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Orthonormal wave functions for periodic fermionic states under an applied magnetic field

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We report an infinite number of orthonormal eigenfunction bases for the quantum problem of a free electron in presence of an applied external magnetic field, suitable to describe *doubly periodic* electronic densities. The completeness of these bases is shown here and so, a single basis, labeled by the number of magnetic fluxons trapped in the unit cell (integer p), expands any function in the unit cell. The present framework unveils for the electronic density an egg-box pattern that displays fractional charge and magnetic flux. In case of electrons confined to the lowest Landau level we obtain an analytic expression for the local magnetic field created by their own motion and find that it yields an attractive magnetic interaction. The well-known de Haas–van Alphen oscillations are retrieved, thus showing the correctness of the present theoretical framework.

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I. INTRODUCTION

"There is hardly a single-electron problem in quantum mechanics which has attracted so much and continuous interest for about seven decades as that of Bloch electrons in a magnetic field," according to Rössler and Suhrke [1]. The problem of free electrons in a magnetic field has been investigated since the early days of quantum mechanics when Landau explained the diamagnetism of metals [2]. They are part of the educational training of any physics student [3–7] and remain of direct interest to explain many measurements, such as magnetotransport and the de Haas–van Alphen effect, the latter allowing for the determination of Fermi energy properties [8–10]. New and startling phenomena [11], such as the quantum Hall effect [12,13], keep a vivid interest in the understanding of the behavior of electrons in presence of an applied magnetic field. The Bloch electrons are more complex than the free electrons as they fill states that are eigenfunctions of a Hamiltonian with a periodic potential, whose lattice represents an underlying solid. Interestingly, periodic states can be formed from electrons that are not submitted to a periodic potential and are nearly free. They are found in nature, such as the Wigner crystal [14,15] that also exists in the presence of a magnetic field [16].

The search for Wannier functions for electrons in presence of both a periodic potential and of an applied magnetic field has been the subject of investigation in the past [17–21]. The Wannier functions are sums over Bloch states which form a complete set of orthonormal functions associated to the unit cell of the underlying lattice. They are localized in the unit cell and rapidly go to zero away from the cell. The construction of Bloch-Wannier states in the limit of a vanishing periodic potential [17,19,20] is useful since in this limit the task of deriving the states is simplified. From this perspective the free-electron Schrödinger Hamiltonian in a magnetic field is the limiting case, however, not all of its solutions are periodic, only selected ones are. Hence, general properties of the Bloch-Wannier states [21] do not generally apply for the free case as periodicity is absent at the Hamiltonian level.

Wannier [20] and Thouless [19] found that Bloch-Wannier functions for free electrons only exist for selected values of the applied magnetic field, but not in general. Their analysis was done in the symmetric gauge [$\vec{A} = (-H_3x_2/2, H_3x_1/2)$, the magnetic field H_3 is along the direction $\hat{x}_3$, which is perpendicular to the plane of motion (x_1, x_2)]. From this point of view their choice of the symmetric gauge is a natural one since the eigenfunctions are based on theta functions, which feature a Gaussian behavior along both the x_1 and x_2 directions, therefore being localized in the unit cell.

In this paper we obtain the eigenfunctions, orthonormal in the unit cell, using the Landau gauge [$\vec{A} = (-H_3x_2, 0, 0)$]. We do not call them Bloch-Wannier states because our analysis is restricted to free electrons. It is well known that free-electron solutions of the Schrödinger Hamiltonian in the Landau gauge display periodicity only along one of the directions but not in the other one, where they are harmonic oscillator eigenfunctions [see Eq. (9)] and, so, expressed in terms of Hermite polynomials. In other words these eigenfunctions are plane waves along one direction (x_1) and display Gaussian behavior in the other one (x_2). This lack of periodicity along one of the directions has been an impediment to the use of the Landau gauge for the purpose of obtaining Wannier-type functions, but this is just an apparent difficulty which is overcome here.

The orthonormal bases found here can only exist for selected values of the flux trapped in the unit cell, similarly to Wannier's and Thouless' eigenfunctions [19,20]. The Wannier and Thouless investigation is limited to electrons confined to the ground state, namely, to the lowest Landau level, whereas here all Landau levels are included. From this perspective the present orthonormal bases do form complete sets whereas the

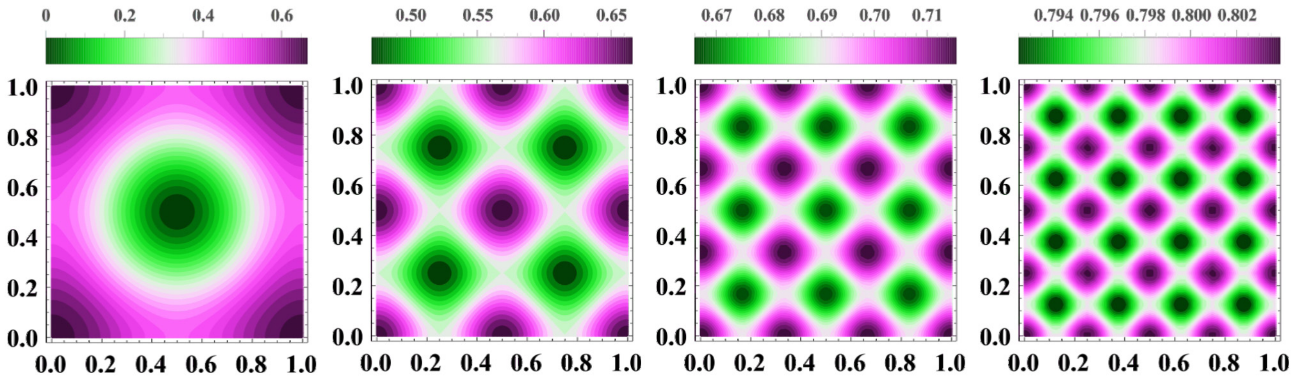

FIG. 1. The spatial electronic density in the unit cell (x_1/L_1, x_2/L_2) for the case of the completely filled first Landau levels ($n = 0$, sum over q). The number of trapped magnetic fluxons is equal to the number of electrons p, and the plots are shown from left to right according to $p = 1,\ 2,\ 3$, and 4.

Wannier and Thouless bases do not since from the very beginning the contributions of higher Landau levels are missing in their work. In power of these bases we unveil properties of the electronic density, i.e., doubly periodic spatial arrangements with fractional charge and flux patterns. Hence, the present Landau gauge orthonormal bases, expressed in terms of Hermite polynomials, bring a different insight into the understanding of doubly periodic free-electron states in a magnetic state that advances over previous studies [17–21].

In this paper we show the remarkable fact that fractional effects arise from free electrons in a magnetic field under the sole hypothesis of double periodicity. These properties have not been reported in previous studies [17–21]. We also report another property, i.e., of an attractive magnetic interaction in case the electrons are confined to the lowest Landau theory. There we show that the local magnetic field also displays fractional patterns. The correctness of the present theoretical approach is confirmed by the derivation of de Haas–van Alphen oscillations from it. The fractional effects can be directly visualized in the spatial electronic density. Figures 1 and 2 summarized these results through the examples of p electrons and a single electron in the unit cell, respectively. The former corresponds to a completely filled lowest Landau level and the latter to a partially filled lowest Landau level with only one electron present. In both situations there are p trapped magnetic fluxons in the unit cell ($\Phi = p\Phi_0$, $\Phi_0 \equiv hc/e$, e the electronic charge); the applied field is $H_3 = \Phi/(L_1L_2)$ and (L_1, L_2) are the lengths of the unit cell. Remarkable egg-box patterns are seen in Figs. 1 and 2 which show the existence of fractional structures, which correspond to the p^2 and p maxima (minima), respectively. Each of the p^2 structures seen in Fig. 1 can be regarded as $1/p$ of an electron (or flux) as their sum must recover the p electrons (fluxons) present in the unit cell. Similarly, the electronic density in Fig. 2 shows p zeros although there is only one electron in the cell. Thus, each zero of the electronic density corresponds to the $1/p$ fraction of an electron and since there are p of them, the single electron is recovered. Figure 3 just extends the results shown in Fig. 1 to higher Landau levels. There are $n + 1$ fully filled Landau levels, and still the electronic density shows p^2 maxima (or minima) independent of n. The total number of electrons (and fluxons) is $N = (n + 1)p$. Hence, the ratio between the total number of electrons and the number of observed electronic density maxima (minima) N/p^2 is $(n + 1)/p$. Notice that the number of electrons N and the number of fluxons in the unit cell p are two independent variables.

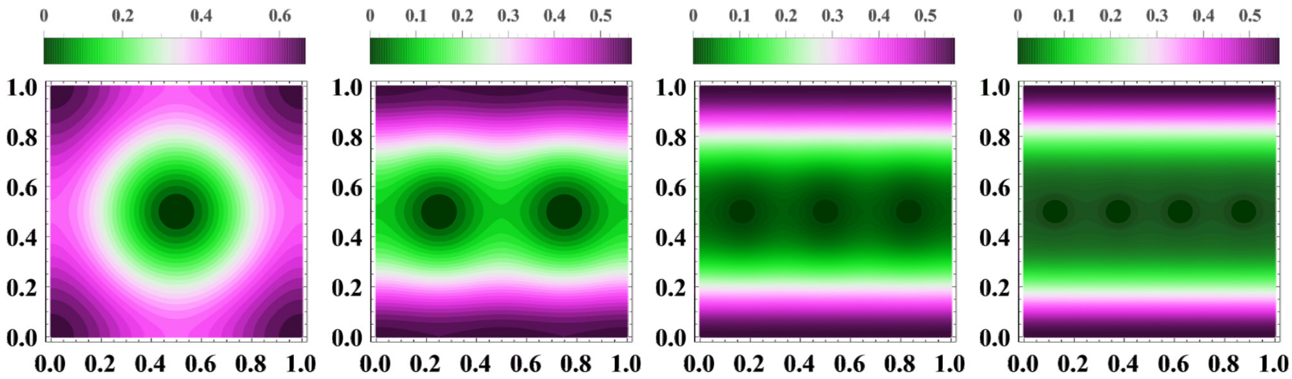

FIG. 2. The spatial electronic density is shown in the unit cell (x_1/L_1, x_2/L_2) for the case of the partially filled first Landau levels ($n = 0$, $q = 0$). The number of trapped magnetic fluxons is p whereas there is only one electron in the unit cell. The plots are shown from left to right according to $p = 1,\ 2,\ 3$, and 4.

