Magnonic thermal transport using the quantum Boltzmann equation

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We present a formula for thermal transport in the bulk of Bose systems based on the quantum Boltzmann equation (QBE). First, starting from the quantum kinetic equation and using the Born approximation for impurity scattering, we derive the QBE of Bose systems and provide a formula for thermal transport subjected to a temperature gradient. Next, we apply the formula to magnons. Assuming a relaxation time approximation and focusing on the linear response regime, we show that the longitudinal thermal conductivity of the QBE exhibits the different behavior from the conventional Boltzmann equation. The thermal conductivity of the QBE reduces to the Drude type in the limit of the quasiparticle approximation, while not in the absence of the approximation. Finally, applying the quasiparticle approximation to the QBE, we find that the correction to the conventional Boltzmann equation is integrated as the self-energy into the spectral function of the QBE, and this enhances the thermal conductivity. Thus, we shed light on the thermal transport property of the QBE beyond the conventional.

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I. INTRODUCTION

The last decade has seen a rapid development of magnon-based spintronics, dubbed magnonics, aiming at utilizing the quantized spin waves, magnons, as a carrier of information [1]. The main subject is the realization of efficient transmission of information using spins in insulating magnets. For this purpose, taking into account the fundamental difference of the quantum-statistical properties between electrons and magnons, i.e., fermions and bosons, respectively, many magnonic analogs of electron transport have been established both theoretically and experimentally [2], with a particular focus on thermal transport, e.g., the thermal Hall effect [3–6] and the Wiedemann-Franz law [7–10] for magnon transport.

The key ingredient in the study of thermal transport phenomena is the Boltzmann equation [11]. As well as electron transport, the conventional Boltzmann equation has been playing a central role in the study of magnon transport (e.g., see Refs. [12–21]); the temperature gradient $\boldsymbol{\nabla} T$ drives a magnonic system out of equilibrium. The conventional Boltzmann equation describes the property of the system at time t as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{\mathbf{k}} \cdot \boldsymbol{\nabla} T \frac{\partial}{\partial T}\right) f_{\mathbf{k},\mathbf{r},t} = I_{\mathbf{k},\mathbf{r},t}, \tag{1}$$

where $\mathbf{v}_{\mathbf{k}} := \partial \omega_{\mathbf{k}}/(\partial \mathbf{k})$ is the magnon velocity for the energy dispersion relation $\hbar\omega_{\mathbf{k}}$ in the wave-number space $\mathbf{k}$, $\hbar$ represents the Planck constant, $f_{\mathbf{k},\mathbf{r},t}$ is the nonequilibrium Bose distribution function of the absolute temperature T, and $I_{\mathbf{k},\mathbf{r},t}$ is the collision integral for a position $\mathbf{r}$. Within the relaxation time approximation, the longitudinal thermal conductivity of magnons in the bulk of magnets is proportional to the relaxation time [9,19,21]. Under some conditions, the relaxation time coincides with the lifetime of magnons, which is proportional to the inverse of the Gilbert damping constant α [22]. Thus the magnonic thermal conductivity of the conventional Boltzmann equation reduces to the Drude type [23] as a function of α in that it is proportional to $1/\alpha$.

From the viewpoint of quantum field theory, the conventional Boltzmann equation [Eq. (1)] is derived from the quantum kinetic equation [24] by taking several approximations [25–28]. Assuming that the variation of the center-of-mass coordinates is slow compared with that of the relative coordinates, Kadanoff and Baym [29] applied an approximation, called the gradient expansion, to the quantum kinetic equation for the nonequilibrium Green's function [30]. The quantum kinetic equation of the lowest-order gradient approximation becomes the quantum Boltzmann equation (QBE). The QBE describes the equation of motion for the lesser Green's function. The spectral function is assumed to be the Dirac delta function in the quasiparticle approximation [19,25–27,31]. In the limit of the quasiparticle approximation, the QBE reduces to the conventional Boltzmann equation [Eq. (1)] for the nonequilibrium distribution function of three variables $(\mathbf{k}, \mathbf{r}, t)$.

This hierarchical structure of quantum field theory indicates that by relaxing some of the approximations, quantum-mechanical corrections to the conventional Boltzmann equation can be evaluated [26]. Such sound development has been made successfully as to electrons [27]. The lifetime of electrons in metals subjected to a strong impurity potential becomes substantial, and the quasiparticle approximation is not applicable. To solve the issue, Prange and Kadanoff introduced an alternative approach [32], which was developed for the application to superconductors and superfluids [33–36], e.g., the Eilenberger equation [33]. However, those are for

Fermi systems. Since the approach is based on the assumption that [27] there is a Fermi surface and the Fermi energy, the developed formula is not applicable to Bose systems, e.g., magnons. Thus, to the best of our knowledge [37], as for magnons in the bulk of magnets, the thermal transport property beyond the conventional Boltzmann equation remains an open issue.

In this paper, we provide a solution to this fundamental challenge by starting from the quantum kinetic equation and developing the QBE for Bose systems. The purpose of any useful formalism is to provide a method for calculation of measurable quantities. First, using the QBE we develop a formula for thermal transport in the bulk of Bose systems, including the nonlinear response to the temperature gradient. Next, as a platform, we apply it to magnons. In the conventional spintronics study, the Landau-Lifshitz-Gilbert equation is playing the central role [38]. To develop a relation with it, using the Gilbert damping constant, we describe the spectral function of magnons and study the longitudinal thermal conductivity of the QBE. Finally, by applying the quasiparticle approximation to the thermal conductivity of the QBE, we find the correction to the conventional Boltzmann equation and discuss thermal transport properties beyond the conventional.

