Non-Hermitian flat-band generator in one dimension

Wulayimu Maimaiti and Alexei Andreanov
Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, Korea
and Basic Science Program (IBS School), Korea University of Science and Technology (UST), Daejeon 34113, Korea

(Received 1 August 2020; revised 8 May 2021; accepted 25 June 2021; published 7 July 2021)

Flat bands—dispersionless bands—have been actively studied thanks to their sensitivity to perturbations, which makes them natural candidates for hosting novel and exotic states of matter. In parallel, non-Hermitian systems have attracted much attention in recent years as a simplified description of open system with gain and loss, motivated by potential applications. In particular, non-Hermitian system with dispersionless energy bands in their spectrum have been studied theoretically and experimentally. In general, flat bands require fine tuning of the Hamiltonian or protection by a symmetry. A number of methods was suggested to construct non-Hermitian flat bands relying either on the presence of symmetries, or specific frustrated geometries, often inspired by Hermitian models featuring flat bands. We introduce a systematic method to construct non-Hermitian flat bands using one-dimensional two-band tight-binding networks as an example, extending the methods used to construct systematically Hermitian flat bands. We show that the non-Hermitian case admits fine-tuned, nonsymmetry protected flat bands and provides more types of flat bands than the Hermitian case.

DOI: 10.1103/PhysRevB.104.035115

I. INTRODUCTION

Macroscopically degenerate flat bands (FB)—dispersionless energy bands in translational invariant tight-binding networks—have been an active area of research due to their sensitivity to perturbations and interactions that results in quantum phases with interesting and novel properties [1–3]. In general, flat-band models require fine tuning of the Hamiltonian parameters and multiple works have explored various method for the construction of flat-band Hamiltonians [4–17]. The effect of different perturbations on FB systems were also studied: itinerant ferromagnetism [11,13–15,18], disorder [8,19–26], external fields [27,28], nonlinearity [29–33], and interactions [34–37], including superconductivity [38–40]. Flat bands have been also experimentally realized in various Hermitian settings: photonic systems [41–43], ultracold atoms [44], and exciton-polariton condensates [45].

Recently, non-Hermitian systems [46,47] have attracted attention in different areas of physics due to their exciting properties, such as complex spectrum, nonorthogonal eigenstates [48,49], and exceptional points [50–55]. Moreover, non-Hermitian Hamiltonians can account for coupling with the environment in open quantum systems and simplifying the analysis reducing the large number of degrees of freedom associated with the environment [56–58]. After the discovery that parity-time ($\mathcal{PT}$)-symmetric non-Hermitian Hamiltonians have real eigenvalues [59–61], $\mathcal{PT}$ symmetry has been realized in optical systems with gain/loss [62], leading to an explosion of studies in $\mathcal{PT}$-symmetric non-Hermitian photonics [63–73]. Recently, non-Hermitian systems with nontrivial topological properties started to attract attention [74–77].

As the interest to non-Hermitian systems is increasing, flat bands have been predicted in various non-Hermitian systems [78–83], including optical lattices using destructive interference [84], where compact localized states are present. Non-Hermitian flat bands have also been realized experimentally [85]. Non-Hermitian flat bands (NHFB) exhibit many interesting phenomena such as polynomial power increase of flat-band eigenstates [82] and manipulation of light localization [86]. In all of these studies the existence of non-Hermitian flat bands was a consequence of either an existing Hermitian flat band, a specific frustrated geometry, or the presence of a specific symmetry in the model, such as $\mathcal{PT}$ symmetry [79,86], or non-Hermitian particle-hole symmetry [80,82]. An important question is whether it is possible to construct NHFBs from fine-tuning systematically, similarly to the Hermitian case [4,6]. The examples of Hermitian flat bands strongly suggest that the NHFB should exist in a generic non-Hermitian system without protection by any specific symmetry, merely due to fine tuning. It is also natural to ask whether such generic, nonsymmetry protected NHFB have similar properties to their Hermitian counterparts, like existence of compact localized states (CLS), etc. It is therefore important to devise a systematic construction and classification of non-Hermitian flat bands, by analogy with the Hermitian case [4,6,9].

In this work we report definition and systematic construction of non-Hermitian flat-band Hamiltonians in one-dimension (1D). For convenience we focus on the case of networks with only two bands by analogy with our previous work on systematic classification of Hermitian flat bands [4]. In Secs. II and III we introduce the classification of NHFB: perfectly flat or partially flat—either the real or imaginary part of the band is flat, or the modulus of the eigenenergy is flat. Based on this classification we explore the various NHFB models and discuss the extension of the generator approach for Hermitian systems from Refs. [4,6] in Sec. IV followed by conclusions.

II. SETTING THE STAGE

We follow the notation of our previous works, Refs. [4,6], which we adapt to the non-Hermitian setting. We consider a one-dimensional ($d = 1$) translational invariant non-Hermitian lattice with $\nu = 2$ sites per unit cell. We group the amplitudes of a wave function Ψ by unit cells: $\Psi = (\vec{\psi}_1, \dots, \vec{\psi}_i, \dots)$ where $\vec{\psi}_i$ is a two-component vector. We use the two notations, $\vec{\psi}_i$ and $|\psi_i\rangle$, interchangeably throughout the text. For convenience we only consider nearest neighbor hopping between unit cells—extension to further range hopping is straightforward. In the Hermitian case the Hamiltonian is parametrized by intracell hopping matrix H_0 and intercell hopping matrix H_1. The intracell hopping matrix H_0 is Hermitian and the intercell hopping matrix H_1 describes hopping to the left unit cell, while $H_1^\dagger$ describes hopping to the right unit cell. In the non-Hermitian case H_0 no longer has to be Hermitian and hoppings to the right and to the left neighbor unit cells need not be Hermitian conjugates of one another. Therefore we replace $H_1 \to H_l$ and $H_1^\dagger \to H_r$. Then the eigenvalue problem for the Hamiltonian $\mathcal{H}$ giving the spectrum of the model reads

$$H_0\vec{\psi}_n + H_r\vec{\psi}_{n-1} + H_l\vec{\psi}_{n+1} = E\vec{\psi}_n, \quad n \in \mathbb{Z}. \tag{1}$$

Discrete translational invariance and the Bloch theorem imply the eigenvectors of the equations are plane waves. After the Fourier transform we find

$$\begin{aligned}\mathcal{H}_k u_k &= E_k u_k,\\ \mathcal{H}_k &= H_0 + H_l e^{-ik} + H_r e^{ik}.\end{aligned} \tag{2}$$

Diagonalization of this Hamiltonian, analysis of its band structure, and identification of the flat bands is the main aim of the following sections.

We also use the notion of a compact localized state (CLS) in the text. In flat-band Hamiltonians with short-range hoppings a CLS is an eigenstate with strictly compact support, i.e., these states have nonzero amplitudes in a finite number of unit cells only. In the 1D case the CLS are conveniently classified by their size U, that is how many unit cells they occupy. We will refer to CLS of size U (occupying exactly U unit cells) as CLS of class U in what follows.

III. THE GENERATOR

The Hermitian flat-band generator with $H_l = H_r^\dagger = H_1$ was introduced in Refs. [4,6]. It relied on compact localized states as the main building block. It can be extended straightforwardly to some (but not all) non-Hermitian cases as we show below. An alternative method can be used when the spectrum of the Hamiltonian (2) is known analytically: one requires that one or more eigenvalues of $\mathcal{H}_k$ (2) are k independent. Solving the resulting equations on the matrix elements one recovers the hopping matrices H_0, H_l, H_r. This method is practical in

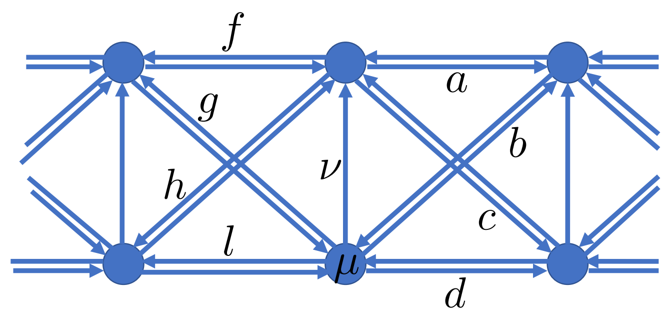

FIG. 1. Hoppings of the non-Hermitian lattice (1): The non-Hermiticity comes from the gain and loss in the hoppings, which results in the asymmetry of the hoppings represented by the arrows.

the case of a small number of bands, $\nu < 5$, where the entire spectrum can be computed analytically, at least in principle.