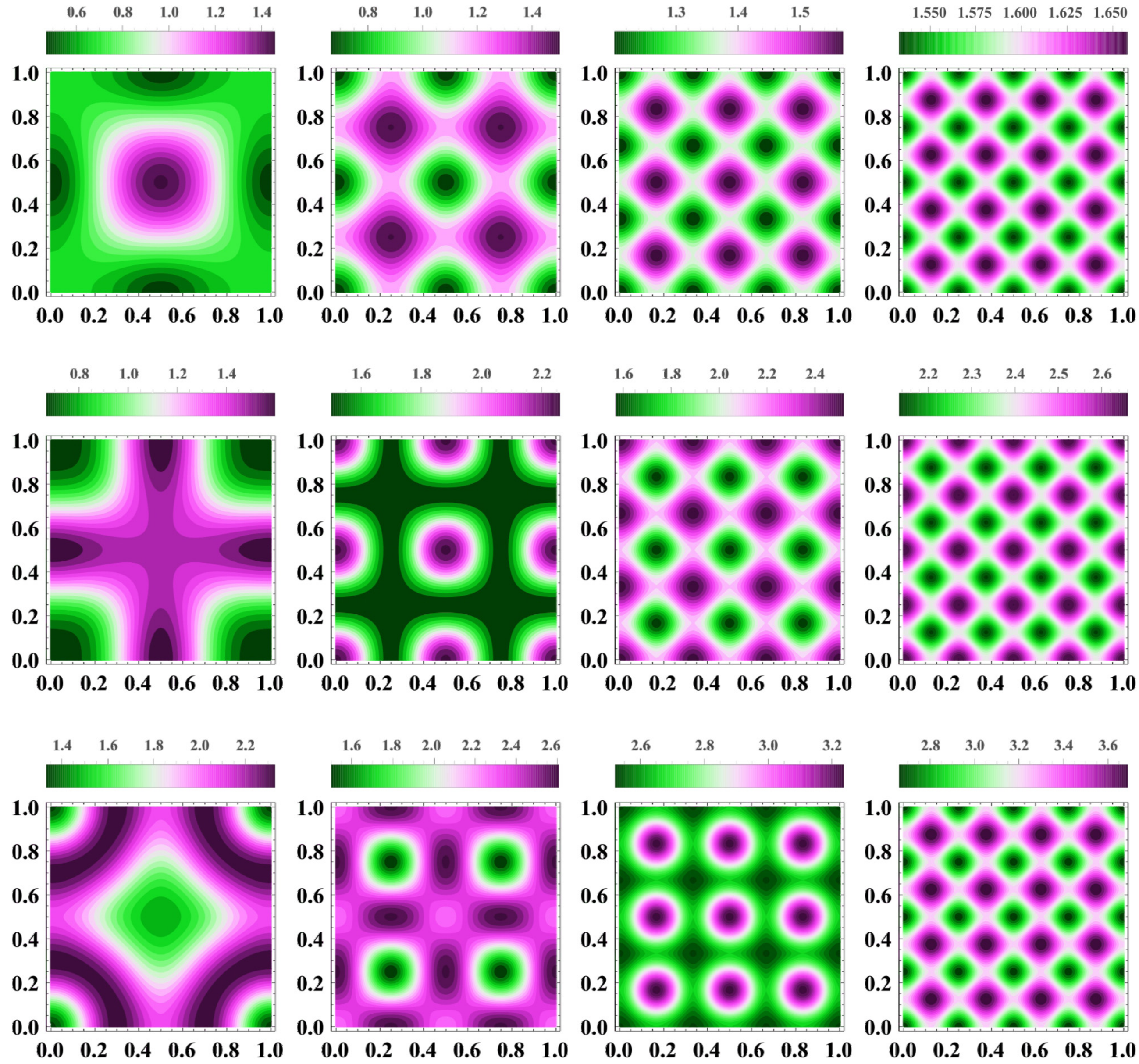

FIG. 3. The spatial electronic density $\langle\Phi|\rho_p|\Phi\rangle$, obtained from Eq. (41), is shown in the unit cell x_1/L_1, (x_2/L_2) for the case of completely filled Landau levels. The rows correspond to the sum over Landau levels $n' = 0, 1$, $n' = 0, 1, 2$, and $n' = 0, 1, 2, 3$ from top to bottom, and are associated to the maximum Landau level $n = 1,\ 2$, and 3, respectively. The columns run from left to right the number of electrons in the unit cell, $p = 1,\ 2,\ 3$, and 4, which is the same as the number of trapped flux. Then the plots are labeled by (p, n) where p and n are integers and the total number of electrons associated to each one is $N(p, n) = (n + 1)p$. Interestingly, the density shows p^2 maxima (or minima) independent of n, the number of filled Landau levels. Hence, the ratio between the number of electrons and the number of density maxima (minima) is given by $(n + 1)/p$. The ratios of each of the plots displayed are given in Table I.

An interesting attempt to understand periodic single-electron solutions in presence of an applied field and of a periodic potential has been developed by Harper [22], who proposed a tight-binding model, which is a simplified version of the Schrödinger equation. The position space is discretized and one seeks solutions just at the points of this tight-binding lattice. The tight-binding lattice adds an extra parameter independent of the periodic length of a possible periodic potential. The Harper's tight-binding lattice can be regarded as a way to solve the Schrödinger equation in the limit that the involved distances are much larger than the tight-binding lattice. This means that the true Schrödinger equation solution must be retrieved, in the limit that the tight-binding cell shrinks to zero, but this limit is not well defined in Harper's equation. In case the magnetic flux trespassing the tight-binding cell is a rational number, this equation yields the so-called Hofstadter's butterfly spectrum [23,24]. The obtainment of the Hofstadter spectrum directly from the Schrödinger equation in the presence of a periodic potential is possible [8] but this hardly proves that the study of solutions of Harper's equation exhausts those of the Schrödinger equation.

The paper is organized as follows. In Sec. II we obtain the orthonormal in the unit-cell eigenfunctions of the Scrödinger equation. Section III provides the diagonalization of the free-electron Hamiltonian using the second quantization formalism. The second quantization formalism is the venue to obtain the spatial electronic density. In Sec. III A the many-electron wave function is obtained and formulas for the spatial electronic density are given. Section IV studies the curious case when the local magnetic field created by the motion of the electrons is taken into account. In case the electrons are confined to the lowest Landau level ($n = 0$), the magnetic energy due to their motion is found to give a negative contribution. This shows that for this situation there is a magnetic attraction between the electrons. For this study we apply the so-called first-order equations [25], first used by Abrikosov to discover vortices in superconductors [26]. In fact, the present orthonormal functions stem from Abrikosov's solution for the vortex lattice [26]. Finally, in Sec. V we present our conclusions. We leave for the Appendices the treatment of some aspects of the problem. The proof that the eigenfunctions form a complete set is given in Appendix A. In Appendix B we show that the boundaries of the unit cell do not contribute to the kinetic energy due to the periodicity of the lattice. Finally, in Appendix C we retrieve here the well-known properties of the de Haas–van Alphen effect, such as the periodicity of the energy with respect to the applied field H_3, and also with respect to $1/H_3$, that allows for the measurement of the Fermi surface area [27,28]. We also retrieve the magnetization M, which is a thermodynamic function of state [4–7].

II. PERIODIC SOLUTIONS FOR THE SCHRÖDINGER EQUATION OF A FREE ELECTRON IN A MAGNETIC FIELD

In this section we obtain the doubly periodic Landau gauge orthonormal eigenfunctions for free electrons in presence of a magnetic field H_3. Electrons are taken as spinless fermions for simplicity. The Schrödinger Hamiltonian is well known and given by

$$\frac{1}{2m}\vec{D}^2\psi = E\psi, \tag{1}$$

where $\vec{D} = \hat{x}_1 D_1 + \hat{x}_2 D_2 + \hat{x}_3 D_3$. The covariant derivative is $D_j \equiv -i\hbar\nabla_j - (e/c)A_j$, $j = 1$, 2, and 3, e is the electron charge, and A_j is the vector potential $\vec{\nabla}\times\vec{A} = \hat{x}_3 H_3$. Uniaxial symmetry is assumed and so, there is no derivative along $\hat{x}_3$ since $A_3 = 0$. Only the derivatives D_1 and D_2 remain and so, the index is limited to $j = 1$ and 2. We assume the simplest possible behavior along the third direction, namely, of no dependence on x_3: $D_3 = 0$. Hence, states are solely described by the coordinates perpendicular to the direction of the applied field $\psi = \psi(x_1, x_2)$. A rectangular unit cell in this plane (x_1, x_2) is assumed with dimensions L_1 and L_2. We seek periodic states on this lattice and for this reason the translations $x_1 \to x_1 + L_1$ and $x_2 \to x_2 + L_2$ are studied such that the electronic density $|\psi(x_1, x_2)|^2$ be periodic. This means to impose quasiperiodicity along the coordinate x_1,

$$\psi(x_1 + L_1, x_2) = e^{i\eta_1}\psi(x_1, x_2), \tag{2}$$

as the phase $e^{i\eta_1}$ does not affect the periodicity condition $|\psi(x_1 + L_1, x_2)|^2 = |\psi(x_1, x_2)|^2$. Similarly the periodicity along the coordinate x_2, $|\psi(x_1, x_2 + L_2)|^2 = |\psi(x_1, x_2)|^2$, is a consequence of the demand that

$$\psi(x_1, x_2 + L_2) = e^{i\eta_2}\psi(x_1, x_2), \tag{3}$$

where $e^{i\eta_2}$ is an arbitrary phase.