We remark that the conventional Boltzmann equation [12–21], i.e., the transport theory based on the quasiparticle approximation, cannot describe paramagnons in the bulk of paramagnets [25]. The quasiparticle approximation assumes that the spectrum has the form of the Dirac delta function. However, the spectrum of paramagnons is broad, in general, and has a peak with a sufficient width of a nonzero value associated with the inverse of the finite lifetime [39–41]; the spectrum cannot be approximated by the Dirac delta function. Therefore, the conventional Boltzmann equation cannot describe paramagnons. In this paper, we also shed light on this issue.

This paper is organized as follows. In Sec. II starting from the quantum kinetic equation of the lowest-order gradient approximation and using the Born approximation for impurity scattering, we derive the QBE for Bose systems. In Sec. III, first, using the QBE and assuming a steady state in terms of time, we provide a formula for thermal transport in the bulk of Bose systems subjected to a temperature gradient, including the nonlinear response. Next, we apply the formula to magnons in Sec. III A. To develop a relation with the conventional spintronics study, we describe the spectral function of magnons in terms of the Gilbert damping constant. Then, assuming a relaxation time approximation and focusing on the linear response regime, we evaluate the longitudinal thermal conductivity of magnons in the bulk of magnets based on the QBE. Finally, in Sec. III B, applying the quasiparticle approximation to the magnonic thermal conductivity of the QBE, we discuss the difference from that of the conventional Boltzmann equation. Comparing also with the linear response theory, we comment on our formula in Sec. IV. We remark on open issues in Sec. V and give some conclusions in Sec. VI. Technical details are deferred to Appendices A and B.

II. QUANTUM BOLTZMANN EQUATION FOR BOSE SYSTEM

We consider a Bose system where the center-of-mass coordinates, the position and time in center-of-mass $(\mathbf{r}, t)$, respectively, vary slowly compared to the relative coordinates. Up to the lowest order of the gradient expansion, the quantum kinetic equation [25–28] for the system reduces to

$$-i\left(\frac{\partial \mathscr{H}_{\mathbf{k},\omega}}{\partial t}\frac{\partial}{\partial \omega} - \frac{\partial \mathscr{H}_{\mathbf{k},\omega}}{\partial \omega}\frac{\partial}{\partial t} - \frac{\partial \mathscr{H}_{\mathbf{k},\omega}}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{k}} + \frac{\partial \mathscr{H}_{\mathbf{k},\omega}}{\partial \mathbf{k}}\frac{\partial}{\partial \mathbf{r}}\right)G^<_{\mathbf{k},\omega,\mathbf{r},t} = (G^<_{\mathbf{k},\omega,\mathbf{r},t}\Sigma^>_{\mathbf{k},\omega,\mathbf{r},t} - G^>_{\mathbf{k},\omega,\mathbf{r},t}\Sigma^<_{\mathbf{k},\omega,\mathbf{r},t}), \tag{2}$$

where $\mathscr{H}_{\mathbf{k},\omega} := \hbar\omega - \hbar\omega_{\mathbf{k}}$ for a frequency ω and the functions $G^{<(>)}_{\mathbf{k},\omega,\mathbf{r},t}$ and $\Sigma^{<(>)}_{\mathbf{k},\omega,\mathbf{r},t}$ are the lesser (greater) component of the bosonic nonequilibrium Green's function and that of the self-energy, respectively; the variables $(\mathbf{k}, \omega)$ arise from the Fourier transform of the relative coordinates. Following Ref. [25], we refer to Eq. (2) as the QBE. The QBE is the equation of motion for the lesser Green's function $G^<_{\mathbf{k},\omega,\mathbf{r},t}$ and consists of four variables $(\mathbf{k}, \omega, \mathbf{r}, t)$, while the conventional Boltzmann equation is for the nonequilibrium Bose distribution function $f_{\mathbf{k},\mathbf{r},t}$ and consists of three variables $(\mathbf{k}, \mathbf{r}, t)$. The QBE in the limit of the quasiparticle approximation reduces to the conventional Boltzmann equation. In this paper we study thermal transport of the QBE for Bose systems and find the properties beyond the conventional Boltzmann equation. Then, applying the quasiparticle approximation to the thermal conductivity of the QBE, we discuss the difference from the conventionalBoltzmann equation.

Within the Born approximation, the self-energy due to impurity scattering of the impurity potential $V_{\mathbf{k},\mathbf{k}'}$ is given as $\Sigma_{\mathbf{k},\omega,\mathbf{r},t} = \sum_{\mathbf{k}'} G_{\mathbf{k}',\omega,\mathbf{r},t}|V_{\mathbf{k},\mathbf{k}'}|^2$. Assuming that the function $\mathscr{H}_{\mathbf{k},\omega}$ is time independent and spatially uniform, the QBE becomes

$$(\partial_t + \mathbf{v}_{\mathbf{k}} \cdot \partial_{\mathbf{r}})G^<_{\mathbf{k},\omega,\mathbf{r},t} = \frac{1}{i\hbar}\sum_{\mathbf{k}'}|V_{\mathbf{k},\mathbf{k}'}|^2(G^<_{\mathbf{k},\omega,\mathbf{r},t}G^>_{\mathbf{k}',\omega,\mathbf{r},t} - G^>_{\mathbf{k},\omega,\mathbf{r},t}G^<_{\mathbf{k}',\omega,\mathbf{r},t}). \tag{3}$$

