Chiral kinetic theory of anomalous transport induced by torsion

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In Weyl semimetals subjected to torsion, there are two different kinds of chirality: (i) The (coordinate-space) shape of the twisted crystal is chiral and (ii) the momentum space contains chiral quasiparticles. Here we construct a general kinetic theory of anomalous transport using the *phase space* (coordinate and momentum spaces combined) Berry curvature induced by torsion in Weyl systems. We describe how torsion generates the chiral chemical potential and thus leads to the chiral magnetic effect (CME) in the presence of a background magnetic field. We propose to measure the CME current induced by the torsion as a way to detect the anomalous coupling between the coordinate-space and momentum-space chiralities.

I. INTRODUCTION

The defining property of Dirac and Weyl semimetals is the existence of gapless chiral quasiparticles. The spectral flow induced by external parallel electric and magnetic fields leads to the imbalance between the chemical potentials for right- and left-handed chiral quasiparticles and thus to nonconservation of chiral charge known as the chiral anomaly [1,2]. The chiral anomaly induces a number of phenomena in chiral materials, including the chiral magnetic effect (CME) [3] that manifests itself through the negative longitudinal magnetoresistance [4,5] in Dirac materials such as $ZrTe_5$ [6] and Na_3Bi [7] and Weyl materials such as TaAs [8]. Weyl materials also exhibit chirality-dependent optical effects [9–12], some of which are driven by the chiral anomaly [13–15].

It is interesting to consider the effects of crystal deformation on anomalous transport, since deformations are known to lead to strong "synthetic" gauge fields [16–18]; for example, in graphene nanobubbles, the synthetic magnetic field in excess of 300 T has been reported [19]. The geometrical torsional response of Weyl fermions has been addressed in Refs. [20–29], including its relations to the Neih-Yan anomaly [30] and chiral torsional effect [31].

A natural question to ask is whether these synthetic gauge fields can be used to source the chiral anomaly and thus drive the CME and other anomalous phenomena. At first glance, it may appear that deformations should be irrelevant for the CME—indeed, the CME is captured [32] by the topological Chern-Simons term $\sim \mu_5 \int \epsilon_{ijk} A^i F^{jk}$ in the effective action (where $\mu_5 = \mu_R - \mu_L$ is the chiral chemical potential and A^i and F^{jk} are the gauge potential and field strength tensor, respectively) that does not depend on the space-time metric $g_{\mu\nu}$. Therefore, if one describes the deformation as a change in an effective metric, the chiral anomaly and thus the CME should seemingly not be affected.

However, this conclusion is premature since a time-dependent, inhomogeneous deformation can induce a change in the momentum space distribution of chiral quasiparticles, e.g., by deviating the chiral chemical potential μ_5 away from its equilibrium value $\mu_5 = 0$ [33,34]. In this case, the CME current will be induced by the deformation in the presence of a background magnetic field [33,34]. The collective excitations mixing sound and chirality have been considered in Refs. [35–37].

Kinetic theory provides a convenient framework for describing transport phenomena. For chiral fermions, the effect of chiral anomaly has been incorporated [38,39] in this theory via the Berry curvature; see Ref. [40] for a review of earlier work on Berry curvature effects on transport. In the resulting chiral kinetic theory, Weyl cones are described as
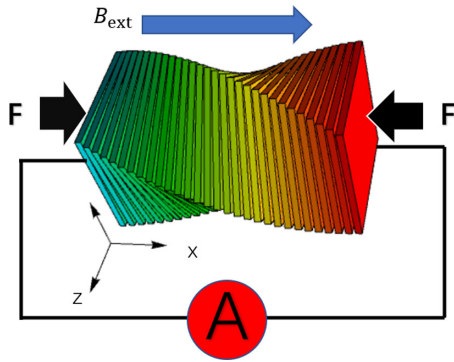
FIG. 1. An illustration of how the chiral imbalance and anomalous transport can be manipulated by deformations in an external magnetic field. The semimetal is twisted and compressed by the force F; this generates an electric current that is measured by an ammeter A.

momentum-space monopoles of Berry curvature. In the presence of an external magnetic field, the combined effect of the momentum-space monopoles and magnetic field is the modification of the density matrix in the phase space, resulting in the anomalous Hall effect and the CME [38,39,41–48].

In this paper, our goal is to construct a chiral kinetic theory for Weyl materials under a mechanical twist, i.e., Saint-Venant torsion. We focus on a twisted crystal under nonuniform strain, similar to the case studied in Ref. [60]. We will first show that the effect of dynamical, time-dependent deformations on chiral fermions can be captured by Berry curvature in phase space (coordinate and momentum spaces combined). This quantity has been introduced and used before in a variety of problems involving chirality and spatially inhomogeneous backgrounds [49–52]. In our problem, Berry curvature in phase space emerges because the spatially inhomogeneous, time-dependent deformations change the distribution of the fermions both in coordinate and momentum spaces. We then derive the generalized chiral kinetic equations (10) describing the effect of deformations on transport of chiral fermions, both with and without external electromagnetic fields. The corresponding anomaly equation (12) has, as a source of chirality, an exterior derivative of the phase-space Berry curvature. Since it measures the charge of the Berry monopole in phase space, we call it the "monopole charge function."

The physical meaning of the anomaly equation is as follows: As shown in Fig. 1, the angular twist creates a synthetic magnetic field B_{eff}, the time-dependent strain creates a synthetic electric field E_{eff}, and they combine to yield a source $E_{\mathrm{eff}}^i B_{\mathrm{eff}}^i$ for the chiral charge generation in the nth cone. Although the anomalous current induced by strain alone cancels out after summation over all Weyl cone pairs, the strain does provide a total chiral imbalance μ_5—therefore, in the presence of an external magnetic field, the chiral magnetic current appears. We then use the generalized chiral kinetic theory to evaluate the magnitude of the CME current induced by torsion in the presence of magnetic field. The current has a linear dependence both on the chiral chemical potential and the magnetic field, similar to the usual CME. It is observable and provides a way to discover the generation of chiral imbalance through synthetic

gauge fields arising from the phase-space Berry curvature. Throughout the paper, we use as an example a concrete tight-binding Hamiltonian (1); however, all of our derivations apply to any Hamiltonian of type $H = \vec{\sigma} \cdot \vec{p}(k, x, t) + \phi(k, x, t)$ without an explicit cutoff.

Our work focuses only on elastic deformations without the defects, e.g., screw dislocations. Screw dislocation is a topological defect in the crystal structure and hence also in the synthetic gauge field, which acquires a δ function analogous to the Abrikosov vortex in a type II superconductor. The elastic torsion we consider deforms the lattice, and therefore the Hamiltonian, continuously, whereas the defects induce singularities and thus potential sensitivity to the UV cutoff [53].

Let us consider an angular twisted Weyl semimetal with time-dependent compression along the axis of twist. The twist is chiral; the compression and twist together generate chirality and produce a chiral imbalance for the Weyl quasiparticles, as we will demonstrate. In order to solve this problem within the framework of chiral kinetic theory, we need to do the following. First, we need to understand how the elastic deformation in position space affects the Berry curvature, in both position and momentum spaces. Second, we need to formulate and solve the kinetic equation, and calculate the anomalous current and chiral charge generation.