The general solution of Eq. (1) in the Landau gauge is well known [3–7]: $\psi \equiv \psi_{n,k}(x_1, x_2) = e^{ikx_1} f_n(x_2)$ where $f_n(x_2)$ is obtained below. We briefly review its derivation for later purposes. There are two quantum indices, k and n, where the latter index defines the Landau level. Equation (1) acquires the form

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m\omega_c^2(x_2 - x_2')^2\right] f_n(x_2) = E_n f_n(x_2), \tag{4}$$

where $x_2' = -(\hbar c k)/(eH_3)$ and ω_c is the Larmor frequency given by

$$\omega_c = \frac{eH_3}{mc}. \tag{5}$$

The analogy of Eq. (4) with the harmonic oscillator yields the eigenvalues as

$$E_n = \hbar\omega_c\left(n + \tfrac{1}{2}\right), \quad n = 0, 1, 2, 3, \ldots \tag{6}$$

and shows that the Landau levels are equally spaced in energy and separated by $\hbar\omega_c$. The solution for $f_n(x_2)$ is given by

$$f_n(x_2) = A_n H_n(\bar{x}_2) e^{-\frac{1}{2}\bar{x}_2^2}, \tag{7}$$

where A_n is a constant to be determined, $H_n(\bar{x}_2)$ are the Hermite polynomials, and the variable $\bar{x}_2$ is defined by

$$\bar{x}_2 \equiv \sqrt{\frac{eH_3}{\hbar c}}\left(x_2 + \frac{c\hbar}{eH_3}k\right). \tag{8}$$

Hence, the wave function is given by

$$\psi_{n,k}(x_1, x_2) = A_n e^{ikx_1} H_n(\bar{x}_2) e^{-\frac{1}{2}\bar{x}_2^2}. \tag{9}$$

The wave functions $\psi_{n,k}(x_1, x_2)$ [Eq. (9)] are orthogonal, and satisfy the following condition:

$$\int d^2x\, \psi^*_{m,k'}\psi_{n,k} = \delta(k - k')\delta_{nm}. \tag{10}$$

Notice that the orthogonality expressed in Eq. (10) refers to integration in the whole plane space and not to the unit cell. Nevertheless, we check its validity by taking $\psi_{m,k'}$ and $\psi_{n,k}$, and computing the integral

$$\int d^2x\, \psi^*_{m,k'}\psi_{n,k} = A_m A_n I_{x_1} I_{x_2}, \tag{11}$$

where

$$I_{x_1} = \int_{-\infty}^{+\infty} dx_1 e^{i(k-k')x_1} \tag{12}$$

and

$$I_{x_2} = \int_{-\infty}^{+\infty} dx_2 H_m(\bar{x}_2') H_n(\bar{x}_2) e^{-\frac{1}{2}(\bar{x}_2'^2 + \bar{x}_2^2)}, \tag{13}$$

such that $\bar{x}_2$ is given by Eq. (8) and $\bar{x}_2'$ is equivalent to Eq. (8) for k'. The integration in x_1 yields that $I_{x_1} = 2\pi\delta(k - k')$. Then, $\bar{x}_2' = \bar{x}_2$ and deriving both sides of Eq. (8) gives that $dx_2 = \sqrt{\frac{\hbar c}{eH_3}} d\bar{x}_2$. Hence, $I_{x_2} = \sqrt{\frac{\hbar c}{eH_3}} \int_{-\infty}^{\infty} d\bar{x}_2 H_m(\bar{x}_2) H_n(\bar{x}_2) e^{-\bar{x}_2^2}$. One obtains that $I_{x_2} = 0$ if

$m \neq n$ and $I_{x_2} = \sqrt{\frac{\hbar c}{eH_3}} 2^n n! \sqrt{\pi}$ if $m = n$. Combining these results in Eq. (11) gives the constant of Eq. (9):

$$A_n = \left(\frac{eH_3}{4\pi^2 \hbar c} \right)^{1/4} (2^n n! \sqrt{\pi})^{-1/2}. \tag{14}$$

We search for periodic eigenfunctions in both directions that must stem from linear combinations of Eq. (9) and for this we must introduce the free parameters $\alpha_{n,k}$. Along x_1 the quasiperiodicity is easily checked: $\psi_n(x_1 + L_1, x_2) = e^{ikL_1} \psi_n(x_1, x_2) = A_n e^{ikL_1} \sum_k \alpha_{n,k} e^{ikx_1} H_n(\bar{x}_2) e^{-\frac{1}{2}\bar{x}_2^2}$, and $e^{i\eta_1} = e^{ikL_1}$. The quasiperiodicity along x_2 is only possible in case there is a sum over k,

$$\psi_n(x_1, x_2) = \sum_k \alpha_{n,k} \psi_{n,k}(x_1, x_2). \tag{15}$$

However, the quasiperiodicity along x_1 limits k to the following values:

$$k = \frac{2\pi}{L_1} l, \quad l = 0, \pm 1, \pm 2, \dots . \tag{16}$$

In this case, $e^{i\eta_1} = 1$, and the coefficients are reduced to the discrete set $\alpha_{n,k} = \alpha_{n,l}$.

Remarkably, the quasiperiodicity condition along x_2 demands the quantization of the magnetic flux in the unit cell and, so, the integer p enters into the theory at this point. Then, the number of coefficients in Eq. (15) becomes limited to just p free coefficients since it must hold that

$$\alpha_{n,l+p} = \alpha_{n,l}. \tag{17}$$

To check this, write the wave function as

$$\psi_n(x_1, x_2) = \sum_l \alpha_{n,l} \psi_{n,l}(x_1, x_2), \tag{18}$$

where

$$\psi_{n,l}(x_1, x_2) = A_n e^{i\frac{2\pi}{L_1} l x_1} H_n(g_l(x_2)) e^{-\frac{1}{2}(g_l(x_2))^2} \tag{19}$$

with $g_l(x_2) \equiv \sqrt{\frac{2\pi H_3}{\Phi_0}} (x_2 + \frac{\Phi_0}{H_3 L_1} l)$. To calculate $\psi_n(x_1, x_2 + L_2)$, first notice that under the flux quantization condition one obtains that $L_2 + (\Phi_0/H_3 L_1) l = (\Phi_0/H_3 L_1)(l + p)$. Next, write $e^{i\frac{2\pi}{L_1} l x_1}$ as $e^{-i\frac{2\pi}{L_1} p x_1} e^{i\frac{2\pi}{L_1}(l+p)x_1}$ to obtain that $\psi_n(x_1, x_2 + L_2) = e^{-i\frac{2\pi}{L_1} p x_1} A_n \sum_l c^n_{l+p} e^{i\frac{2\pi}{L_1}(l+p)x_1} H_n(\bar{g}_l(x_2)) e^{-\frac{1}{2}(\bar{g}_l(x_2))^2}$ with $\bar{g}_l(x_2) = \sqrt{\frac{2\pi p}{L_1 L_2}} (x_2 + \frac{L_2}{p}(l + p))$. Define $l' = l + p$ to retrieve the original form, namely, $\psi_n(x_1, x_2 + L_2) = e^{-i\frac{2\pi}{L_1} p x_1} \psi_n(x_1, x_2)$. Thus, it holds that $\eta_2 = -\frac{2\pi}{L_1} p x_1$. For this reason we introduce p and $l' \to l$ into the wave function. Thus, under the assumption that the coefficients are limited to a set, as stated in Eq. (17), the wave function can be expressed in terms of p instead of H_3. Then the wave function, as given by Eq. (18), becomes

$$\psi_n(x_1, x_2, p) = \sum_l \alpha_{n,l} \psi_{n,l}(x_1, x_2, p),$$

where

$$\psi_{n,l}(x_1, x_2, p) = A_n e^{i\frac{2\pi}{L_1} x_1} H_n(\bar{g}_l(x_2)) e^{-\frac{1}{2}(\bar{g}_l(x_2))^2},$$

and $\bar{g}_l(x_2) = \sqrt{\frac{2\pi p}{L_1 L_2}} (x_2 + \frac{L_2 l}{p})$.

Next, Eq. (17) must be solved in order to to fully determine the set of p wave functions in a given Landau level n. Notice that p is fixed and this defines $\phi_{n,q}(x_1, x_2, p)$, such that $q = 0, \dots, p-1$. For $p = 1$, Eq. (17) becomes $\alpha_{n,l+1} = \alpha_{n,l}$, and this means that all coefficients are equal regardless of l. We choose to express all of them in terms of $\alpha_{n,0}$, namely, $\alpha_{n,l} = \alpha_{n,0}$. For $p = 2$, Eq. (17) becomes $\alpha_{n,l+2} = \alpha_{n,l}$ and in this case there are two free coefficients, chosen to be $\alpha_{n,0}$ and $\alpha_{n,1}$. Hence, solutions split into two sets for l even and odd ones, respectively. Therefore, $\alpha_{n,l} = \alpha_{n,0}$ for $l = 0, \pm 2, \pm 4, \pm 6, \dots$ and $\alpha_{n,l} = \alpha_{n,1}$ for $l = \pm 1, \pm 3, \pm 4, \dots$. For the next case $p = 3$, Eq. (17) becomes $\alpha_{n,l+3} = \alpha_{n,l}$, and there are three free coefficients at each Landau level n. Similarly, the choices of free coefficients are $\alpha_{n,0}$, $\alpha_{n,1}$, and $\alpha_{n,2}$. This gives that $\alpha_{n,l} = \alpha_{n,0}$ for $l = 0, \pm 3, \pm 6, \dots$, $\alpha_{n,l} = \alpha_{n,1}$ for $l = \pm 1, \pm 4, \pm 7, \dots$, and $\alpha_{n,l} = \alpha_{n,2}$ for $l = \pm 2, \pm 5, \pm 8, \dots$. The last case explicitly treated here is $p = 4$, and in this case, Eq. (17) becomes $\alpha_{n,l+4} = \alpha_{n,l}$. There are four free coefficients, namely, $\alpha_{n,0}$, $\alpha_{n,1}$, $\alpha_{n,2}$, and $\alpha_{n,3}$. Hence, we introduce the general notation $\alpha_{n,q}$ for the coefficients, such that $q = 0, 1, 2, \dots, p-1$, represent the p free and independent coefficients. The coefficients $\alpha_{n,q}$ satisfy conditions. Let us look for the first four cases, namely, $p = 1$, 2, 3, and 4. For $p = 1$ there is only one ($q = 0$) free and independent coefficient: $\alpha_{n,l} = \alpha_{n,0}$. For $p = 2$ there are ($q = 0$, $q = 1$) two coefficients: $\alpha_{n,2l} = \alpha_{n,0}$, $\alpha_{n,2l+1} = \alpha_{n,1}$. For $p = 3$ the three free coefficients are associated to $q = 0$, 1, and 2: $\alpha_{n,3l} = \alpha_{n,0}$, $\alpha_{n,3l+1} = \alpha_{n,1}$, $\alpha_{n,3l+2} = \alpha_{n,2}$. Lastly, for $p = 4$ the four free coefficients are associated to $q = 0$, 1, 2, and 3: $\alpha_{n,4l} = \alpha_{n,0}$, $\alpha_{n,4l+1} = \alpha_{n,1}$, $\alpha_{n,4l+2} = \alpha_{n,2}$, $\alpha_{n,4l+3} = \alpha_{n,3}$. In power of such information we write the most general quasiperiodic ψ_n function as

$$\psi_{n,p}(x_1, x_2) = \sum_{q=0}^{p-1} \alpha_{n,q} \phi_{n,q}(x_1, x_2, p), \tag{20}$$

where the functions $\phi_{n,q}(x_1, x_2, p)$ are defined below,

$$\phi_{n,q}(x_1, x_2, p) = A_n e^{i\frac{2\pi}{L_1} q x_1} \sum_{l=-\infty}^{+\infty} e^{i\frac{2\pi}{L_1} p l x_1} H_n(f_{lq}) e^{-\frac{1}{2} f_{lq}^2}, \tag{21}$$

such that f_{lq} is defined by

$$f_{lq} \equiv \sqrt{\frac{2\pi p}{L_1 L_2}} \left[x_2 + \frac{L_2}{p}(pl + q) \right]. \tag{22}$$