Since we aim for a systematic construction of NHFB, we discuss a convenient parametrization of the hopping matrices $H_{\alpha=0,l,r}$. In the Hermitian case [4] the matrix H_0 is always Hermitian and therefore can be taken diagonal in general, since Hamiltonians are equivalent up to a unitary transformation. In the non-Hermitian case that is no longer true: H_0 needs not be Hermitian and might not be diagonalizable. Still for any generic intracell hopping matrix H_0', e.g., a combination of on-site energies and asymmetric hoppings, we can parametrize H_0 by its Jordan normal form [87]

$$H_0(\nu, \mu) = \begin{pmatrix} 0 & \nu \\ 0 & \mu \end{pmatrix}, \tag{3}$$

related to the original matrix by a similarity transformation $H_0' = P^{-1}H_0P$ with invertible square matrix P. Here $\mu, \nu \in \{0, 1\}$ and $\mu\nu = 0$, i.e., they cannot be equal to 1 simultaneously. We have also used the freedom to shift and rescale the spectrum of $\mathcal{H}$, to ensure that eigenvalues of H_0 are either 0 or 1. This corresponds to three distinct possible forms of H_0:

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mu = \nu = 0 \quad \text{degenerate}, \tag{4}$$

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu = 0, \mu = 1 \quad \text{nondegenerate}, \tag{5}$$

$$H_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \nu = 1, \mu = 0 \quad \text{abnormal}. \tag{6}$$

The first two cases correspond to diagonalizable H_0, while the third one corresponds to nondiagonalizable H_0 and is special to the non-Hermitian case. To get back to the generic case of H_0' one needs to rescale and shift H_0 and then apply a similarity transformation P. Therefore, generic H_0 will have all four matrix elements nonzero, corresponding to on-site energies and asymmetric hoppings between the two sites in the unit cell.

We use the following parametrization for the intercell hopping matrices:

$$H_l = \begin{pmatrix} f & g \\ h & l \end{pmatrix}, \quad H_r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7}$$

Graphical illustration of the non-Hermitian lattice corresponding to the above hopping matrices is shown in Fig. 1. The non-Hermiticity is represented by unidirectional hopping links inside the unit cell. In some cases a suitable similarity transformation might convert these to on-site terms, e.g., gain and loss.

We emphasize here that the three cases of the intracell hopping matrix H_0 result from our freedom to shift and rescale the Hamiltonian and change the wave function basis. This is done for the convenience of systematic classification of flat bands only: other flat-band Hamiltonians can be generated from the solutions $\mathcal{H}$ presented below as follows:

(1) rescale and shift the spectrum: $\mathcal{H}_1 = a\mathcal{H} + b\mathbb{I}$ where $a, b \in \mathbb{C}$,

(2) apply a similarity transformation $\mathcal{H}_2 = P\mathcal{H}_1P^{-1}$ where P is an invertible square matrix.

In particular, with this algorithm one can choose an almost arbitrary intracell hopping matrix H_0.

The k-space Hamiltonian $\mathcal{H}_k$ (2) is a 2×2 matrix whose eigenvalues x_k, y_k (band energies) can be computed explicitly. For our purposes it is more convenient to analyze the equations

$$\begin{aligned} x_k + y_k &= \mu + e^{ik}(a+d) + e^{-ik}(f+l),\\ x_k y_k &= e^{2ik}\det H_r + e^{-2ik}\det H_l\\ &\quad + (\mu f - \nu h)e^{ik} + (\mu a - \nu c)e^{-ik}\\ &\quad + df - cg - bh + al, \end{aligned} \tag{8}$$

which follow from the Vieta's formulas for the characteristic equation of $\mathcal{H}_k$ (2).

Our goal is to analyze the solutions of these two equations with respect to the coefficients of H_r, H_l (a, b, c, d, e, f, g, h) under different constraints on the k dependence of x_k and/or y_k. We refer to this as the band calculation method. At this point it is important to note that the non-Hermitian case is richer than the Hermitian one. In the Hermitian case we can only require either of x_k, y_k or both of them to be k independent to generate *perfect flat band(s)*. For a non-Hermitian system, where x_k, y_k are not necessarily real, there are additional options of requiring only real or only imaginary parts of x_k, y_k to be k independent producing what we coin as *partial flat band(s)*. The third option is to request the modulus of x_k, y_k to be k independent.

IV. RESULTS

In this section we discuss one by one the solutions of the system (8) under the constraints on x_k, y_k discussed above. Since the complete solutions for all the cases are quite lengthy, especially for the partial flat bands, we only discuss selected cases, referring the reader to the Appendices for the complete list of solutions.

A. Perfect flat bands

For perfect flat bands we impose k independence of band energies x_k, y_k. This case is the direct extension of the Hermitian cases and we also find that flat bands support compact localized states. Therefore we adopt the language of the Hermitian flat bands: we refer to a CLS that occupies U unit cells as the class U CLS, and to the respective FB as FB of class U. The appearance of one or two FB in the cases below is by construction, e.g., due to fine tuning of the models. In general there is no simple physical argument predicting the existence of a single or multiple flat bands. One of the exceptions being the presence of a symmetry, like chiral or $\mathcal{PT}$, that can enforce flat band(s) in some situations. This is similar to the Hermitian case where most flat-bands systems result from fine tuning. We note however that a necessary condition suggesting a possibility of one or several flat bands is the degeneracy of the hopping matrices $H_{r,l}$, i.e., they should have a zero mode to expect a single flat band or several modes in case of two or more flat bands.

1. Two flat bands (all bands flat)

In this case we enforce $x_k = x$, $y_k = y$ in Eq. (8): requiring all the k-dependent terms to vanish we get (see Appendix B 1 for technical details of the derivation)

$$\begin{aligned} &a + d = 0, \quad f + l = 0,\\ &\det H_r = ad - bc = 0,\\ &\det H_l = fl - hg = 0,\\ &\mu f - \nu h = 0, \quad \mu a - \nu c = 0,\\ &xy = df - cg - bh + al,\\ &x + y = \mu. \end{aligned} \tag{9}$$

From these it follows $d = -a$, $l = -f$. Since $\det H_r = \det H_l = 0$ the hopping matrices $H_{l,r}$ are singular similarly to the Hermitian case. Solving Eqs. (9) for different choices of H_0 [Eqs. (6)]—abnormal case: $f = a = 0$, or nondegenerate case: $h = c = 0$, or degenerate case: $a, c, f, h \neq 0$—we find the hopping matrices H_l, H_r. As an illustration we provide the solution for the degenerate $\mu = \nu = 0$ case (see Appendix B 1 for the other cases):

$$\begin{aligned} H_r &= \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix},\\ H_l &= \begin{pmatrix} f & g \\ -\frac{f^2}{g} & -f \end{pmatrix},\\ E_{\rm FB}^2 &= 2af - \frac{a^2 g}{b} - \frac{bf^2}{g}, \end{aligned} \tag{10}$$

This solution has four free complex parameters a, b, f, g. Both flat bands support CLS. Using the CLS tester introduced in Ref. [4], we find that these CLS have size $U = 2$ in general. Unlike the Hermitian case [4] we were not able to reduce this model to noninteracting sites for generic values of a, b, f, g through a suitable similarity transformation. This is again in contrast with the Hermitian case, where 1D models with all bands flat can be always reduced to decoupled sites by a suitable sequence of *local* unitary transformations [88]. We picked a specific example with $a = 1 + i$, $b = 1$, $f = -1 - i$, $g = 1$, and two flat bands at $E_{\rm FB} = \pm 2(1 - i)$. The lattice and the CLS of the two flat bands are shown in Fig. 2.

Previous studies of non-Hermitian flat bands [79,86] relied on the $\mathcal{PT}$ symmetry to ensure real eigenenergies. Importantly, the above non-Hermitian Hamiltonian (10) can have two real flat bands in the absence of $\mathcal{PT}$ symmetry by a suitable choice of the free parameters: One possibility is to take $a, b, f \in \mathbb{R}$ and set $g = -b$ in Eq. (10). Then the Hamiltonian

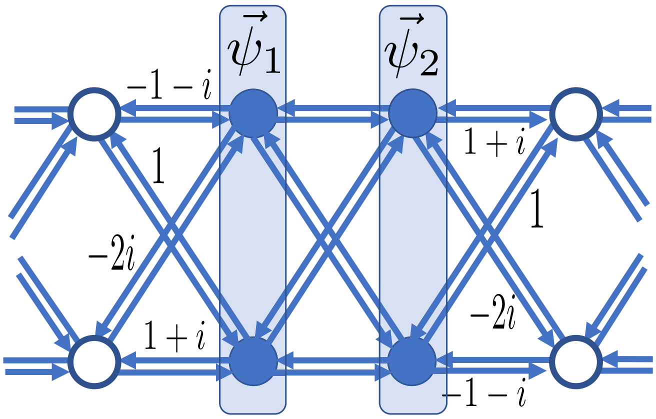

FIG. 2. Non-Hermitian lattice with all bands flat: the hopping strengths are as shown next to the links. $E_{\text{FB}} = \pm 2(1-i)$ and CLS size is $U = 2$, filled circles indicate the sites of the CLS with nonzero amplitudes $\vec{\psi}_1 = (-1, -1-i)$, $\vec{\psi}_2 = \mp(i, 1-i)$, and empty circles indicate all the other sites with zero wave amplitudes.