For the first problem, because the Hamiltonian in momentum space now depends on the spatial coordinates, it is impossible to separate the position space and momentum space components of Berry curvature. Therefore, we will address this problem from the perspective of the phase-space path integral; our starting point is a phase-space Lagrangian, which we will obtain from the coordinate-dependent Hamiltonian. To solve the second problem, we start from the definition of the Berry connection in phase space and derive the corresponding kinetic equation.

II. HAMILTONIAN IN DEFORMED WEYL SEMIMETAL

Since much work on Hamiltonian under deformation is derived from a tight-binding model [54–57], we will use a simple model Hamiltonian of an anisotropic Weyl semimetal proposed in Ref. [58] as an example. This model has a simple tetragonal lattice with lattice constants a and b. The tight-binding Hamiltonian of the model is given by

$$
\begin{aligned}
H = & -\sum_i (c_{i-1x'}^+ c_i + c_{i+1x'}^+ c_i) t_1 \sigma_x \\
& + 2\cos(k_0 a) \sum_i c_i^+ c_i \, t_1 \sigma_x \\
& + i \sum_i (c_{i+1\perp'}^+ c_i - c_{i-1\perp'}^+ c_i) t_2 \sigma_\perp \\
& + \sum_i (c_{i+1\perp'}^+ c_i + c_{i-1\perp'}^+ c_i) t_3,
\end{aligned} \tag{1}
$$

where a is the lattice spacing in the x direction, and c^+ and c are creation and annihilation operators. The parameters t_1, t_2, and t_3 represent the strength of tight-binding interaction, and k_0 is the location of Weyl points in momentum space. The $\perp$ denotes the directions perpendicular to the specific x axis, so

it corresponds to the indices in y-z plane, which can be taken as $\hat{y}$ or $\hat{z}$. For example, $c_{i+1\perp'}\sigma_\perp$ means $c_{i+1y'}\sigma_y + c_{i+1z'}\sigma_z$.

This model possesses a fourfold rotation symmetry around the x axis, and inversion symmetry but not the time-reversal symmetry. The deformation can be described by the following parameter change:

$$(c^+_{i-1x'}c_i + c^+_{i+1x'}c_i)t_1\sigma_x$$
$$\rightarrow c^+_{i-1x'}c_i + c^+_{i+1x'}c_i)(t_1\sigma_x - \beta_1 u_{xx}\sigma_x) + i(c^+_{i-1x'}c_i$$
$$- c^+_{i+1x'}c_i)\beta_2 u_{x\perp}\sigma_\perp, \quad (2)$$

$$t_2\sigma_\perp \rightarrow t_2\sigma_\perp - \beta_3 u_{\perp\perp}\sigma_\perp, \quad (3)$$

where β is the anisotropic Gruneisen parameter and u_{ab} is the strain tensor. We set the twist to be along the x axis, $u_{x\perp} = \gamma\varepsilon_{ij}r_j$; γ describes the twist angle gradient; and ε_{ij} is the rank-two antisymmetric tensor in y-z plane. We also apply a time-dependent compression $\lambda = -u_{xx}$.

With these substitutions, the strain-modified Hamiltonian in momentum space becomes

$$H = -2t_1[\cos(k'_x a) - \cos(k_{0x}a)]\sigma_x$$
$$- 2t_2 \sin(k'_y b)\sigma_y - 2t_2 \sin(k'_z b)\sigma_z$$
$$- 2\beta_2\gamma y \sin(k'_x a)\sigma_z + 2\beta_2\gamma z \sin(k'_x a)\sigma_y$$
$$+ 2\beta_1 \cos(k'_x a)\lambda\sigma_x + 2[\cos(k'_y b) + \cos(k'_z b)]t_3, \quad (4)$$

where the primed symbols denote the local lattice vector direction. The transformation of direction from global vierbein to the local one is given by

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 1-\lambda(t) & 0 & 0 \\ 0 & 1 & -\gamma x' \\ 0 & \gamma x' & 1 \end{pmatrix} \begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix}; \quad (5)$$

we will denote the transformation matrix as $M^j_i = \frac{dx^j}{dx'^i}$.

Expanding the effective Hamiltonian in the vicinity of the Weyl point at momentum K_i, we get a familiar form,

$$H_{\text{eff}} = e'^i_a\sigma_a(k'_i - K'_i) + W'^i(k'_i - K'_i) + \mathcal{E}$$
$$= e^i_a\sigma_a(k_i - K_i) + W^i(k_i - K_i) + \mathcal{E}, \quad (6)$$

where the local Weyl points K', dreibein e', tilt vector W', and energy of Weyl points $\mathcal{E}$ can be derived from Eq. (4). The corresponding dreibein and Weyl points in global coordinates, which are defined to be consistent with the untwisted lattice, are given by $K_i = (M^{-1})^j_i K'_j$, $e^i_a = e'^j_a M^i_j$, and $W^i = W'^j M^i_j$.

In the next section, we will prove that the elastic chiral anomaly is independent of e^i_a and W^i, so there is no need to give an explicit form of these quantities here. All we need is the elastic gauge field that can be read off the form of the effective Hamiltonian. There are eight Weyl cones. In the lowest order,

$$K_x = s_x\left[k_{0x} + \frac{\beta_1}{t_1 a}\cot(k_{0x}a)\lambda\right],$$

$$K_y = s_y\frac{\beta_2\gamma z'}{t_2 b}s_x \sin(k_{0x}a) + \Theta(-s_y)\frac{\pi}{b}, \quad (7)$$

$$K_z = -s_z\frac{\beta_2\gamma y'}{t_2 b}s_x \sin(k_{0x}a) + \Theta(-s_z)\frac{\pi}{b},$$

where $s_x = \text{sgn}(\sin(K_x a))$ and $s_{y,z} = \text{sgn}(\cos(K_{y,z}a))$, E and the chirality χ is $s_x s_y s_z$. The velocity of fermions in the x direction is $v^x = 2at_1\cos(k_{0x}a)$ and in the y, z direction is $v^{y,z} = 2t_2 b$.

III. KINETIC EQUATIONS FOR THE HAMILTONIAN $\sigma \cdot p(k, x, t) + \phi(k, x, t)$.

In the above analysis, we find that the elastic deformation affects the dreibein, tilt vector, and position of Weyl points, but the Hamiltonian always maintains the form $\sigma \cdot p(k, x, t) + \phi(k, x, t)$. This Hamiltonian can be analyzed by chiral kinetic theory from a path integral perspective [38], where the action is given by

$$I = \int_{t_i}^{t_f} [k \cdot \dot{x} - \epsilon(p) - a_k \cdot \dot{k} - a_x \cdot \dot{x} - a_t]dt; \quad (8)$$

$\epsilon = \pm|p| + \phi$ is the eigenvalue of energy.

