, we could consider U_R, U_R' inserted at some large (relative to the correlation length) but fixed distance $|x - x'|$, and this would be an order parameter for a broken symmetry. As long as the gap remains open and all symmetries are unbroken, these charges give topological invariants for 1d gapped phases.

Another useful fact is that if U and V are commuting symmetries with U_R having charge q under V and V_R having charge q' under U, then $q = -q'$ (i.e., there is charge reciprocity). To see this, consider

$$\left(\prod_{y \leqslant m} V_y\right)\left(\prod_{x \geqslant n} U_x\right)|0\rangle \tag{B8}$$

with $m \gg n$. We have

$$\left(\prod_{y \leqslant m} V_y\right)\left(\prod_{x \geqslant n} U_x\right)|0\rangle = \left(\prod_{y \leqslant m} V_y\right) U_{R,n}^\dagger |0\rangle$$
$$= e^{-iq} U_{R,n}^\dagger V_{R,m} |0\rangle \tag{B9}$$

On the other hand,

$$\left(\prod_{y \leqslant m} V_y\right)\left(\prod_{x \geqslant n} U_x\right)|0\rangle = \left(\prod_{x \geqslant n} U_x\right)\left(\prod_{y \leqslant m} V_y\right)|0\rangle$$
$$= \left(\prod_{x \geqslant n} U_x\right) V_{R,m} |0\rangle$$
$$= e^{iq'} V_{R,m} U_{R,n}^\dagger |0\rangle. \tag{B10}$$

The claim follows.

For Abelian discrete symmetries, the charges of these string operators (subject to no self-charges and charge reciprocity) are known to characterize all SPTs. For example with $\mathbb{Z}_2^U \times \mathbb{Z}_2^V$, there are two possibilities consistent with charge

reciprocity: either both strings have trivial charges or the U string is odd under V and even under U and vice versa. Thus there is a $\mathbb{Z}_2$ classification. Meanwhile for $\mathbb{Z}_4$, the symmetry class considered in Secs. II and III, there are no possible nontrivial charges.

The existence of a charged string with long-range order implies an exponentially localized edge mode. This is often phrased in terms of projective symmetry representations. Indeed, if we have a boundary, the symmetry will fractionalize on it as before, and now the charges of the U_R define the commutation relations of a projective representation, which we identify with the "anomalous" symmetry action at the edge. A more concrete way to see the edge degeneracy is to consider our system defined on an interval. The string operator stretching across the whole system satisfies

$$0 \neq \left\langle \mathcal{O}_0^\dagger \left(\prod_{0 \leqslant x \leqslant N} U_x \right) \mathcal{O}_N \right\rangle = \langle \mathcal{O}_0^\dagger \mathcal{O}_N \rangle, \qquad \text{(B11)}$$

where we used the global symmetry. If the end point operator is charged, it therefore acts as a symmetry breaking order parameter at the boundary.

In gapless systems, we still have symmetry fractionalization for gapped symmetries (i.e., symmetries which act nontrivially only on gapped degrees of freedom) and many of the previous arguments apply. For gapless symmetries (i.e., symmetries which act nontrivially on the gapless degrees of freedom) it does not hold. Nevertheless, one can still study string operators $(\prod_{x<n} U_x)\mathcal{O}_n$, which will always have algebraic decay for a gapless symmetry. In Ref. [33], it was argued that the string operators with the slowest such decay define topological invariants for gapless phases, and can lead to edge modes, with either exponential or algebraic localization at the edge. In that setting, the string operators can actually be degenerate and have different charges, which complicates the bulk-boundary correspondence. We will see that for systems with an emergent anomaly, however, the charges are much more regular, since they have to satisfy the anomaly. Indeed, we will argue that we always have exponentially localized edge modes.

2. String operators in anomalous theories

Let us first consider 1d bosonic anomalies for a cyclic group $G = \mathbb{Z}_n$. It is a consequence of the classification of anomalies that for any Abelian group, the anomaly can be determined by its finite cyclic subgroups. For these, there is a $\mathbb{Z}_n$ classification. Let $k \in \mathbb{Z}_n$ be the level of the anomaly. In terms of the 2d SPT bulk, the gauge fluxes have fractional statistics with topological spin $\theta = 2\pi i k/n^2$ [41]. This also can be derived from the Chern-Simons form of the associated topological term [68]

$$\omega(A) = \frac{k}{n^2} A \cup dA. \qquad \text{(B12)}$$

See Appendix G. Below we mention a direct connection between the U(1) chiral anomaly and the $\mathbb{Z}_n$ anomaly.

This topological spin translates into a certain spin-selection rule for the boundary [96,97]. One finds that in the g-twisted sector, where g is the generator of $\mathbb{Z}_n$, all states have fractional

spin (i.e. momentum around the circle)

$$S \in \frac{k}{n^2} + \frac{1}{n}\mathbb{Z}. \qquad \text{(B13)}$$

Since the spin in the twisted sector is also the self-charge of the string operator, we find all string operators have *fractional* self-charge with fractional part k/n. This spin-selection rule is both necessary and sufficient to diagnose the $\mathbb{Z}_n$ anomaly. We note a possible confusion which is that the $\mathbb{Z}_n$ charge of the bulk $\mathbb{Z}_n$ flux (that is, in the 2d SPT) is $2k/n$ [98], which differs by a factor of two from the charge of the boundary string operator, see below.

We can see this spin-selection rule and fractional charge from the field theory of Sec. III. The shift symmetry $\Phi_2 \mapsto \Phi_2 + \pi$ (respectively, $\Theta_2 \mapsto \Theta_2 + \pi$) is generated by $\exp \frac{i}{2} \int_x \partial_x \Theta_2$ (respectively, $\exp \frac{i}{2} \int_x \partial_x \Phi_2$). The string operators for these symmetries thus have the form

$$\exp \left(\frac{i}{2} \int_{-\infty}^x \partial_x \Theta_2 \right) \mathcal{O}_x = \exp \left(\frac{i}{2} \Theta_2(x) \right) \mathcal{O}_x, \quad \text{(B14)}$$

$$\exp \left(\frac{i}{2} \int_{-\infty}^x \partial_x \Phi_2 \right) \mathcal{O}'_x = \exp \left(\frac{i}{2} \Phi_2(x) \right) \mathcal{O}'_x, \quad \text{(B15)}$$

respectively, where $\mathcal{O}_x$ and $\mathcal{O}'_x$ are local operators. Observe that these operators are not mutually local, but have a braiding phase of $\pm i$ (the two shift symmetries have a mutual anomaly). Thus their fusion products, which are the string operators for the diagonal shift symmetry R_x, have spin $\pm 1/4$ mod 1, hence also fractional self-charge $\pm i$.

This highlights several recurring features of anomalous symmetries. First, unlike SPT phases where the charges of the string operators are ordinary linear charges, the charges of string operators in an anomalous theory are fractional or projective. This allows us to quickly see that our phases are not of the familiar SPT $\times$ gapless type.

Second, there is no nondegenerate, gapped symmetric phase, since by modularity the spin is related to the conformal dimension, which must thus be nonzero if the spin is nonzero. (In a fermionic system, the constraints of modular invariance are slightly relaxed, and allow for half-integer spins in a gapped phase, but the anomaly fractional spins which occur are always $\leqslant 1/4$.) For even n, the fractional charge leads to degenerate string operators, which is impossible in a gapped phase (without symmetry breaking) [36]. For example, if $n = 2$, then any string operator has charge $\pm i$ by the spin selection rule. Taking its Hermitian conjugate we get a degenerate string operator of charge $\mp i$.

This raises a third point, which is that all string operators in the anomalous theory are charged. Indeed, one can consider an anomaly to be an obstruction to gauging the symmetry [68,99]. The fact that all string operators are charged means that there are no gauge-invariant states in the twisted sector of the gauge theory, which is a pathology. For edge modes, this means that the energy splitting is always exponentially small, with localization length set by the gap of the fundamental charges. We return to these edge modes in Appendix C 6 below.

Several 1d anomalies can be understood in terms of the familiar chiral anomaly [73]. For a U(1) symmetry, there is a continuous family of twisted boundary conditions on a circle,

which we think of as the magnetic flux of a U(1) gauge field passing through it. As we vary the magnetic flux Φ there is a *spectral flow*, where the energy levels of the system in the flux background move continuously as a function of Φ [100].

The meaning of the chiral anomaly is that as Φ is taken from 0 to 2π, the spin of all operators shifts by an integer $k \in \mathbb{Z}$, which is the level of the anomaly (for fermionic systems k can be a half integer [101]). In a conformal field theory, if we begin with a neutral spin-zero state at $\Phi = 0$, the spins of the corresponding family of states along the spectral flow have the universal form [97]

$$S = \frac{k\Phi^2}{4\pi^2}. \tag{B16}$$

If $\Phi = 2\pi/n$, we can identify the ground states in the flux sector with the string operators for the $\mathbb{Z}_n$ subgroup of U(1) and we see the form of S above matches our spin selection rule (B13) for the $\mathbb{Z}_n$ anomaly.

The spectral flow above can be related to a bulk Hall current. As we increase the flux adiabatically from 0 to 2π, we end up producing charge. In a bulk+boundary setup, this charge is thought of as coming from the bulk (hence "anomaly in-flow"). If we put our system on a cylinder with two circular boundaries, and thread a flux through the middle, the amount of charge pumped is identified with the bulk Hall current. Thus one would like to identify the charge of the boundary twisted states with the charge of the bulk flux, but the charges of the former are half the charges of the latter. The way this is resolved in the bulk+boundary quantum Hall setup is that the bulk Chern-Simons term itself contributes to the boundary current [102], and makes up for the missing half of the charge, so the bulk Hall current corresponding to the above spectral flow is $2k$.

Another important case for us is $U(1) \times \mathbb{Z}_2$, which can be thought of as a subgroup of a mixed $U(1) \times U(1)$ anomaly such as between the vector and axial symmetries of a 1d compact boson. The mixed anomaly in this symmetry class pumps a $\mathbb{Z}_2$ charge when we thread a 2π flux for the U(1). Equivalently, we find that the $\mathbb{Z}_2$ string carries half-integer charge under the U(1).

Meanwhile, for $U(1) \rtimes \mathbb{Z}_2^T$, the anomaly may be detected by the U(1) π-flux string operator. Indeed, only a 0 and π flux have time reversal symmetry. The anomalous case is where the π flux has a Kramers degeneracy [61].