(2) becomes

$$H_k = \begin{pmatrix} e^{-ik}(e^{2ik}a + f) & be^{-ik}(-1 + e^{2ik}) \\ \frac{e^{-ik}(f^2 - a^2 e^{2ik})}{b} & -e^{-ik}(e^{2ik}a + f) \end{pmatrix}. \tag{11}$$

It has two real flat bands $E_{\text{FB}} = \pm(a + f)$, yet this Hamiltonian is neither Hermitian nor $\mathcal{PT}$ symmetric.

2. *Single flat band*

Requiring any of x_k or y_k in Eq. (8) to be k independent yields singular $H_{l,r}$: $\det H_r = 0$, $\det H_l = 0$. Therefore we can parametrize H_r, H_l as

$$H_r = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}, \quad H_l = \begin{pmatrix} f & g \\ h & \frac{gh}{f} \end{pmatrix}, \tag{12}$$

For concreteness we require $x_k = x$ to be k independent. Substituting $d = \frac{bc}{a}$ in Eq. (8) gives the following solution (see Appendix B 3 for details; note the two solutions corresponding to the $\pm$ signs):

$$\begin{aligned} b &= \frac{\left(-E_{\text{FB}} \pm \sqrt{E_{\text{FB}}^2 - 4af}\right)(f\nu + gE_{\text{FB}}) + 2afg}{2f^2}, \\ c &= \frac{(E_{\text{FB}} - \mu)\left[\left(E_{\text{FB}}^2 \pm E_{\text{FB}}\sqrt{E_{\text{FB}}^2 - 4af}\right) - 2af\right]}{2(f\nu + gE_{\text{FB}})}, \\ h &= \frac{f^2(\mu - E_{\text{FB}})}{f\nu + gE_{\text{FB}}}. \end{aligned} \tag{13}$$

The dispersive band is

$$\begin{aligned} E_k = \mu - E_{\text{FB}} &+ \frac{(ae^{ik} + fe^{-ik})(f\nu + g\mu)}{f\nu + gE_{\text{FB}}} \\ &- \frac{e^{ik}\nu(E_{\text{FB}} - \mu)\left(E_{\text{FB}} \pm \sqrt{E_{\text{FB}}^2 - 4af}\right)}{2(f\nu + gE_{\text{FB}})}. \end{aligned} \tag{14}$$

Substituting all the possible pairs of values of μ, ν into Eqs. (13) and (14) we generate single flat-band Hamiltonians for the degenerate, nondegenerate, and abnormal cases of H_0 and their respective band structures, as illustrated in Fig. 3.

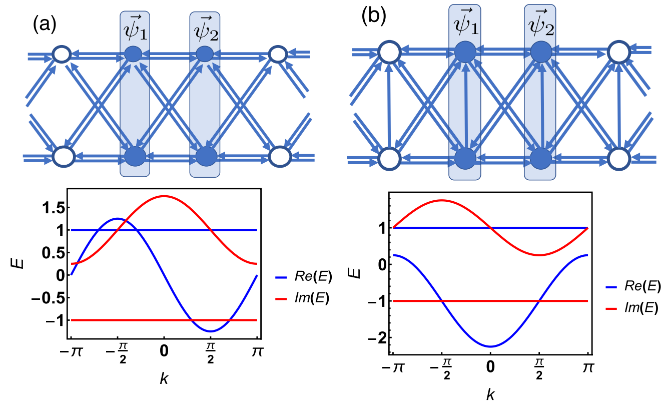

FIG. 3. Single flat band of the Hamiltonian (12): (a) Nondegenerate (canonical) case with parameters [Eq. (13)] $a = 1 + i, x = 1 - i, f = -\frac{1}{4} - \frac{i}{4}$, (b) abnormal case with parameters $a = 1 + i, x = 1 - i, f = -\frac{1}{4} - \frac{i}{4}, g = \frac{1}{4}$. The CLS size is $U = 2$ in both cases. Filled circles indicate the sites with nonzero CLS amplitudes, while the empty circles are the sites with zero amplitudes. The CLS amplitudes are given by Eq. (A5) in Appendix A 2.

Interestingly the degenerate case $\mu = \nu = 0$ reduces to the previously considered case of two flat bands, with $E = \pm x$. There are four free complex parameters: x, a, f, g while in the Hermitian case there are only two real free parameters [4]. The CLS corresponding to this FB are in general of size $U = 2$ (see also Sec. IV B and Appendix A 2). Also unlike the Hermitian case where a FB in a two-band system is always gapped away from the dispersive band, in the non-Hermitian setting, the FB and the dispersive bands can have crossings of the real and imaginary parts of the energy as illustrated in Fig. 3(a).

B. CLS based construction

We rederive the above results with the help of the CLS based method originally developed for the Hermitian models [4,6]. We sketch briefly the straightforward modification of the construction for the Hermitian case and refer the interested reader to Appendix A which contains the full details of the method. We search for a non-Hermitian Hamiltonian with the CLS $\Psi_{\text{CLS}} = (\vec{\psi}_1, \vec{\psi}_2, \ldots, \vec{\psi}_U)$ of size U. This CLS is an eigenvector of the $U \times U$ block tridiagonal matrix

$$\mathcal{H}_U = \begin{pmatrix} H_0 & H_l & 0 & 0 & \ldots & 0 \\ H_r & H_0 & H_l & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & H_r & H_0 & H_l \\ 0 & \ldots & 0 & 0 & H_r & H_0 \end{pmatrix}, \tag{15}$$

with eigenenergy $E_{\text{FB}} \in \mathbb{C}$. Out of the U eigenvectors of this Hamiltonian the CLS is selected by the destructive interference conditions

$$H_l|\psi_1\rangle = H_r|\psi_U\rangle = 0 \tag{16}$$

that ensure that the eigenstate remains compactly localized under the action of the full Hamiltonian $\mathcal{H}$. Therefore a *necessary* condition for existence of a non-Hermitian CLS reads

$$\det H_l = \det H_r = 0, \tag{17}$$

and coincides with the results obtained in the previous section. Similarly to the Hermitian case, the eigenproblem for the CLS (15) and (16) reads

$$\begin{aligned} H_l|\psi_2\rangle &= (E_{\rm FB} - H_0)|\psi_1\rangle, \\ H_r|\psi_{j-1}\rangle + H_l|\psi_{j+1}\rangle &= (E_{\rm FB} - H_0)|\psi_j\rangle, \quad 2 \leqslant j \leqslant U-1, \\ H_r|\psi_{U-1}\rangle &= (E_{\rm FB} - H_0)|\psi_U\rangle, \\ H_l|\psi_1\rangle &= H_r|\psi_U\rangle = 0, \\ |\psi_j\rangle &= 0, \quad j<0,\ j>U. \end{aligned} \tag{18}$$

The non-Hermitian FB generator is the set of all possible matrices H_0, H_r, H_l and CLSs $\Psi_{\rm CLS} = (\vec{\psi}_1, \vec{\psi}_2, \dots, \vec{\psi}_U)$ that satisfy Eq. (18). It is an inverse eigenvalue problem similar to the Hermitian case [6], and is solved by treating H_0, $\vec{\psi}_{i=1,\dots,U}$ and $E_{\rm FB}$ as inputs and reconstructing $H_{l,r}$. It is worth pointing out that H_0 does not have to have one of the three forms (6). This formulation is especially convenient for constructing the most common type of non-Hermitian Hamiltonians, where $H_l = H_r^\dagger$, e.g., the hopping is Hermitian, but the on-site terms are not, i.e., diagonal of H_0 contains gain and loss terms.

As an example we provide a solution for the $U=2$ case. The solution of this eigenproblem in the case of two bands $\nu=2$ and H_0 given by Eq. (3) is presented in Appendix A. In the $U=2$ case the eigensystem (18) simplifies to

$$\begin{aligned} H_l|\psi_2\rangle &= (E_{\rm FB} - H_0)|\psi_1\rangle, \\ H_r|\psi_1\rangle &= (E_{\rm FB} - H_0)|\psi_2\rangle, \\ H_l|\psi_1\rangle &= H_r|\psi_2\rangle = 0. \end{aligned} \tag{19}$$

Unlike the Hermitian case [6], now the $U=2$ case decouples into two independent inverse eigenvalue problems for H_l and H_r. Their solutions are

$$\begin{aligned} H_l &= \frac{(E_{\rm FB} - H_0)|\psi_1\rangle\langle\psi_2|Q_1}{\langle\psi_2|Q_1|\psi_2\rangle}, \\ H_r &= \frac{(E_{\rm FB} - H_0)|\psi_2\rangle\langle\psi_1|Q_2}{\langle\psi_1|Q_2|\psi_1\rangle}. \end{aligned} \tag{20}$$

Similarly to the Hermitian case [4], the maximum size of a CLS is $U=2$. This is confirmed by the consistency of the $U=2$ solution with the solution from the band calculation method that we presented above and which contains all the possible two-band non-Hermitian FB Hamiltonians.

C. Partial flat bands

Partial flat bands when only real, imaginary parts, or modulus of the eigenenergy are k independent only exist for non-Hermitian Hamiltonians. Unlike the case of perfect flat bands, the eigenstates of partial flat bands are not compactly localized, since the bands are not fully flat. Therefore, the CLS construction cannot be employed in this case and we use the band calculation method, which also restricts us to the case of two bands only.