After generalizing the Berry connection and curvature in momentum space [38], the Berry connection and Berry curvature in phase space can be defined as $a_\alpha = i\langle u|\frac{\partial p_i}{\partial\alpha}\partial_p|u\rangle = \frac{\partial p_i}{\partial\alpha}a^p_i$, $\Omega_{\alpha\beta} = \partial_\beta a_\alpha - \partial_\alpha a_\beta = \frac{1}{2}\partial_\alpha p_m \partial_\beta p_l \epsilon_{mln}\Omega^p_n$, where $\Omega^p_n = \frac{\hat{p}_n}{2|p|^2}$. Below, we use the Greek characters as the indices in phase space.

For an arbitrary Hamiltonian, the kinetic equations corresponding to (8) are derived in [59]; see Eqs. (C3), (C8), and (C9) in that paper. For our Hamiltonian, the Berry curvatures satisfy the following identity:

$$\Omega_{\alpha\beta}\Omega_{\gamma\sigma} + \Omega_{\alpha\gamma}\Omega_{\sigma\beta} + \Omega_{\alpha\sigma}\Omega_{\gamma\beta} = 0. \quad (9)$$

Using this identity, the kinetic equations can be brought to the form

$$\sqrt{G}\dot{x}^i = -\Omega_{k_i t} + (\delta^i_j(1 + \Omega_{k_i x^l}) - \Omega_{k_i x^j})\partial_{k_j}\epsilon + \Omega_{k_i k_j}\partial_{x^j}\epsilon, \quad (10)$$

$$\sqrt{G}\dot{k}_i = \Omega_{x^i t} - (\delta^j_i(1 + \Omega_{k_i x^k}) - \Omega_{k_i x^j})\partial_{x^j}\epsilon - \Omega_{x^i x^j}\partial_{k_j}\epsilon, \quad (11)$$

where $\sqrt{G} = 1 + \Omega_{k_i x^i}$ describes the modification of the phase space density.

The corresponding anomaly equation can be written as

$$\frac{\partial\sqrt{G}}{\partial t} + \frac{\partial\sqrt{G}\dot{x}^i}{\partial x^i} + \frac{\partial\sqrt{G}\dot{k}_i}{\partial k_i}$$
$$= \Theta_{k_i x^i t} + \Theta_{k_j x^j x^i}\partial_{k_i}\epsilon + \Theta_{x^j k_j k_i}\partial_{x^i}\epsilon, \quad (12)$$

where

$$\Theta_{\alpha\beta\gamma} = \partial_\alpha\Omega_{\beta\gamma} + \partial_\beta\Omega_{\gamma\alpha} + \partial_\gamma\Omega_{\alpha\beta} \quad (13)$$

is the exterior derivative of the Berry curvature; we will call this quantity the monopole charge function, as it measures the charge of the Berry monopole in phase space.

It is easy to prove that if the Berry connections are all continuous analytical functions, Eq. (12) becomes a classical Liouville equation, because $\Theta_{\alpha\beta\gamma} = 0$. However, the Berry curvature possesses a singularity at the point $p = 0$, where two degenerate bands cross. This is the source for the chiral anomaly.

The chiral anomaly is linked to the topology of fields in the system; to investigate the gravitational topology and gauge topology, we need to understand the source of e_a^i and K_i.

As we showed in the first section, to form the dreibein e_a^i we need the following: (1) the elastic deformation of the lattice that changes the dispersion relation and (2) the frame transformation matrix M_i^j induced by the geometric structure. The gauge-analogy effect is affected only by the existence of the Weyl points K_i, and is independent of the dreibein $(e)_a^i$ and the tilt W^i that describe the shape of the cones. The gravitational "Nieh-yan" anomaly effect is determined by nontrivial structure (for example, dislocation) and the momentum cutoff. The structure is absent in our continuous deformation model. So, the final result for the anomaly is cutoff independent. In other words, the chiral anomaly in our model just contains the gauge-analogy anomaly which depends on the existence of Weyl cones (K_i) but not on their detailed shapes, as we will now demonstrate.

IV. CONSISTENCY CHECK OF KINETIC EQUATIONS

Let us first check that our equations are consistent with Ref. [38] in the absence of deformations. In this case, $p_i = k_i - A_i$; we will use the Coulomb gauge. From (13), we get for the monopole charge function

$$\Theta_{\alpha\beta\gamma} = \varepsilon_{mnl} \frac{\partial p_m}{\partial \alpha} \frac{\partial p_n}{\partial \beta} \frac{\partial p_l}{\partial \gamma} \Theta_p, \quad (14)$$

where $\Theta_p = 2\pi\delta^3(p)$ is the well-known Berry monopole in momentum space. Therefore,

$$\Theta_{k_i x^i t} = 2\pi(\vec{E} \cdot \vec{B})\delta^3(p), \quad (15)$$

$$\Theta_{k_j x^j x^i} \partial_{k_i}\epsilon + \Theta_{x^j k_j k^i} \partial_{x^i}\epsilon = 0. \quad (16)$$

We thus obtain from (12) the same expression for the "Liouville anomaly" as in Ref. [38]:

$$\frac{\partial\sqrt{G}}{\partial t} + \frac{\partial\sqrt{G}\dot{x}_i}{\partial x^i} + \frac{\partial\sqrt{G}\dot{k}_i}{\partial k_i} = 2\pi(\vec{E} \cdot \vec{B})\delta^3(p). \quad (17)$$

Let us now check the consistency of kinetic equation (10). For the Weyl Hamiltonian $H = \sigma_i k_i$, we get

$$\Omega_{k_i t} = -\vec{E} \times \frac{\hat{p}}{2|p|^2}, \qquad \Omega_{k_i x^i} \partial_{k_i}\epsilon = (\vec{B} \cdot \vec{\Omega})\hat{p}, \quad (18)$$

$$-\Omega_{k_i x^j} \partial_{k_j}\epsilon + \Omega_{k_i k_j} \partial_{x^j}\epsilon = (\hat{p} \cdot \vec{\Omega})\vec{B} - (\vec{B} \cdot \vec{\Omega})\hat{p}. \quad (19)$$

The kinetic equation is thus given by

$$\sqrt{G}\dot{x} = \hat{p} + \vec{E} \times \frac{\hat{p}}{2|p|^2} + (\hat{p} \cdot \vec{\Omega})\vec{B}, \quad (20)$$

which is also same with the equation derived by Stephanov and Yin [38].