The QBE consists of the lesser (greater) Green's functions $G^{<(>)}_{\mathbf{k},\omega,\mathbf{r},t}$. The Kadanoff-Baym ansatz ensures that [25–27] the Green's functions are associated with the spectral function $\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}$ and the nonequilibrium distribution function $\phi_{\mathbf{k},\omega,\mathbf{r},t}$ as $G^<_{\mathbf{k},\omega,\mathbf{r},t} = -i\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\phi_{\mathbf{k},\omega,\mathbf{r},t}$ and $G^>_{\mathbf{k},\omega,\mathbf{r},t} = -i\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}(1 + \phi_{\mathbf{k},\omega,\mathbf{r},t})$ for bosons, while $G^<_{\mathbf{k},\omega,\mathbf{r},t} = i\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\phi_{\mathbf{k},\omega,\mathbf{r},t}$ and $G^>_{\mathbf{k},\omega,\mathbf{r},t} = -i\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}(1 - \phi_{\mathbf{k},\omega,\mathbf{r},t})$ for fermions. The nonequilibrium Bose distribution function in the wave-number space is given as $f^{\rm QBE}_{\mathbf{k},\mathbf{r},t} := \int[\hbar d\omega/(2\pi)]\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\phi_{\mathbf{k},\omega,\mathbf{r},t}$. Using the Kadanoff-Baym ansatz for bosons, finally, we obtain the QBE of the functions $\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}$ and $\phi_{\mathbf{k},\omega,\mathbf{r},t}$ as

$$(\partial_t + \mathbf{v}_\mathbf{k} \cdot \partial_\mathbf{r})(\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\phi_{\mathbf{k},\omega,\mathbf{r},t}) = -\frac{1}{\hbar}\sum_{\mathbf{k}'} |V_{\mathbf{k},\mathbf{k}'}|^2 \mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\mathcal{A}_{\mathbf{k}',\omega,\mathbf{r},t}(\phi_{\mathbf{k},\omega,\mathbf{r},t} - \phi_{\mathbf{k}',\omega,\mathbf{r},t}). \tag{4}$$

The QBE is useful to a wide range of Bose systems subjected to impurity scattering. For convenience, we define the collision integral as $\mathcal{I}_{\mathbf{k},\omega,\mathbf{r},t} := -\sum_{\mathbf{k}'} |V_{\mathbf{k},\mathbf{k}'}|^2 \mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}\mathcal{A}_{\mathbf{k}',\omega,\mathbf{r},t}(\phi_{\mathbf{k},\omega,\mathbf{r},t} - \phi_{\mathbf{k}',\omega,\mathbf{r},t})/\hbar$. Hereafter, for simplicity, we drop the indices $(\mathbf{r}, t)$ when those are not important.

III. THERMAL TRANSPORT IN THE BOSE SYSTEM

The temperature gradient drives the Bose system out of equilibrium and generates a heat current. The QBE [Eq. (4)] describes the transport property of a steady state in terms of time as

$$\mathbf{v}_\mathbf{k} \cdot \nabla T \frac{\partial}{\partial T}(\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega}) = \mathcal{I}_{\mathbf{k},\omega}, \tag{5}$$

where we assume that the temperature gradient is spatially uniform $\nabla T = (\text{const.})$. In this section, first, using the functions $\mathcal{A}_{\mathbf{k},\omega}$ and $\phi_{\mathbf{k},\omega}$ of the QBE [Eq. (5)], we provide a formula for the heat current in the bulk of two-dimensional Bose systems. Next, as a platform, in Sec. III A we apply the formula to magnons in two-dimensional insulating magnets. Assuming a relaxation time approximation for the function $\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega}$ and focusing on the linear response regime, we evaluate the thermal conductivity in the bulk of the magnet. Finally, in Sec. III B, applying the quasiparticle approximation to the magnonic thermal conductivity of the QBE, we discuss the difference from that of the conventional Boltzmann equation.

The applied temperature gradient drives the system out of equilibrium and the Bose distribution function $\phi_{\mathbf{k},\omega}$ deviates from the one, $\phi_0 = (e^{\beta\hbar\omega} - 1)^{-1}$, in equilibrium, where $\beta := 1/(k_\mathrm{B}T)$ is the inverse temperature and k_B represents the Boltzmann constant. The deviation is characterized as the function $\delta\phi_{\mathbf{k},\omega} := \phi_{\mathbf{k},\omega} - \phi_0$. Since the self-energy arises from impurity scattering, we assume that the spectral function $\mathcal{A}_{\mathbf{k},\omega}$ is little influenced by temperature and we neglect the temperature dependence. Therefore the nonequilibrium Bose distribution function in the wave-number space is given as $f_\mathbf{k}^{\mathrm{QBE}} = f_0^{\mathrm{QBE}} + \delta f_\mathbf{k}^{\mathrm{QBE}}$, with $f_0^{\mathrm{QBE}} := \int [\hbar d\omega/(2\pi)]\mathcal{A}_{\mathbf{k},\omega}\phi_0$ and $\delta f_\mathbf{k}^{\mathrm{QBE}} := \int [\hbar d\omega/(2\pi)]\mathcal{A}_{\mathbf{k},\omega}\delta\phi_{\mathbf{k},\omega}$. The function consists of two parts: the equilibrium component f_0^{QBE} and the nonequilibrium one $\delta f_\mathbf{k}^{\mathrm{QBE}}$. Since each mode ω subjected to a chemical potential μ carries the energy $\hbar\omega$, the heat current density in the bulk of two-dimensional Bose systems, $\mathbf{j}_Q = (j_{Q_x}, j_{Q_y})$, is given as

$$\mathbf{j}_Q = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\mathbf{v}_\mathbf{k} \int \frac{\hbar d\omega}{2\pi}(\hbar\omega - \mu)\mathcal{A}_{\mathbf{k},\omega}\delta\phi_{\mathbf{k},\omega}. \tag{6}$$

This is the formula for the heat current density of the QBE, including the nonlinear response to the temperature gradient. The formula [Eq. (6)] is useful to Bose systems (e.g., insulators and metals) with the spectral function of arbitrary shape.