We write $x_k = x_1 + ix_2, y = y_1 + iy_2$, where $x_i, y_i \in \mathbb{R}$, $i = 1, 2$ and write the real and imaginary parts of Eq. (8) separately:

$$\begin{aligned} x_1 + y_1 &= \mu + (a+d+f+l)\cos k, \\ x_2 + y_2 &= -\ (a+d-f-l)\sin k, \\ x_1y_1 - x_2y_2 &= al - bh - cg + df \\ &\quad + (a\mu - c\nu + f\mu - h\nu)\cos k \\ &\quad + (\det H_l + \det H_r)\cos 2k, \\ x_2y_1 + x_1y_2 &= (-a\mu + c\nu + f\mu - h\nu)\sin k \\ &\quad + (\det H_l - \det H_r)\sin 2k. \end{aligned} \tag{21}$$

Requiring different subsets of x_1, x_2, y_1, y_2 to be k independent, and solving the above set of equations with respect to the hopping matrices H_r, H_r, we can find Hamiltonians with partial flat bands. There are many possible subsets for partial flat bands: real (imaginary) parts of both bands are flat, real (imaginary) part of only one band is flat, real (imaginary) part of one band and imaginary (real) part of the other band are flat, etc. Also since the band energies x_k, y_k are in general complex, they can also be parametrized by their moduli and phases, e.g., $x_k = r_k e^{i\theta_k}$ and similarly for y_k. We note that the case of flat phase of the band simply corresponds to real band energy. We only consider two specific cases of partial flat bands below—flat real parts and modulus of the band is flat, while all the other cases as well as the full derivations are provided in Appendix C. This choice is dictated by convenience—the derivations are quite lengthy for all these cases.

a. Real parts of two bands are flat. In this case x_1, y_1 in Eq. (21) are k independent. We consider here the abnormal H_0, i.e., $\nu = 1, \mu = 0$. Solving the system (21) we find the following solution (see Appendix C 1 for further details):

$$\begin{aligned} H_0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ H_l &= \begin{pmatrix} f & \frac{h^2 - x_1^2(d+f-x_1)^2}{4hx_1^2} \\ h & x_1 - d \end{pmatrix}, \\ H_r &= \begin{pmatrix} -f - x_1 & \frac{x_1^2(d+f+x_1)^2 - h^2}{4hx_1^2} \\ -h & d \end{pmatrix}, \end{aligned} \tag{22}$$

The corresponding band structure is

$$\begin{aligned} E_1 &= -x_1 + i\sin(k)\left(d - f + \frac{h}{x_1} - x_1\right), \\ E_2 &= x_1 - i\sin(k)\frac{[x_1(-d+f+x_1)+h]}{x_1}, \end{aligned} \tag{23}$$

where the parameters x_1, d, f, h are real. Figure 4 shows a specific example with $x_1 = 1, d = -1, f = 1, h = 1$. As the real part of the eigenenergy is flat, it might seem that kinetic energy is quenched and no transport is possible in that band. However, one might observe an apparent spreading of an initially localized state due to the k dependence of the imaginary part of the eigenenergy. If the original state overlaps with several eigenmodes, the mode with the largest imaginary part would be amplified (or decay the slowest if all the eigenmodes are decaying). As the eigenmodes are extended this might lead to an apparent spreading of the initial state.

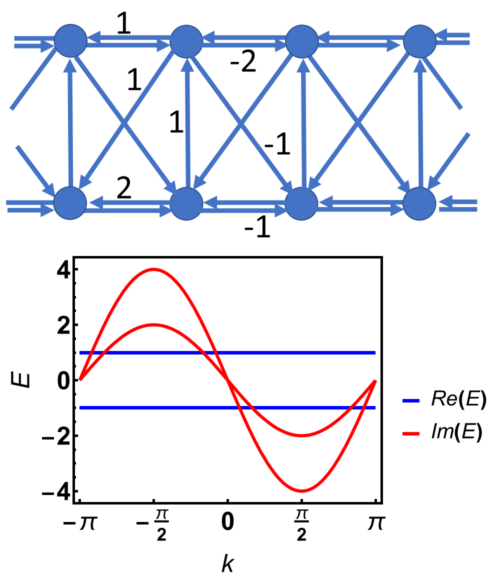

FIG. 4. Example of real parts of both bands flat case with abnormal H_0. The parameters in these examples are $x_1 = 1, d = -1, f = 1, h = 1$. Corresponding hopping strengths are also given in the figure.

b. Modulus of a band is flat. We assume that one of the bands has a constant modulus, but allow the phase to be k dependent: $x = re^{i\theta_k}, r \in \mathbb{R}$ in Eq. (8). Since Eq. (8) only involves integer powers of e^{ik}, this implies that the phase is also an integer multiple of k, i.e., $\theta_k = mk$, and m can only take two values $m = \pm 1$ (see Appendix C 5). For each value of m we solve Eq. (8) and get the hopping matrices H_r, H_l that gives the band structure $E = re^{i\theta_k}, r \in \mathbb{R}$. As an example we present results for $m = 1$ and abnormal H_0, i.e., $\mu = 0, \nu = 1$:

$$H_r = \begin{pmatrix} r & b \\ 0 & d \end{pmatrix},$$
$$H_l = \begin{pmatrix} 0 & g \\ 0 & l \end{pmatrix}, \tag{24}$$

which has the following band structure:

$$E_1 = e^{ik} r,$$
$$E_2 = de^{ik} + e^{-ik} l + 1. \tag{25}$$

Similarly, other cases of H_0 and $m = -1$ case are solved (see details in Appendix C 5). Interestingly the above band E_1 has a nontrivial winding number.

V. CONCLUSIONS AND DISCUSSIONS

We presented a systematic way of constructing flat bands in non-Hermitian networks with two or more bands. The existence of these flat bands relies on fine tuning of the hoppings and on-site energies rather than specific symmetry, like chiral or parity-time symmetries. We considered two distinct cases of perfect and partial flat bands. The former are straightforward extensions of Hermitian flat bands and also host compact localized states that can be used to construct and classify such models. We focused in this work on the case of two band networks and we discovered that the maximum possible size of the CLS is $U = 2$ for the perfect flat bands, similarly to the Hermitian case. We have also provided examples of real valued perfect flat bands in non-Hermitian models not enforced by any symmetry. The other case considered was partial flat bands that are absent in the Hermitian models and are special to the non-Hermitian systems with either a real or imaginary part or modulus of the eigenenergies becoming dispersionless. Understanding their physical properties is an interesting direction of future research. Interestingly, in the case of constant modulus of the eigenenergy we observed that the respective band has a nontrivial winding number.

Our work is only a first step in systematic understanding of non-Hermitian flat bands. The obvious open questions are the connection to Hermitian flat bands—how could one perturb them systematically with gain/loss terms and preserve the flat band? What is the effect of various perturbations—disorder and interaction—on the non-Hermitian flat bands? The partial flat bands do not have analogies in Hermitian models and are therefore of special interest. The case of the flat imaginary part of eigenergies (see Appendix C 4) might be interesting, as it might exhibit dynamics that is sensitive to initial conditions, due to the presence of gain/loss and in contrast to the real part of the eigenenergy flat case. It might be interesting to study this case further, to see whether the propagation dynamics remain qualitatively similar to the Hermitian case, or if new features emerge due to non-Hermiticity. In cases of real perfect flat bands we might expect an appearance in finite chains of edge states with complex energies, or more dramatically, non-Hermitian skin effect, in which the bulk eigenvalues are highly sensitive to the boundary conditions.

Another interesting aspect of non-Hermitian systems is the appearance of exceptional points in their spectra, where several eignvectors coalesce. We have identified two models where exceptional points occur, both corresponding to an all bands flat case: the degenerate case (10) and the abnormal case (B6) in Appendix B. In both models the flat-band energy has to be zero and to be doubly degenerate for the exceptional points to occur. The exceptional points occupy the entire Brillouin zone in both models. We note that these two cases provide examples of all bands flat systems with two bands and doubly degenerate flat-band energy, yet nontrivial Hamiltonian: a situation impossible for any Hermitian model. The presence of an exceptional point in the abnormal case suggests that its appearance in the spectrum might be triggered by imposing nondiagonalizable intracell hopping H_0. The other case, the degenerate all band flat system, suggests to search for exceptional points in non-Hermitian flat-band systems with doubly or higher degenerate flat bands. The models with a single perfect flat band do not show any exceptional points in their spectra. Understanding further the properties of these two all bands flat cases related to their exceptional points is an interesting open problem, as well as studying further the relation between flat bands and exceptional points in the non-Hermitian systems.