V. DERIVATION OF TOPOLOGICALLY PROTECTED CME AND CHIRAL ANOMALY FOR A GENERAL DEFORMED WEYL SEMIMETAL

Let us now apply the kinetic equations (10)–(12) to the Weyl semimetal under torsion. We first use the anomaly equation (12) to identify the quantities responsible for the

topological configuration of the system. In our case, by substituting $p_a = e_a^i(x)(k_i - A_i - K_i)$, and $\phi = W^i(x)(k_i - A_i - K_i) + \mathcal{E}(x)$ into (12), we get

$$\Theta_{k_i x^i t} = e_a^i e_b^j \frac{\partial(A_j + K_j)}{\partial x_i} e_c^l \frac{\partial(A_l + K_l)}{\partial t} \varepsilon_{abc} 2\pi\delta^3(p)$$

$$- e_a^i \frac{\partial e_b^j}{\partial x^i} T_j e_c^l \frac{\partial(A_l + K_l)}{\partial t} \varepsilon_{abc} 2\pi\delta^3(p) \quad (21)$$

$$= 2\pi \det(e) \left(\frac{d(\vec{A} + \vec{K})}{dt} \cdot [\nabla \times (\vec{A} + \vec{K})] \right) \delta^3(p),$$

where $T_i = e_{ij}^{-1} p_j = k_i - A_i - K_i$. The second line above vanishes by $p_i \delta^3(p) = 0$.

The monopole charge functions Θ only depend on p, so (21) is tilt independent. One can further prove that the last two terms in (12) are also the same:

$$\Theta_{k_j x^j x^i} \partial_{k_i}\epsilon + \Theta_{x^i k_j k_i} \partial_{x^i}\epsilon$$

$$= 2\pi \det(e)(\nabla\mathcal{E} \cdot [\nabla \times (\vec{A} + \vec{K})])\delta^3(p). \quad (22)$$

Therefore, the anomaly equation for an isolated Weyl cone is

$$\partial_\mu j^\mu = \int \frac{d^3 k}{(2\pi)^3} \left(\frac{\partial\sqrt{G}}{\partial t} + \frac{\partial\sqrt{G}\dot{x}^i}{\partial x^i} + \frac{\partial\sqrt{G}\dot{k}_i}{\partial k_i} \right) f$$

$$= \int \text{sgn}(J) \frac{J^{-1} d^3 p}{(2\pi)^3} (\Theta_{k_i x^i t} + \Theta_{k_j x^j x^i} \partial_{k_i}\epsilon + \Theta_{x^i k_j k_i} \partial_{x^i}\epsilon) f$$

$$= \frac{\text{sgn}(J)}{4\pi^2} \det(e)^{-1} \det(e) \vec{B}_{\text{eff}} \cdot \vec{E}_{\text{eff}}$$

$$= \frac{\chi \vec{B}_{\text{eff}} \cdot \vec{E}_{\text{eff}}}{4\pi^2}, \quad (23)$$

where the effective electric and magnetic fields are

$$\vec{E}_{\text{eff}} = \frac{d(\vec{A} + \vec{K})}{dt} + \nabla\mathcal{E}, \quad \vec{B}_{\text{eff}} = \nabla \times (\vec{A} + \vec{K}), \quad (24)$$

and $J = \det(e)$ is the Jacobian; its sign corresponds to the chirality of the Weyl cone.

It is clear from (23) that the anomaly equation is independent of dreibein and tilt vector—it depends only on the existence of the Weyl point, due to its topological nature.

Although the anomaly is not affected by dreibein and tilt vector, the general expression for the current still depends on them:

$$j^i = \int \frac{d^3 k}{(2\pi)^3} \sqrt{G}\dot{x}^i f = \int \text{sgn}(J)J^{-1} \frac{d^3 p}{(2\pi)^3} \sqrt{G}\dot{x}^i f$$

$$= \int \text{sgn}(J)J^{-1}[-\Omega_{k_i t} + (\delta_j^i(1 + \Omega_{k_l x^l}) - \Omega_{k_i x^j})\partial_{k_j}\epsilon$$

$$+ \Omega_{k_i k_j} \partial_{x^j}\epsilon] f \frac{d^3 p}{(2\pi)^3}. \quad (25)$$

The first term $\Omega_{k_i t}$ describes a transverse current analogous to the anomalous Hall effect. The second term $\partial_{k_j}\epsilon$ corresponds to the anomalous velocity induced by lattice motion. To study the CME, we focus on the last three terms:

$$\Omega_{k_j x^j} \partial_{k_i}\epsilon = \varepsilon_{abc} \left(e_a^j \Delta_{bj} e_d^i \frac{p_c p_d}{2|p|^4} + e_a^j \Delta_{bj} \frac{W_i p_c}{2|p|^3} \right), \quad (26)$$

$$\Omega_{k_j x^j} \partial_{k_j} \epsilon = \varepsilon_{abc} \left(e_a^i \Delta_{bj} e_d^j \frac{p_c p_d}{2|p|^4} + e_a^i \Delta_{bj} \frac{W_j p_c}{2|p|^3} \right), \quad (27)$$

$$\Omega_{k_i k_j} \partial_{x^j} \epsilon = \varepsilon_{abc} \left(e_a^i e_b^j \Delta_{dj} \frac{p_c p_d}{2|p|^4} + e_a^i e_b^j \frac{\Gamma_j p_c}{2|p|^3} \right), \quad (28)$$

where $\Delta_{ai} = \frac{\partial p_a}{\partial x^i} = \omega_{ai}^j T_j - e_a^j \frac{\partial A_j}{\partial x^j}$. $\Gamma_i = \frac{\partial W^j}{\partial x^i} T_j - W^j \frac{\partial A_j}{\partial x^i} + \frac{\partial \mathcal{E}}{\partial x^i}$, and we have defined $\omega_{aj}^i = \frac{\partial e_a^i}{\partial x^j}$.

Using the symmetry relation $\epsilon_{abc} p_d - \epsilon_{dbc} p_a - \epsilon_{adc} p_b - \epsilon_{abd} p_c = 0$, one can simplify these equations, and get the following expression for the anomalous CME current:

$$
\begin{aligned}
j^i &= \int \text{sgn}(J) J^{-1} \frac{d^3 p}{(2\pi)^3} \\
&\quad \times [\Omega_{k_j x^j} \partial_{k_i} \epsilon - \Omega_{k_i x^j} \partial_{k_j} \epsilon + \Omega_{k_i k_j} \partial_{x^j} \epsilon] f \\
&= \int \text{sgn}(J) J^{-1} \frac{d^3 p}{(2\pi)^3} \left[e_c^i e_b^j \omega_{na}^j \frac{T_a \epsilon_{bcn}}{2|p|^2} + \det(e) \frac{B_{\text{eff}}^i}{2|p|^2} \right. \\
&\quad + \varepsilon_{abc} \left(e_a^k \omega_{bk}^j T_j W^i - e_a^i \omega_{bk}^j T_j W^k + e_a^i e_b^j \Gamma_j \right) \frac{p_c}{2|p|^3} \\
&\quad \left. + \frac{1}{2} \varepsilon_{abc} \varepsilon_{jlm} e_a^m e_b^l W^j \frac{p_c}{2|p|^3} B_{\text{eff}}^i \right] f. \quad (29)
\end{aligned}
$$

It is easy to derive from Eq. (4) that $W^i = [0, -2t_3 s_x \frac{\beta_2}{t_2} \gamma z \sin(k_0 a), 2t_3 s_x \frac{\beta_2}{t_2} \gamma y \sin(k_0 a)]$ and $\gamma y, \gamma z \ll 1$, so we will ignore it in our calculations and treat the cones as untilted. In such a cone, in a equilibrium distribution, $f(|p|)$ can be assumed to be an even function of p_a. So, most terms in Eq. (29) should vanish except the second term. It is easy to show that the second term is also independent of dreibein. Therefore, the CME current is topologically protected too. The contribution of each cone is

$$
\begin{aligned}
j_n^i &= \int \text{sgn}(J) J^{-1} \frac{d^3 p}{(2\pi)^3} \det(e) \frac{B_{\text{eff}}^i}{2|p|^2} f \\
&= \chi_n \frac{(\mu - \mathcal{E}_n) B_{\text{eff}}^i}{4\pi^2},
\end{aligned}
\quad (30)
$$

where $\chi_n, \mathcal{E}_n$ are the chirality and energy of the nth Weyl point. In this expression, as explained in Ref. [44], both μ and $\mathcal{E}_n$ should be counted from the bottom of the filled band.