A. Magnonic thermal conductivity

As a platform, we apply the formula for the heat current of the QBE [Eq. (6)] to magnons in a two-dimensional insulating magnet where time-reversal symmetry is broken, e.g., due to an external magnetic field. At sufficiently low temperatures, the effect of magnon-magnon interactions and that of phonons are negligibly small, and impurity scattering makes a major contribution to the self-energy. Therefore, we work under the assumption that the spectral function $\mathcal{A}_{\mathbf{k},\omega}$ is little influenced by temperature, and we neglect the temperature dependence.

First, we comment on the chemical potential of magnons subjected to the temperature gradient [7–9]. The applied temperature gradient induces magnon transport, which leads to an accumulation of magnons at the boundaries and builds up a nonuniform magnetization in the sample. This magnetization gradient plays the role of an effective magnetic field gradient and works as the gradient of a nonequilibrium spin chemical potential [15,42–44] for magnons. This generates a countercurrent of magnons and thus the nonequilibrium spin chemical potential contributes to the thermal conductivity. See Ref. [9] for details. In this paper, for simplicity, we consider a sufficiently large system and work under the assumption that the effect of the boundaries is negligibly small. Consequently, the nonequilibrium magnon accumulation becomes negligible and the nonequilibrium spin chemical potential of magnons vanishes. In the magnonic system, the heat current is identified with the energy current [5,8]. Note that [15,42–44] the spin chemical potential of magnons is peculiar to the system out of equilibrium [45].

Then, we consider thermal transport carried by magnons with the energy dispersion relation $\hbar\omega_\mathbf{k} = Dk^2 + \Delta$, where D represents the spin stiffness constant, $k := |\mathbf{k}|$ denotes the magnitude of the wave number, and Δ is the magnon energy gap, e.g., due to an external magnetic field and a spin anisotropy, etc. Assuming a relaxation time approximation for the function $\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega}$ of

$$\mathcal{I}_{\mathbf{k},\omega} = -\frac{\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega} - \mathcal{A}_{\mathbf{k},\omega}\phi_0}{\tau^{\mathrm{R}}_{\mathbf{k},\omega}} \tag{7}$$

and focusing on the linear response regime, from Eq. (5) the nonequilibrium component $\delta\phi_{\mathbf{k},\omega}$ is given as

$$\delta\phi_{\mathbf{k},\omega} = -\tau^{\mathrm{R}}_{\mathbf{k},\omega}\mathbf{v}_\mathbf{k} \cdot \nabla T \frac{\partial\phi_0}{\partial T} + O((\nabla T)^2), \tag{8}$$

where $\tau^{\mathrm{R}}_{\mathbf{k},\omega}$ is the relaxation time for the magnonic system. Under the assumption that impurity scattering is elastic and that the relaxation time depends solely on the magnitude of the wave number, it is evaluated as $1/\tau^{\mathrm{R}}_{\mathbf{k},\omega} = \sum_{\mathbf{k}'} |V_{\mathbf{k},\mathbf{k}'}|^2 \mathcal{A}_{\mathbf{k}',\omega}(1 - \mathbf{v}_\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}'}/|\mathbf{v}_\mathbf{k}|^2)/\hbar$. We remark that the relaxation time $\tau^{\mathrm{R}}_{\mathbf{k},\omega}$ is different from the magnon lifetime $\tau^{\mathrm{L}}_{\mathbf{k},\omega}$ in general. Those are distinct quantities. However, when the impurity potential is localized in space, the Fourier component $V_{\mathbf{k},\mathbf{k}'}$ becomes independent of the wave number, and the relaxation time coincides with the magnon lifetime, which takes a wave-number-independent value as $\tau^{\mathrm{R}}_\omega = \tau^{\mathrm{L}}_\omega$. From the Landau-Lifshitz-Gilbert equation [38], the magnon lifetime is associated with the inverse of the Gilbert damping constant α and it is described as $\hbar/(2\tau^{\rm L}_{\omega}) = \alpha\hbar\omega$ [21,46–50]. Therefore, under the assumption that the real part of the self-energy is negligibly small compared with the magnon energy gap, the spectral function is given as $\mathcal{A}_{\mathbf{k},\omega} = 2\alpha\hbar\omega/[(\mathscr{H}_{\mathbf{k},\omega})^2 + (\alpha\hbar\omega)^2]$. See Appendices A and B for details [22].

Finally, from Eqs. (6) and (8), we obtain the longitudinal thermal conductivity of magnons in the bulk of the two-dimensional insulating magnet, $\kappa_{xx} := -j_{Q_x}/(\partial_x T)$, as

$$\kappa_{xx} = \frac{1}{2}\left(\frac{D}{\pi}\right)^2 \frac{1}{k_{\rm B}T^2}\int_0^\infty dk k^3 \times \int d\omega \frac{(\hbar\omega)^2}{(\mathscr{H}_{\mathbf{k},\omega})^2 + (\alpha\hbar\omega)^2}\frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}, \tag{9}$$

where we assume that the temperature gradient is applied along the x axis. In contrast to the conventional Boltzmann equation (cf., Sec. I), the thermal conductivity of the QBE in the absence of the quasiparticle approximation does not reduce to the Drude type [23], as a function of the Gilbert damping constant α, in that it is not proportional to $1/\alpha$. The factor $1/\alpha$ arises from the relaxation time $\tau^{\rm R}_{\omega}$ in $\delta\phi_{\mathbf{k},\omega}$ [Eq. (8)] as $\tau^{\rm R}_{\omega} = \tau^{\rm L}_{\omega} = 1/(2\alpha\omega)$. However, it cancels out by the factor $2\alpha\hbar\omega$ of the spectral function $\mathcal{A}_{\mathbf{k},\omega} = 2\alpha\hbar\omega/[(\mathscr{H}_{\mathbf{k},\omega})^2 + (\alpha\hbar\omega)^2]$ [Eq. (6)]. Therefore the integrand remains the Lorentz type [51], and the thermal conductivity of the QBE does not reduce to the Drude type in the absence of the quasiparticle approximation.

