Constitutive relations and adiabatic invariants for electromagnetic waves in a dynamic Lorentz medium

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Constitutive relations for Lorentz media with a time-varying density of oscillators and a time-varying intrinsic frequency of oscillators are derived *ab initio*. For an electromagnetic wave propagating in a dynamic Lorentz medium, combinations of the wave energy and frequency (adiabatic invariants) that are conserved during slow time variations of the medium are obtained. If the oscillator density varies, the invariants differ from the known ones for nondispersive dielectrics and plasmas and have a different form for increasing and decreasing oscillator densities. In both cases, however, they predict that the frequency shift is accompanied by a decrease of the wave energy. If the intrinsic frequency varies, the invariant coincides with the one for a nondispersive dielectric and predicts the wave amplification and attenuation with the increase and decrease of the frequency, respectively. It is also shown that a medium with a decreasing density of oscillating dipoles cannot be described by a time-varying dielectric permittivity $\varepsilon(t)$ even in the nondispersive limit.

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I. INTRODUCTION

Electromagnetic phenomena in dynamic media have gained renewed interest in recent years as a platform for testing fundamental physics effects [1–3] and developing novel devices of active photonics and plasmonics [4–8]. An attractive feature of dynamic media is the possibility to convert the frequency of electromagnetic radiation without the need to fulfil any phase-matching conditions, contrary to the nonlinear optical techniques [9–12]. Using time as a new degree of freedom allows one also to convert bulk radiation into localized modes [13–16] and vice versa [17–19] without any spatial modulation of the medium. Dynamic media are energetically active, exchanging energy with electromagnetic waves even in the absence of ordinary loss or gain mechanisms [20–22].

Understanding the peculiarities of electromagnetic phenomena in dynamic media has been mainly based on the models of a nondispersive material with a time-varying dielectric permittivity [11,23–26] and plasma with a varying density of free particles (a dispersive material) [27–31]. These models cannot account for the electromagnetic wave interaction with atomic (molecular) resonances in a dynamic medium. Meanwhile, such resonances can play an important role, in particular, in dynamic metamaterials with externally activated meta-atoms [6] or in a mixture of chemically reacting molecules [32].

Recently [33], a time-variant Lorentz medium was proposed as a model of resonant dynamic media (transformation of an electromagnetic wave in a suddenly created Lorentz medium was considered for the first time in Ref. [34]). The Lorentz medium represents a real material as a collection of classical oscillators with intrinsic frequency. In this model, the electron (a particle of a small mass) is bound to the nucleus (a much heavier particle) by a springlike force. An applied electric field acts as a driving force that sets the electron into oscillating motion. The corresponding oscillations of the dipole moment generate electromagnetic fields. In Refs. [33–35], the idealistic case of a steplike increase in time of the oscillator density was treated by using initial conditions on the density jump. To study a more realistic situation when the medium varies on a timescale comparable to or even longer than the wave period, a constitutive relation of such a medium is required.

In this paper, we derive constitutive relations for a dynamic Lorentz medium whose oscillator density increases or decreases in time according to an arbitrary law and also for a medium with an arbitrarily varying intrinsic frequency of the oscillators. In contrast to recent papers [36,37] in which a (time-invariant) Lorentz medium with a negative oscillator strength was used as a model of an active material with a population inversion (gain), we consider the classical Lorentz medium with no population inversion. In the gainless medium, there can be no instabilities similar to those in Refs. [36,37]. By using the obtained constitutive relations, we study the energetics of an electromagnetic wave in a slowly varying Lorentz medium. In particular, we derive adiabatic invariants, i.e., combinations of the wave frequency and energy that are conserved during slow time variations of the medium. Adiabatic invariants are a useful tool for studying wave modulation in temporally dynamic systems. They allow one to find the change in the wave energy (and therefore the wave amplitude) directly from the frequency shift.