The evolution of the chiral charge can be determined in the relaxation time approximation:

$$\partial_i j_n^i + \dot{\rho}_n = \chi_n \frac{e^3 \vec{E} \cdot \vec{B}}{4\pi^2} - \frac{e}{\tau} [\rho_n - (\rho_n)_{\text{eq}}], \quad (31)$$

where $(\rho_n)_{\text{eq}}$ denotes the equilibrium chiral charge density in position space and τ is the chirality relaxation time. So the chemical potential μ_n associated with the chiral charge is [3,60]

$$
\begin{aligned}
\delta \mu_n &= \left(\frac{(\mu - \mathcal{E}_n)^2}{2\pi^2 v^x v^y v^z} \right)^{-1} \delta \rho_n \\
&= \chi \frac{16\pi^2 ab^2 \cos(k_{x0} a) t_1 t_2^2}{(\mu - \mathcal{E}_n)^2} \frac{\vec{E}_{\text{eff}} \cdot \vec{B}_{\text{eff}}}{4\pi^2} \tau,
\end{aligned}
\quad (32)
$$

where μ is the chemical potential. The contribution of each cone to the chiral magnetic current is given by

$$\vec{j}_n = \chi_n \frac{e^2}{4\pi^2} \delta \mu_n \vec{B}; \quad (33)$$

therefore, the total CME current is determined by the chiral imbalance [3]. The general form of the anomalous CME current in a deformed Weyl semimetal is given by (29).

The current in the model with torsion

Following the general results derived above, we will now focus on the topology effect by elastic torsion—effective gauge field induced on the Weyl points by torsion. Since the effective electric field transforms as a 2-form under arbitrary diffeomorphisms, the effective electric and magnetic fields (24) are

$$\vec{E}_{\text{eff}} = s_x \frac{\beta_1}{t_1 a} \cot(k_{0x} a) \dot{\lambda} \hat{x}; \quad (34)$$

$$\vec{B}_{\text{eff}} = -s_x (s_y + s_z) \frac{\beta_2 \gamma}{t_2 b} \sin(k_{0x} a) \hat{x}. \quad (35)$$

The chiral anomaly at each cone is

$$\vec{E}_{\text{eff}} \cdot \vec{B}_{\text{eff}} = -(s_y + s_z) \frac{\beta_1 \beta_2 \cos(ak_{0x}) \gamma \dot{\lambda}}{abt_1 t_2}. \quad (36)$$

The total charge is conserved, as expected:

$$
\begin{aligned}
\sum_n (\partial_i j^i + \dot{\rho})_n &= \sum_n \frac{\chi_n}{4\pi^2} (E_{\text{eff}} \cdot B_{\text{eff}})_n \\
&= \sum_{s_{x,y,z}=\pm 1} -s_y s_z (s_y + s_z) \frac{\beta_1 \beta_2 \cos(ak_{0x}) \gamma \dot{\lambda}}{abt_1 t_2} \\
&= 0.
\end{aligned}
\quad (37)
$$

The energies of the Weyl points are

$$\mathcal{E}_n = t_3 (s_y + s_z);$$

therefore, combining Eqs. (30) and (32) we get

$$
\begin{aligned}
j_{\text{total}}^i &= \sum_n (j^i)_n = \sum_n \frac{\chi_n \delta \mu_n e^2 B_{\text{total}}^i}{4\pi^2} \\
&= -4e^2 \frac{\beta_1 \beta_2 bt_2 \cos(ak_{0x}) \gamma \dot{\lambda}}{\pi^2 \hbar^2} \\
&\quad \times \left(\frac{1}{(\mu - 4t_3)^2} - \frac{1}{(\mu + 4t_3)^2} \right) \tau B_{\text{ext}}^i. \quad (38)
\end{aligned}
$$

This shows that the current is linear in the external magnetic field [3], which is different from the CME induced in parallel (weak) electric and magnetic fields, where $j \propto B^2$ [4,5].

For a numerical estimate, let us assume a statically twisted and dynamically compressed (along the axis of twist) crystal with a square cross section 2×2 mm, and take parameters $t_1 = 2$ eV, $t_2 = 0.5$ eV, $t_3 = 0.05$ eV, $\beta_1 = 1.5$ eV, $\beta_2 = 1$ eV, $a = 0.3$ nm, $b = 0.4$ nm, $\tau = 10^{-10}$ s, $\mu = 0.15$ eV, $ak_{0x} = \pi/3$. Let us also assume an external magnetic field of 10 T, with a twist parameter of $\gamma = 10$ m^{-1} and compression rate $\dot{\lambda} = 10$ s^{-1}. The

anomalous current density would be ≈ 470 $\mu A/m^2$, and the total current would be about 2 nA. Note that the current has an inverse square dependence on $\mu - \mathcal{E}_n$; it will be maximized in materials with a Fermi surface very close to a Weyl point. For example, for our parameters, if we take $\mu - 4t_3$ to be 0.01 eV instead of 0.05 eV, our current will increase by a factor of 25 to approximately 50 nA.

VI. SUMMARY AND OUTLOOK

Our generalized chiral kinetic equations (10) and (11) and the anomaly equation (23) apply to any Weyl system with the Hamiltonian of the form $\sigma \cdot p(k, x, t) + \phi(k, x, t)$. In the anomaly equation (23), torsion creates a synthetic magnetic field, while the time-dependent compressive strain creates a synthetic electric field—so no external electric field is necessary to generate the chiral chemical potential. Detecting the resulting torsion-induced chiral magnetic current would thus allow us to establish the anomalous coupling between the spatial and momentum-space chiralities, without a background from the Ohmic current that exists in the longitudinal magnetoresistance measurements.

The current measured in the corresponding experiment will also not get contributions from the piezoelectric effect (because the material is inversion symmetric and has no piezoelectricity) or from eddy currents (because the compression is along the magnetic field).

It has been suggested that lattice defects can act as sources of emergent gravity: a disclination induces a curvature and a dislocation induces torsion [61]. It would be interesting to extend our method to computing the anomalous currents resulting from the combination of chiral and gauge-gravitational anomalies in this case.