The functions $\phi_{n,q}$ form a set orthonormal in the rectangular unit cell for a given p:

$$\int_{L_1 L_2} d^2x \, \phi^*_{m,q'}(x_1, x_2, p) \phi_{n,q}(x_1, x_2, p) = \delta_{qq'} \delta_{nm}. \tag{23}$$

To prove it we introduce the explicit representation of the functions $\phi_{m,q'}$ and $\phi_{n,q}$:

$$\begin{aligned} I_{L_1 L_2} &= \int_{L_1 L_2} d^2x \, \phi^*_{m,q'} \phi_{n,q} \\ &= A_m A_n \sum_{l'=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} I_1 I_2, \end{aligned} \tag{24}$$

where

$$I_1 = \int_0^{L_1} dx_1 e^{i\frac{2\pi}{L_1}[(pl+q)-(pl'+q')]x_1} \tag{25}$$

and

$$I_2 = \int_0^{L_2} dx_2 H_m(f'_{l'q'}) H_n(f_{lq}) e^{-\frac{1}{2}(f'^2_{l'q'}+f^2_{lq})}, \tag{26}$$

such that $f'_{l'q'}$ is given by Eq. (22) for l' and q'. The integral I_1 differs from zero only in case $l = l'$ and $q = q'$. Then, one obtains that $I_1 = L_1\delta_{qq'}$ and this yields that $f'_{l'q'} = f_{lq}$. A change of variables $y = f_{lq}$ in Eq. (22) brings a change in the integration variable $dx_2 = \sqrt{\frac{L_1L_2}{2\pi p}}dy$ and, so, $I_2 = \sqrt{\frac{L_1L_2}{2\pi p}}\int_{y_i}^{y_f} dy\, H_m(y)H_n(y)e^{-y^2}$, with the limits of integration given by

$$y_i = \sqrt{\frac{2\pi L_2}{pL_1}}(pl+q) \quad \text{and} \quad y_f = \sqrt{\frac{2\pi L_2}{pL_1}}[p(l+1)+q].$$

Hence, the integral of Eq. (24) is expressed as

$$I_{L_1L_2} = A_m A_n \sqrt{\frac{L_1^3 L_2}{2\pi p}} \sum_{l=-\infty}^{+\infty} \int_{y_i}^{y_f} dy\, H_m(y)H_n(y)e^{-y^2}. \tag{27}$$

Through this trick, the integral along the x_2 direction, limited to the unit cell, is extended to the whole axis. According to the change of variable, y_i and y_f run from l until $l+1$, that is, the integrand becomes independent of l. Therefore, we write that $\sum_{l=-\infty}^{+\infty}\int_{y_i}^{y_f} dy = \int_{-\infty}^{+\infty} dy$, and in this way, $\int_{-\infty}^{\infty} dy\, H_m(y)H_n(y)e^{-y^2} = 2^n n!\sqrt{\pi}\delta_{nm}$. Substituting this result into Eq. (27) leads to the conclusion that the functions of Eq. (21) are orthogonal. They become orthonormal according to Eq. (23) by choice of

$$A_n = \left(\frac{2\pi p}{L_1^3 L_2}\right)^{1/4} (2^n n!\sqrt{\pi})^{-1/2}. \tag{28}$$

The set of orthonormal functions $\phi_{n,q}$ carry the index p and correspond to degenerate eigenfunctions of the free Hamiltonian with eigenvalues labeled by n, the Landau level index: $\frac{1}{2m}\vec{D}^2\phi_{n,q} = \hbar\omega_c(n+\frac{1}{2})\phi_{n,q}$. For the first Landau level $n = 0$, $\phi_{0,q}$ satisfies the condition $D_+\phi_{0,q} = 0$, where this operator is defined by Eq. (45).

The orthonormality relation was also found to hold through a direct numerical verification of Eq. (23). We stress that any two functions with distinct values of the trapped flux, say p_1 and p_2, do not belong to the same set of orthonormal functions and so are not orthogonal, as numerically checked. In fact, they belong to two distinct sets of orthonormal functions.

III. KINETIC ENERGY OF FREE FERMIONS IN A MAGNETIC FIELD

In this section we introduce the second-quantization formalism to treat the free fermions in a magnetic field under the second-quantized field $\Psi_p(x_1, x_2)$, which carries the index p. Thus, there are p fluxons trapped in the unit-cell area L_1L_2. This formalism, although of straightforward derivation, is important for the obtainment of the doubly periodic electronic density which shows fractional effects:

$$\Psi_p(x_1, x_2) = \sum_{n=0}^{\infty}\sum_{q=0}^{p-1} c_{n,q}\phi_{n,q}(x_1, x_2, p). \tag{29}$$

The $c_{n,q}$ are operators intimately connected to the eigenfunctions $\phi_{n,q}(x_1, x_2, p)$ and, so, carry the index p: $c_{n,q}^{(p)}$. However, this dependence is hidden in order to keep the notation simple. The second-quantized field $\Psi_p(x_1, x_2)$ satisfies the anticommutation relation

$$\{\Psi_p(x_1, x_2),\, \Psi_p^\dagger(x_1', x_2')\} = \delta(x_1 - x_1')\delta(x_2 - x_2') \tag{30}$$

and

$$\{\Psi_p(x_1, x_2),\, \Psi_p(x_1', x_2')\} = 0, \tag{31}$$

assuming that the operators $c_{n,q}$ obey the conditions

$$\{c_{n,q}\, ,\, c_{m,q'}^\dagger\} = \delta_{qq'}\delta_{nm}, \tag{32}$$

$$\{c_{n,q}\, ,\, c_{m,q'}\} = 0. \tag{33}$$

The completeness of the $\phi_{n,q}(x_1, x_2, p)$ set is the key ingredient to prove that the anticommutation relations hold in real space, as shown in Appendix A.

For noninteracting fermions the Hamiltonian is given by the kinetic energy $\mathcal{H} = K$, where K is

$$K = \int d^2x \frac{1}{2m}(\vec{D}\Psi)^\dagger(\vec{D}\Psi). \tag{34}$$

For simplicity we drop the index p on the second-quantized field. In Appendix B we show that the boundaries of the unit cell do not contribute, and using this, the Hamiltonian acquires diagonal form

$$\mathcal{H} = \frac{1}{2m}\int d^2x(\Psi^\dagger\vec{D}^2\Psi) = \sum_{n=0}^{\infty}\sum_{q=0}^{p-1} E_n c_{n,q}^\dagger c_{n,q}, \tag{35}$$

where E_n is given by Eq. (6). The number of electrons is also diagonal and given by

$$\mathcal{N} = \int d^2x\, \Psi^\dagger\Psi = \sum_{n=0}^{\infty}\sum_{q=0}^{p-1} c_{n,q}^\dagger c_{n,q}. \tag{36}$$

Although the above expressions are the standard ones, the electrons are associated to the new functions $\phi_{n,q}(x_1, x_2, p)$.

A. Wave function and the spatial electronic density

The fermions fill the first n' Landau levels, $n = 0, 1, 2, \ldots, n'-1$, each with p electrons, and the last Landau level n' is partially filled with p' electrons, hence $p' < p$. Therefore, the wave function $|\Phi\rangle$ representing this state has the form $|\Phi\rangle = \prod_{n,q} c_{n,q}^\dagger|0\rangle$ where $c_{n,q}^\dagger$ is a creation operator, the index $n \in [0, n']$ represents the Landau levels, and the index q, $0 \leqslant q \leqslant \tilde{p} - 1$, runs over $\tilde{p}$, which is the number of electrons in each Landau level n. An explicit version of the wave function is given as follows:

$$|\Phi\rangle = \underbrace{c^\dagger_{n',p'-1}\cdots c^\dagger_{n',1}c^\dagger_{n',0}}_{\text{(level } n=n' \text{ incomplete)}}\overbrace{c^\dagger_{n'-1,p-1}\cdots c^\dagger_{n'-1,1}c^\dagger_{n'-1,0}}^{\text{(level } n=n'-1 \text{ complete)}}\cdots\underbrace{c^\dagger_{1,p-1}\cdots c^\dagger_{1,1}c^\dagger_{1,0}}_{\text{(level } n=1 \text{ complete)}}\overbrace{c^\dagger_{0,p-1}\cdots c^\dagger_{0,1}c^\dagger_{0,0}}^{\text{(level } n=0 \text{ complete)}}|0\rangle. \tag{37}$$

The state $|\Phi\rangle$ is automatically normalized, $\langle\Phi|\Phi\rangle = 1$ and has a fixed number of electrons [Eq. (36)] since

$$\mathcal{N}|\Phi\rangle = N|\Phi\rangle, \quad \text{where}\, N = n'p + p'. \tag{38}$$

From the second-quantized formalism we obtain the electronic density, which is given by the expectation value $\langle\Phi|\rho_p|\Phi\rangle$, where $\rho_p \equiv \Psi_p^\dagger\Psi_p$ and Ψ_p is the operator associated to the periodic density, that acquires the following form:

$$\rho_p(x_1, x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q'=0}^{p-1}\sum_{q=0}^{p-1} c^\dagger_{m,q'}c_{n,q}\phi^*_{m,q'}(x_1, x_2)\phi_{n,q}(x_1, x_2). \tag{39}$$

Then, the expectation value is obtained used the anticommutation condition of Eq. (32):

$$\langle\Phi|\rho_p(x_1, x_2)|\Phi\rangle = \sum_{n=0}^{n'-1}\sum_{q=0}^{p-1}|\phi_{n,q}(x_1, x_2, p)|^2 + \sum_{q=0}^{p'-1}|\phi_{n',q}(x_1, x_2, p)|^2. \tag{40}$$

In power of the above expressions one can numerically calculate the electronic density and observe some original features of free-electron states.