APPENDIX C: EMERGENT ANOMALIES

1. Anomaly vanishing

In this Appendix, we describe the anomaly vanishing equation which must be satisfied by an emergent anomaly which appears in a system with a microscopic on-site symmetry. We will show how the equation is solved in our $\mathbb{Z}_4$ example as well as other symmetry classes which have appeared in the literature, namely $U(1) \times \mathbb{Z}_2$ and $U(1) \rtimes T$. We also describe how anomaly vanishing applies to systems with Lieb-Schultz-Mattis constraints and the 2d deconfined quantum critical point. We end with a discussion where we argue that any global anomaly may be realized as an emergent anomaly in a system with on-site symmetry action. Some details on discrete gauge fields may be found in Appendix G.

Let G be the microscopic symmetry group, G_{low} be the quotient of G which is realized on the low-energy degrees of freedom, and G_{gap} the (normal) subgroup of G which acts only on the gapped degrees of freedom. We have $G_{\text{low}} = G/G_{\text{gap}}$. Let $\pi : G \to G_{\text{low}}$ be the quotient map. One considers the anomaly as classified by an SPT phase in one higher dimension, which is in turn associated with an element $[\omega]$ in a cohomology theory such as group cohomology [42] or spin cobordism [60,103]. There is an associated map π^* from the group of G_{low} SPT phases to G SPT phases. Since in a system with on-site microscopic symmetry, there is no anomaly, we must have

$$\pi^*[\omega] = 0, \tag{C1}$$

which we refer to as the anomaly vanishing equation. See Appendix G for a proof.

For the examples we study, it suffices to consider this equation in group cohomology, where $[\omega]$ is represented by some group cocycle $\omega(A_{\text{low}})$, which can be thought of as the effective action of the higher dimensional SPT controlling the anomaly, coupled to a background G_{low} gauge field A_{low} [68]. The anomaly vanishing equation may be rewritten

$$\exists \alpha \qquad \omega(A_{\text{low}}) = d\alpha(A_{\text{low}}, A_{\text{gap}}), \tag{C2}$$

where $\alpha(A_{\text{low}}, A_{\text{gap}})$ can be thought of as a boundary counterterm involving background gauge fields A_{low} for G_{low} *and* A_{gap} for G_{gap}. We will see that this counterterm has a physical interpretation that allows us to reason about edge modes. In a way, it is like the topological term for the background gauge fields generated by integrating out the gapped degrees of freedom. Indeed, if $Z(A_{\text{low}})$ is the partition function of the gapless theory coupled to background G_{low} gauge field, then although this is not gauge invariant, because of the anomaly, the anomaly vanishing equation is equivalent to saying that

$$Z_{\text{low}}(A_{\text{low}})e^{2\pi i \int \alpha(A_{\text{low}}, A_{\text{gap}})} \tag{C3}$$

is gauge invariant when $(A_{\text{low}}, A_{\text{gap}})$ is interpreted as a G gauge field. Up to terms exponentially small in the gap, this is the partition function of the full theory.

The background gauge field A_{low} is a usual background gauge field (see Appendix G for a review) and satisfies

$$dA_{\text{low}} = 0 \tag{C4}$$

when G_{low} is discrete, but when the group extension

$$G_{\text{gap}} \to G \to G_{\text{low}} \tag{C5}$$

is nontrivial, then A_{gap} sees a flux background defined by A_{low}, meaning

$$dA_{\text{gap}} = c(A_{\text{low}}), \tag{C6}$$

where $c \in H^2(BG_{\text{low}}, Z(G_{\text{gap}}))$ is a 2-cocycle associated with the group extension and $Z(G_{\text{gap}})$ is the center of G_{gap} (see below) [104]. Note that if G is not a central extension, then G_{low} acts on G_{gap} and this cohomology must be considered to be twisted by this action. See Ref. [105] for an introduction. Likewise, in expressions such as $dA_{\text{gap}} = c(A_{\text{low}})$, dA_{gap} must be considered the A_{low}-twisted differential of A_{gap}. Even when $c = 0$, this can lead to solutions to the anomaly vanishing equation [106]. However, in the case of central extensions,

which includes all the examples in this work, if $c = 0$, then there are no nontrivial solutions to the anomaly vanishing equation for bosonic extensions (although see Appendix C 2 e below).

To derive (C6), we write the G gauge field as

$$A = j(A_{\text{gap}}) + s(A_{\text{low}}), \tag{C7}$$

where $j : G_{\text{gap}} \to G$ is the inclusion map, and $s : G_{\text{low}} \to G$ is a section (not necessarily a group homomorphism) of the quotient $\pi : G \to G_{\text{low}}$, meaning $\pi \circ s$ is the identity on G_{low}. We have

$$dA = 0 \Leftrightarrow j(dA_{\text{gap}}) + ds(A_{\text{low}}) = 0. \tag{C8}$$

The extension cocycle is defined by

$$j(c(A_{\text{low}})) = -ds(A_{\text{low}}). \tag{C9}$$

It is independent of s up to gauge transformations. Thus, we find (C6). Since a gauge field A on a manifold X can be interpreted as an element in $Z^1(X, G)$, we see that $c : Z^1(X, G_{\text{low}}) \to Z^2(X, Z(G_{\text{gap}}))$. By the classification of such maps discussed in Appendix G, we can identify c with an element in $H^2(BG_{\text{low}}, Z(G_{\text{gap}}))$, as claimed above.

Let us note that when the system is completely gapped, so $G = G_{\text{gap}}$, then necessarily $\omega = 0$, so the anomaly vanishing equation is simply

$$d\alpha = 0. \tag{C10}$$

In this case, we identify α with the SPT cocycle which characterizes this gapped phase. This illustrates that α is indeed part of the physical data which defines the 1d phase. For a given ω, solutions to the anomaly vanishing equation form a torsor over the G SPT classes. Physically this means that different solutions to the anomaly vanishing equation correspond with stacking an SPT phase on top of our system.

2. Examples

Let us show (C1) holds for the symmetries and anomalies we have considered.

a. $G = \mathbb{Z}_4$

In the case $G = \mathbb{Z}_4$, $G_{\text{gap}} = \mathbb{Z}_2$, $G_{\text{low}} = \mathbb{Z}_2$, working in group cohomology, the anomaly can be written in the Chern-Simons-like form [68]

$$\omega = \frac{1}{2} A_{\text{low}} \cup \frac{dA_{\text{low}}}{2}. \tag{C11}$$

Note that $dA_{\text{low}} = 0 \mod 2$, so $\frac{dA_{\text{low}}}{2}$ is an integer class (it is equivalent to the Bockstein of A_{low}). Meanwhile

$$c(A_{\text{low}}) = \frac{dA_{\text{low}}}{2}, \tag{C12}$$

so we can write

$$\pi^* \omega = d\left(\frac{1}{2} A_{\text{low}} \cup A_{\text{gap}}\right), \tag{C13}$$

which means (C1) is satisfied in cohomology. Indeed, using the product rule, (C4), (C6), and (C12),

we have

$$d\left(\frac{1}{2} A_{\text{low}} \cup A_{\text{gap}}\right) = \frac{1}{2} dA_{\text{low}} \cup A_{\text{gap}} + \frac{1}{2} A_{\text{low}} \cup dA_{\text{gap}}$$

$$= \frac{1}{2} A_{\text{low}} \cup c(A_{\text{low}})$$

$$= \frac{1}{2} A_{\text{low}} \cup \frac{dA_{\text{low}}}{2}. \tag{C14}$$

See Ref. [77] for another perspective on the trivialization of this 2d SPT class.

The meaning of the term in parentheses is that the G_{low} string is charged under G_{gap}, indeed as it must be to match the fractional charge. We return to this point in Appendix C 6 below.

b. $G = \mathbb{Z}_2 \times U(1)$

The case with $G = \mathbb{Z}_2 \times U(1)$, $G_{\text{gap}} = \mathbb{Z}_2$ generated by the order two element of U(1) is particularly common in the literature, especially where the U(1) is particle number, so G_{gap} is the fermion parity, meaning we are in a gapless phase where the fermion is gapped. For such models, we write A_{low} as a pair of a $U(1) = U(1)/\mathbb{Z}_2$ gauge field $A_{\text{low}}^{U(1)}$ and a $\mathbb{Z}_2$ gauge field $A_{\text{low}}^{\mathbb{Z}_2}$. The extension class is the first Chern class

$$c(A_{\text{low}}) = \frac{dA_{\text{low}}^{U(1)}}{2\pi}. \tag{C15}$$

The effective anomaly is

$$\omega = \frac{1}{2} A_{\text{low}}^{\mathbb{Z}_2} \cup \frac{dA_{\text{low}}^{U(1)}}{2\pi}. \tag{C16}$$

This anomaly is realized for instance via the 1d chiral anomaly, where $A_{\text{low}}^{U(1)}$ couples to the vector current and $A_{\text{low}}^{\mathbb{Z}_2}$ to the axial current. When we consider (C6) however, we find

$$\pi^* \omega = d\left(\frac{1}{2} A_{\text{low}}^{\mathbb{Z}_2} \cup A_{\text{gap}}\right). \tag{C17}$$

When G is broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the term in parentheses is a nontrivial SPT class, so the topology in such gapless phases is essentially due to an SPT sector. Extending the parity symmetry to U(1), however, requires a vanishing gap because of the anomaly (C16). We note that if one considers the $\mathbb{Z}_4$ subgroup generated by the generator of the U(1) times the generator of the $\mathbb{Z}_2$, then such examples reduce to the calculation above, and in this symmetry class the topological phase is intrinsically gapless.

c. $G = U(1) \rtimes T$

The case $G_{\text{gap}} = \mathbb{Z}_2$, $G_{\text{low}} = U(1) \rtimes \mathbb{Z}_2^T$ is also quite interesting, although similar. Because of the time reversal symmetry, we must work with the first Stiefel-Whitney class $w_1 \in Z^1(X, \mathbb{Z}_2)$, which plays the role of the time-reversal gauge field [60]. We also must treat the U(1) background A_{low} as a background U(1) gauge field with curvature F_{low}. The formula for the emergent anomaly can be found in Ref. [61]. It is

$$\omega = \frac{1}{2} w_1 \cup \frac{F_{\text{low}}}{2\pi}. \tag{C18}$$

The physics of this term is that the π flux of U(1)$_{\text{low}}$ carries a Kramers doublet under T.

Since both the low-energy U(1) and time reversal symmetries are extended by fermion parity (to a group called $Pin(2)^-$), the extension class may be written

$$c(A_{\text{low}}, w_1) = w_1 \cup w_1 + \frac{F_{\text{low}}}{2\pi}. \quad (C19)$$

We see

$$d\left(w_1 \cup \frac{1}{2} A_{\text{gap}}\right) = \frac{1}{2} w_1 \cup w_1 \cup w_1 + \pi^* \omega. \quad (C20)$$

The first term is actually exact by a Wu relation [80]. Thus we find (C1) is satisfied.