Previously, adiabatic invariants were obtained for the waves in time-varying nondispersive dielectrics [20–22], isotropic and magnetized plasmas [22,30,38], and surface waves guided by time-varying plasma slabs [39] and graphene sheets [40]. The adiabatic invariants obtained here for a

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Lorentz medium with a time-varying oscillator density differ from the known invariants but agree with them in the limits of very high and very low frequency. These invariants have a different form in the cases of increasing and decreasing oscillator density. In both cases, however, they predict that frequency shift is accompanied by a decrease of the wave energy. For a Lorentz medium with a time-varying intrinsic frequency of the oscillators, the invariant coincides with the one for a nondispersive dielectric and predicts the wave amplification and attenuation with the increase and decrease of the frequency, respectively.

II. LORENTZ MEDIUM WITH A GROWING OSCILLATOR DENSITY

Let us start by considering the electromagnetic properties of a Lorentz medium with the density of the oscillators growing over time.

A. Constitutive relations

We assume that the oscillator density grows in time according to an arbitrary function $N(t)$. The response of an oscillator to a driving electric field $\mathbf{E}(t)$ is determined by the equation

$$\frac{d^2\mathbf{p}}{dt^2}+\Omega_0^2\mathbf{p}=\frac{e^2}{m}\mathbf{E}, \tag{1}$$

where e and m are the electron charge and mass; $\mathbf{p}=-e\xi$ is the dipole moment of the oscillator, with ξ being the electron displacement; and Ω_0 is the intrinsic oscillator frequency. The oscillators are assumed to appear with zero displacement ξ and zero velocity $\mathbf{v}=d\xi/dt$, i.e.,

$$\mathbf{p}(t=\tilde{t})=0, \quad d\mathbf{p}/dt(t=\tilde{t})=0, \tag{2}$$

where $\tilde{t}$ denotes the instant of time at which an oscillator appears. Equation (1) with initial conditions (2) can be integrated for an arbitrary time dependence of the driving field $\mathbf{E}(t)$ [41]:

$$\mathbf{p}(t)=\frac{e^2}{m\Omega_0}\mathrm{Im}\int_{\tilde{t}}^{t}\mathbf{E}(t')e^{i\Omega_0(t-t')}dt' \tag{3}$$

(Im means the imaginary part). The polarization $\mathbf{P}(t)$ of the medium at an instant t can be represented as

$$\mathbf{P}(t)=\int_{-\infty}^{t}\frac{dN(\tilde{t})}{d\tilde{t}}\mathbf{p}(t,\tilde{t})d\tilde{t}, \tag{4}$$

where $\mathbf{p}(t,\tilde{t})$ is the dipole moment at instant t of an oscillator appearing at instant $\tilde{t}$. According to Eq. (4), the polarization $\mathbf{P}(t)$ is a superposition of contributions from the oscillators appearing at different instants of time $-\infty<\tilde{t}<t$. In other words, there exists a multistream motion of the oscillators in the medium.

By substituting Eq. (3) into Eq. (4) and integrating by parts, with the assumption that $N(-\infty)=0$, one can obtain

$$\mathbf{P}(t)=\frac{e^2}{m\Omega_0}\mathrm{Im}\int_{-\infty}^{t}N(\tilde{t})\mathbf{E}(\tilde{t})e^{i\Omega_0(t-\tilde{t})}d\tilde{t}. \tag{5}$$

Taking into account that $\mathbf{E}(t)$ is a real physical field, Eq. (5) can also be presented in the form

$$\mathbf{P}(t)=\frac{e^2}{m\Omega_0}\int_{-\infty}^{t}N(\tilde{t})\mathbf{E}(\tilde{t})\sin\Omega_0(t-\tilde{t})d\tilde{t}. \tag{6}$$

Equations (5) and (6) represent the integral forms of the constitutive relation of a Lorentz medium with growing oscillator density. For practical use, however, it is convenient to reduce them to a differential form. Differentiating Eq. (5) or Eq. (6) twice with respect to t yields

$$\frac{\partial^2\mathbf{P}}{\partial t^2}+\Omega_0^2\mathbf{P}=\frac{\Omega_p^2(t)}{4\pi}\mathbf{E}, \tag{7}$$

where a partial derivative is used to account for the possible spatial dependence of $\mathbf{E}$ and $\mathbf{P}$ and the parameter

$$\Omega_p(t)=\sqrt{4\pi e^2N(t)/m} \tag{8}$$

is analogous to the plasma frequency of freely moving electrons.