Figure 1 shows the spatial electronic density of the completely filled first Landau levels ($n = 0$). In this case the wave function (37) is $|\Phi\rangle = c^\dagger_{0,p-1}\cdots c^\dagger_{0,1}c^\dagger_{0,0}|0\rangle$, and so the spatial density becomes $\langle\Phi|\rho_p|\Phi\rangle = \sum_{q=0}^{p-1}|\phi_{0,q}(p)|^2$, obtained from Eq. (41). The number of trapped flux and the number of electrons is the same and equal to p. Interestingly, the density displays an egg-box pattern with p^2 maxima (minima). Hence, each maxima (minima) can be associated to a fraction $1/p$ of a electron. Notice that as p increases, the difference between the maximum and the minimum density shrinks. This is a consequence of the sum over the $|\phi_{0,q}(p)|^2$ wave functions which adds more positive contributions as p increases. For this reason the first plot, which corresponds to $p = 1$, is able to reach zero density, as the sum over q is absent.

Figure 2 shows the spatial electronic density for the partially filled first Landau levels ($n = 0$) such that only the $q = 0$ state is present. In this case the wave function (37) contains a single electron $|\Phi\rangle = c^\dagger_{0,0}|0\rangle$, and the spatial density becomes $\langle\Phi|\rho_p|\Phi\rangle = |\phi_{0,0}(p)|^2$, obtained from Eq. (41). The number of trapped flux and the number of electrons is not the same: the former is p whereas the latter is one. The density displays p zeros, which is the number of trapped flux in the unit cell. Thus, each zero in the density is associated to $1/p$ electrons. The first plots of Figs. 1 and 2 coincide as both correspond to $p = 1$ and $q = 0$.

Figure 3 shows the remarkable fact that the electronic density of $N = (n + 1)p$ electrons displays an egg-box pattern with p^2 maxima (minima). There are p electrons (and fluxons) in each Landau level and n filled Landau levels, thus making a total of $N = (n + 1)p$ electrons (fluxons). From top to bottom the sum is over the Landau levels $n' = 0, 1$, $n' = 0, 1, 2$, and $n' = 0, 1, 2, 3$, thus defining the maximum Landau level $n = 1,\ 2,$ and 3. Hence, the unit cell is shown in units of $(x_1/L_1, x_2/L_2)$, such that the last level of Eq. (41), $n' = n$, is also filled, $p' = p$. The columns run from left to right and the number of electrons in the unit cell ranges $p = 1,\ 2,\ 3,$ and 4. Notice that the scales of the plots do not coincide although the same set of colors is used. The case where only the lowest Landau level is considered, $n = n' = 0$, is treated in Fig. 1. Therefore, the first, second, and third rows correspond to the densities $\langle\Phi|\rho_p|\Phi\rangle = |\phi_{0,0}(p)|^2 + |\phi_{1,0}(p)|^2$, $\langle\Phi|\rho_p|\Phi\rangle = |\phi_{0,0}(p)|^2 + |\phi_{1,0}(p)|^2 + |\phi_{2,0}(p)|^2$, and $\langle\Phi|\rho_p|\Phi\rangle = |\phi_{0,0}(p)|^2 + |\phi_{1,0}(p)|^2 + |\phi_{2,0}(p)|^2 + |\phi_{3,0}(p)|^2$, respectively. Remarkably, the ratio between the total number of electrons and the number of maxima (minima) is fractional, and given by $(n + 1)/p$, as shown in Table I.

IV. ATTRACTIVE MAGNETIC INTERACTION AMONG ELECTRONS CONFINED TO THE LOWEST LANDAU LEVEL

In this section we include interaction among electrons confined to the lowest Landau level. The motion of the electrons due to the external field makes them generate currents which create a local magnetic field causing a mutual interaction among them. We prove here that this magnetic interaction is attractive and to treat it the field energy is incorporated into

TABLE I. The (n, p) above elements are associated to the Landau level n, and the number of electrons in each Landau level p. Indices run $n = 0, 1, 2, 3$ and $p = 1, 2, 3, 4$, respectively. Notice that all levels are filled up to n which gives for the total number of electrons $p(n + 1)$. The number of density maxima (minima) observed in Figs. 1 and 3 is p^2. Therefore, the ratio between the total number of electrons and the observed maxima (minima) is fractional and given by $(n + 1)/p$.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$n = 0$	1	1/2	1/3	1/4
$n = 1$	2	1	2/3	1/2
$n = 2$	3	3/2	1	3/4
$n = 3$	4	2	4/3	1

the Hamiltonian, and added to the kinetic energy such that

$$\mathcal{H} = K + F. \tag{41}$$

We emphasize that the present approach only applies in case the system has uniaxial symmetry which means that the kinetic energy K is

$$K = \int d^3x \frac{1}{2m} \sum_{j=1}^{2} (D_j\Psi)^\dagger (D_j\Psi), \tag{42}$$

and the field energy F is

$$F = \int d^3x \frac{1}{8\pi} [h_3(\Psi) - H_3]^2. \tag{43}$$

The local magnetic field $\vec{h}(\Psi)$ is also a second-quantized field solution of Ampère's law,

$$\vec{\nabla} \times \vec{h}(\Psi) = \frac{4\pi}{c} \vec{J}(\Psi), \quad \vec{J}(\Psi) = \frac{e}{2m} (\Psi^\dagger \vec{D}\Psi + \text{c.c.}), \tag{44}$$

by knowledge of the current $\vec{J}$. A special attention must be paid to the dimensionality of the fields, as the magnetic field and the the current are tied to each other through Ampère's law, which assigns to Ψ the dimensionality $[\Psi] = 1/\sqrt{V}$, where $V = AL_3$ is a volume, the product of the unit-cell area in the plane, A times a length along the direction of the applied field L_3. This is consistent with the energy being an integral over three-dimensional space, but uniaxial symmetry along the direction of the external applied field renders the problem two dimensional. All the fields only depend on (x_1, x_2), including the local field $\vec{h}(\Psi) = \hat{x}_3 h_3(\Psi)$, $h_3 = \nabla_1 A_2 - \nabla_2 A_1$. For the applied field we choose, as before, that $A_3 = 0$ and $\vec{H} = \hat{x}_3 H_3$.

Remarkably, Ampère's law is solved exactly and the local magnetic field $h_3(\Psi)$ fully determined under the lowest Landau level condition. To show this consider a dual view of the kinetic energy that requires the operators

$$D_\pm \equiv D_1 \pm iD_2. \tag{45}$$

We use the identity

$$(D_+\Psi)^\dagger (D_+\Psi) = \sum_{j=1}^{2} (D_j\Psi)^\dagger (D_j\Psi) + i\Psi^\dagger [D_1, D_2]\Psi - \frac{\hbar m}{e} (\nabla_1 J_2 - \nabla_2 J_1), \tag{46}$$

where the current components J_i, $i = 1$ and 2, are given by Eq. (44), and the commutation relation is given by

$$[D_1, D_2] = -\frac{e\hbar}{ic} h_3. \tag{47}$$

Combining Eqs. (42), (46), and (47), and using that

$$\frac{\hbar m}{e} (\nabla_1 J_2 - \nabla_2 J_1) = \frac{\hbar^2}{2} \vec{\nabla}^2 (\Psi^\dagger \Psi), \tag{48}$$

one obtains that

$$K = \int d^3x \left(\frac{|D_+\Psi|^2}{2m} + \frac{e\hbar}{2mc} h_3 \Psi^\dagger \Psi \right) + \frac{\hbar^2}{4m} \int d^3x \, \vec{\nabla}^2 (\Psi^\dagger \Psi), \tag{49}$$

where the last term is a surface term. The dual formulation of the kinetic energy also leads to a dual formulation of the current, which is obtained by varying the kinetic energy with respect to the vector potential $\delta F_k = \cdots - \frac{1}{c} \int d^2x \, \vec{J} \cdot \delta\vec{A}$. Therefore, one obtains the components of the current in the plane:

$$J_1 = \frac{e}{2m} [\Psi^\dagger (D_+\Psi) + (D_+\Psi)^\dagger \Psi] - \frac{e\hbar}{2m} \nabla_2 (\Psi^\dagger \Psi) \tag{50}$$

and

$$J_2 = \frac{e}{2im} [\Psi^\dagger (D_+\Psi) - (D_+\Psi)^\dagger \Psi] + \frac{e\hbar}{2m} \nabla_1 (\Psi^\dagger \Psi). \tag{51}$$

The confinement to the lowest Landau leads corresponds to

$$D_+\Psi = 0. \tag{52}$$

Under this condition the current components become $J_1 = -\frac{e\hbar}{2m} \nabla_2 (\Psi^\dagger \Psi)$ and $J_2 = \frac{e\hbar}{2m} \nabla_1 (\Psi^\dagger \Psi)$. Ampère's law, given by $\nabla_2 h_3 = (4\pi/c) J_1$ and $\nabla_1 h_3 = -(4\pi/c) J_2$, is solved and $h_3 + (2\pi e\hbar/mc)|\Psi|^2 =$ constant obtained. The constant is determined by the condition that the local field must be equal to the applied field $h_3 = H_3$ in case that $\Psi = 0$. Hence, there are first-order equations that link Ψ to h_3, namely, Eq. (52) and

$$h_3(\Psi) = H_3 - 4\pi \mu_B \Psi^\dagger \Psi, \tag{53}$$

where μ_B is Bohr's magneton

$$\mu_B = \frac{e\hbar}{2mc}. \tag{54}$$

Next, we find approximate solutions for these first-order equations that result in the determination of Ψ and h_3. First, Eq. (52) is solved for Ψ assuming that the vector potential is only due to H_3. Second, Eq. (53) is solved. The general solution for Ψ, given by Eq. (29), is the $n = 0$ part of the general solution since $D_+\Psi_{0,p} = 0$:

$$\Psi_{0,p}(x_1, x_2) = \frac{1}{\sqrt{L_3}} \sum_{q=0}^{p-1} c_{0,q} \phi_{0,q}(x_1, x_2, p), \tag{55}$$

where

$$\phi_{0,q}(x_1, x_2, p) = \left(\frac{2\pi p}{L_1^3 L_2} \right)^{\frac{1}{4}} \sum_{l=-\infty}^{+\infty} e^{i \frac{2\pi}{L_1} (pl+q) x_1} e^{-\frac{1}{2} f_{lq}^2}, \tag{56}$$

with f_{lq} defined in (22) since $D_+\phi_{0,q} = 0$. Once in power of Ψ the local field follows from (53) by computing the expectation value $\langle \Phi | h_3(\Psi_{0,p}) | \Phi \rangle$ for a state $|\Phi\rangle$ constrained to the first Landau level. Hence, one obtains that

$$\langle \Phi | h_3(\Psi_{0,p}) | \Phi \rangle = H_3 - \mu_B \frac{4\pi}{L_3} \sum_{q=0}^{p'-1} |\phi_{0,q}(x_1, x_2, p)|^2. \tag{57}$$