ACKNOWLEDGMENTS

APPENDIX A: THE DETAILED DERIVATION OF EQS. (10)–(12)

From the Euler-Lagrange equation for the momenta, we get

$$\frac{\partial I}{\partial k_i} - \frac{d}{dt}\frac{\partial I}{\partial \dot{k}_i} = 0, \tag{A1}$$

and we derive

$$\dot{x}_i = \partial_{k_i}\epsilon - \Omega_{k_i x_j}\dot{x}_j - \Omega_{k_i k_j}\dot{k}_j. \tag{A2}$$

The Euler-Lagrange equation for the coordinates yields

$$\frac{\partial I}{\partial x_i} - \frac{d}{dt}\frac{\partial I}{\partial \dot{x}_i} = 0, \tag{A3}$$

from which we derive

$$\dot{k}_i = -\partial_{x_i}\epsilon + \Omega_{x_i k_j}\dot{k}_j + \Omega_{x_i x_j}\dot{x}_j. \tag{A4}$$

Since these two equations are coupled, the solution can be found using a method based on differential geometry. This work has already been done in Ref. [59]. Those results are

$$\sqrt{G} = 1 + \Omega_{k_i x_i} - \Omega_{x_i x_j}\Omega_{k_i k_j} + [(\Omega_{k_i x_i})^2 - \Omega_{k_i x_j}\Omega_{k_j x_i}]/2$$
$$- \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_k k_l}\Omega_{k_i x_j}\Omega_{x_m x_n}/4$$
$$+ \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_j x_i}\Omega_{k_m k_k}\Omega_{k_n x_l}/6, \tag{A5}$$

$$\sqrt{G}\dot{x} = (\delta_{ij}(1 + \Omega_{k_k x_k}) - \Omega_{k_i x_j} - \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m k_n}\Omega_{x_k x_l}/4$$
$$+ \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m x_k}\Omega_{k_n x_l}/2)(\partial_{k_j}\epsilon - \Omega_{k_j t})$$
$$+ (\Omega_{k_i k_j} + \epsilon_{kls}\epsilon_{ijm}\Omega_{k_k x_m}\Omega_{k_l k_s}/2)(\partial_{x_j}\epsilon - \Omega_{x_j t})$$
$$\tag{A6}$$

$$\sqrt{G}\dot{k} = \delta_{ij}(1 + \Omega_{k_k x_k}) - \Omega_{k_j x_i} - \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m k_n}\Omega_{x_k x_l}/4$$
$$+ \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m x_k}\Omega_{k_n x_l}/2)(-\partial_{x_j}\epsilon + \Omega_{x_j t})$$
$$- (\Omega_{x_i x_j} + \epsilon_{kls}\epsilon_{ijm}\Omega_{k_k x_m}\Omega_{k_l k_s}/2)(\partial_{k_j}\epsilon - \Omega_{k_j t}). \tag{A7}$$

Here we will consider the case of a Weyl-type Hamiltonian: $H = \sigma \cdot p + \phi$. Equation (9) is a key point for deriving (10), (11), and (12).

Since

$$\Omega_{\alpha\beta} = \partial_b a_\alpha - \partial_a a_\beta = \frac{\partial p_j}{\partial \beta}\frac{\partial p_i}{\partial \alpha}\frac{\partial a_p^i}{\partial p_j} - \frac{\partial p_j}{\partial \alpha}\frac{\partial p_i}{\partial \beta}\frac{\partial a_p^i}{\partial p_j}$$
$$= \frac{1}{2}\partial_\alpha p_m \partial_\beta p_l \epsilon_{mln}\Omega_n^p, \tag{A8}$$

we have the following relation:

$$\Omega_{\alpha\beta}\Omega_{\gamma\sigma} = \varepsilon_{mnl}\varepsilon_{m'n'l'}\frac{\partial p_m}{\partial \alpha}\frac{\partial p_n}{\partial \beta}\frac{\partial p_m'}{\partial \gamma}\frac{\partial p_n'}{\partial \sigma}\Omega_l^p\Omega_{l'}^p$$
$$= \left(\frac{\partial p \cdot \partial p}{\partial \alpha \partial \gamma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \sigma} - \frac{\partial p \cdot \partial p}{\partial \alpha \partial \sigma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \gamma}\right)\Omega^2$$
$$- \frac{\Omega \cdot \partial p}{\partial \alpha}\left(\frac{\Omega \cdot \partial p}{\partial \gamma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \sigma} - \frac{\Omega \cdot \partial p}{\partial \sigma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \gamma}\right)$$
$$+ \frac{\Omega \cdot \partial p}{\partial \beta}\left(\frac{\Omega \cdot \partial p}{\partial \gamma}\frac{\partial p \cdot \partial p}{\partial \alpha \partial \sigma} - \frac{\Omega \cdot \partial p}{\partial \sigma}\frac{\partial p \cdot \partial p}{\partial \alpha \partial \gamma}\right). \tag{A9}$$

Therefore, the following cancellation takes place, as shown in (9):

$$\Omega_{\alpha\beta}\Omega_{\gamma\sigma} + \Omega_{\alpha\gamma}\Omega_{\sigma\beta} + \Omega_{\alpha\sigma}\Omega_{\beta\gamma}$$
$$= \left(\frac{\partial p \cdot \partial p}{\partial \alpha \partial \gamma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \sigma} - \frac{\partial p \cdot \partial p}{\partial \alpha \partial \sigma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \gamma}\right)\Omega^2$$
$$+ \left(\frac{\partial p \cdot \partial p}{\partial \alpha \partial \sigma}\frac{\partial p \cdot \partial p}{\partial \gamma \partial \beta} - \frac{\partial p \cdot \partial p}{\partial \alpha \partial \beta}\frac{\partial p \cdot \partial p}{\partial \sigma \partial \gamma}\right)\Omega^2$$
$$+ \left(\frac{\partial p \cdot \partial p}{\partial \alpha \partial \beta}\frac{\partial p \cdot \partial p}{\partial \sigma \partial \gamma} - \frac{\partial p \cdot \partial p}{\partial \alpha \partial \gamma}\frac{\partial p \cdot \partial p}{\partial \sigma \partial \beta}\right)\Omega^2$$
$$- \frac{\Omega \cdot \partial p}{\partial \alpha}\left(\frac{\Omega \cdot \partial p}{\partial \gamma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \sigma} - \frac{\Omega \cdot \partial p}{\partial \sigma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \gamma}\right)$$
$$+ \frac{\Omega \cdot \partial p}{\partial \sigma}\frac{\partial p \cdot \partial p}{\partial \beta \partial \gamma} - \frac{\Omega \cdot \partial p}{\partial \beta}\frac{\partial p \cdot \partial p}{\partial \gamma \partial \sigma}$$

$$+ \frac{\Omega \cdot \partial p}{\partial \beta} \frac{\partial p \cdot \partial p}{\partial \gamma \partial \sigma} - \frac{\Omega \cdot \partial p}{\partial \gamma} \frac{\partial p \cdot \partial p}{\partial \sigma \partial \beta} \Bigg)$$