The meaning of the term in parentheses is that the π flux of U(1)$_{\text{low}}$ carries odd fermion parity, indeed as it must be to match the Kramers degeneracy. We return to this point in Appendix C 6 below.

d. 2+1D deconfined quantum critical point

These calculations also apply to higher dimensions, as we demonstrate in the example of the SO(3) × U(1) anomaly of the $\mathbb{CP}^1$ model, associated to various deconfined quantum critical points (DQCPs). The anomaly is [48,63,64]

$$\omega = \frac{1}{2} w_2\left(A_{\text{low}}^{\text{SO(3)}}\right) \cup \frac{dA_{\text{low}}^{U(1)}}{2\pi}, \quad (C21)$$

where $A_{\text{low}}^{\text{SO(3)}}$ is the background SO(3) gauge field, w_2 is its 2nd Stiefel-Whitney class, which obstructs lifting it to an SU(2) gauge field, and $A_{\text{low}}^{U(1)}$ is the background U(1) gauge field. If fundamental fermions are gapped where the SO(3) is spin rotation and the U(1) is particle number, then both groups get extended by $G_{\text{gap}} = \mathbb{Z}_2$ according to

$$c\left(A_{\text{low}}^{SO(3)}, A_{\text{low}}^{U(1)}\right) = w_2\left(A_{\text{low}}^{SO(3)}\right) + \frac{dA_{\text{low}}^{U(1)}}{2\pi}. \quad (C22)$$

We find

$$d\left(\frac{1}{2} A_{\text{gap}} \cup \frac{dA_{\text{low}}^{U(1)}}{2\pi} + \frac{1}{4\pi} A_{\text{low}}^{U(1)} \wedge \frac{dA_{\text{low}}^{U(1)}}{2\pi}\right) = \pi^* \omega. \quad (C23)$$

We note the second term on the left-hand side is a *half-quantized* U(1)$_{\text{low}}$ Chern-Simons term (since the low-energy theory is bosonic).

Another way we can cure the anomaly is by considering

$$d\left(\frac{1}{2} A_{\text{gap}} \cup w_2\left(A_{\text{low}}^{\text{SO(3)}}\right) + \frac{1}{2} CS\left(A_{\text{low}}^{\text{SO(3)}}\right)\right) = \pi^* \omega, \quad (C24)$$

where $CS(A_{\text{low}}^{\text{SO(3)}})$ is the SO(3) Chern-Simons term of smallest level [it looks like level 4 for SU(2)], which satisfies

$$\int dCS\left(A_{\text{low}}^{\text{SO(3)}}\right) = \int w_2\left(A_{\text{low}}^{\text{SO(3)}}\right) \cup w_2\left(A_{\text{low}}^{\text{SO(3)}}\right) \mod 2. \quad (C25)$$

We see the two solutions to the anomaly vanishing equation differ by a Chern-Simons term for the full group $G = \text{Spin}^c(3)$.

A similar result holds for the largest possible unitary symmetry SO(5) of the DQCP. In this case the emergent anomaly is

$$\omega = \frac{1}{2} w_4\left(A_{\text{low}}^{\text{SO(5)}}\right), \quad (C26)$$

where w_4 is the fourth Stiefel-Whitney class. The group extension by fermion parity $G_{\text{gap}} = \mathbb{Z}_2$ is

$$\mathbb{Z}_2 \rightarrow \text{Spin}(5) \rightarrow \text{SO}(4), \quad (C27)$$

again classified by the second Stiefel-Whitney class. The integer cohomology of $B\text{Spin}(5)$ is torsion-free (for instance, using the isomorphism $\text{Spin}(5) = Sp(2)$), so since ω is associated with a 2-torsion class, $\pi^* \omega = 0$ in cohomology. This cancellation underlies the fermionic model for DQCP with on-site Spin(5) symmetry in Ref. [66]. By naturality of the pullback, the anomaly is therefore microscopically trivial (admitting an on-site representation) for any subgroup $G \subset \text{Spin}(5)$.

e. Fermionic trivializations

In the case of fermionic systems, because fermionic SPTs are classified by spin cobordism, occasionally the group cohomology incarnation of the anomaly vanishing equation we used above is insufficient. This can lead to some new mechanisms for anomaly vanishing which work even when the group extension is trivial, so $c = 0$.

For instance, on spin manifolds, the second Stiefel-Whitney class of the tangent bundle is exact: $w_2 = d\eta$. In some sense η can be thought of as the spin structure [105,107]. Some bosonic anomalies have an ω which is "proportional" to w_2 and in this case we can write

$$\omega(A_{\text{low}}) = d\alpha(A_{\text{low}}, \eta). \quad (C28)$$

For example, there is time-reversal-protected 2+1D anomaly/3+1D SPT with

$$\omega = \frac{1}{2} w_1^4 = \frac{1}{2} w_2 w_1^2 \quad (C29)$$

(known as $eTmT$ in the notation of Ref. [108]), where we have used the Wu formula. This becomes trivial with the addition of $T^2 = 1$ fermions because

$$\omega = d\left(\frac{1}{2} \eta w_1^2\right). \quad (C30)$$

This anomaly is not cancellable with $T^2 = (-1)^F$ fermions however since in that case $d\eta = w_2 + w_1^2$ [109].

An example which does not involve time reversal is that of a $\mathbb{Z}_2$ 1-form symmetry anomaly in 2+1D/SPT in 3+1D with

$$\omega = \frac{1}{2} B^2 = \frac{1}{2} B w_2, \quad (C31)$$

where B is a background $\mathbb{Z}_2$ 2-form gauge field. This anomaly is again canceled by adding fermions:

$$\omega = d\left(\frac{1}{2} B\eta\right). \quad (C32)$$

This particular anomaly vanishing actually underlies bosonization in 2+1D [109,110].

3. Solvability of the anomaly vanishing equation

Let us discuss when the anomaly vanishing equation (C1) can be solved in group cohomology. For a given G_{low} and ω, there is usually a G and π, where ω is in $\text{Ker}(\pi^*)$. For group cohomology SPTs, this is always the case. In Refs. [78,79],

the authors considered a technique for constructing *gapped* boundaries of SPTs, where G_{gap} plays the role of a gauge symmetry, and G_{low} that of the microscopic symmetry, where the same mathematical problem appears.[8] They showed that for any finite G_{low}, there is a corresponding extension G for which the SPT may be disentangled in this way.

Applied to our setting, these results imply that any system with finite anomalous G_{low} symmetry is realized in a system with some on-site G, of which G_{low} is a quotient. Using the results of Ref. [80] (where one of us studied gapped boundaries for SPTs protected by Lie groups), we can extend this to the case of *global* anomalies of continuous symmetry groups, meaning we have to exclude anomalies such as the chiral anomaly, which can be diagnosed by local correlation functions, for which no gapped charges can cure the anomaly.

For fermion SPTs/anomalies beyond group cohomology, the conditions are more subtle. For example, a unitary symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ protects a $\mathbb{Z}_8$ classification of fermion SPTs in 2d, or equivalently anomalies in 1d. The generators of this group cannot be trivialized by any extension of the $\mathbb{Z}_2$ symmetry (fermion parity cannot be extended). Indeed, when $\mathbb{Z}_2$ is extended to $\mathbb{Z}$, a $\mathbb{Z}_2$ classification remains, where the domain wall carries a Kitaev wire. Another example is the $\mathbb{Z}_{16}$ classification of $T^2 = (-1)^F$ fermion SPT phases in 3d. Again under symmetry extension at most we can reduce $\mathbb{Z}_{16} \to \mathbb{Z}_2$, since the domain wall on the boundary will always carry a chiral mode of $c_- = 1/2$.

4. Construction of instrinsically gapless SPTs

Let us discuss how to construct intrinsically gapless SPT phases. From the point of view of field theory, the topological term α we get from solving the anomaly vanishing equation is enough to give an effective action for this phase, but what if we want a lattice model?

Suppose we begin with a d-space-dimensional lattice model for the anomalous G_{low} theory as a boundary of a $d + 1$-dimensional G_{low} SPT, so that G_{low} acts on-site in the whole system. Since we want to construct a d-dimensional system, we suppose that the bulk SPT has only some finite width L, forming a slab geometry $Y_d \times [0, L]$, with our anomalous theory localized to $Y_d \times 0$. See Fig. 4.

On the other boundary $Y_d \times L$, we place microscopic degrees of freedom transforming in an on-site representation of G. Because of the nontrivial extension, these degrees of freedom look projective from the point of view of G_{low}. The anomaly vanishing equation tells us that we can create a featureless edge along $Y_d \times L$ without breaking the G symmetry (which acts through its quotient $G \to G_{low}$ in the bulk and along $Y_d \times 0$) (compare Refs. [78,79]). Different featureless edges will correspond to different solutions of the anomaly vanishing equation by layering d-dimensional G SPTs along $Y_d \times L$. See Appendix C 5 for a general construction for finite G.

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[8]Our constructions can be used to create anomalous theories with microscopic G_{low} symmetry by gauging G_{gap}, but of course they will not be gapped.
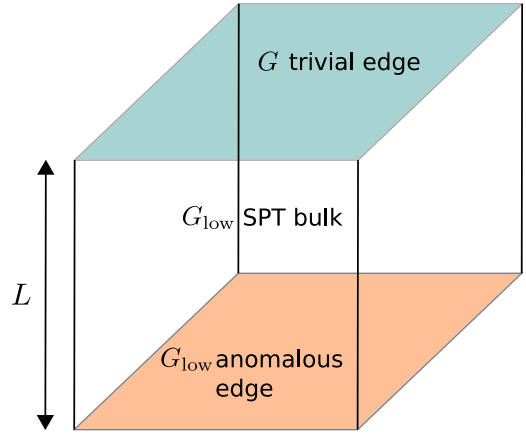

FIG. 4. The "slab construction" of intrinsically gapless SPT phases. The G symmetry acts on-site everywhere, but inside the SPT bulk and along the anomalous edge it acts by the quotient $G \to G_{low}$. In terms of the partition function (6), the bottom edge contributes $Z_{low}(X, A_{low})$ and the top edge contributes $e^{2\pi i \int_X \alpha(A)}$. To define the purely d-dimensional system, we keep the width of the slab L fixed and take the horizontal directions to be infinite.

Once we have "trivialized" the $Y_d \times L$ edge we may be worried that we have gapped out the anomalous $Y_d \times 0$ edge. However, this coupling can at most introduce local G_{low}-symmetric perturbations of the low-energy modes, and these can be canceled by tuning parameters along the $Y_d \times 0$ edge. These perturbations can be controlled by taking L much larger than the bulk correlation length.