Next, we consider the attraction among electrons confined to the lowest Landau level that stems from the first-order equations. Introduce Eq. (52) into the kinetic energy, given by Eq. (49). The Hamiltonian (41) becomes $\mathcal{H} = \mu_B \int d^3x \, h_3 \Psi^\dagger \Psi + \frac{\hbar^2}{4m} \int d^3x \, \vec{\nabla}^2 (\Psi^\dagger \Psi) + \frac{1}{8\pi} \int d^3x \, (h_3 - H_3)^2$, that once combined with $\Psi^\dagger \Psi = -(1/4\pi \mu_B)(h_3 - H_3)$

from Eq. (53) renders that

$$\mathcal{H} = \int d^3x\left(\frac{1}{8\pi}H_3^2 - \frac{1}{8\pi}h_3(\Psi)^2 + \frac{\hbar^2}{4m}\vec{\nabla}^2(\Psi^\dagger\Psi)\right). \quad (58)$$

Next, we express $\mathcal{H}$ by introducing Eq. (53) into Eq. (58):

$$\mathcal{H} = 2\pi\mu_B^2\int d^3x\left[\frac{H_3\Psi^\dagger\Psi}{2\pi\mu_B} - (\Psi^\dagger\Psi)^2 + \frac{\vec{\nabla}^2(\Psi^\dagger\Psi)}{2\pi r_e}\right], \quad (59)$$

where Bohr's magneton is related to the electron's classical radius by $\hbar^2/4m = \mu_B^2/r_e \rightarrow r_e = e^2/mc^2$. Equation (58) shows that the magnetic field energy is negative $-h_3^2/8\pi$ and so able to lower the energy by becoming more intense. In other words, it causes attraction among the electrons. Notice that the total energy must remain positive, as initially assumed according to Eq. (41). For the periodic state the last term vanishes $\int d^2x\vec{\nabla}^2\Psi^\dagger\Psi = 0$, as previously shown. Since $\mu_B H_3 = \hbar\omega_c/2$, the Hamiltonian can also be expressed as

$$\mathcal{H} = \int d^3x\left[\frac{\hbar\omega_c}{2}\Psi^\dagger\Psi - 2\pi\mu_B^2(\Psi^\dagger\Psi)^2\right]. \quad (60)$$

The second term is evidently attractive although, in comparison with the first, very small.

By turning the three-dimensional integration into a two-dimensional integration, a new parameter L_3 is introduced: $\int d^3x/L_3 = \int d^2x$. For this reason Eq. (55) carries L_3 that also appears in the local field, given by Eq. (57). Hence, L_3 is a phenomenological parameter that sets the scale of the difference $h_3 - H_3$, and so, cannot be determined within the present approach. L_3 limits the validity of the first-order equation approach since $h_3 - H_3$ must be very small, and according to Eq. (53) this requires that $H_3 \gg 4\pi\mu_B\Psi^\dagger\Psi$ or, equally,

$$L_3 \gg \sqrt{\frac{2\pi}{p}}r_e. \quad (61)$$

This is easily reachable for any reasonable L_3, considering that $r_e = 2.8\times\ 10^{-15}$ m. This relation is obtained by assuming that $\Psi^\dagger\Psi \sim \sqrt{2\pi p}/(L_3A)$, where the area $A \equiv L_1L_2$ is associated to the flux quantization in the plane $H_3A = p\Phi_0$. Bohr's magneton, the magnetic flux, and the classical radius are connected through $\mu_B = \frac{\Phi_0 r_e}{4\pi}$, and from it the above condition for L_3 is straightforwardly obtained. In summary, once the deviation $H_3 - h_3$ is adjusted by an external argument, such as a measurement, the parameter L_3 becomes known. Apart from the choice of L_3 the theory can be treated two dimensionally in the plane (x_1, x_2) due to its uniaxial symmetry along the applied field.

The deviation of the local field to the applied field is shown in Fig. 4, and is given by $-\langle\Phi|h_3(\Psi_{0,p}) - H_3|\Phi\rangle L_3/\mu_B$. This deviation shows the existence of maxima and minima in the unit cell $(x_1/L_1, x_2/L_2)$. The lowest Landau level is assumed totally filled and the figure shows the cases $p = 1$ (left, up), $p = 2$ (right, up), $p = 3$ (left, down), and $p = 4$ (right, down). Notice that while this figure shows the density, Fig. 4 displays the local field h_3, and they are connected to each other by Eq. (57). An egg-box pattern emerges with p^2 maxima (pink) and p^2 minima (black) in the magnetic field pattern. The ratio between the maxima (minima) of the local field and the number of electrons is fractional, and given by $1/p$, as found in Table I.

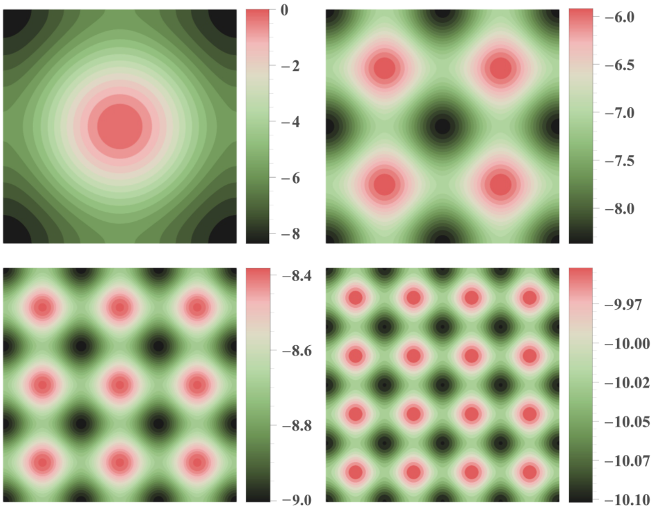

FIG. 4. The deviation of the local magnetic field to the applied field $-\langle\Phi|h_3(\Psi_{0,p}) - H_3|\Phi\rangle L_3/\mu_B$ is shown here in a unit cell. The unit cell $(x_1/L_1, x_2/L_2)$ is displayed for the cases $p = 1$ (left, up), $p = 2$ (right, up), $p = 3$ (left, down), and $p = 4$ (right, down), respectively. The number of electrons for the totally filled lowest Landau level $n = 0$ is $N = p$. Thus, there are 1 (top-left), 2 (top-right), 3 (bottom-left), and 4 (bottom-right) electrons (fluxons), respectively. Remarkably, the ratio between the number of maxima (minima) of the local field, as seen in the above plots, and the number of electrons is 1, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, respectively.

V. CONCLUSION

The set of orthonormal bases developed here yield wave functions for the well-known Schrödinger equation problem of a electron in a magnetic field that features a periodic density probability. These wave functions display a fixed number of p trapped magnetic flux per Landau level. The second-quantization study reveals that the number of electrons N and the number of fluxons trapped in the unit cell p are independent. For N fermions distributed among $n + 1$ Landau energy levels we find the remarkable property that the density of electrons presents p^2 spatial maxima (minima) in the unit cell for any n. In case the highest Landau level is completely filled, and so with $N = (n + 1)p$ electrons, there are $(n + 1)/p$ electrons per maxima (minima). We have shown that in case that electrons fall in the lowest Landau level ($n = 0$), the magnetic field produced by the electrons yields a residual attractive interaction among them. The present set of orthonormal functions retrieves the de Haas–van Alphen effect results, namely, the total magnetization displays a periodicity with respect to the inverse of the applied field which is proportional to the area of the Fermi surface.

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APPENDIX A: COMPLETENESS OF THE ORTHONORMAL WAVE FUNCTIONS IN THE UNIT CELL

In this Appendix we prove the completeness of the orthonormal set defined by p, which is given by

$$\sum_{n=0}^{\infty}\sum_{q=0}^{p-1}\phi_{n,q}(x_1,x_2,p)\phi^*_{n,q}(x_1',x_2',p)$$
$$=\delta(x_1-x_1')\delta(x_2-x_2'). \tag{A1}$$

This completeness relation is behind Eq. (30) through Eq. (29). Using Eq. (32) it follows that

$$\{\Psi_p(x_1,x_2),\,\Psi_p^\dagger(x_1',x_2')\}$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{p-1}\phi_{n,q}(x_1,x_2,p)\phi^*_{n,q}(x_1',x_2',p). \tag{A2}$$

To prove it one must calculate

$$\{\Psi_p(x_1,x_2),\,\Psi_p^\dagger(x_1',x_2')\}$$
$$=\sqrt{\frac{2\pi p}{L_1^3L_2}}\sum_{q=0}^{p-1}\sum_{l=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}(pl+q)x_1}\sum_{l'=-\infty}^{+\infty}e^{-i\frac{2\pi}{L_1}(pl'+q)x_1'}$$
$$\times\sum_{n=0}^{\infty}b_n^2H_n(f_{lq})H_n(f'_{l'q})e^{-\frac{1}{2}(f_{lq}^2+f'^2_{l'q})}. \tag{A3}$$

The latter is obtained by using $\phi_{n,q}(x_1,x_2,p)$, defined by Eqs. (21) and (28), where $b_n=(2^nn!\sqrt{\pi})^{-1/2}$. First, consider the sum in n, which is essentially done over the normalized wave functions of the harmonic oscillator $\psi_n(x)=(2^nn!\sqrt{\pi})^{-1/2}H_n(x)e^{\frac{1}{2}x^2}$, where $H_n(x)$ are the Hermite polynomials. Hence, the $\psi_n(x)$ satisfy $\int_{-\infty}^{\infty}dx\,\psi_m^*(x)\psi_n(x)=\delta_{nm}$ and the corresponding completeness relation is $\sum_{n=0}^{\infty}\psi_n(x)\psi_n(y)=\delta(x-y)$. Thus, the sum over n in Eq. (A2) becomes

$$\sum_{n=0}^{\infty}b_n^2H_n(f_{lq})H_n(f'_{l'q})e^{-\frac{1}{2}(f_{lq}^2+f'^2_{l'q})}=\delta(f_{lq}-f'_{l'q}),$$

and using Eq. (22), one obtains that

$$\sum_{n=0}^{\infty}b_n^2H_n(f_{lq})H_n(f'_{l'q})e^{-\frac{1}{2}(f_{lq}^2+f'^2_{l'q})}$$
$$=\delta\left(\sqrt{\frac{2\pi p}{L_1L_2}}(x_2-x_2')+L_2(l-l')\right).$$

Considering that x_2 and l are independent variables it holds that the delta function is centered at $x_2=x_2'$ and $l=l'$. Therefore, we have the following completeness relation:

$$\sum_{n=0}^{\infty}b_n^2H_n(f_{lq})H_n(f'_{l'q})e^{-\frac{1}{2}(f_{lq}^2+f'^2_{l'q})}=\sqrt{\frac{L_1L_2}{2\pi p}}\delta(x_2-x_2'), \tag{A4}$$

where we have used the property $\delta(a(x_2-x_2'))=\frac{1}{a}\delta(x_2-x_2')$, $a=\sqrt{\frac{2\pi p}{L_1L_2}}$. Using Eq. (A4), Eq. (A2) becomes

$$\{\Psi_p(x_1,x_2),\,\Psi_p^\dagger(x_1',x_2')\}=\frac{1}{L_1}S_p(x_1-x_1')\delta(x_2-x_2'), \tag{A5}$$

where $S_p(x_1-x_1')=\sum_{q=0}^{p-1}\sum_{l=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}(pl+q)(x_1-x_1')}$ must be analyzed.