It should be emphasized that the constitutive relations Eqs. (5)–(7) are valid for an arbitrary law of the oscillator density growth, including abrupt and slow density variations. In particular, integrating Eq. (7) twice over the infinitely small time of a density jump, one can obtain the conditions of continuity of $\mathbf{P}$ and $\partial\mathbf{P}/\partial t$ at the jump, which were derived earlier from physical assumptions [33–35].

B. Adiabatic invariant

Now let us consider an electromagnetic wave propagating in a Lorentz medium with a growing oscillator density. Assuming the growth rate is much smaller than the wave frequency, the electric (E_x) and magnetic (H_y) fields of the wave may be written in a quasimonochromatic form,

$$\begin{Bmatrix}E_x(z,t)\\H_y(z,t)\end{Bmatrix}=\begin{Bmatrix}\mathcal{E}(t)\\\mathcal{H}(t)\end{Bmatrix}e^{i\varphi(t)-ikz}, \tag{9}$$

with slowly varying frequency $\omega(t)=d\varphi/dt$ and amplitudes $\mathcal{E}(t)$ and $\mathcal{H}(t)$. The wave number k is conserved due to the translational invariance of the medium. The wave evolves in time adiabatically following the time variations in the oscillator density. Its frequency and field amplitudes can change substantially on timescales longer than the period $2\pi/\omega$.

To find the wave evolution, we use the Maxwell equations

$$ikE_x=\frac{1}{c}\frac{\partial H_y}{\partial t}, \tag{10a}$$

$$ikH_y=\frac{1}{c}\frac{\partial E_x}{\partial t}+\frac{4\pi}{c}J_x, \tag{10b}$$

where the current density $J_x=\partial P_x/\partial t$ and c is the speed of light.

The system of Eqs. (10) and (7) can be reduced to a single fourth-order differential equation for the electric field,

$$\frac{\partial^4E_x}{\partial t^4}+(\Omega_0^2+\Omega_p^2+c^2k^2)\frac{\partial^2E_x}{\partial t^2}+2\frac{d\Omega_p^2}{dt}\frac{\partial E_x}{\partial t}+\left(\frac{d^2\Omega_p^2}{dt^2}+\Omega_0^2c^2k^2\right)E_x=0. \tag{11}$$

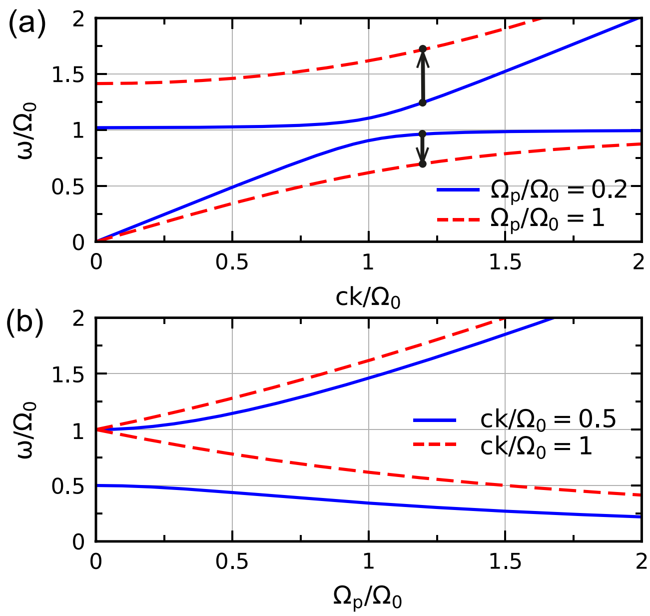

FIG. 1. Adiabatic frequency shift in a Lorentz medium with growing oscillator density. (a) As the density parameter Ω_p increases, the dispersion curves $\omega(k)$ shift apart from the resonance frequency Ω_0. The wave number k remains unchanged, while the frequency ω shifts downward for the lower curve and upward for the upper curve. (b) The frequencies of the lower and upper curves as functions of Ω_p for two values of ck/Ω_0.