In the end, we produce a system on the slab $Y_d \times [0, L]$ with global on-site G symmetry, and low-energy theory described by our starting G_{low}-anomalous theory. Since L remains fixed, this is a d-dimensional system (although one which has a very large local Hilbert space). It shows that to any system with G_{low} anomaly and a solution to the anomaly vanishing equation, so long as we can construct a lattice model for this anomalous system, we can construct a lattice model for the associated intrinsically gapless SPT.

5. Trivialization of the SPT boundary in a fixed point model

To show how a solution to the anomaly vanishing equation $\omega(A_{low}) = d\alpha(A)$ allows us to define a trivial G-symmetric edge of the G_{low} SPT classified by ω, we can adapt a construction of Ref. [42] for finite G. First let us recall that construction. We will take our local Hilbert space to be associated with the vertices of a triangulated spatial manifold Y, with the local Hilbert space at a single vertex spanned by basis vectors labeled by elements of G_{low}. G_{low} acts on these vectors in the so-called regular representation, i.e., by multiplication (and later G will act by its quotient $G \to G_{low}$). Thus the whole Hilbert space is spanned by states $|\phi\rangle$ associated to G_{low}-valued 0-cochains $\phi \in C^0(Y, G_{low})$ (see Appendix G for notation and concepts used below).

We define the operator Z_y^g near a vertex y that multiplies the G_{low} label at y by g and also produces the following phase

factor:

$$e^{i\beta(\phi,y,g)} = \exp\left(2\pi i \int_{Y\times[0,1]} \omega(A(\phi,y,g))\right), \quad \text{(C33)}$$

where $A(\phi, y, g)$ is a gauge field on the "spacetime" $Y \times [0, 1]$, which restricts to 1^ϕ on $Y \times 0$ and $1^{\phi'}$, where ϕ' is the new ϕ' on $Y \times 1$. Thus, along the "vertical" edge connecting $y \times 0$ to $y \times 1$, A has label g, while on other "vertical edges" (i.e., those connecting $Y \times 0$ and $Y \times 1$) it has the identity label. Thus the phase factor $\beta(\phi, y, g)$ although we have expressed it as an integral over all of $Y \times [0, 1]$, only receives nontrivial contributions from simplices touching y, so Z_y^g so defined is a local operator. Writing it as an integral over all of $Y \times [0, 1]$ however makes it clear that since ω is gauge invariant (by virtue of $d\omega = 0$), the Z_y^g's satisfy the group algebra at y and commute at separated points. Thus the Hamiltonian

$$H = -\sum_y \left(\frac{1}{|G_{\text{low}}|} \sum_g Z_y^g\right) \quad \text{(C34)}$$

is a commuting projector Hamiltonian. Its unique ground state on a closed Y may be written

$$|\omega\rangle = \sum_\phi e^{2\pi i \int_Y \nu(\phi)} |\phi\rangle, \quad \text{(C35)}$$

where

$$d\nu(\phi) = \omega(1^\phi). \quad \text{(C36)}$$

This reproduces the SPT ground state in Ref. [42] which corresponds to ω.

Now suppose we have a solution to the anomaly vanishing equation $\omega(A_{\text{low}}) = d\alpha(A)$. We define a Hilbert space on a spatial manifold Y with boundary in much the same way as above, but now vertices along ∂Y have Hilbert spaces spanned by elements of the bigger group G. We can write this as a combination of a G_{low}-valued 0-cochain $\phi_{\text{low}} \in C^0(Y, G_{\text{low}})$ and a G-valued 0-cochain $\phi \in C^0(\partial Y, G)$, satisfying the condition $\phi_{\text{low}}|_{\partial Y} = \pi(\phi)$, where $\pi : G \to G_{\text{low}}$ is the quotient map.

We define Z_y^g to be associated with a group element $g \in G$. On general vertices of Y, it acts by multiplication on ϕ_{low} by the image of g in G_{low}, while on boundary vertices it also acts by multiplication in G on ϕ. We see it preserves the boundary condition $\phi_{\text{low}}|_{\partial Y} = \pi(\phi)$. The associated phase factor is now

$$e^{i\beta(\phi,y,g)} = \exp\left(2\pi i \int_{Y\times[0,1]} \omega(A_{\text{low}}(\phi_{\text{low}}, y, g)) - 2\pi i\right.$$
$$\left. \times \int_{\partial Y\times[0,1]} \alpha(A(\phi, y, g))\right), \quad \text{(C37)}$$

where $A_{\text{low}}(\phi_{\text{low}}, y, g)$ is a G_{low} gauge field on $Y \times [0, 1]$ defined as above to restrict to $1^{\phi_{\text{low}}}$ on $Y \times 0$ and $1^{\phi'_{\text{low}}}$ on $Y \times 1$, where ϕ'_{low} is the transformed ϕ'_{low}. Likewise, $A(\phi, y, g)$ is a G gauge field on $\partial Y \times [0, 1]$, which restricts to 1^ϕ on $\partial Y \times 0$ and to $1^{\phi'}$ on $\partial Y \times 1$. This phase factor reduces to local contributions from simplices touching y and by the anomaly vanishing equation, which implies gauge invariance of this integral so long as ϕ_{low} and ϕ are fixed along $Y \times 0$ and $Y \times 1$, the Z_y^g so defined satisfy the group algebra at y and commute at separated points. Thus we can define the commuting projector

Hamiltonian

$$H = -\sum_y \left(\frac{1}{|G|} \sum_g Z_y^g\right), \quad \text{(C38)}$$

which has a unique G-symmetric ground state.

We expect a similar construction for supercohomology phases can be adapted from Refs. [111,112]. It would be very interesting to understand such constructions for continuous symmetry groups.

6. Bulk-boundary correspondence

We have seen in the above examples that in 1d the emergent anomaly is often associated with a fractional charge or projective symmetry representation of a string operator for G_{low}. From the perspective of the full microscopic symmetry G, however, these fractional charges or projective representations are actually integral or linear. This necessitates that certain G_{low} string operators are charged under G_{gap}. We will argue this and show it leads to edge modes.

Let $g \in G$ be a gapless symmetry. Observe that all g algebraic string operators must have the same G_{gap} charge (which thus must be Abelian). Otherwise, by fusion of these string operators, we would obtain a local algebraic operator with nontrivial G_{gap} charge, contradicting the definition of G_{gap}.

We will show that if $h \in G_{\text{gap}}$ acts nontrivially in any of these twisted sectors, that the unique h-string with long-range order has nontrivial G charge. In a certain sense, we will discuss, at any symmetric boundary condition this string operator has a vev. Since it defines a local order parameter at the boundary, there must be spontaneous symmetry breaking at the edge.

Let us be more precise. Suppose algebraic g strings have charge $\chi_g : G_{\text{gap}} \to U(1)$. This charge is multiplicative in g, so it defines a bicharacter (a homomorphism) $\chi : G \times G_{\text{gap}} \to U(1)$. We will argue that for $h \in G_{\text{gap}}$, the h string with long-range order has G charge $\chi(-, h) : G \to U(1)$. This charge is uniquely defined because G is unbroken.

The $g \in G$ charge of the $h \in G_{\text{gap}}$ string operator with longest range order is captured by the leading term of the partition function on a rectangular torus of aspect ratio β, with a g twist in the time direction and an h twist in the space direction, in the "low temperature" limit $\beta \to \infty$. We denote this partition function

$$Z(h/g, \beta) = \text{Tr}_{\mathcal{H}_h} g e^{-\beta H}, \quad \text{(C39)}$$

where $\mathcal{H}_h$ is the Hilbert space in the h-twisted sector and H is the Hamiltonian. In a conformal invariant IR limit, if we perform a modular S transformation, we will find

$$Z(h/g, \beta) = Z(g/h, 1/\beta) = \text{Tr}_{\mathcal{H}_g} h e^{-H/\beta}. \quad \text{(C40)}$$

Because all g-twisted states of the CFT limit have the same G_{gap} charge, in this case $\chi(g, h)$, this partition function is

$$e^{i\chi(g,h)} Z_{\text{low}}(1/\beta, g/1), \quad \text{(C41)}$$

where $Z_{\text{low}}(1/\beta, g/1)$ is positive for all β (however, if there is spontaneous symmetry breaking, it can be exponentially small see Appendix C 7 below). Thus, the phase of the leading term in the limit $\beta \to \infty$ of $Z(h/g, \beta)$ is $e^{i\chi(g,h)}$, proving the claim

$$\left\langle U_L^\dagger \; \prod_n U_n \; 1 \right\rangle \neq 0$$
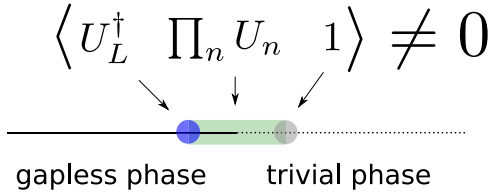
gapless phase   trivial phase

FIG. 5. A gapped string operator straddling an interface between the topological phase and the trivial phase has a nonzero vev. It plays the role of a symmetry breaking order parameter at the boundary, signaling the presence of edge modes.

of reciprocity, at least in the CFT case. We believe this claim holds in full generality.

Thus, in a system with an emergent anomaly, there will be a long-range ordered string of a gapped symmetry with non-trivial charge. If we consider symmetry fractionalization at the interface between our system and a trivial phase, we then see that something must happen at the boundary. Indeed, a string which crosses the interface, with one end in the topological phase and the other in the trivial phase, will have some vev, yet carry a global charge since it has different end point in each phase. This signals a kind of spontaneous symmetry breaking at the boundary. See Fig. 5.

For a finite G, one can construct such boundaries as follows. First we break the G_{low} symmetry by a generic boundary perturbation to define a "fixed" boundary condition. G will permute these boundary conditions through the action of its quotient G_{low}. We can thus restore the symmetry by taking a direct sum of each of these $|G_{\text{low}}|$-many boundary conditions.

In more familiar situations, such a "spontaneously fixed" boundary condition would be unstable since one can add boundary-condition-changing (bcc) operators to the Hamiltonian and induce a flow to a "free" boundary condition. These bcc operators are not local bulk operators but instead are associated with the G_{low} twisted sectors. As we have seen however, in the presence of an emergent anomaly, at least some of these bcc operators will be charged under G_{gap}, so we cannot perturb by them. Some symmetry will thus remain broken at the boundary.

The sense in which the symmetries are broken is that if we consider an interval with the spontaneously fixed boundary condition on either end, the edges will polarize eachother but there will be an overall degeneracy equal to the number of broken symmetry generators. In the case of a nontrivial anomaly then, there will be a nontrivial degeneracy, which we associate to edge modes. A similar analysis was presented in Ref. [33].