For $p=1,\ 2$, and 3 the sum becomes

$$S_1=\sum_{l=-\infty}^{+\infty}\left(e^{i\frac{2\pi}{L_1}l(x_1-x_1')}\right),$$
$$S_2=\sum_{l=-\infty}^{+\infty}\left(e^{i\frac{2\pi}{L_1}(2l)(x_1-x_1')}+e^{i\frac{2\pi}{L_1}(2l+1)(x_1-x_1')}\right),$$
$$S_3=\sum_{l=-\infty}^{+\infty}\Bigg(e^{i\frac{2\pi}{L_1}(3l)(x_1-x_1')}+e^{i\frac{2\pi}{L_1}(3l+1)(x_1-x_1')}$$
$$+\sum_{l=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}(3l+2)(x_1-x_1')}\Bigg),\ \text{respectively.}$$

Notice that the total sum runs over all integers, and in this way, one can write that

$$S_p=\sum_{q=0}^{p-1}\sum_{l=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}(pl+q)(x_1-x_1')}=\sum_{r=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}r(x_1-x_1')}.$$

Hence, we have reached the second completeness relation

$$S_p=\sum_{r=-\infty}^{+\infty}e^{i\frac{2\pi}{L_1}r(x_1-x_1')}=L_1\delta(x_1-x_1'), \tag{A6}$$

that once added to Eq. (A5) gives the anticommuting relation given by Eq. (A1) and so to Eq. (30).

APPENDIX B: KINETIC ENERGY AND THE BOUNDARY OF THE UNIT CELL

The kinetic energy is expressed in a more convenient way,

$$K=\frac{1}{2m}\int d^2x(\Psi^\dagger\vec{D}^2\Psi)+\frac{\hbar^2}{4m}\int d^2x\,\vec{\nabla}^2\rho, \tag{B1}$$

where $\rho\equiv\Psi^\dagger\Psi$. In this Appendix we show that the second integral vanishes due to the periodicity of the state. To derive Eq. (B1) we write $(\vec{D}\Psi)^\dagger(\vec{D}\Psi)=(i\hbar\vec{\nabla}\Psi^\dagger-(e/c)\vec{A}\Psi^\dagger)\cdot\vec{\alpha}$, with $\vec{\alpha}\equiv\vec{D}\Psi$. Using that $\vec{\nabla}\Psi^\dagger\cdot\vec{\alpha}=\vec{\nabla}\cdot(\Psi^\dagger\vec{\alpha})-\Psi^\dagger(\vec{\nabla}\cdot\vec{\alpha})$, one gets that

$$(\vec{D}\Psi)^\dagger(\vec{D}\Psi)=\Psi^\dagger(-i\hbar\vec{\nabla}-(e/c)\vec{A})\cdot\vec{\alpha}+i\hbar\vec{\nabla}\cdot(\Psi^\dagger\vec{\alpha}).$$

The kinetic energy becomes

$$K=\frac{1}{2m}\int d^2x(\Psi^\dagger\vec{D}^2\Psi)+\frac{i\hbar}{2m}\int d^2x\,\vec{\nabla}\cdot(\Psi^\dagger\vec{D}\Psi), \tag{B2}$$

and its complex conjugate

$$K^*=\frac{1}{2m}\int d^2x(\vec{D}^2\Psi)^\dagger\Psi-\frac{i\hbar}{2m}\int d^2x\,\vec{\nabla}\cdot((\vec{D}\Psi)^\dagger\Psi). \tag{B3}$$

The kinetic energy is real, thus summing Eqs. (B2) and (B3) and dividing by two gives that

$$K=\frac{1}{4m}\int d^2x\,\Psi^\dagger\vec{D}^2\Psi+(\vec{D}^2\Psi)^\dagger\Psi$$
$$+\frac{i\hbar}{4m}\int d^2x\,\vec{\nabla}\cdot(\Psi^\dagger\vec{D}\Psi-\Psi(\vec{D}\Psi)^\dagger).$$

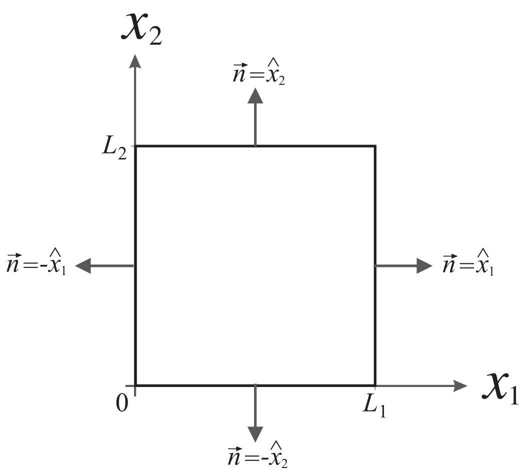

FIG. 5. The integration scheme and the definition of the vector $\vec{n}$ perpendicular to the surface.

The two terms in the first integral are shown to be equal $\Psi^\dagger\vec{D}^2\Psi = (\vec{D}^2\Psi)^\dagger\Psi$, and in the second integral it holds that $\Psi^\dagger\vec{D}\Psi - (\vec{D}\Psi)^\dagger\Psi = \vec{\nabla}(\Psi^\dagger\Psi)$. In this way we obtain the symmetrized kinetic energy given in Eq. (B1).

The second integral (B1) simply vanishes in case of a periodic state, as shown below. We write it as a surface integral

$$\frac{\hbar^2}{4m}\int d^2x\,\vec{\nabla}^2\rho = \frac{\hbar^2}{4m}\oint(\vec{\nabla}\rho)\cdot\vec{n}\,dl, \tag{B4}$$

where $\vec{n}$ is a vector normal to the border lines of the unit cell where the above integration is taken and performed, using the theorem $\oint_c \vec{F}\cdot\vec{n}\,dl = \int_S \vec{\nabla}\cdot\vec{F}\,d^2x$ for a vector function $\vec{F}$ in the plane (x_1, x_2). The integral of Eq. (B4) is illustrated in Fig. 5. Notice that the line integrals along the path $dl = dx_1$ are obtained by taking $\vec{n} = -\hat{x}_2$ for $x_2 = 0$, and $\vec{n} = \hat{x}_2$ for $x_2 = L_2$. The integrals along the path $dl = dx_2$ have $\vec{n} = -\hat{x}_1$ for $x_1 = 0$ and $\vec{n} = \hat{x}_1$ for $x_1 = L_1$, then Eq. (B4) is written as

$$\begin{aligned}&\frac{\hbar^2}{4m}\oint(\vec{\nabla}\rho)\cdot\vec{n}\,dl\\ &= \frac{\hbar^2}{4m}\left\{\int_0^{L_1} dx_1\left(\frac{\nabla\rho(x_1,L_2)}{\nabla x_2} - \frac{\nabla\rho(x_1,0)}{\nabla x_2}\right)\right.\\ &\quad\left. + \int_0^{L_2} dx_2\left(\frac{\nabla\rho(L_1,x_2)}{\nabla x_1} - \frac{\nabla\rho(0,x_2)}{\nabla x_1}\right)\right\}.\end{aligned} \tag{B5}$$

Recall that ρ is periodic and, so, the integrals in opposite sides of the rectangular unit cell annihilate each other, resulting that

$$\frac{\hbar^2}{4m}\oint(\vec{\nabla}\rho)\cdot\vec{n}\,dl = 0. \tag{B6}$$

APPENDIX C: de HAAS–van ALPHEN OSCILLATIONS

In this Appendix we show that the present formalism describes the well-known results of de Hass–van Alphen oscillations, which essentially describe how the energy and the magnetization change according to the magnetic field. There are N spinless fermions distributed in $n' + 1$ Landau levels, the first n' ones, $n = 0, 1, 2, \ldots, n' - 1$, are totally filled and the highest one n' can be partially filled with p' electrons. Previously we have developed a set of orthonormal wave functions $\phi_{n,q}(x_1, x_2, p)$, defined by Eqs. (21) and (22), where p is the number of trapped magnetic fluxons in the unit cell. The second-quantization formalism shows that there are p available states in the Landau level since $q = 0, \ldots, p-1$, but not all of them have to be filled. Therefore, the number of fermions is a free parameter and for this reason λ is introduced and is associated to the highest Landau level n', that is only partially filled:

$$p' = \lambda p, \quad \text{where } \lambda \in [0, 1). \tag{C1}$$

We define a filling variable

$$\nu \equiv \frac{N}{p} \tag{C2}$$

that determines the number of occupied Landau levels according to the number of electrons

$$\nu = n' + \lambda. \tag{C3}$$

Notice that ν and λ are continuous variables whereas n' is a discrete variable. A new critical field H_3' is defined associated to the density of electrons N/A,

$$H_3' \equiv \frac{N}{A}\Phi_0, \tag{C4}$$

where $A = L_1L_2$ is the unit-cell area. As the applied field H_3 is associated to p, one obtains that

$$\nu = \frac{H_3'}{H_3}. \tag{C5}$$

A new way is obtained to write the filling factor ν:

$$\nu = \frac{2\pi N\hbar^2}{Am}\frac{1}{\hbar\omega_c}. \tag{C6}$$

The Landau levels are filled in multiples of H_3' since $n' = [\frac{H_3'}{H_3}]$, and the incomplete filling λ becomes

$$\lambda = \frac{H_3'}{H_3} - \left[\frac{H_3'}{H_3}\right]. \tag{C7}$$

The notation $[\alpha]$ means to take the largest equal or smaller integer contained in a number α.