Finally, our work focused on identifying flat bands in the parameter space of Hamiltonians, and therefore we worked in the momentum space for convenience. We expect that non-Hermiticity might affect the behavior of the flat bands in real space, i.e., in the presence of open boundaries. First we note that perfect flat bands that feature compact localized states should not be strongly affected by the presence of the boundaries—the absolute majority of the CLS remain eigenstates of the Hamiltonian in this case due to their compactness, and only those CLS at the boundary or cut by the boundary might be replaced by edge modes as it happens in the Hermitian case. The situation is different for the partial flat bands which do not feature compact eigenstates and therefore might be strongly affected by the presence of a boundary. Understanding their fate in this case is an interesting open question requiring further investigation, in particular a possibility of non-Hermitian skin effect in partial flat bands, or in proximity to perfect flat bands.

ACKNOWLEDGMENTS

We are grateful to S. Flach and D. Leykam for their valuable feedback on the manuscript. This work was supported by the Institute for Basic Science in Korea (IBS-R024-D1).

APPENDIX A: CLS BASED GENERATOR

In this Appendix we show the CLS based derivation of the perfect flat bands.

1. The $U = 1$ case

In this case the eigenvalue problem (18) becomes

$$\begin{aligned} H_0\vec{\psi} &= E_{\rm FB}\vec{\psi}, \\ H_r\vec{\psi} &= H_l\vec{\psi} = 0. \end{aligned} \tag{A1}$$

Parametrizing $\vec{\psi} = (x, y) \neq 0$, Eq. (A1) becomes

$$\begin{aligned} \nu y &= E_{\rm FB}x, \\ \mu y &= E_{\rm FB}y, \\ ax + by &= E_{\rm FB}x, \\ cx + dy &= E_{\rm FB}y, \\ fx + gy &= E_{\rm FB}x, \\ hx + ly &= E_{\rm FB}y. \end{aligned} \tag{A2}$$

Next we solve this system for all the possible choices of H_0.

c. Degenerate H_0. In this case $\mu = \nu = 0$ and Eq. (A2) yields the following solution:

$$\begin{aligned} &E_{\rm FB} = 0, \\ &a = -\frac{y}{x}b, \quad c = -\frac{y}{x}d, \\ &f = -\frac{y}{x}g, \quad h = -\frac{y}{x}l. \end{aligned}$$

Therefore the hopping matrices read

$$\begin{aligned} &H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ &H_l = \begin{pmatrix} -\frac{y}{x}g & g \\ -\frac{y}{x}l & l \end{pmatrix}, \quad H_r = \begin{pmatrix} -\frac{y}{x}b & b \\ -\frac{y}{x}d & d \end{pmatrix}. \end{aligned}$$

d. Abnormal H_0. In this case $\mu = 0, \nu = 1$ and Eq. (A2) gives the following solution $y = a = c = f = h = 0$. Therefore the hopping matrices, CLS, and FB energy are

$$\begin{aligned} &H_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ &H_l = \begin{pmatrix} 0 & g \\ 0 & l \end{pmatrix}, \quad H_r = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \\ &\vec{\psi} = (x, 0), \quad E_{\rm FB} = 0. \end{aligned}$$

e. Nondegenerate H_0. In this case $\mu = 1, \nu = 0$ and Eq. (A1) have the following solution:

$$\begin{aligned} &H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ &H_l = \begin{pmatrix} f & 0 \\ g & 0 \end{pmatrix}, \quad H_r = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \\ &\vec{\psi} = (0, 1), \quad E_{\rm FB} = 1. \end{aligned}$$

Here y has been normalized to 1.

2. $U = 2$ case

In this case the eigenvalue problem (18) becomes

$$\begin{aligned} &H_0\psi_1 + H_l\psi_2 = E_{\rm FB}\psi_1, \\ &H_0\psi_2 + H_r\psi_1 = E_{\rm FB}\psi_2, \\ &H_l\psi_1 = 0, \quad H_r\psi_2 = 0. \end{aligned} \tag{A3}$$

We can parametrize H_r, H_l as follows:

$$H_r = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}, \quad H_l = \begin{pmatrix} f & g \\ h & \frac{gh}{f} \end{pmatrix}, \tag{A4}$$

which satisfy destructive interference conditions by definition. Then we can choose ψ_1, ψ_2 to be zero eigenvector of H_l, H_r respectively:

$$\psi_1 = (-g\ f), \quad \psi_2 = \alpha\begin{pmatrix} -b \\ a \end{pmatrix}. \tag{A5}$$

Then Eqs. (A3) become

$$\begin{aligned} &\begin{pmatrix} \alpha ag - \alpha bf + f\nu \\ \frac{\alpha agh}{f} - \alpha bh + f\mu \end{pmatrix} = E_{\rm FB}\begin{pmatrix} -g \\ f \end{pmatrix}, \\ &\begin{pmatrix} \alpha a\nu - ag + bf \\ \alpha a\mu + \frac{bcf}{a} - cg \end{pmatrix} = \alpha E_{\rm FB}\begin{pmatrix} -b \\ a \end{pmatrix}. \end{aligned} \tag{A6}$$

Solving Eqs. (A6) above we get ($\lambda = E_{\rm FB}$)

$$\begin{aligned} b &= \frac{(-\lambda \pm \sqrt{\lambda^2 - 4af})(f\nu + g\lambda) + 2afg}{2f^2}, \\ c &= \frac{(\lambda - \mu)[(\lambda^2 \pm \lambda\sqrt{\lambda^2 - 4af}) - 2af]}{2(f\nu + g\lambda)}, \\ h &= \frac{f^2(\mu - \lambda)}{f\nu + g\lambda}, \\ \alpha &= -\frac{\lambda \pm \sqrt{\lambda^2 - 4af}}{2a}. \end{aligned} \tag{A7}$$

Putting the corresponding values of μ, ν in Eq. (A7) we get solutions for degenerate, nondegenerate, and abnormal cases.

Obviously the above solution (A7) is the same with the solution (13) from band calculation method. Since the solution of band calculation method contains all possible U class, $U = 2$ is the maximum CLS class. The band structure corresponding to above solution (A7) is

$$E_{\rm FB} = \lambda,$$
$$E_k = \mu - \lambda + \frac{(ae^{ik} + fe^{-ik})(f\nu + g\mu)}{f\nu + g\lambda} - \frac{e^{ik}\nu(\lambda - \mu)(\lambda \pm \sqrt{\lambda^2 - 4af})}{2(f\nu + g\lambda)}.$$

APPENDIX B: PERFECT FLAT BANDS: BAND CALCULATION METHOD

A completely flat band has k-independent real and imaginary parts. Our starting point solving this case is Eq. (8), which is

$$\begin{aligned} x_k + y_k &= \mu + e^{ik}(a+d) + e^{-ik}(f+l),\\ x_k y_k &= e^{2ik}\det H_r + e^{-2ik}\det H_l\\ &\quad + (\mu f - \nu h)e^{ik} + (\mu a - \nu c)e^{-ik}\\ &\quad + df - cg - bh + al. \end{aligned} \tag{B1}$$

We assume one of x_k, y_k or both are k independent and solve the above system to find the flat-band Hamiltonian.

1. All bands flat

In this case both x_k, y_k in Eqs. (B1) are k independent. We assume $x_k = x$ and $y_k = y$, then Eq. (B1) becomes

$$\begin{aligned} x + y &= \mu + e^{ik}(a+d) + e^{-ik}(f+l),\\ xy &= e^{2ik}\det H_r + e^{-2ik}\det H_l + (\nu f - \mu h)e^{ik}\\ &\quad + (\nu a - \mu c)e^{-ik} + df - cg - bh + al. \end{aligned} \tag{B2}$$

Requiring polynomial of e^{ik} to vanish gives the following equations:

$$\begin{aligned} &a + d = 0,\\ &f + l = 0,\\ &\det H_r = ad - bc = 0,\\ &\det H_l = fl - hg = 0,\\ &\mu f - \nu h = 0,\\ &\mu a - \nu c = 0,\\ &xy = df - cg - bh + al,\\ &x + y = \mu. \end{aligned} \tag{B3}$$

From these it follows $d = -a$, $l = -f$ and either $f = a = 0$, or $h = c = 0$, or none of a, c, f, h is zero.

Solving Eq. (B3) for degenerate, nondegenerate, and abnormal cases separately, we can get H_l, H_r that gives both bands flat.

f. Degenerate case. In this case we have $\mu = \nu = 0$ and $bc + a^2 = 0,\ hg + f^2 = 0,\ y = -x$. Therefore, $c = -a^2/b$, $h = -f^2/g$:

$$H_r = \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix},\quad H_l = \begin{pmatrix} f & g \\ -\frac{f^2}{g} & -f \end{pmatrix},$$
$$x^2 = 2af - \frac{a^2 g}{b} - \frac{bf^2}{g}. \tag{B4}$$

Then the corresponding Hamiltonian has two FBs at energies $E_{\rm FB} = \pm x$.

g. Nondegenerate H_0. In this case $\mu = 1, \nu = 0$ in Eqs. (B3). Then Eq. (B3) give $a = d = f = l = 0$, implying $bc = gh = 0$, and $y = 1 - x$. Therefore either b or c are nonzero and either h or g are nonzero:

$$\begin{aligned} H_r &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},\\ H_l &= \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix},\\ x(1-x) &= -cg - bh. \end{aligned} \tag{B5}$$

There are four possible solutions of $bc = 0, hg = 0$. Interestingly, two out of the four imply $x = 0, y = 1$ or $x = 1, y = 0$. Since the zero eigenvectors of H_l, H_r are proportional and they are also an eigenvector of H_0 corresponding to eigenvalue 1 and they satisfy Eq. (A1), the CLS class is $U = 1$.

h. Abnormal H_0. In this case $\mu = 0$, $\nu = 1$ and $h = c = 0$, and $y = -x$. Therefore $a = d = f = l = 0$, and

$$\begin{aligned} H_r &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\quad H_l = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix},\\ x &= y = 0. \end{aligned} \tag{B6}$$

Similarly to the nondegenerate case in the previous paragraph, the above solution also has $U = 1$ class. It is also straightforward to observe that every point in the Brillouin spectrum in this case is exceptional.