Let us relate this argument to the one given in Sec. IV. We would like to interpret the counterterm α on the RHS (C2) [see also (C13), (C17), and (C20)] as defining the string charges χ. Indeed, consider the partition function (C3):

$$Z_{\text{low}}(A_{\text{low}})e^{2\pi i \int \alpha(A_{\text{low}}, A_{\text{gap}})}. \tag{C42}$$

We have argued that the topological term α is necessary for gauge invariance. When we compute torus partition functions in the twisted sector, as we did above to measure the charges of string operators, this term contributes to the phase of the partition function, and therefore helps to encode this informa-

tion. In particular, since $Z_{\text{low}}(A_{\text{low}})$ is independent of A_{gap}, α completely encodes the G_{gap} charges of the different twisted sectors.

To encode such a charge, α contains a term schematically like $qA_{\text{gap}}A_{\text{low}}$. This term has a reciprocal interpretation that says the G_{gap} string also carries charge. This charge is what is picked up in the boundary variation when we do a G_{gap} gauge transformation $A_{\text{gap}} \mapsto A_{\text{gap}} + dg$, $\delta\alpha \ni d(qgA_{\text{low}})$. To cancel this gauge variation, we can restrict A_{low} at the boundary so that $qA_{\text{low}} \sim 0$. This amounts to the symmetry breaking we argued above.

For instance, with $G = U(1) \times \mathbb{Z}_2$ with the anomaly as in Appendix C 2 b, in light of (C17) the nontrivial G_{gap} charges are in the $\mathbb{Z}_2$ twisted sector, so we expect there is a stable boundary condition which is "free" from the perspective of the U(1) but spontaneously fixed for the $\mathbb{Z}_2$, associated with a twofold degeneracy on an interval. The same applies to $G = U(1) \rtimes T$ of Appendix C 2 c with T spontaneously broken on an interval.

7. Partition function vanishing, symmetry breaking, and topological order

Analysis based on the factorized partition function

$$Z(X, A) = Z_{\text{low}}(X, A)e^{2\pi i \int_X \alpha(A)} + \cdots \tag{C43}$$

is only valid as long as $Z_{\text{low}}(X, A)$ is nonvanishing. Otherwise, the partition function is dominated by exponentially small subleading corrections hidden in $\cdots$, which are contributions purely from the gapped sector, which we expect are nonuniversal. In particular, the topological data in the counterterm $e^{2\pi i \int_X \alpha(A)}$ can only be extracted for backgrounds with $Z_{\text{low}}(X, A) \neq 0$.

This is a regular occurrence at SPT transitions. For instance, take $G_{\text{gap}} = 1$ so $d\alpha = \omega = 0$ is an arbitrary SPT class in the same dimension as our system. Suppose we study a theory which is a phase transition between the trivial SPT and the SPT classified by α. As we tune through the transition, the phase of $Z(X, A)$ jumps from 1 to $e^{2\pi i \int_X \alpha(A)}$, so for *any* background with $e^{2\pi i \int_X \alpha(A)} \neq 1$, $Z(X, A)$ vanishes at the transition. This phenomenon is crucial to understand the fact that some gapless systems can "absorb" stacked SPTs [33]. However, if there is a G_{gap}, this can only happen for α which only depend on G_{low}.

The partition functions we are most interested in are those of the form $Y \times S^1$ where A has G_{low} holonomy around the spatial slice Y and G_{gap} holonomy around the temporal S^1, since this computes the G_{gap} charge of a G_{low} symmetry defect. For these partition functions, as long as the symmetry in unbroken and there is no topological order, $Z_{\text{low}}(X, A)$ will be nonzero, since the magnitude of such partition functions computes the spectrum of the low-energy theory in these backgrounds, which is manifestly positive (compare Appendix C 6).

In the presence of symmetry breaking or topological order, it can happen that even with just spatial components of A nonzero, $Z(X, A)$ may be exponentially small, and thus we will not be able to extract a universal part of (C43). In the case of spontaneous symmetry breaking, this is because the domain walls are tensionful (or massive in 1+1D). In

the case of topological order, it can happen because symmetry fractionalization may bind nontrivial anyons to the symmetry defects, and these anyons are massive [98,113,114].

In these cases, one must be more clear what is meant by emergent anomaly. With SSB, we can simply restrict our attention to the unbroken subgroup. With topological order however, it is less obvious how to proceed, and we mostly leave this case to future work. Let us just comment that in a purely gapped system with topological order, there is no sense of an emergent anomaly. One might like to say perhaps that G acts on the anyons and that there is some quotient $G_{\text{"low"}}$ for which this action may be considered projective. That would be wrong though, since the anyons, being defined only up to multiplication by local operators, do not carry well-defined G representations (although G may permute the superselection sectors).

APPENDIX D: NEARBY PHASE DIAGRAM OF THE TOPOLOGICAL PHASE

In this Appendix, we discuss the nearby phase diagram of our topological gapless phase protected by $\mathbb{Z}_4$ symmetry R_x. We use the field theory approach of Sec. III.

To examine the nearby phase diagram, we must first study the marginal parameters. The $c = 2$ mother theory has several such parameters. It is convenient to think of the bosonized $c = 2$ theory as a sigma model with 2-torus target and these parameters as the shape of this torus (see Appendix A of Ref. [117] for a review). There are actually two dual tori we can think about, depending on whether we look at the coordinates $(\varphi_\uparrow, \varphi_\downarrow)$ or $(\theta_\uparrow, \theta_\downarrow)$. At the free fermion point, the torus is rectangular in $(\varphi_\uparrow, \varphi_\downarrow)$ coordinates with radii $R_\uparrow$ and $R_\downarrow$ along the $\varphi_\uparrow$ and $\varphi_\downarrow$ directions, respectively. For our models, these radii must be equal: $R_\uparrow = R_\downarrow = R = 1/\sqrt{K}$ to have R_x or T symmetry. These radii can be tuned together by the symmetric marginal perturbation

$$(\partial_x \varphi_\uparrow)^2 + (\partial_x \varphi_\downarrow)^2, \tag{D1}$$

which represents a separate Luttinger interaction in each spin channel, but of the same strength. There is one more symmetric marginal perturbation:

$$\partial_x \varphi_\uparrow \partial_x \varphi_\downarrow, \tag{D2}$$

which can be tuned by the Hubbard interaction. This corresponds to squashing the torus in the diagonal direction. We therefore write the (constant) metric as

$$g = \begin{bmatrix} R^2 & S \\ S & R^2 \end{bmatrix}. \tag{D3}$$

The Hamiltonian term corresponding to this metric is

$$\partial_x \vec{\varphi}^T g \partial_x \vec{\varphi}. \tag{D4}$$

The dimension of the φ vertex operator $e^{in_\uparrow \varphi_\uparrow + in_\downarrow \varphi_\downarrow}$ is

$$\Delta_{\vec{n}} = \vec{n}^T g^{-1} \vec{n}, \tag{D5}$$

where $\vec{n} = (n_\uparrow, n_\downarrow)^T$. Thus $\mathcal{O}_{zz} = \cos(\varphi_\uparrow - \varphi_\downarrow)$, which corresponds to a lattice vector $(1, -1)$ has dimension $2/(R^2 - S)$. For it to be relevant, we need $R^2 - S > 1$.

To determine the effective radius R_{eff} (or effective Luttinger parameter K_{eff}) of the topological gapless phase in
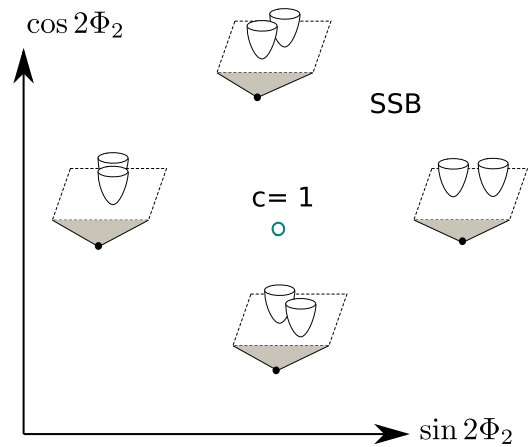

FIG. 6. The gapless $\mathbb{Z}_4$-SPT phase has two symmetric relevant operators, $\cos 2\Phi_2$ and $\sin 2\Phi_2$ in the region of the marginal parameter $1/8 < K_{\text{eff}} < 1/2$, which describe a 2-parameter nearby phase diagram where the gapless phase is an isolated point (open blue circle) in a sea of a gapped phase where $\mathbb{Z}_4$ is broken to its $\mathbb{Z}_2$ subgroup. It is a diabolical point in the sense of [115], protected by a nontrivial vacuum crossing [116]: as one adiabatically traverses a loop encompassing the gapless point, the two SSB vacua are exchanged. A cartoon of how the mean field potential looks is illustrated at the black circles in the figure. The emergent anomaly further protects this point from deformation into an island of the trivial phase.

terms of R and S, we must simply see the length of the circle of minima of the potential $\mathcal{O}_{zz} = \cos(\varphi_\uparrow - \varphi_\downarrow)$. This has minima along the circle $\varphi_\uparrow = \varphi_\downarrow + \pi$ embedded in the 2-torus. The length of this circle is $R_{\text{eff}} = \sqrt{2R^2 + 2S}$ or

$$K_{\text{eff}} = 1/(2R^2 + 2S), \tag{D6}$$

which for the free fermion $R = 1$ and $S = 0$ is the SU(2) point $K_{\text{eff}} = 1/2$. At this free point, the key perturbation $\mathcal{O}_{zz}$ [with dimension $2/(R^2 - S)$] is marginal. To make it relevant, we can either decrease S, which results in $K_{\text{eff}} > 1/2$, or we can increase R (i.e., turn on a repulsive Luttinger interaction as in Appendix A 2), which results in $K_{\text{eff}} < 1/2$. These represent two phenomenologically distinct regimes of the $c = 1$ topological phase. Below we will mainly focus on the region $K_{\text{eff}} < 1/2$, which is realized by the Ising-Hubbard chain in Sec. II as observed in Appendix A.

Recall in the new variables $\Phi_2 = \varphi_\downarrow$, $\Theta_2 = \theta_\uparrow + \theta_\downarrow - \varphi_\uparrow/2 + \varphi_\downarrow/2$, we have the symmetry action

$$R_x : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle \\ \Theta_2 \mapsto \Theta_2 + \langle \Phi_1 \rangle \end{cases}, \tag{D7}$$

where in the topological phase we have $\langle \Phi_1 \rangle = \pi$. In this case, the most relevant symmetric operators are $\cos 2\Theta_2$, $\sin 2\Theta_2$, $\cos 2\Phi_2$, $\sin 2\Phi_2$ (there are also $\cos \Theta_2 \cos \Phi_2$, etc. but these operators cannot open a gap). The first two are relevant for $K_{\text{eff}} > 1/2$ and the second two for $K_{\text{eff}} < 1/2$. As long as K_{eff} is within the window (1/8, 2), these are no other symmetric relevant operators. For any combination of each pair, the system flows to an SSB state where $\mathbb{Z}_4$ is broken to $\mathbb{Z}_2^F$. The nearby phase diagram for $1/8 < K_{\text{eff}} < 1/2$ is drawn in Fig. 6.