Further physical insight into the filling factor ν is achieved by introducing the magnetic length $l_0 = \sqrt{\hbar c/eH_3}$. This length describes the classical orbit of the electron in presence of the magnetic field. It stems from the origin of the centripetal force that renders $mv^2/r = qvH_3/c$, where v is the velocity and r the radius of the circular orbit. Added to Bohr's angular momentum condition, $mvr = n_0\hbar$, where n_0 is an integer, it leads to the quantization of the orbits $r = \sqrt{n_0}l_0$. There are two possible areas associated to the disk defined by magnetic orbit πl_0^2 and to the electronic density A/N. The filling factor corresponds to their ratio $\nu = 2\pi l_0^2/(A/N)$. In case that many Landau levels are filled, $n' \gg 1$, the filling factor is large, $\nu \gg 1$. In this limit, the electronic orbits overlap, $l_0 \gg \sqrt{A/N}$, and the classical picture of an electronic orbit in this collective state no longer holds.

The total energy of the fermionic electrons requires the summation over all levels and this is readily obtained from the

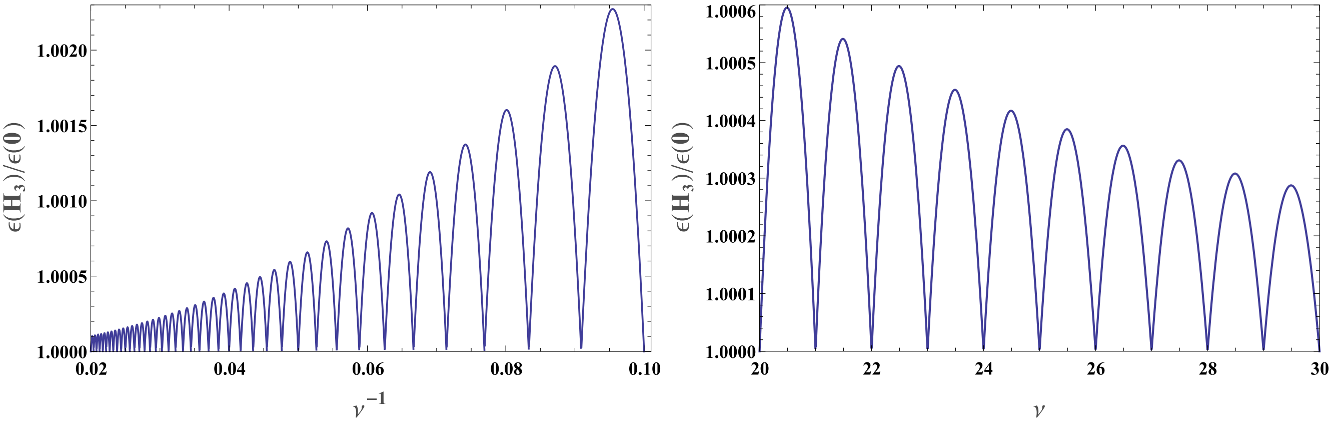

FIG. 6. The oscillations of the energy, normalized by its zero magnetic field, are shown here, as obtained from Eq. (C14). The left plot shows this energy ratio as a function of the inverse of the filling factor $\nu^{-1} = H_3/H_3'$, whereas the right one is with respect to the filling factor $\nu = H_3'/H_3$. ν describes the number of filled Landau levels and H_3' is required to locate all electrons in the lowest Landau level. Notice the periodicity in both H_3 and $1/H_3$.

Hamiltonian of Eq. (35) by computing the energy expectation value

$$E_p = \langle\Phi|\mathcal{H}|\Phi\rangle = (E_0 p + E_1 p + \cdots + E_{n'-1} p) + E_{n'} p' \\ = p \sum_{n=0}^{n'-1} E_n + E_{n'} p', \tag{C8}$$

where $E_n = \hbar\omega_c(n + \frac{1}{2})$ is given by Eq. (6), and $E_{n'} = \hbar\omega_c(n' + \frac{1}{2})$. In summary, total energy E_p takes the contribution of n' fully filled Landau levels $p\sum_{n=0}^{n'-1} E_n$, added to the highest level n', that may be partially filled $E_{n'}p'$. Introducing Eq. (C1) and the sum $\sum_{n=0}^{n'-1}(n + \frac{1}{2}) = n'^2/2$, one obtains that

$$E_p = \left[\frac{n'^2}{2} + \left(n' + \frac{1}{2}\right)\lambda\right]\hbar\omega_c p. \tag{C9}$$

The energy E_p is shown to be explicitly periodic with respect to the filling factor ν, or to its inverse ν^{-1}, the latter being proportional to the applied field H_3. We combine Eqs. (C9) and (C2) to obtain the energy per electron as a function of the applied field:

$$\epsilon(H_3) \equiv \frac{E_p}{N} = \left[\frac{n'^2}{2} + \left(n' + \frac{1}{2}\right)\lambda\right]\frac{\hbar\omega_c}{\nu}. \tag{C10}$$

Using Eq. (C6) one obtains that

$$\epsilon(H_3) = \epsilon(0) + \frac{\hbar\omega_c}{2\nu}\lambda(1 - \lambda), \tag{C11}$$

where $\epsilon(0)$ is the energy per electron in the absence of field

$$\epsilon(0) = \frac{N\pi\hbar^2}{Am}. \tag{C12}$$

This is done by first casting the energy as $\epsilon(H_3) = (n'^2 + 2n'\lambda + \lambda)\frac{\hbar\omega_c}{2\nu}$, and then adding a term $\pm\lambda^2$ inside the parentheses to obtain that $\epsilon(H_3) = [(n' + \lambda)^2 + \lambda - \lambda^2]\frac{\hbar\omega_c}{2\nu}$. Next, using that $\nu = n' + \lambda$ [Eq. (C3)], one obtains that

$$\epsilon(H_3) = \frac{1}{2}\nu\hbar\omega_c + \frac{\hbar\omega_c}{2\nu}\lambda(1 - \lambda). \tag{C13}$$

We briefly review the derivation of the average energy per electron without the applied field, given by Eq. (C12). This can be achieved by direct arguments without the inclusion of the applied field. By integrating over the Fermi surface disk, one obtains that $N/A = k_F^2/4\pi$ and that the total energy per electron is $E_T/N = E_F/2$, $E_F = (\hbar k_F)^2/2m$. Hence, one finds the zero-field energy per electron of Eq. (C12), that is, $E_T/N = \epsilon(0)$. Hence, the ratio between the energy in presence of a field $\epsilon(H_3)$ and without a field $\epsilon(0)$ becomes

$$\frac{\epsilon(H_3)}{\epsilon(0)} = 1 + \left(\frac{H_3}{H_3'}\right)^2 \lambda(1 - \lambda). \tag{C14}$$

The above expression is plotted in Fig. 6 and shows the oscillations with respect to the filling factor. Since $\nu^{-1} = H_3/H_3'$ the left plot shows that the period of oscillations increases for increasing H_3. The amplitude of oscillations smoothly increases proportional to H_3^2 according to Eq. (C14), while λ is a periodic function of $1/H_3$. The same data are also plotted in the right plot, in this case with respect to ν, and both plots show a periodicity.

The magnetization M is obtained from

$$M = -\frac{\partial\epsilon(H_3)}{\partial H_3}, \tag{C15}$$

where $\epsilon(H_3)$ is given by Eq. (C11), that once combined with Eqs. (5) and (C7) gives that

$$\epsilon(H_3) = \mu_B\left\{\frac{2\pi\hbar c N}{Ae} + \frac{H_3^2}{H_3'}\left(\frac{H_3'}{H_3} - \left[\frac{H_3'}{H_3}\right]\right)\left(1 - \frac{H_3'}{H_3} - \left[\frac{H_3'}{H_3}\right]\right)\right\}. \tag{C16}$$

M/μ_B is obtained by deriving Eq. (C16) with respect to H_3:

$$\frac{M}{\mu_B} = 1 - 2\lambda - \frac{2\lambda(1 - \lambda)}{\lambda + n'}. \tag{C17}$$

Figure 7 plots the magnetization versus $\nu = \lambda + n'$ as given by Eq. (C17). Notice that although the energy is a continuous function of H_3, its derivative is discontinuous when the Landau level is totally filled. Consider the field H_3 near to a multiple of H_3'. In this neighborhood there is a change in $\nu = \lambda + n'$ by going from $\lambda = 1^-$ (slightly below 1.0) to $\lambda = 0^+$

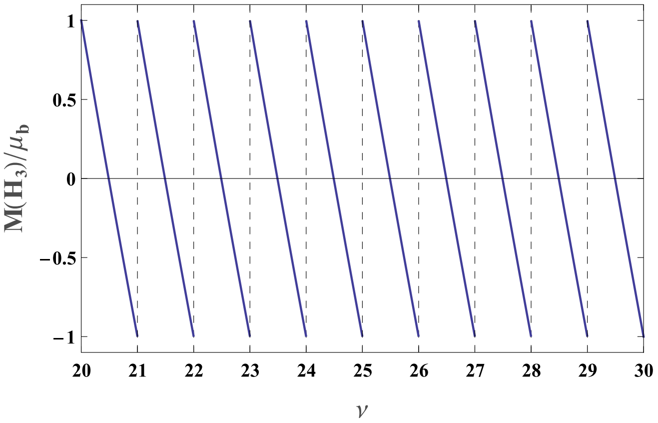

FIG. 7. The magnetization, given by Eq. (C17), as a function of $\nu = H_3'/H_3$, where $H_3' = N/(A\Phi_0)$, according to Eq. (C4). Notice the periodicity of the magnetization with respect to $1/H_3$.

(slightly above 0). A discontinuity arises in the magnetization since $\frac{M(\lambda=1^-)}{\mu_B} \approx -1.0$ and $\frac{M(\lambda=0^+)}{\mu_B} \approx 1.0$, as shown in Fig. 7.

In case many Landau levels are filled, namely, case $n' \gg 1$, Eq. (C17) becomes

$$\frac{M}{\mu_B} \approx 1 - 2\lambda, \tag{C18}$$

which shows that the magnetization M is a periodic function of $1/H_3$ with the period given by

$$\Delta\left(\frac{1}{H_3}\right) = \frac{1}{H_3'} = \frac{A}{N\Phi_0}. \tag{C19}$$

Using that $N = 4\pi/k_F^2$ (spinless fermions) we retrieve the de Haas–van Alphen result that the oscillations are inversely proportional to the Fermi surface area $A_F \equiv \pi k_F^2$:

$$\Delta\left(\frac{1}{H_3}\right) = \frac{2\pi e}{\hbar c}\frac{1}{A_F}. \tag{C20}$$

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