B. Comparison: Conventional Boltzmann equation

The QBE in the limit of the quasiparticle approximation reduces to the conventional Boltzmann equation, which provides the thermal conductivity of the Drude type [23] in terms of the Gilbert damping constant α (cf., Sec. I). This agrees with our formula of the QBE [Eq. (6)]; when we employ the quasiparticle (qp) approximation $\mathcal{A}_{\mathbf{k},\omega} \approx \mathcal{A}^{\rm qp}_{\mathbf{k},\omega} := 2\pi\delta(\mathscr{H}_{\mathbf{k},\omega})$ for Eq. (6), the thermal conductivity of the QBE reduces to $\kappa_{xx} \approx \kappa^{\rm qp}_{xx}$ as

$$\kappa^{\rm qp}_{xx} = \frac{1}{2}\left(\frac{D}{\pi}\right)^2 \frac{1}{k_{\rm B}T^2}\frac{\pi}{\alpha}\int_0^\infty dk k^3 \omega_{\mathbf{k}}\frac{e^{\beta\hbar\omega_{\mathbf{k}}}}{(e^{\beta\hbar\omega_{\mathbf{k}}}-1)^2}. \tag{10}$$

This is consistent with Eq. (9) in the limit of $\alpha \to 0$. Thus, we find in the limit of the quasiparticle approximation that the thermal conductivity of the QBE becomes the Drude type, as a function of α, in that it is proportional to $1/\alpha$ as $\kappa_{xx} \approx \kappa^{\rm qp}_{xx} \propto 1/\alpha$. The factor $1/\alpha$ arises from the relaxation time $\tau^{\rm R}_{\omega} = \tau^{\rm L}_{\omega} = 1/(2\alpha\omega)$ in $\delta\phi_{\mathbf{k},\omega}$ [Eq. (8)].

In conclusion, using the QBE we have developed the formula for thermal transport in the bulk of Bose systems, with a particular focus on magnons. As a function of the Gilbert damping constant, the thermal conductivity of the QBE reduces to the Drude type [23] in the limit of the quasiparticle approximation, while not in the absence of the approximation. This is the main difference in the thermal conductivity between the QBE and the conventional Boltzmann equation.

We remark that our formula based on the QBE reduces to the Drude type [23] in the limit of the quasiparticle approximation. This means that, by relaxing the quasiparticle approximation, the correction to the conventional Boltzmann equation is integrated as the self-energy into the spectral function $\mathcal{A}_{\mathbf{k},\omega,\mathbf{r},t}$ of the QBE; Fig. 1 shows that the correction to the conventional Boltzmann equation enhances the thermal conductivity of the QBE. Thus our thermal transport theory of the QBE with four variables ($\mathbf{k}$, ω, $\mathbf{r}$, t) is identified as an appropriate extension of that of the conventional Boltzmann equation with three variables ($\mathbf{k}$, $\mathbf{r}$, t).

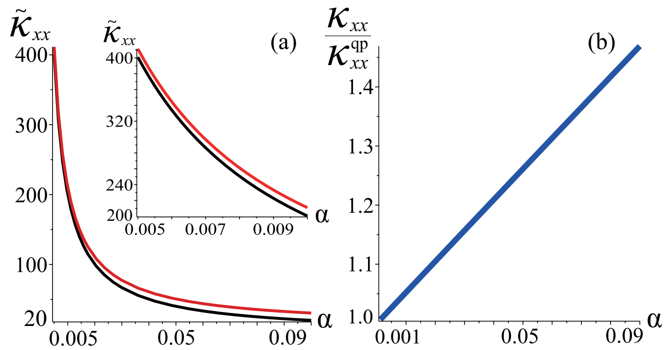

FIG. 1. (a) Plots of the rescaled thermal conductivity of magnons, $\tilde{\kappa}_{xx} := [2\pi^2\hbar/(k_{\rm B}^2 T)]\kappa_{xx}$, as a function of the Gilbert damping constant α [52,53] obtained by numerically solving Eqs. (9) and (10) for $\Delta = k_{\rm B}T$ with a fixed temperature. The red line denotes Eq. (9) of the QBE. The black line denotes Eq. (10) for $\kappa_{xx} \approx \kappa^{\rm qp}_{xx}$ in the limit of the quasiparticle approximation, i.e., the conventional Boltzmann equation. The correction to the conventional Boltzmann equation enhances the thermal conductivity. Inset: The enhancement still works and remains significant even for small values of the parameter α. (b) The plot of the ratio $\kappa_{xx}/\kappa^{\rm qp}_{xx} = 1 + (\kappa_{xx} - \kappa^{\rm qp}_{xx})/\kappa^{\rm qp}_{xx}$ as a function of the Gilbert damping constant α, i.e., the ratio of the magnonic thermal conductivity κ_{xx} of the QBE to the magnonic thermal conductivity $\kappa^{\rm qp}_{xx}$ of the conventional Boltzmann equation (i.e., the QBE in the limit of the quasiparticle approximation). In units of $\kappa^{\rm qp}_{xx}$, the correction to the conventional Boltzmann equation and the resulting enhancement of the magnonic thermal conductivity increase as the value of α becomes large.