In the region $K_{\text{eff}} < 1/2$, at generic filling, translation symmetry stabilizes the gapless phase. In the region $K_{\text{eff}} > 1/2$, it is stabilized by particle number U(1) symmetry. In this latter case, the $\mathbb{Z}_2$ symmetry generated by the product of an order four element of U(1) and R_x forms an SPT phase with $\mathbb{Z}_2^F$. To have a stable gapless phase but still be in a symmetry class with no gapped SPTs, we can take our symmetry group to be $\mathbb{Z}_{12} = \mathbb{Z}_4 \times \mathbb{Z}_3$ generated by R_x and the $\mathbb{Z}_6$ subgroup of U(1).

APPENDIX E: FIELD THEORY FOR OTHER SYMMETRY CLASSES

In Sec. III, we studied an intrinsically gapless $\mathbb{Z}_4$-SPT phase in proximity to a free spinful fermion. In this Appendix, we will show that the same field theory captures topological phases in the symmetry classes U(1) $\times \mathbb{Z}_2$ and U(1) $\rtimes T$ as well.

1. Other fermionic classes

a. Symmetries

Besides our $\mathbb{Z}_4$ symmetry R_x, we are also interested in time reversal and particle number conservation. These act on the bosonic variables as follows:

$$
R_x : \begin{cases} \varphi_s \mapsto \varphi_{-s} \\ \theta_\uparrow \mapsto \theta_\downarrow + \pi/2 \\ \theta_\downarrow \mapsto \theta_\uparrow - \pi/2 , \\ U_\uparrow \mapsto U_\downarrow \\ U_\downarrow \mapsto -U_\uparrow \end{cases} \quad T : \begin{cases} \varphi_s \mapsto \varphi_{-s} \\ \theta_s \mapsto -\theta_{-s} \\ U_\uparrow \mapsto U_\downarrow \\ U_\downarrow \mapsto -U_\uparrow , \end{cases}
$$

$$
\text{U(1)} : \begin{cases} \varphi_s \mapsto \varphi_s \\ \theta_s \mapsto \theta_s + \alpha . \\ U_s \mapsto U_s . \end{cases} \tag{E1}
$$

We have $T^2 = (-1)^F$ and fermion parity $(-1)^F$ is a U(1) rotation with $\alpha = \pi$. For convenience, we also collect

$$
e^{i\alpha S_z} : \begin{cases} \varphi_s \mapsto \varphi_s \\ \theta_\uparrow \mapsto \theta_\uparrow + \alpha/2 \\ \theta_\downarrow \mapsto \theta_\downarrow - \alpha/2 , \\ U_s \mapsto U_s \end{cases}
$$

$$
\text{Trans} : \begin{cases} \varphi_s \mapsto \varphi_s + k_R - k_L \\ \theta_s \mapsto \theta_s + k_R/2 + k_L/2 . \\ U_s \mapsto U_s \end{cases} \tag{E2}
$$

We see the operator $\mathcal{O}_{zz} = \cos(\varphi_\uparrow - \varphi_\downarrow)$ which tunes between the trivial and topological gapless phases remains symmetric under U(1), T, and translation.

b. The topological phase

The variables of Sec. III can be derived by first performing the following $SL(2, \mathbb{Z})$ change of variables on the bosonic variables:

$$
\Phi_1 = \varphi_\uparrow - \varphi_\downarrow, \quad \Theta_1 = \theta_\uparrow,
$$
$$
\Phi_2 = \varphi_\downarrow, \quad \Theta_2' = \theta_\uparrow + \theta_\downarrow, \tag{E3}
$$

and then defining the gauge-invariant $\Theta_2 = \Theta_2' + \Phi_1/2$, which is conjugate to Φ_2. After turning on the perturbation

$\mathcal{O}_{zz}$, we find the effective symmetry action

$$
T : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle \\ \Theta_2 \mapsto -\Theta_2 \end{cases} ,
$$

$$
\text{U(1)} : \Theta_2 \mapsto \Theta_2 + 2\alpha,
$$

$$
R_x : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle \\ \Theta_2 \mapsto \Theta_2 + \langle \Phi_1 \rangle \end{cases} ,
$$

$$
\text{Trans} : \begin{cases} \Phi_2 \mapsto \Phi_2 + k_R - k_L \\ \Theta_2 \mapsto \Theta_2 + k_R + k_L . \end{cases} \tag{E4}
$$

We see in the topological phase with $\langle \Phi_1 \rangle = \pi$, the effective U(1) $\rtimes T$ symmetry has an emergent anomaly that matches the edge of a 2d bosonic topological insulator [73] (cf. Appendix C 2 c). In this symmetry class, the topological phase has no symmetric relevant operators, and defines a stable gapless phase.

With just T symmetry, and $1/2 < K_{\text{eff}} < 2$ (see Appendix D) there are two symmetric relevant operators, $\cos \Theta_2$ and $\cos 2\Theta_2$. From the perspective of the low-energy theory, as we tune $\cos \Theta_2$, the gapless topological phase looks like a codimension 2 critical point between a trivial phase and the bosonic T SPT. However, since the microscopic symmetry has $T^2 = (-1)^F$, these phases are actually equivalent (and actually realize a different nontrivial SPT, see below). Thus, from this point of view, our topological phase is reminiscent of an "unnecessary" (multi)critical point in the language of [118]. See Figs. 7 and 8.

We note that the T symmetry class is enough to project the edge modes of the topological phase. The trick is to realize that if we include the gapped sector, then the full $T^2 = (-1)^F$ symmetry is in an SPT phase (such phases are classified by a $\mathbb{Z}_2$ invariant). This follows from our calculation in Appendix C 2 c as well as the form of the string operator from Sec. II, namely $S_m^z(\prod_{m<k<n} P_k) S_n^z$, since S^z is odd under T.

We can also define a $\mathbb{Z}_2$ symmetry Q by R_x times the U(1) rotation with $\alpha = \pi/2$. The effective U(1) $\times \mathbb{Z}_2^Q$ symmetry of the topological gapless phase has the emergent anomaly of Appendix C 2 b, where $Q : \Phi_2 \mapsto \Phi_2 + \pi$ and U(1) : $\Theta_2 \mapsto \Theta_2 + 2\alpha$. In the symmetry class, we see there no symmetric relevant operators, and therefore our topological gapless phase is stable.

As in the case of T symmetry, if we break the U(1) down to fermion parity, the edge mode of the topological phase is protected by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phase between Q and $(-1)^F$, the latter of which acts only in the gapped sector. This follows from our calculation in Appendix C 2 b as well as the form of the string operator $S_m^z(\prod_{m<k<n} P_k) S_n^z$, since S^z is Q-odd.

2. A field theory for bosons

This field theory can be easily adapted to describe intrinsically gapless phases of bosons as well. For symmetries, we take

$$
U : \begin{cases} \varphi_{1,2} \mapsto \varphi_{2,1} \\ \theta_{1,2} \mapsto \theta_{2,1} + \pi/2 \end{cases} , \quad T : \begin{cases} \varphi_{1,2} \mapsto \varphi_{2,1} \\ \theta_\uparrow \mapsto -\theta_\downarrow + \pi/2 , \\ \theta_\downarrow \mapsto -\theta_\uparrow - \pi/2 \end{cases}
$$

$$
\text{U(1)} : \begin{cases} \varphi_{1,2} \mapsto \varphi_{1,2} \\ \theta_{1,2} \mapsto \theta_{1,2} + \alpha . \end{cases} \tag{E5}
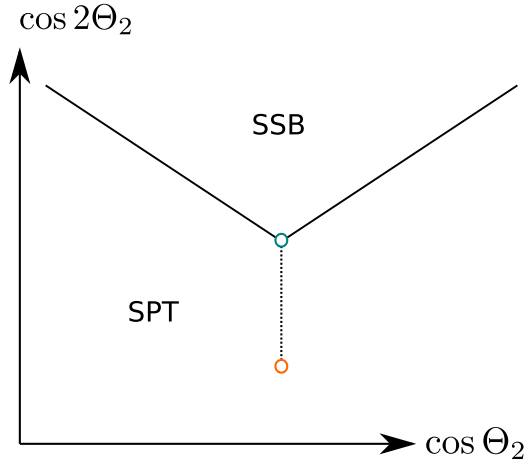$$

FIG. 7. The phase diagram near the topological gapless phase (blue circle) in the symmetry class $T^2 = (-1)^F$, for which there is no emergent anomaly, in a region of parameter space with effective marginal parameter $1/2 < K_{\text{eff}} < 2$. In the SSB phase, T is broken to fermion parity, while in the nondegenerate gapped phase, which is the SPT phase for T, there is a spurious first order line, explained that in the effective $T^2 = 1$ symmetry of the gapless phase (where fermions are gapped), the two nearby gapped phases look like distinct SPTs. We expect this first order line to end in an Ising critical point where the fermion gap closes. One way to account for the topological nontriviality of this critical point is to note that it has no nearby trivial phase. See Ref. [34] for a discussion of this philosophy and how it results in edge modes. The black lines are (SPT-twisted) Ising phase transitions controlling the spontaneous symmetry breaking of T, and are also expected to have edge modes [33]. The edge mode at the multicritical point where the two lines meet can be considered the continuation of the edge modes from either side. R_x acts by reflecting this phase diagram over the first order line, i.e., $\cos \Theta_2 \mapsto -\cos \Theta_2$, and is spontaneously broken along it. In the symmetry class $\mathbb{Z}_4^{R_x} \times \mathbb{Z}_2^{R_x T}$ generated by R_x and T therefore this critical point is like a deconfined quantum critical point between two different ordered phases with an emergent anomaly for $\mathbb{Z}_2^{R_x} \times \mathbb{Z}_2^{T}$.
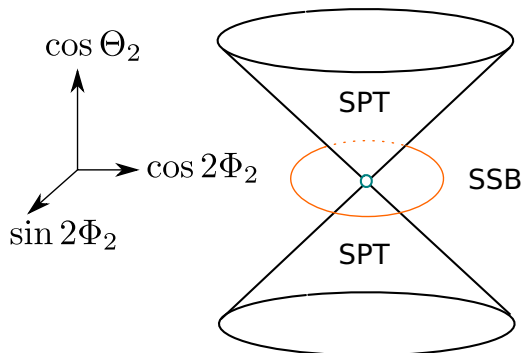

FIG. 8. The phase diagram near the topological gapless phase (blue circle) in the symmetry class $T^2 = (-1)^F$ again, but this time in a region of parameter space with effective marginal parameter $1/8 < K_{\text{eff}} < 1/2$. In the SSB phase, T is broken to fermion parity, and both SPT phases are the unique $\mathbb{Z}_4^T$ SPT phase. Along the orange circle, we have vacuum crossing as in Fig. 6, which is reproduced along the plane where we turn off the $\cos \Theta_2$ perturbation, which is odd under R_x.