2. Checking the CLS class of both bands are completely flat case

The solutions (B4) for the degenerate all bands flat case do not tell the CLS class. Here we show a test procedure to check the CLS class of the given solution.

According to the destructive interference condition (18), the zero eigenvectors of H_l, H_r are $\vec{\psi}_1, \vec{\psi}_U$, respectively. Thus we define

$$\vec{\psi}_1 = \begin{pmatrix} -g \\ f \end{pmatrix},\quad \vec{\psi}_U = \alpha\begin{pmatrix} -b \\ a \end{pmatrix}.$$

Suppose the CLS class is $U = 2$, then we have $\vec{\psi}_2 = \vec{\psi}_U$. Then the eigenvalue Eq. (A3) are always satisfied with $\alpha = \pm\frac{i\sqrt{g}}{\sqrt{b}}$. Therefore the CLS class is $U = 2$.

3. Single perfect flat band

Requiring any of x or y in Eq. (B1) will yield $\det H_r = 0$, $\det H_l = 0$, therefore we can parametrize H_r, H_l as

$$H_r = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix},\quad H_l = \begin{pmatrix} f & g \\ h & \frac{gh}{f} \end{pmatrix},$$

which makes H_r, H_l to be singular by definition. I assume that only $x_k = x$ is flat, then Eq. (B1) becomes

$$x + y_k = \frac{bce^{ik}}{a} + ae^{ik} + e^{-ik}\left(\frac{gh}{f} + f\right) + \mu,$$
$$xy_k = \frac{(ag - bf)(ah - cf)}{af} + e^{ik}(a\mu - c\nu) + e^{-ik}(f\mu - h\nu). \tag{B7}$$

This results into the following equations:

$$y_k = \frac{bce^{ik}}{a} + ae^{ik} + e^{-ik}\left(\frac{gh}{f} + f\right) + \mu - x,$$
$$y_k = \frac{e^{ik}(a\mu - c\nu) + e^{-ik}(f\mu - h\nu)}{x} + \frac{(ag - bf)(ah - cf)}{afx}. \tag{B8}$$

Consequently, equating powers of e^{ik}, I find

$$\frac{bc}{a} + a = \frac{a\mu - c\nu}{x},$$
$$\frac{gh}{f} + f = \frac{f\mu - h\nu}{x}, \tag{B9}$$
$$\mu - x = \frac{(ag - bf)(ah - cf)}{x(af)}.$$

Solving Eq. (B9)

$$b = \frac{(-x \pm \sqrt{x^2 - 4af})(f\nu + gx) + 2afg}{2f^2},$$
$$c = \frac{(x - \mu)[(x^2 \pm x\sqrt{x^2 - 4af}) - 2af]}{2(f\nu + gx)}, \tag{B10}$$
$$h = \frac{f^2(\mu - x)}{f\nu + gx}.$$

Then the band structure is

$$E_{\rm FB} = x,$$
$$E_k = \mu - x + \frac{(ae^{ik} + fe^{-ik})(f\nu + g\mu)}{f\nu + gx} - \frac{e^{ik}\nu(x - \mu)(x \pm \sqrt{x^2 - 4af})}{2(f\nu + gx)}. \tag{B11}$$

Putting corresponding values of μ, ν into Eqs. (B10) and (B11), we can get solutions for degenerate, nondegenerate, and abnormal cases and the corresponding band structures.

APPENDIX C: SOLVING PARTIAL FLAT BANDS

We assume that $x_k = x_1 + ix_2$, $y = y_1 + iy_2$, and $x_1, x_2, y_1, y_2 \in \mathcal{R}$, then complex expanding Eq. (B1) yields

$$x_k + y_k = x_1 + y_1 + i(x_2 + y_2) = \mu + \cos(k)(a + d + f + l) - i\sin(k)(a + d - f - l),$$
$$x_k y_k = x_1y_1 - x_2y_2 + i(x_2y_1 + x_1y_2) = al - bh - cg + df + (a\mu - c\nu + f\mu - h\nu)\cos(k) + (\det H_l + \det H_r)\cos(2k) + i[(-a\mu + c\nu + f\mu - h\nu)\sin(k) + (\det H_l - \det H_r)\sin(2k)]. \tag{C1}$$

Equating real and imaginary parts of Eq. (C1):

$$x_1 + y_1 = \mu + (a + d + f + l)\cos(k), \tag{C2}$$
$$x_2 + y_2 = -(a + d - f - l)\sin(k), \tag{C3}$$
$$x_1y_1 - x_2y_2 = al - bh - cg + df + (a\mu - c\nu + f\mu - h\nu)\cos(k) + (\det H_l + \det H_r)\cos(2k), \tag{C4}$$
$$x_2y_1 + x_1y_2 = (-a\mu + c\nu + f\mu - h\nu)\sin(k) + (\det H_l - \det H_r)\sin(2k). \tag{C5}$$

Solving Eqs. (C2)–(C5) under the condition that some of x_1, x_2, y_1, y_2 to be k independent, we get the solution for partially flat bands.

1. Real parts of both bands are flat

In this case x_1, y_1 are k independent, then Eqs. (C2) and (C3) gives

$$y_1 = \mu - x_1 + (a + d + f + l)\cos(k),$$
$$y_2 = -x_2 - (a + d - f - l)\sin(k). \tag{C6}$$

We put Eq. (C6) into Eqs. (C4) and (C5), and solve for x_2. Then requiring x_1 to be k independent, by which making coefficients of k-independent terms to be zero, we got the following equations:

$$a + d + f + l = 0,$$
$$a - c + f - h = 0,$$
$$ad + fl - bc - gh = \frac{(X + Y)(X + Z)}{2(\mu - 2x_1)^2}, \tag{C7}$$
$$-ad + bc + fl - gh = 0,$$

where $X = x_1(-a - d + f + l)$, $Y = a\mu - c\nu - f\mu + h\nu$, $Z = c\nu + d\mu - h\nu - l\mu$.

Solving Eqs. (C7) for degenerate, nondegenerate, and abnormal cases separately, we got the H_r, H_l that gives the real part of both bands are flat.

i. Degenerate case. In this case $\mu = \nu = 0$, and Eqs. (C7) becomes

$$a + d + f + l = 0,$$
$$\tfrac{1}{8}(a + d - f - l)^2 + bc + gh = ad + fl,$$
$$-ad + bc + fl - gh = 0,$$
$$\tfrac{1}{8}(a + d - f - l)^2 + al + df + x_1^2 = bh + cg. \tag{C8}$$

If we consider x_1 as a parameter, then the solution for Eq. (C8) is

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$H_l = \begin{pmatrix} -f & \frac{\pm 2\sqrt{A}+B}{(a-d)^2} \\ -\frac{\pm 2\sqrt{A}+B}{4b^2} & -a-d-f \end{pmatrix},$$
$$H_r = \begin{pmatrix} a & b \\ -\frac{(a-d)^2}{4b} & d \end{pmatrix},$$

where $A = b^2x_1^2[(d-a)(a+d+2f)+x_1^2]$, $B = b(a-d)(a+d+2f) - 2bx_1^2$. Then the band structure is

$$x_k = -x_1 + i(a+d)\sin(k),$$
$$y_k = x_1 + i(a+d)\sin(k).$$

If we consider x_1 as a function of H_r, H_l, then the solution for Eqs. (C8) is

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$H_l = \begin{pmatrix} f & -\frac{(a+d+2f)^2}{4h} \\ h & -a-d-f \end{pmatrix},$$
$$H_r = \begin{pmatrix} a & b \\ -\frac{(a-d)^2}{4b} & d \end{pmatrix},$$
$$x_1 = \pm\frac{(a-d)(a+d+2f)+4bh}{4\sqrt{b}\sqrt{h}}.$$

The band structure is

$$x_k = -\frac{\sqrt{bh[(a-d)(a+d+2f)+4bh]^2}}{4bh} + \frac{4ibh(a+d)\sin(k)}{4bh},$$
$$y_k = \frac{\sqrt{bh[(a-d)(a+d+2f)+4bh]^2}}{4bh} + \frac{4ibh(a+d)\sin(k)}{4bh}.$$

j. Nondegenerate case. In this case $\mu = 1, \nu = 0$, and Eqs. (C7) become

$$a+d+f+l = 0,$$
$$a+f = 0,$$
$$\frac{(X+a-f)(X+d-l)}{2(1-2x_1)^2} + bc + gh = ad + fl, -ad + bc + fl - gh = 0,$$
$$\frac{x_1(1-x_1)(a-d-f+l)^2}{2(1-2x_1)^2} = -(bh+cg+x_1) + al + df + x_1^2,$$

where $X = x_1(-a-d+f+l)$. Then the solution is

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$H_r = \begin{pmatrix} -f & -\frac{(x_1-1)x_1(f-l)^2}{c(1-2x_1)^2} \\ c & -l \end{pmatrix},$$
$$H_l = \begin{pmatrix} f & \frac{(x_1-1)x_1(C+D)}{2c(1-2x_1)^2} \\ \frac{c(C-D)}{2(f-l)^2} & l \end{pmatrix},$$

where $C = \sqrt{4(f-l)^2 + (1-2x_1)^2}(1-2x_1)$, $D = 2(f-l)^2 + (1-2x_1)^2$.