$$+ \frac{\partial p}{\partial \alpha} \cdot \Bigg(\frac{\partial p}{\partial \sigma} \frac{\Omega \cdot \partial p}{\partial \beta} \frac{\Omega \cdot \partial p}{\partial \gamma} - \frac{\partial p}{\partial \gamma} \frac{\Omega \cdot \partial p}{\partial \sigma} \frac{\Omega \cdot \partial p}{\partial \beta}$$

$$+ \frac{\partial p}{\partial \gamma} \frac{\Omega \cdot \partial p}{\partial \beta} \frac{\Omega \cdot \partial p}{\partial \sigma} - \frac{\partial p}{\partial \sigma} \frac{\Omega \cdot \partial p}{\partial \beta} \frac{\Omega \cdot \partial p}{\partial \gamma}$$

$$+ \frac{\partial p}{\partial \sigma} \frac{\Omega \cdot \partial p}{\partial \gamma} \frac{\Omega \cdot \partial p}{\partial \beta} - \frac{\partial p}{\partial \beta} \frac{\Omega \cdot \partial p}{\partial \sigma} \frac{\Omega \cdot \partial p}{\partial \gamma} \Bigg)$$

$$= 0. \tag{A10}$$

Equation (9) makes the following terms vanish:

$$(\epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m x_k}\Omega_{k_n x_l}/2 - \epsilon_{ikl}\epsilon_{jmn}\Omega_{k_m k_n}\Omega_{x_k x_l}/4) = 0,$$

$$\epsilon_{kls}\epsilon_{ijm}\Omega_{k_k x_m}\Omega_{k_l k_s}/2 = 0, \tag{A11}$$

$$-(\Omega_{k_k x_k} - \Omega_{k_i x_j})\Omega_{k_j t} - \Omega_{k_i k_j}\Omega_{x_j t} = 0. \tag{A12}$$

Finally, Eqs. (A5)–(A7) can be simplified to

$$\sqrt{G} = 1 + \Omega_{k_i x_i}, \tag{A13}$$

$$\sqrt{G}\dot{x} = -\Omega_{k_i t} + [\delta_{ij}(1 + \Omega_{k_k x_k}) - \Omega_{k_i x_j}]\partial_{k_j}\epsilon + \Omega_{k_i k_j}\partial_{x_j}\epsilon, \tag{A14}$$

$$\sqrt{G}\dot{k} = \Omega_{x_i t} - [\delta_{ij}(1 + \Omega_{k_k x_k}) - \Omega_{k_i x_j}]\partial_{x_j}\epsilon - \Omega_{x_i x_j}\partial_{k_j}\epsilon, \tag{A15}$$

The Liouville anomaly equation is easily derived from above three equations:

$$\frac{\partial \sqrt{G}}{\partial t} + \frac{\partial \sqrt{G}\dot{x}_i}{\partial x_i} + \frac{\partial \sqrt{G}\dot{k}_i}{\partial k_i} = \Theta_{x_i k_i t} + \Theta_{k_j x_j x_i}\partial_{k_i}\epsilon - \Theta_{x_j k_j k_i}\partial_{x_i}\epsilon. \tag{A16}$$

APPENDIX B: THE SUMMATION OVER CONES

The chirality and effective gauge fields depend on s_x, s_y and s_z as we explained in Eq. (7). To determine the total anomalous current and the total chiral charge generation, we should take into account the following summation rule:

$$\sum_n s_i = 0 (i = x, y, z), \tag{B1}$$

$$\sum_n s_i s_j = 0 (i \neq j), \tag{B2}$$

$$s_i s_i = 1. \tag{B3}$$

Using this summation rule, we find that the current j_{tor} originating from torsion alone and the total current j_{tot} resulting

from the combination of the torsional electric field and an external magnetic field vanish after summation over all cones:

$$j_{\text{tor}} \propto \sum_n \frac{1}{(\mu - \mathcal{E}_n)^2}(E_{\text{eff}} \cdot B_{\text{eff}})B_{\text{eff}}$$

$$= \sum_n \frac{s_x}{[\mu - (s_y + s_z)t_3]^2} \frac{\beta_1}{t_1 a} \cot(k_{0x}a)\dot{\lambda}|B_{\text{eff}}|^2$$

$$= \sum_n \frac{s_x(s_y + s_z)^2}{[\mu - (s_y + s_z)t_3]^2} \frac{\beta_1}{t_1 a} \cot(k_{0x}a)\dot{\lambda}(\frac{\beta_2\gamma}{t_2 b})^2\sin^2(k_{0x}a)$$

$$= 0, \tag{B4}$$

$$j_{\text{tot}} \propto \sum_n \frac{1}{(\mu - \mathcal{E}_n)^2}(E_{\text{eff}} \cdot B_{\text{ext}})B_{\text{ext}}$$

$$= \sum_n \frac{s_x}{[\mu - (s_y + s_z)t_3]^2} \frac{\beta_1}{t_1 a} \cot(k_{0x}a)\dot{\lambda}B_x\vec{B} = 0. \tag{B5}$$

Therefore, the only current that survives the summation over the cones is the chiral magnetic current resulting from the chiral chemical potential induced by strain in an external magnetic field.

APPENDIX C: CUTOFF INDEPENDENCE OF THE MONOPOLE CHARGE FUNCTIONS

In this section, we show that the monopole charge function computed from the entire tight-binding Hamiltonian (4) coincides with the one computed for an effective low-energy Weyl Hamiltonian.

From Eq. (4), we get

$$p_1 = -2t_1[\cos(k'_x a) - \cos(k'_{x0}a)] + 2\beta_1\lambda\cos(k'_x a), \tag{C1}$$

$$p_2 = -2t_2\sin(k'_y b) + 2\beta_2\gamma z\sin(k'_x a), \tag{C2}$$

$$p_3 = -2t_2\sin(k'_z b) - 2\beta_2\gamma y\sin(k'_x a), \tag{C3}$$

$$\phi = 2[\cos(k'_y b) + \cos(k'_z b)]t_3, \tag{C4}$$

where $k'_i = M_i^j k_j$, so the dependence of the p_a on the variables is given by $p_1(k_x, \lambda(t))$, $p_2(k_x, k_y, k_z, x, z, \lambda(t))$, $p_3(k_x, k_y, k_z, x, y, \lambda(t))$.