The heat current density in the bulk of d-dimensional Bose systems ($d = 1$, 2, and 3) is given as $\mathbf{j}_Q = \int[d^d\mathbf{k}/(2\pi)^d]\mathbf{v}_{\mathbf{k}}\int[\hbar d\omega/(2\pi)](\hbar\omega-\mu)\mathcal{A}_{\mathbf{k},\omega}\delta\phi_{\mathbf{k},\omega}$. Using this equation we have numerically confirmed after a straightforward calculation that the behavior holds for d-dimensional magnonic systems in that the correction to the conventional Boltzmann equation enhances the thermal conductivity.

IV. DISCUSSION

To conclude, a few comments on our approach are in order. First, the conventional Boltzmann equation [12–21] [Eq. (1)], i.e., the transport theory based on the quasiparticle approximation, cannot describe paramagnons [25]. The quasiparticle approximation assumes that the spectrum has the form of the Dirac delta function. However, the spectrum of paramagnons is broad, in general, and has a peak with a sufficient width of a nonzero value associated with the inverse of the finite lifetime [39–41]; the spectrum of paramagnons cannot be approximated by the Dirac delta function. Therefore, the conventional Boltzmann equation cannot describe paramagnons. On the other hand, our formula based on the QBE is applicable to magnons with the spectrum of arbitrary shape [Eq. (6)]. In that sense, it is expected that our thermal transport theory is useful also to paramagnons in the bulk of paramagnets [54].

We remark that as well as insulators, our formula [Eq. (6)] is applicable, in principle, also to metals; the spectral function is described in terms of the Gilbert damping constant [22], and the value for metals (e.g., transition metal ferromagnets) [55], $\alpha = O(10^{-2})$, is large compared with that for insulators, $\alpha = O(10^{-3})$, in general [38,52,53]. Figure 1 shows the behavior of the magnonic thermal conductivity in the region $O(10^{-3}) \leqslant \alpha \leqslant O(10^{-2})$. As seen in Fig. 1(b), $\kappa_{xx}/\kappa_{xx}^{\rm qp} = 1 + (\kappa_{xx} - \kappa_{xx}^{\rm qp})/\kappa_{xx}^{\rm qp}$, in units of $\kappa_{xx}^{\rm qp}$ the correction to the conventional Boltzmann equation and the resulting enhancement of the magnonic thermal conductivity increase as the value of α becomes large. The damping constant is associated with the inverse of the magnon lifetime [22].

Next, our formula based on the QBE has an advantage over the linear response theory in that it includes the nonlinear response; Eq. (6) is the formula for the heat current density including the nonlinear response to the temperature gradient. Here, using the QBE we develop an analysis on the nonlinear response. We apply the relaxation time approximation [Eq. (7)] for the function $\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega}$ to the QBE [Eq. (5)]. Since the relaxation is induced by impurity scattering, we assume that the relaxation time is little influenced by temperature, and we neglect the temperature dependence. Using the method of successive substitution, the nonequilibrium component of the Bose distribution function $\delta\phi_{\mathbf{k},\omega}$ [Eq. (6)] is evaluated, beyond the liner response regime, as $\delta\phi_{\mathbf{k},\omega} = \sum_{n=1}^{\infty}(-\tau_{\mathbf{k},\omega}^{\rm R}\mathbf{v}_{\mathbf{k}}\cdot\nabla T)^n[\partial^n\phi_0/(\partial T^n)]$, where the nonequilibrium component $\delta\phi_{\mathbf{k},\omega}$ is arranged in terms of the function $(\nabla T)^n$. Last, combining this equation with Eq. (6), we can obtain each coefficient of the nonlinear response to the temperature gradient. The formula is not restricted to magnonic systems, and it is useful to a wide range of Bose systems.

Finally, we add a comment to the linear response theory. According to Ref. [49], which focuses on the region $\alpha \ll 1$, the linear response theory (i.e., Kubo formula) results in the magnonic thermal conductivity of the Drude type [23], the same as the conventional Boltzmann equation, in that it is proportional to the inverse of the Gilbert damping constant. Note that the temperature gradient is not mechanical force but thermodynamic force; the temperature gradient is not described by a microscopic Hamiltonian. Therefore, it is not straightforward to integrate the temperature gradient into the linear response theory. Reference [49] described the effect of the temperature gradient with the help of the thermal vector potential [56] associated with the Luttinger's approach [57], i.e., a fictitious scalar field called a gravitational potential. For the details, see Ref. [49]. We remark that the Boltzmann equation has no difficulty in integrating the temperature gradient into the formalism [e.g., see Eq. (1)].

Note that Ref. [49] explains that there is the problem associated with the Luttinger's gravitational potential formalism (i.e., the scalar potential formalism) also in magnonic systems. The problem that there are unphysical equilibrium contributions in the Luttinger's gravitation formalism is caused by the issue that the gravitational scalar potential couples to the total energy. To overcome this problem, Ref. [49] introduced the thermal vector potential by means of rewriting Luttinger's gravitational scalar potential and developed the formalism so that the thermal vector potential couples to the energy current. Thus, the problem is solved in that the unphysical equilibrium contributions are automatically canceled by diamagnetic currents associated with the vector potential. See Ref. [49] for details. The above issue is irrelevant to our work based on the Boltzmann equation.

V. OUTLOOK

We give a few perspectives on further research. As a platform, in Secs. III A and III B focusing on insulating magnets at sufficiently low temperature, we have studied thermal transport of magnons under the assumption that the effect of magnon-magnon interactions and that of phonons are negligibly small and that elastic impurity scattering makes a major contribution to the self-energy. It is of interest to study those effects on magnonic thermal transport of the QBE, with considering also the case that impurity scattering is inelastic [58]; while the details of the relaxation time vary, the form of the nonequilibrium component [Eq. (8)] remains unchanged and the formula for the heat current [Eq. (6)] still holds even if impurity scattering is inelastic. Therefore, we expect that the result qualitatively remains unchanged.