Then the band structure is

$$x_k = x_1 - \frac{2i\sin(k)[x_1(f+l)-f]}{2x_1-1},$$
$$y_k = -\frac{2i\sin(k)[x_1(f+l)-l]}{2x_1-1} - x_1 + 1.$$

Obviously the real parts x_1, $1-x_1$ are k independent, i.e., flat.

k. Abnormal case. In this case $\mu = 0, \nu = 1$, and Eqs. (C7) becomes

$$a+d+f+l = 0,$$
$$-c-h = 0,$$
$$\frac{(X+c-h)(X-c+h)}{8x_1^2} = ad + fl - bc - gh,$$
$$-ad + bc + fl - gh = 0,$$
$$bh + cg + \frac{(c-h)^2}{8x_1^2} = \frac{1}{8}(a+d-f-l)^2 + al + df + x_1^2,$$

where $X = x_1(-a-d+f+l)$. Then the solution is

$$H_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$H_l = \begin{pmatrix} f & \frac{h^2 - x_1^2(d+f-x_1)^2}{4hx_1^2} \\ h & x_1 - d \end{pmatrix},$$
$$H_r = \begin{pmatrix} -f-x_1 & \frac{x_1^2(d+f+x_1)^2-h^2}{4hx_1^2} \\ -h & d \end{pmatrix}.$$

Then the band structure is

$$x_k = -x_1 + i\sin(k)\left(d - f + \frac{h}{x_1} - x_1\right),$$
$$y_k = x_1 - i\sin(k)\frac{[x_1(-d+f+x_1)+h]}{x_1}.$$

2. Real part of one band is flat

In this case either x_1 or y_1 is k independent in Eqs. (C2)–(C5). If we assume x_1 is k independent and solve Eqs. (C2)–(C5) for x_1, then according to Eq. (C2) we have

$$x_1 = \mu,$$
$$y_1 = (a+d+f+l)\cos(k).$$

[In a similar way we can assume y_1 is k independent, then $y_1 = \mu, x_1 = (a+d+f+l)\cos(k)$.] Then Eqs. (C2)–(C5) become

$$\begin{aligned} x_2+y_2 &= \sin(k)(a+d-f-l),\\ x_2y_2 &= -al+bh+\cos(k)[\nu(c+h)+d\mu+l\mu]\\ &\quad +cg-df-(\det H_l+\det H_r)\cos(2k),\\ \mu y_2 &= \sin(k)(a\mu-c\nu-f\mu+h\nu)\\ &\quad -x_2\cos(k)(a+d+f+l)\\ &\quad +(\det H_r-\det H_l)\sin(2k). \end{aligned} \tag{C9}$$

For convenience we make the following replacement of variables:

$$\begin{aligned} \det H_r &= ad-bc,\\ \det H_l &= fl-gh,\\ V &= a+d+f+l,\\ X &= a+d-f-l,\\ Y &= a\mu-c\nu+f\mu-h\nu,\\ Z &= a\mu-c\nu-f\mu+h\nu,\\ W &= al-bh-cg+df. \end{aligned} \tag{C10}$$

Then Eqs. (C9) become

$$x_1+y_1=V\cos(k)+\mu, \tag{C11}$$

$$x_2+y_2=X\sin(k), \tag{C12}$$

$$\begin{aligned} x_1y_1-x_2y_2 &= (\det H_l+\det H_r)\cos(2k)\\ &\quad +Y\cos(k)+W, \end{aligned} \tag{C13}$$

$$\begin{aligned} x_2y_1+x_1y_2 &= -(\det H_l-\det H_r)\sin(2k)\\ &\quad +Z\sin(k). \end{aligned} \tag{C14}$$

We can solve Eq. (C14) three variables: x_2, y_2, and third variable is one of the X, Y, Z, U, V, W. Then we require the third variable to be k independent by zeroing coefficients of all k-dependent terms. This gives a set of equations, and solving it gives the solution for the real part of one band is flat case. Following are our results.

l. Degenerate cases: In this case $\mu=0, \nu=0$, and the solution is

$$\begin{aligned} H_r &= \begin{pmatrix} a & b\\ \frac{(a+f)(b(d+f)+(d-a)g)}{(b+g)^2} & d\end{pmatrix},\\ H_l &= \begin{pmatrix} f & g\\ \frac{(a+f)(g(a+l)+b(l-f))}{(b+g)^2} & l\end{pmatrix}. \end{aligned} \tag{C15}$$

The band structure is

$$\begin{aligned} E_1 &= -\frac{2i\sin(k)(bf-ag)}{b+g},\\ E_2 &= \frac{\{e^{ik}[b(a+d+f)+dg]\}}{b+g}\\ &\quad +\frac{[e^{-ik}(ag+bl+fg+gl)]}{b+g}. \end{aligned}$$

m. Abnormal case. In this case $\mu=0, \nu=1$, and the solution is

$$H_r=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix},\quad H_l=\begin{pmatrix} -a & g\\ 0 & l\end{pmatrix}. \tag{C16}$$

The band structure is

$$\begin{aligned} E_1 &= 2ia\sin(k),\\ E_2 &= de^{ik}+e^{-ik}l+1. \end{aligned}$$

Note that when $a=-f$, the solution (C15) for the degenerate case reduces to abnormal case (C16).

n. Nondegenerate case. In this case $\mu=1, \nu=0$, and the solution is

$$H_r=\begin{pmatrix} a & 0\\ c & d\end{pmatrix},\quad H_l=\begin{pmatrix} f & 0\\ h & -d\end{pmatrix}.$$

The band structure is

$$\begin{aligned} E_1 &= 1+2id\sin(k),\\ E_2 &= e^{-ik}(f+ae^{2ik}). \end{aligned}$$

3. Imaginary part of both bands are flat

In this case x_2, y_2 are k independent in Eqs. (C2)–(C5). Using the same procedure as in Appendix C 1, solving Eqs. (C2)–(C5) for y_2, and requiring y_1 to be k independent, we find

$$\begin{aligned} a+d &= f+l,\\ bc+fl &= ad+gh,\\ 16x_2^2(bc-ad) &= -[a\mu+\nu(h-c)-f\mu]^2\\ &\quad -x_2^2(a+d+f+l)^2,\\ \mu x_2(a+d+f+l) &= 2x_2[a\mu-\nu(c+h)+f\mu],\\ 8x_2^4+2\mu^2x_2^2 &= 8x_2^2(al-bh-cg+df)\\ &\quad -x_2^2(a+d+f+l)^2\\ &\quad +[a\mu+\nu(h-c)-f\mu]^2. \end{aligned} \tag{C17}$$

o. Degenerate case. In this case $\mu=\nu=0$, and Eq. (C17) becomes

$$\begin{aligned} a+d &= f+l,\\ bc+fl &= ad+gh,\\ 16x_2^2(bc-ad) &= -x_2^2(a+d+f+l)^2,\\ x_2^2(a+d+f+l)^2 &= 8x_2^2(al-bh-cg+df)-8x_2^4. \end{aligned}$$

The solution is

$$\begin{aligned} H_0 &= \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix},\\ H_l &= \begin{pmatrix} f & \frac{b(F+2x_2^2)}{(a-d)^2}\\ \frac{F-2x_2^2}{4b} & a+d-f\end{pmatrix},\\ H_r &= \begin{pmatrix} a & b\\ -\frac{(a-d)^2}{4b} & d\end{pmatrix}, \end{aligned}$$

where $F = -2\sqrt{x_2^2(d-a)(a+d-2f)+x_2^4}+(d-a)(a+d-2f)$. Then the band structure is

$$x_k = \frac{2b(a-d)^2(a+d)\cos k}{2b(a-d)^2} - \frac{2\sqrt{-b^2x_2^2(a-d)^4}}{2b(a-d)^2},$$
$$y_k = \frac{2b(a+d)(a-d)^2\cos k}{2b(a-d)^2} + \frac{2\sqrt{-b^2x_2^2(a-d)^4}}{2b(a-d)^2}.$$

p. Nondegenerate case. In this case $\mu = 1, \nu = 0$, and Eq. (C17) becomes

$$a+d = f+l,$$
$$bc+fl = ad+gh,$$
$$x_2^2[16(bc-ad)+(a+d+f+l)^2]+(a-f)^2 = 0,$$
$$x_2(a-d+f-l) = 0,$$
$$x_2^2[8(al-bh-cg+df)-(a+d+f+l)^2-2] + (a-f)^2 = 8x_2^4.$$