Our perturbation into the topological phase is $\cos(\varphi_\uparrow - \varphi_\downarrow)$, which we analyze by passing to the variables

$$\begin{aligned} \Phi_1 &= \varphi_\uparrow - \varphi_\downarrow, & \Theta_1 &= \theta_\uparrow, \\ \Phi_2 &= \varphi_\downarrow, & \Theta_2 &= \theta_\uparrow + \theta_\downarrow. \end{aligned} \tag{E6}$$

The effective symmetry action is

$$T : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle, \\ \Theta_2 \mapsto -\Theta_2 \end{cases},$$

$$U(1) : \Theta_2 \mapsto \Theta_2 + 2\alpha,$$

$$U : \begin{cases} \Phi_2 \mapsto \Phi_2 + \langle \Phi_1 \rangle \\ \Theta_2 \mapsto \Theta_2 + \pi \end{cases}. \tag{E7}$$

We recognize the $\mathbb{Z}_2^U$ action as the Levin-Gu/CXZ anomaly. $U(1) \rtimes T$ matches the edge of a 2d bosonic topological insulator [73], and $\mathbb{Z}_2^U \times U(1)$ a kind of bosonic quantum spin Hall SPT.

APPENDIX F: NOTE ON BOSONIZATION

Bosonization takes many forms. Perhaps the most familiar is the Jordan-Wigner transformation, which abstractly describes a fermionic theory as a bosonic theory ("the bosonization") with a $\mathbb{Z}_2$ global symmetry. The fermionic theory is reconstructed by gauging this global symmetry in the presence of a modified Gauss law, such that we project out $\mathbb{Z}_2$ charged local operators, but keep $\mathbb{Z}_2$ charged string operators (these become the local fermions) and project out $\mathbb{Z}_2$ neutral strings. See for instance [119,120]. This fermionization transformation can also be thought of as gauging the $\mathbb{Z}_2$ symmetry in the presence of a special discrete torsion which couples the gauge field to the spin structure [109,110]. It can be confusing but it is useful to be clear about such issues when questions of global topological properties are important. An advantage of JW bosonization is that all fermion operators anticommute by virtue of being charged endpoints of the same string.

The bosonization used in Sec. III is a little bit different from the familiar Jordan-Wigner transformation. Instead, the fermionization transformation involves gauging a $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry. For the $c = 2$ theory, these symmetries are

$$U_B : \theta_\uparrow \mapsto \theta_\uparrow + \pi, \quad U_C : \theta_\downarrow \mapsto \theta_\downarrow + \pi. \tag{F1}$$

We project out all local operators which are charged under them, but we keep certain charged twist operators. From the three twisted sectors, corresponding to U_A, U_B, and $U_A U_B$, respectively, the operators we keep are those of the form

$$e^{i\varphi_\uparrow/2 + i\theta_\uparrow} \mathcal{O}, \quad e^{i\varphi_\downarrow/2 + i\theta_\downarrow} \mathcal{O}', \quad e^{i\varphi_\uparrow/2 + i\theta_\uparrow} e^{i\varphi_\downarrow/2 + i\theta_\downarrow} \mathcal{O}'', \tag{F2}$$

where $\mathcal{O}$ etc are gauge invariant local operators. The operators $e^{i\varphi_j/2 + i\theta_j}$ are the two fermions. This rule may be stated that an operator in the $U_A^m U_B^n$ twisted sector is local iff it has U_A charge $(-1)^m$ and U_B charge $(-1)^n$.

Fermion parity in this theory is the diagonal magnetic symmetry $\varphi_j \mapsto \varphi_j + 2\pi$, which is equivalent to $\theta_j \mapsto \theta_j + \pi$ by the selection rules. The selection rules mean that certain innocent-looking operators such as $\cos(\theta_\uparrow + \theta_\downarrow)$, which appear to be even under fermion parity, are actually not local because they do not obey the gauge constraints. The bosonic operator $\cos(\varphi_\uparrow/2 + \varphi_\downarrow/2) \cos(\theta_\uparrow + \theta_\downarrow)$ however, is local.

In the language of Refs. [109,121], we are gauging the $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry of the product of the two bosonized theories, in the presence of the discrete torsion $(-1)^{Q_\eta(B)+Q_\eta(C)}$, where B and C are $\mathbb{Z}_2$ gauge fields, η is the spin structure, and Q_η is the quadratic form associated to it.

APPENDIX G: DISCRETE GAUGE FIELD PRIMER

Here we collect some basic definitions and facts about discrete gauge fields. A good reference for the general topological notions is Ref. [122].

1. Discrete gauge fields

Let G be a finite group, possibly non-Abelian. To describe a G gauge field on a space X, we choose a triangulation of X. A flat G gauge field is a 1-cocycle $A \in Z^1(X, G)$ meaning to each oriented edge (ij) ("i to j") it assigns an element $A(ij) \in G$ such that $A(ji) = A(ij)^{-1}$ and for every triangle (ijk) with ordered vertices ("i to j to k"), it satisfies the cocycle condition

$$A(ij)A(jk) = A(ik), \tag{G1}$$

which says that the magnetic flux through this triangle vanishes. A gauge transformation is parametrized by a 0-cochain $g \in C^0(X, G)$ meaning that to each vertex i it assigns an element $g(i) \in G$. It acts on A by

$$A(ij) \mapsto A^g(ij) = g(i)^{-1}A(ij)g(j). \tag{G2}$$

Since we have written A multiplicatively, the Wilson loop along some oriented cycle $\gamma = (i_1 i_2) + (i_2 i_3) + \cdots + (i_n i_1)$ is

$$W_R(\gamma, A) = \text{Tr}\left(\prod_j \rho(A(i_j i_{j+1}))\right), \tag{G3}$$

where ρ is a representation of G.

We will be interested in studying the Euclidean partition function on an arbitrary spacetime X equipped with a flat G gauge field. It is so far unclear how to do this for a general lattice model, especially when X has interesting topology. In field theory, it is more-or-less clear however what one needs to do. That is, we think about the gauge field, defined on a triangulation, in terms of a Poincaré dual network of topological defects [121]. That is, in the cell structure dual to the triangulation of X, there are hypersurface associated to each edge (ij). We choose boundary conditions for the fields across these hypersurfaces such that when a field crosses the hypersurface along the direction $i \to j$, it satisfies the gluing condition $\phi_L \cdot A(ij) = \phi_R$, where $\phi_L \cdot A(ij)$ is the action of the global symmetry. The cocycle condition says that at a codimension 2 junction of hypersurfaces, the boundary condition is well-posed. Otherwise there would be a singularity in the fields and we would have a G flux. The triangulation considered above should be considered on a length scale where the continuum field theory reigns.

For more general "quantum symmetries," whose action cannot be seen on the fields, we associate a topological defect to each of these hypersurfaces, defined by the group element $A(ij)$. If the topological defect associated to $g \in G$ is placed along a spacelike slice of the theory, then in the partition function we apply the global g symmetry at that moment in time.

If the topological defect is placed along some hypersurface with one coordinate in the time direction and the rest along the spatial directions, then where the spatial slice intersects it we are considering a g-twisted sector of the Hilbert space.

Gauge transformations allow us to freely move and recombine this network of topological defects without changing the value of the partition function. In the case of an anomaly, however, the phase of the partition function will change under certain gauge transformations. In $1+1$D, the basic recombination of the network of topological line defects is the "F-move" or "crossing relation." The anomaly in this case is captured by the F symbols of the $2+1$D Dijkgraaf-Witten theory obtained by gauging the global symmetry of the associated bulk SPT. See Ref. [98] for more details.

2. Simplicial cohomology

We will need some basic constructions from simplicial cohomology. First, a k-simplex is determined by its $k+1$ vertices. If these vertices are ordered $v_0 < \cdots < v_k$ we write $(v_0 \cdots v_k)$ to denote the corresponding k-simplex with this vertex ordering. When we are talking about a single k-simplex, we will just write it with the shorthand $(0 \cdots k)$.

We define the group of k-chains $C_k(X, \mathbb{Z})$ on a triangulated space X to be integer combinations of k-simplices with ordered vertices, modulo reordering of the vertices: $(\sigma(0) \cdots \sigma(k)) = (-1)^\sigma (0 \cdots k)$.

Let M be an Abelian group. We define a k-cochain $\alpha \in C^k(X, M)$ to assign an element $\alpha(0 \cdots k) \in M$ to every k-simplex $(0 \cdots k)$ with ordered vertices such that $\alpha(\sigma(0) \cdots \sigma(k)) = (-1)^\sigma \alpha(0 \cdots k)$ for any permutation $\sigma \in S_{k+1}$ of the vertices with sign $(-1)^\sigma$. From now on we just refer to $(1 \cdots k)$ as a k-simplex.

For $\Gamma \in C_k(X, \mathbb{Z})$, $\alpha \in C^k(X, M)$ integral

$$\int_\Gamma \alpha \in M \tag{G4}$$

is defined to be the sum of α evaluated on these k-simplices, weighted by those integers. Thus by definition

$$\int_{(0 \cdots k)} \alpha = \alpha(0 \cdots k), \tag{G5}$$

so α is determined by its integrals. In this sense, α is like a discrete analog of a differential k-form.

Taking the boundary of a k-simplex with ordered vertices, and passing that ordering onto its boundary $k-1$-simplices produces a $k-1$ chain, and extending that function by linearity defines the boundary map $\partial : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z})$. We define the group of k-cycles as $Z_k(X, \mathbb{Z}) := \ker(d)$ and the group of k-boundaries as $B_k(X, \mathbb{Z}) := \text{im}(d)$. We have $\partial^2 = 0$, so $B_k(X, \mathbb{Z}) \subset Z_k(X, \mathbb{Z})$. The homology is defined by $H_k(X, \mathbb{Z}) := Z_k(X, \mathbb{Z})/B_k(X, \mathbb{Z})$.