Through the interaction with impurities, magnons acquire the self-energy and thus carry more energy. The effect is integrated into the thermal conductivity through the spectral function via the self-energy, while it is neglected in the quasiparticle approximation. Therefore the thermal conductivity of the QBE takes a larger value than the one in the limit of the quasiparticle approximation. We intuitively understand the result as explained above. Still, to further develop the qualitative understanding will be of significance, particularly in terms of the quantum interference effect intrinsic to magnons (i.e., spin waves).

In this paper relaxing the quasiparticle approximation and using the QBE for Bose systems, we have found that the correction to the conventional Boltzmann equation enhances the thermal conductivity. Therefore, based on the QBE, it is intriguing to study the effect of the correction on the magnonic Wiedemann-Franz law [7–10]; at low temperatures magnon transport obeys a magnonic analog of the Wiedemann-Franz law [59], a universal law, in that the ratio of heat to spin conductivity is linear in temperature and does not depend on material parameters except the g factor. Note that the magnonic Wiedemann-Franz law for the bulk of magnets has been proposed in Ref. [9] based on the conventional Boltzmann equation. For the magnonic Wiedemann-Franz law of the QBE, the spin conductivity and the off-diagonal elements of the Onsager coefficient [9] remain to be obtained. We believe it can be evaluated by following the study [25] on the electrical conductivity of the QBE and developing it into the Bose system. Using the QBE, it will be intriguing also to study the magnonic Hall coefficients in topologically nontrivial magnonic systems [4–6,8–10]. We leave the advanced studies for future work.

VI. CONCLUSION

Developing the quantum Boltzmann equation, we have provided the formula for thermal transport in the bulk of Bose systems, including the nonlinear response to the temperature gradient. We have then applied the formula to magnons and have shown that thermal transport of the quantum Boltzmann equation exhibits behavior different from that of the conventional Boltzmann equation. The longitudinal thermal conductivity of the quantum Boltzmann equation reduces to the Drude type in the limit of the quasiparticle approximation, while not in the absence of the approximation. Relaxing the quasiparticle approximation, we have found that the correction to the conventional Boltzmann equation is integrated as the self-energy into the spectral function of the quantum Boltzmann equation, and this enhances the thermal conductivity. Our formula is useful to Bose systems, including metals as well as insulators, with the spectral function of arbitrary shape. Using the quantum Boltzmann equation we have found the thermal transport property beyond the conventional.

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APPENDIX A: RELAXATION TIME AND MAGNON LIFETIME

In this Appendix, we show that the relaxation time coincides with the magnon lifetime and takes a wave-number-independent value under the assumptions that the relaxation time depends solely on the magnitude of the wave number, the impurity scattering is elastic, and the impurity potential is localized in space. We remark that at sufficiently low temperatures, the effect of magnon-magnon interactions and that of phonons are negligibly small, and impurity scattering makes a major contribution to the self-energy. Therefore, we assume that the spectral function is little influenced by temperature and neglect the temperature dependence.

We consider magnons with the energy dispersion relation of $\hbar\omega_{\mathbf{k}} = Dk^2 + \Delta$, where $|\mathbf{k}| =: k$ denotes the magnitude of the wave number. First, assuming the steady state in terms of time and applying the relaxation time approximation for the function $\mathcal{A}_{\mathbf{k},\omega}\phi_{\mathbf{k},\omega}$ to the QBE (see the main text), within the linear response regime we obtain the nonequilibrium component $\delta\phi_{\mathbf{k},\omega}$ as $\delta\phi_{\mathbf{k},\omega} = -\tau^{\mathrm{R}}_{\mathbf{k},\omega}\mathbf{v}_{\mathbf{k}}\cdot\nabla T[\partial\phi_0/(\partial T)] + O((\nabla T)^2)$. Next, using the relaxation time approximation for the collision integral $\mathcal{I}_{\mathbf{k},\omega} = -\sum_{\mathbf{k}'}|V_{\mathbf{k},\mathbf{k}'}|^2\mathcal{A}_{\mathbf{k},\omega}\mathcal{A}_{\mathbf{k}',\omega}(\phi_{\mathbf{k},\omega} - \phi_{\mathbf{k}',\omega})/\hbar$, we reach $\delta\phi_{\mathbf{k},\omega}/\tau^{\mathrm{R}}_{\mathbf{k},\omega} = \sum_{\mathbf{k}'}|V_{\mathbf{k},\mathbf{k}'}|^2\mathcal{A}_{\mathbf{k}',\omega}(\delta\phi_{\mathbf{k},\omega} - \delta\phi_{\mathbf{k}',\omega})/\hbar$. Finally, combining the equations under the assumption that the relaxation time depends solely on the magnitude of the wave number and that the impurity scattering is elastic, we obtain the relaxation time as $1/\tau^{\mathrm{R}}_{\mathbf{k},\omega} = \sum_{\mathbf{k}'}|V_{\mathbf{k},\mathbf{k}'}|^2\mathcal{A}_{\mathbf{k}',\omega}(1 - \mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{\mathbf{k}'}/|\mathbf{v}_{\mathbf{k}}|^2)/\hbar$.