The solution is

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$H_l = \begin{pmatrix} f & \frac{-G}{(a-f)^2} \\ \frac{(4x_2^2+1)(a-f)^4}{16x_2^2 G} & a \end{pmatrix},$$
$$H_r = \begin{pmatrix} a & b \\ -\frac{(4x_2^2+1)(a-f)^2}{16bx_2^2} & f \end{pmatrix},$$

where $G = 2\sqrt{b^2x_2^2[x_2^2-(a-f)^2]}+b(a-f)^2-2bx_2^2$. Then the band structure is

$$x_k = -\frac{K}{4bGx_2^2(a-f)^2}+(a+f)\cos(k)+\frac{1}{2},$$
$$y_k = \frac{K}{4bGx_2^2(a-f)^2}+(a+f)\cos(k)+\frac{1}{2},$$

where

$$K = \big[bGx_2^2(a-f)^4\big(4x_2^2\{b^2(a-f)^4-4ibG(a-f)\sin(k) + bG[1-2(a-f)^2]+G^2\}+b^2(a-f)^4 - 2bG(a-f)^2\cos(2k)+G^2\big)\big]^{1/2}.$$

q. Abnormal case. In this case $\mu = 0, \nu = 1$, and Eq. (C17) becomes

$$a+d = f+l,$$
$$bc+fl = ad+gh,$$
$$(c-h)^2 = -16x_2^2(bc-ad)-x_2^2(a+d+f+l)^2,$$
$$x_2(c+h) = 0,$$
$$(c-h)^2-8x_2^4 = x_2^2(a+d+f+l)^2 - 8x_2^2(al-bh-cg+df).$$

The solution is

$$H_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$H_l = \begin{pmatrix} d-ix_2 & \frac{c}{4x_2^2}-\frac{x_2^2}{c} \\ -c & d+ix_2 \end{pmatrix},$$
$$H_r = \begin{pmatrix} d & -\frac{c}{4x_2^2} \\ c & d \end{pmatrix}.$$

The band structure is

$$x_k = 2d\cos k - \frac{\sqrt{c^2x_2^2[c\sin(k)+ix_2^2]^2}}{cx_2^2},$$
$$y_k = 2d\cos k + \frac{\sqrt{c^2x_2^2[c\sin(k)+ix_2^2]^2}}{cx_2^2}.$$

4. Imaginary part of one band is flat

Using the same method with the real part of one band is flat case, we can also solve the case of the imaginary part of one band is flat case. In this case we require x_2 (y_2) to be k independent in Eqs. (C2)–(C5). The only possibility is $x_2 = 0$, $y_2 = (a+d-f-l)$, then following the same steps with the real part of one band case we got the following results.

r. Degenerate case.

$$H_r = \begin{pmatrix} a & b \\ \frac{(a-f)[g(a-d)+b(d-f)]}{(b-g)^2} & d \end{pmatrix},$$
$$H_l = \begin{pmatrix} f & g \\ \frac{(a-f)[g(a-l)+b(l-f)]}{(b-g)^2} & l \end{pmatrix}.$$

The band structure is

$$E_1 = \frac{2\cos(k)(bf-ag)}{b-g},$$
$$E_2 = \frac{e^{ik}[b(a+d-f)-dg]}{b-g} + \frac{e^{-ik}(ag+bl-fg-gl)}{b-g}.$$

s. Abnormal case.

$$H_r = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad H_l = \begin{pmatrix} a & g \\ 0 & l \end{pmatrix}.$$

The band structure is

$$E_1 = 2a\cos(k),$$
$$E_2 = e^{-ik}(l+de^{2ik}).$$

t. Nondegenerate case. In this case the solution is the same with abnormal case, and the band structure is

$$E_1 = 2a\cos(k),$$
$$E_2 = de^{ik}+e^{-ik}l+1.$$

5. Modulus of a band is flat

We suppose $x = re^{i\theta_k}$ in Eq. (B2), then

$$y = e^{ik}(a+d) + e^{-ik}(f+l) + \mu - re^{i\theta_k},$$
$$y = \frac{1}{r}e^{-i\theta_k}[e^{ik}(a\mu - c\nu) + al - bh - cg + df + \det H_l e^{-2ik} + \det H_r e^{2ik} + e^{-ik}(f\mu - h\nu)].$$

Since there only integer powers of e^{ik} present in the above, $\theta_k = mk,\ m \in \mathcal{Z}$, and

$$e^{i(m-1)k}(f+l) + e^{i(m+1)k}(a+d) + \mu e^{imk} - re^{2imk} = \frac{1}{r}[e^{-ik}(f\mu - h\nu) + al - bh - cg + df + e^{ik}(a\mu - c\nu) + \det H_l e^{-2ik} + \det H_r e^{2ik}]. \quad \text{(C18)}$$

By equating the same powers of e^{ik} in Eq. (C18), we can show that, when $m > 1$ or $m < -1$, the only possible solution for Eq. (C18) is $r = 0$. Therefore, only when $m = 0, \pm 1$ do we have a nontrivial solution. Note that the $m = 0$ case corresponds to one band is completely flat, which is solved in Appendix B 3.

a. m = 1 case

In this case equating the coefficients of the same powers of e^{ik} on two sides of Eq. (C18) we get

$$f + l = \frac{1}{r}(al - bh - cg + df),$$
$$\mu = \frac{1}{r}(a\mu - c\nu),$$
$$a + d - r = \frac{1}{r}\det H_r, \quad \text{(C19)}$$
$$\det H_l = 0,$$
$$f\mu - h\nu = 0.$$

a. Degenerate case: In this case $\mu = 0, \nu = 0$, and Eq. (C19) gives the following solution:

$$H_r = \begin{pmatrix} a & b \\ -\frac{(a-r)(r-d)}{b} & d \end{pmatrix}, \quad H_l = \begin{pmatrix} f & \frac{bf}{a-r} \\ \frac{l(a-r)}{b} & l \end{pmatrix}.$$

Then the band structure is

$$E_1 = e^{ik}r,$$
$$E_2 = e^{-ik}(ae^{2ik} + de^{2ik} + f - e^{2ik}r + l).$$

b. Nondegenerate case. In this case $\nu = 0, \mu = 1$, and Eq. (C19) gives the following solution:

$$H_r = \begin{pmatrix} r & 0 \\ c & d \end{pmatrix}, \quad H_l = \begin{pmatrix} 0 & 0 \\ h & l \end{pmatrix},$$

band structure is

$$E_1 = e^{ik}r,$$
$$E_2 = de^{ik} + e^{-ik}l + 1.$$

c. Abnormal case. In this case $\mu = 0, \nu = 1$, and Eq. (C19) gives following solution:

$$H_r = \begin{pmatrix} r & b \\ 0 & d \end{pmatrix}, \quad H_l = \begin{pmatrix} 0 & g \\ 0 & l \end{pmatrix}.$$

The band structure is

$$E_1 = e^{ik}r,$$
$$E_2 = de^{ik} + e^{-ik}l + 1.$$

b. m = −1 case

Similar to $m = 1$ case, by equating the same powers of e^{ik} in Eq. (C18), we have

$$f + l - r = \frac{1}{r}\det H_l,$$
$$\frac{1}{r}(f\mu - h\nu) = \mu,$$
$$a + d = \frac{1}{r}(al - bh - cg + df),$$
$$\det H_r = 0,$$
$$a\mu - c\nu = 0.$$

a. Degenerate case: In this case $\mu = 0, \nu = 0$, and Eq. (C18) gives the following solution:

$$H_r = \begin{pmatrix} a & b \\ \frac{ad}{b} & d \end{pmatrix}, \quad H_l = \begin{pmatrix} f & \frac{b(f-r)}{a} \\ \frac{a(l-r)}{b} & l \end{pmatrix}.$$

Then the band structure is

$$E_1 = e^{-ik}r,$$
$$E_2 = e^{-ik}\left(ae^{2ik} + de^{2ik} + f + l - r\right).$$

b. Nondegenerate case. In this case $\nu = 0, \mu = 1$, and Eq. (C18) gives the following solution:

$$H_r = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \quad H_l = \begin{pmatrix} r & 0 \\ h & l \end{pmatrix}.$$

Then the band structure is

$$E_1 = e^{-ik}r,$$
$$E_2 = de^{ik} + e^{-ik}l + 1.$$

c. Abnormal case. In this case $\nu = 1, \mu = 0$, and Eq. (C18) gives the following solutions:

$$H_r = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \quad H_l = \begin{pmatrix} r & g \\ 0 & l \end{pmatrix}.$$

Then the band structure is

$$E_1 = e^{-ik}r,$$
$$E_2 = e^{-ik}(l + de^{2ik}).$$

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