We define the differential $d : C^k(X, M) \to C^{k+1}(X, M)$ by Stokes' theorem

$$\int_\Gamma d\alpha = \int_{\partial \Gamma} \alpha, \tag{G6}$$

which translates to

$$(d\alpha)(0\cdots k+1) = \sum_{j=0}^{k+1}(-1)^j\alpha(0\cdots\hat{j}\cdots k+1) \quad \text{(G7)}$$

$$= \alpha(123\cdots k+1) - \alpha(023\cdots k+1) + \alpha(013\cdots k+1)$$

$$- \cdots + (-1)^{k+1}\alpha(0\cdots k), \quad \text{(G8)}$$

where $\hat{j}$ means j is excluded from $(0\cdots j\cdots k+1)$. We define the group of k-cocycles as $Z^k(X,M) :=$ $\ker(d) \subset C^k(X,M)$ and the group of k-coboundaries as $B^k(X,M) := \mathrm{im}(d) \subset C^k(X,M)$. We find $d^2 = 0$ so $B^k(X,M) \subset Z^k(X,M)$. The cohomology group is defined by $H^k(X,M) := Z^k(X,M)/B^k(X,M)$.

We can connect this to two physical objects: gauge fields and partition functions.

Firstly, when G is Abelian, $A \in C^1(X,G)$, and the cocycle condition (G1) for a G gauge field can be rewritten

$$dA = 0, \quad \text{(G9)}$$

and a gauge transformation acts by

$$A \mapsto A^g = A + dg, \quad \text{(G10)}$$

with $g \in C^0(X,G)$ so gauge equivalence classes of flat G gauge fields on X are classified by $H^1(X,G)$. [This in fact also holds for non-Abelian G, although in that case this construction does not generalize to $H^k(X,G)$.]

Secondly, if X is a spacetime of dimension $k = d+1$, then $H^k(X,U(1))$ (where we identify $U(1) \cong \mathbb{R}/\mathbb{Z}$) corresponds to topological actions. More precisely, any $\omega \in H^k(X,U(1))$ defines a partition function $Z = e^{2\pi i \int_X \omega}$ for any closed manifold X. The fact that $d\omega = 0$ physically implies cobordism invariance of Z.

To connect these two, i.e., the gauge field A and the action ω, we turn to group cohomology.

3. Group cohomology and topological terms

There is a useful space called the classifying space, denoted BG, which has the property that a flat G gauge field on X is the same as a map $X \to BG$. A construction can be found in [122]. See also [104]. The group cohomology is defined to be the cohomology of this space: $H^k(BG,M)$. An element of $H^k(BG,M)$ can be represented by an M-valued group k-cocycle which is a function $\Omega : G^k \to M$ that satisfies the group k-cocyle equation

$$\Omega(g_2, g_3, g_4\ldots, g_{k+1}) - \Omega(g_1g_2, g_3, g_4, \ldots, g_{k+1})$$

$$+ \Omega(g_1, g_2g_3, g_4, \ldots, g_{k+1}) - \cdots$$

$$+ (-1)^{k+1}\Omega(g_1, \ldots, g_k) = 0. \quad \text{(G11)}$$

One can use such a k-cocycle to define a map (not necessarily a group homomorphism)

$$\omega : Z^1(X,G) \to Z^k(X,M), \quad \text{(G12)}$$

which satisfies the k-cocycle equation

$$d\omega(A) = 0 \quad \text{(G13)}$$

and the gauge-invariance equation

$$\omega(A^g) = \omega(A) + d\omega_1(A,g) \quad \text{(G14)}$$

for some $\omega_1(A,g) \in C^{k-1}(X,M)$. To do so, we define

$$\omega(0\cdots k) = \Omega(A(01), A(12), \ldots, A(k-1,k)). \quad \text{(G15)}$$

(In terms of A considered as a map $\mathcal{A} : X \to BG$, $\omega(A) = \mathcal{A}^*\Omega$, where $\mathcal{A}^*$ is the pullback [122].) Ω can be reconstructed from ω by evaluating ω on a k-simplex.

For $M = U(1) = \mathbb{R}/\mathbb{Z}$, $k = d+1$ the dimension of spacetime, such a cocycle defines a topological action for the G gauge field [89]:

$$Z_{\text{top}}(X,A) = e^{2\pi i \int_X \omega(A)}. \quad \text{(G16)}$$

We think of this partition function as arising from a d-space-dimensional SPT coupled to the background gauge field A. It turns out that in a certain sense gauge invariance (G14) is equivalent to the cocycle equation (G13) (see below), and moreover shifts $\omega(A) \mapsto \omega(A) + d\chi(A)$ do not change the partition function on closed spacetime. Thus the gauge invariant partition functions of this form on closed spacetimes are classified by $H^{d+1}(BG,U(1))$. If we assume that these partition functions capture all the topological features of the G symmetry of the underlying model, we reproduce the group cohomology classification of SPTs from Ref. [42]. See also Refs. [60,61] for generalizations.

The equivalence between gauge invariance (G14) and the cocycle equation (G13) is very important. It underlies our anomaly vanishing equation as well as the calculations of anomalies in Ref. [68]. A mathematical proof can be found in Ref. [105], which we reproduce here.

The proof relies on being able to study ω on different X, so that ω is not allowed to depend on precise details of X. Intuitively it should be a function of the gauge field only. The precise condition is that ω is a natural transformation of functors from $Z^1(-,G)$ to $Z^{d+1}(-,U(1))$.

To show that gauge invariance (G14) implies the cocycle equation (G13), we consider $X = \Delta^{d+2}$ a $d+2$ simplex with flat gauge field A and

$$\int_{\Delta^{d+2}} d\omega(A) = \int_{\partial\Delta^{d+2}} \omega(A). \quad \text{(G17)}$$

Because Δ^{d+2} is contractible, $A = 1^g$ for some g (i.e. it is a gauge transformation of the trivial gauge field $A = 1$, written in the multiplcative notation). Using (G14), we then complete the proof

$$\int_{\Delta^{d+2}} d\omega(A) = \int_{\partial\Delta^{d+2}} \omega(A)$$

$$= \int_{\partial\Delta^{d+2}} \omega(1) - d\omega_1(1,g) = 0. \quad \text{(G18)}$$

To show the other direction, for any $A \in Z^1(X,G)$, $g \in C^0(X,G)$, we can define a flat gauge field $\hat{A}$ on $X \times [0,1]$ such that $A|_{X\times 0} = A$ and $A|_{X\times 1} = A^g$. Using the cocycle equation, we have

$$0 = \int_{X\times[0,1]} d\omega(\hat{A}) = \int_{\partial(X\times[0,1])} \omega(\hat{A})$$

$$= \int_X \omega(A^g) - \int_X \omega(A) + \int_{\partial X\times[0,1]} \omega(\hat{A}|_{\partial X\times[0,1]}). \quad \text{(G19)}$$

The last term can be integrated over the interval to define

$$\omega_1(A, g) := -\int_{[0,1]} \omega(\hat{A}|_{\partial X \times [0,1]}). \tag{G20}$$

Gauge invariance (G14) follows.

This argument is very general, and can be easily adapted to *relative* cohomology, in the form it is used to prove the anomaly vanishing equation. That is, if we have a G_{low} gauge field A_{low} on X and a G gauge field A on ∂X, we are interested in invariants of the form

$$\int_X \omega(A_{\text{low}}) - \int_{\partial X} \alpha(A). \tag{G21}$$

We find that gauge invariance of such expressions is equivalent to $d\omega(A_{\text{low}}) = 0$ and the anomaly vanishing equation $\omega(A_{\text{low}}) = d\alpha(A)$.

4. Example: $\mathbb{Z}_n$

Let us discuss the special case $G = \mathbb{Z}_n$ and describe the classification of its 2+1D SPTs using the method of topological terms.

To do so we will need one more definition. Let R be a ring. We define the cup product $\cup : C^j(X, R) \times C^k(X, R) \to C^{j+k}(X, R)$ by

$$(\alpha \cup \beta)(0 \cdots j + k) = \alpha(0 \cdots j)\beta(j \cdots j + k). \tag{G22}$$

It satisfies

$$d(\alpha \cup \beta) = (d\alpha) \cup \beta + (-1)^k \alpha \cup (d\beta), \tag{G23}$$

and if $d\alpha = 0$ and $d\beta = 0$, then

$$\alpha \cup \beta - (-1)^{jk}\beta \cup \alpha = d(\cdots), \tag{G24}$$

where $\cdots$ define higher "cup-i products" of cochains.

It turns out that $H^3(B\mathbb{Z}_n, U(1)) = \mathbb{Z}_n$. The generator of this cyclic group defines, according to Eq. (G15), a map $\omega : H^1(X, \mathbb{Z}_n) \to H^3(X, U(1))$ given by

$$\omega(A) = \frac{1}{n^2} A \cup dA. \tag{G25}$$

Note that because the denominator is n^2 and not n, we need to lift A to an integer-valued cochain to define this quantity. We

can do this by expressing $A(ij) \in \mathbb{Z}_n$ as an integer $\tilde{A}(ij) \in [0, n)$, so that $\tilde{A} = A \mod n$. We see the cocycle equation for A implies $d\tilde{A} = n\beta$ for some integer valued 2-cocycle β. β can be thought of as the density of 2π fluxes. Thus the form above may be rewritten

$$\omega(A) = \frac{1}{n} A \cup \beta. \tag{G26}$$

We see that if we change the lift $\tilde{A} \mapsto \tilde{A} + n\alpha$, $\beta \mapsto \beta + d\alpha$ and so

$$\omega \mapsto \omega + \frac{1}{n} A \cup d\alpha. \tag{G27}$$

Integrating the second term by parts we find it is

$$\delta\omega = -\frac{1}{n} dA \cup \alpha + d(\cdots) \tag{G28}$$

which is an integer on a closed spacetime and does not contribute to the phase of the partition function.

We see

$$d\omega = \frac{1}{n^2} dA \cup dA = \beta \cup \beta \in \mathbb{Z}, \tag{G29}$$

which implies gauge invariance up to a boundary term. Indeed, if we shift $A \mapsto A + dg$, $\tilde{A} \mapsto \tilde{A} + \widetilde{dg}$, so

$$\delta\omega = \frac{1}{n}(dg \cup \beta + A \cup d\widetilde{dg} + dg \cup d\widetilde{dg})$$
$$= \frac{1}{n} d(g \cup \beta - A \cup \widetilde{dg} + g \cup d\widetilde{dg}). \tag{G30}$$

To get a handle on what this term means for the boundary anomaly, we can consider a single 3-simplex (0123) with $\tilde{A}(01) = 1$, $\tilde{A}(12) = n - 1$ $\tilde{A}(23) = 1$. We see $n\beta(123) = \tilde{A}(12) + \tilde{A}(23) - \tilde{A}(13) = n - 1 + 1 - 0 = n$, so

$$\omega(A)(0123) = \frac{1}{n}. \tag{G31}$$

Thus, this configuration is in some sense the minimal configuration which yields a nontrivial contribution from the topological term. It can be related to the nontrivial F symbol of 2+1D $\mathbb{Z}_n$ Dijkgraaf-Witten theory [89], and thus to the $\mathbb{Z}_n$ anomaly.

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