Let us use the above equations to compute the monopole charge functions responsible for the chiral anomaly:

$$\Theta_{k_i x^i t} = \varepsilon_{mnl}\frac{\partial p_m}{\partial k_i}\frac{\partial p_n}{\partial x^i}\frac{\partial p_l}{\partial t}\Theta(p)$$

$$= \Bigg[-\frac{\partial p_1}{\partial \lambda}\dot{\lambda}\bigg(\frac{\partial p_2}{\partial z}\frac{\partial p_3}{\partial k_z} + \frac{\partial p_2}{\partial x}\frac{\partial p_3}{\partial k_x}\bigg)$$

$$+ \frac{\partial p_1}{\partial \lambda}\dot{\lambda}\bigg(\frac{\partial p_3}{\partial y}\frac{\partial p_2}{\partial k_y} + \frac{\partial p_3}{\partial x}\frac{\partial p_2}{\partial k_x}\bigg)$$

$$+ \frac{\partial p_2}{\partial \lambda}\dot{\lambda}\bigg(\frac{\partial p_1}{\partial x}\frac{\partial p_3}{\partial k_x} - \frac{\partial p_3}{\partial x}\frac{\partial p_1}{\partial k_x}\bigg)$$

$$- \frac{\partial p_3}{\partial \lambda}\dot{\lambda}\bigg(\frac{\partial p_1}{\partial x}\frac{\partial p_2}{\partial k_x} - \frac{\partial p_2}{\partial x}\frac{\partial p_1}{\partial k_x}\bigg) \Bigg]\Theta(p)$$

$$= C\Theta(p), \tag{C5}$$

where, for simplicity, we use C to denote the part inside the square bracket. C contains four terms separately displayed by the four lines above. For the first term, which we denote as C_1, we have

$$C_1 = -\frac{\partial p_1}{\partial \lambda}\dot{\lambda}\left(\frac{\partial p_2}{\partial z}\frac{\partial p_3}{\partial k_z} + \frac{\partial p_2}{\partial x}\frac{\partial p_3}{\partial k_x}\right)$$

$$= -2\beta_1\cos(k_x'a)\dot{\lambda}(-4t_2\beta_2\gamma\sin(k_x'a)b\cos(k_z'b)$$

$$+ 4t_2\beta_2ba\cos(k_y'b)\frac{dk_y'}{dx}\gamma y\cos(k_x'a)\frac{dk_x'}{dk_x})$$

$$= -2\beta_1\cos(k_x'a)\dot{\lambda}[-4t_2\beta_2\gamma\sin(k_x'a)b\cos(k_z'b)$$

$$+ 4t_2\beta_2ba\cos(k_y'b)\gamma k_z\gamma y\cos(k_x'a)(1-\lambda)].$$

Since $\Theta_{kxt} = C\Theta(p)$, where $\Theta(p) = 2\pi\delta^3(p)$, we are only interested in the value of C_1 at the point $p = 0$:

$$C_1|_{p=0} = C_1|_{k=K} = 2\beta_1\cos(K_{0x}a)\dot{\lambda}$$

$$\times[-4t_2\beta_2\gamma\sin(K_{0x}a)bs_z + 4t_2\beta_2ba\gamma K_z\gamma y(1-\lambda)]$$

$$= -8\beta_1\cos(K_{0x}a)\dot{\lambda}t_2\beta_2\gamma\sin(K_{0x}a)bs_z + 0(\gamma^2).$$

The same way, one can derive the second term of C to be

$$C_2|_{k=K} = 2\beta_1\cos(K_{0x}a)\dot{\lambda}$$

$$\times[-4t_2\beta_2\gamma\sin(K_{0x}a)bs_y - 4t_2\beta_2ba\gamma K_y\gamma z(1-\lambda)]$$

$$= -8\beta_1\cos(K_{0x}a)\dot{\lambda}t_2\beta_2\gamma\sin(K_{0x}a)bs_y + 0(\gamma^2).$$

The third term is of higher order in deformation:

$$C_3 = 4\beta_1\gamma z\cos(k_x'a)a\frac{dk_x'}{d\lambda}\dot{\lambda}t_2\cos(k_z'b)b\frac{dk_z'}{dx}(2t_1 - 2\beta_1\lambda)\sin(k_x'a)a(1-\lambda),$$

$$C_3|_{k=K} = -8\beta_1\gamma z\cos(K_{x0}'a)aK_x\dot{\lambda}t_2b\gamma K_y(t_1 - \beta_1\lambda)\sin(K_x'a)a(1-\lambda) = 0(\gamma^2).$$

Similarly, the last term C_4 is also a higher order correction.

So, the integration of monopole charge function yields

$$\int \frac{d^3k}{2\pi}\Theta_{k_ix^it} = \int \frac{d^3p}{(2\pi)^3}[J^{-1}C]2\pi\delta^3(p)$$

$$= \frac{1}{4\pi^2}[\det(J)^{-1}C]|_{k_i=K_i}, \tag{C6}$$

where J is a Jacobian. If we define $e_a^i = \frac{dp_a}{dk_i}$, $J = \det(e)$. Since $e_a^i = (e')_a^j M_j^i$,

$$\det(J)^{-1}C|_{k_i=K_i} = \det(e')^{-1}\det(M)^{-1}C|_{k_i=K_i}. \tag{C7}$$

From (C1)–(C4),

$$(e')_a^i = \frac{dp_a}{dk_j'} = \begin{bmatrix} (2t_1 - 2\beta_1\lambda)\sin(k_x'a)a & 0 & 0 \\ 2\beta_2\gamma z\sin(k_x'a)a & -2t_2\cos(k_y'b)b & 0 \\ -2\beta_2\gamma y\sin(k_x'a)a & 0 & -2t_2\cos(k_y'b)b \end{bmatrix}.$$

To count the lowest order, we should take the lowest order of $\det(e')^{-1}|_{k_i=K_i}$, $\det(M)^{-1}|_{k_i=K_i}$, and $C|_{k_i=K_i}$ separately as below:

$$\det(e')^{-1}|_{k_i=K_i} = \frac{1}{8t_1t_2^2\sin(K_x'a)ab^2}, \tag{C8}$$

$$\det(M)^{-1} = 1, \tag{C9}$$

$$C = C_1 + C_2 + C_3 + C_4$$

$$= -8(s_y + s_z)\beta_1\beta_2\gamma\dot{\lambda}t_2ab\sin(K_{0x}a)\cos(K_{0x}'a). \tag{C10}$$

One can also prove that in the first order of γ we get

$$\Theta_{k_jx_jx_i}\partial_{k_i}\epsilon + \Theta_{x_jk_jk_i}\partial_{x_i}\epsilon = 0. \tag{C11}$$

So, finally let us multiply (C8), (C9), and (C10) to get the result for the anomaly:

$$\partial_\mu j^\mu = \int \frac{d^3k}{2\pi}\Theta_{k_ix^it} = \frac{1}{4\pi^2}[\det(J)^{-1}C]|_{k_i=K_i}$$

$$= -\frac{\cos(K_{0x}'a)\beta_1\beta_2\gamma\dot{\lambda}}{4\pi^2t_1t_2ab}(s_y + s_z). \tag{C12}$$

It coincides with Eq. (36) in the main text, which means that the computation with the effective Hamiltonian does not miss any anomaly terms in lowest order in deformation.

This means that we do not need to impose a cutoff. In principle, we can go beyond the lowest order using the entire Hamiltonian of Eq. (4), but the result becomes cumbersome at higher order. We also do not need a cutoff in computing the monopole charge function in the case of elastic continuous deformation.

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