Since we assume that impurities are dilute, the effect can be taken into account within the Born approximation (see the main text for details). When the impurity potential is localized in space, the Fourier component becomes independent of the wave number and it is described as $|V_{\mathbf{k},\mathbf{k}'}|^2 =: u^2 n_{\mathrm{imp}}$, where n_{imp} is the impurity concentration [26]. Then, the self-energy becomes independent of the wave number $\mathbf{k}$, and it is given as $\Sigma_{\mathbf{k},\omega} = \Sigma_\omega := u^2 n_{\mathrm{imp}}\sum_{\mathbf{k}'}G_{\mathbf{k}',\omega}$. Therefore, the spectral function $\mathcal{A}_{\mathbf{k},\omega}$ depends solely on the magnitude of the wave number $|\mathbf{k}| =: k$ and it is denoted as $\mathcal{A}_{\mathbf{k},\omega} = \mathcal{A}_{k,\omega}$. We remark that the spectral function $\mathcal{A}_{\mathbf{k},\omega}$ consists of the self-energy $\Sigma_{\mathbf{k},\omega} = \Sigma_\omega$ and the function $\mathscr{H}_{\mathbf{k},\omega} = \mathscr{H}_{k,\omega}$; since we assume that the energy dispersion relation of magnons takes the form of $\hbar\omega_{\mathbf{k}} = Dk^2 + \Delta$, the function $\mathscr{H}_{\mathbf{k},\omega} := \hbar\omega - \hbar\omega_{\mathbf{k}}$ depends only on the magnitude of the wave number and it is represented as $\mathscr{H}_{\mathbf{k},\omega} = \mathscr{H}_{k,\omega}$. Thus, the spectral function becomes dependent only on the magnitude of the wave number as $\mathcal{A}_{\mathbf{k},\omega} = \mathcal{A}_{k,\omega}$. Using this result with the relation $\int_0^{2\pi} d\theta\cos\theta = 0$, finally, we obtain the relaxation time as $1/\tau^{\mathrm{R}}_{\mathbf{k},\omega} = 1/\tau^{\mathrm{R}}_{\omega} := u^2 n_{\mathrm{imp}}\sum_{\mathbf{k}'}\mathcal{A}_{\mathbf{k}',\omega}/\hbar$, and we find that it is independent of the wave number $\mathbf{k}$.

The relaxation time coincides with the lifetime for the impurity potential of $|V_{\mathbf{k},\mathbf{k}'}|^2 = u^2 n_{\mathrm{imp}}$. The lifetime $\tau^{\mathrm{L}}_{\mathbf{k},\omega}$ is associated with the imaginary part of the self-energy and it is described as $\hbar/(2\tau^{\mathrm{L}}_{\mathbf{k},\omega}) := -\mathrm{Im}\Sigma^{\mathrm{r}}_{\mathbf{k},\omega}$ in general, where $\Sigma^{\mathrm{r}}_{\mathbf{k},\omega}$ represents the retarded component of the self-energy and it is given as $\Sigma^{\mathrm{r}}_{\mathbf{k}\omega} = \Sigma^{\mathrm{r}}_{\omega} := u^2 n_{\mathrm{imp}}\sum_{\mathbf{k}'}G^{\mathrm{r}}_{\mathbf{k}',\omega}$ for the impurity potential of $|V_{\mathbf{k},\mathbf{k}'}|^2 = u^2 n_{\mathrm{imp}}$. Since the imaginary part of the retarded Green's function is associated with the spectral function as $\mathrm{Im}G^{\mathrm{r}}_{\mathbf{k}',\omega} = -\mathcal{A}_{\mathbf{k}',\omega}/2$, the lifetime becomes $1/\tau^{\mathrm{L}}_{\mathbf{k},\omega} = 1/\tau^{\mathrm{L}}_{\omega} := u^2 n_{\mathrm{imp}}\sum_{\mathbf{k}'}\mathcal{A}_{\mathbf{k}',\omega}/\hbar$ and takes the wave-number-independent value. Thus the lifetime coincides with the relaxation time $1/\tau^{\mathrm{L}}_{\omega} = 1/\tau^{\mathrm{R}}_{\omega}$.

We stress that the relaxation time is different from the lifetime in general. Those are distinct quantities. However, under the assumptions that the relaxation time depends solely on the magnitude of the wave number, the impurity scattering is elastic, and the impurity potential is localized in space, the relaxation time coincides with the lifetime and takes the wave-number-independent value.

APPENDIX B: MAGNON SPECTRAL FUNCTION AND GILBERT DAMPING CONSTANT

In this Appendix, we describe the spectral function of magnons in terms of the Gilbert damping constant α. The Landau-Lifshitz-Gilbert equation is playing the central role in the conventional spintronics study [38]. Therefore to develop a relation with it, it is useful to describe the spectrum as a function of the Gilbert damping constant. First, as seen above, the spectral function $\mathcal{A}_{\mathbf{k},\omega} = \mathcal{A}_{k,\omega}$ consists of the function $\mathscr{H}_{\mathbf{k},\omega} = \mathscr{H}_{k,\omega}$ and the self-energy $\Sigma_{\mathbf{k},\omega} = \Sigma_\omega$, and it is described as $\mathcal{A}_{k,\omega} = -2\mathrm{Im}\Sigma^{\mathrm{r}}_{\omega}/[(\mathscr{H}_{k,\omega})^2 + (\mathrm{Im}\Sigma^{\mathrm{r}}_{\omega})^2]$ [25–27], where we assume that the real part of the self-energy is negligibly small compared with the magnon energy gap.

Next, the lifetime is defined as the imaginary part of the self-energy, in general, as $\hbar/(2\tau^{\mathrm{L}}_{\omega}) := -\mathrm{Im}\Sigma^{\mathrm{r}}_{\omega}$. Since the lifetime of magnons is associated with the inverse of the Gilbert damping constant as [21,46–50] $\hbar/(2\tau^{\mathrm{L}}_{\omega}) = \alpha\hbar\omega$, the imaginary part of the self-energy is characterized in terms of the Gilbert damping constant. Finally, the spectral function is described as a function of the Gilbert damping constant as $\mathcal{A}_{k,\omega} = 2\alpha\hbar\omega/[(\mathscr{H}_{k,\omega})^2 + (\alpha\hbar\omega)^2]$.

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