We substitute E_x in the quasimonochromatic form (9) into Eq. (11) and apply the Wentzel-Kramers-Brillouin (WKB) method.

By neglecting all derivatives of slowly varying quantities, such as $\mathcal{E}(t)$, $\omega(t)$, and $\Omega_p(t)$, we obtain a zeroth-order approximation that determines the evolution of the wave frequency,

$$\omega^4 - \left(\Omega_0^2 + \Omega_p^2 + c^2k^2\right)\omega^2 + \Omega_0^2c^2k^2 = 0. \tag{12}$$

Equation (12) coincides with the dispersion equation for polaritons in a stationary oscillator medium [33–35], but with Ω_p and ω taken as slow functions of time. Two positive roots of Eq. (12), $0 < \omega_1 < \omega_2$, which are given by

$$\omega_{1,2}^2 = q \mp \sqrt{q^2 - \Omega_0^2c^2k^2}, \quad q = \left(\Omega_0^2 + \Omega_p^2 + c^2k^2\right)/2, \tag{13}$$

correspond to the lower and upper dispersion curves in Fig. 1(a) and describe the waves propagating in the $+z$ direction. Two negative roots $-\omega_{1,2}$ correspond to the waves propagating in the $-z$ direction [the dispersion curves for $\omega < 0$ are not shown in Fig. 1(a)]. Only one of the four roots, which coincides with the frequency of the initial wave in the beginning of the medium time variations, is of interest. The other three roots correspond to new waves that can, in principle, be generated by the initial wave due to medium nonstationarity. Since, however, the amplitudes of these waves are proportional to dN/dt [38,39,42] and therefore small in a slowly varying medium, these waves are neglected in the standard WKB approximation. The initial wave evolves adiabatically following $\Omega_p(t)$ [Figs. 1(a) and 1(b)].

In the first-order approximation, i.e., collecting the terms proportional to the first time derivatives of $\mathcal{E}(t)$, $\omega(t)$, and $\Omega_p^2(t)$, we obtain the first-order differential equation for the amplitude $\mathcal{E}$,

$$2\omega\left(\omega^4 - \Omega_0^2c^2k^2\right)\frac{d\mathcal{E}}{dt} + \left(\omega^4 + 3\Omega_0^2c^2k^2\right)\mathcal{E}\frac{d\omega}{dt} = 0. \tag{14}$$

In deriving Eq. (14), we related $d\Omega_p^2/dt$ to $d\omega/dt$ by using Eq. (12). Equation (14) relates the change in the wave amplitude to the change in its frequency. To derive the adiabatic invariant, i.e., a combination of the wave energy density and frequency that is conserved in a dynamic medium, we supplement Eq. (14) with the expression for the wave energy density [35]

$$W = \frac{|\mathcal{E}|^2}{4\pi}\left[1 + \frac{c^2k^2}{\omega^2} + \Omega_p^2\frac{\omega^2 + \Omega_0^2}{\left(\omega^2 - \Omega_0^2\right)^2}\right]. \tag{15}$$

In Eq. (15), the first and second terms represent, respectively, the energies of the electric and magnetic fields, whereas the last term represents the polarization energy, i.e., the potential and kinetic energies of the oscillators.

From Eqs. (14) and (15), by using the procedure described in Ref. [38], one can obtain the desired adiabatic invariant of the form

$$W\frac{\omega^2 - \Omega_0^2}{\omega} = \text{const.} \tag{16}$$

In the limit $\omega \gg \Omega_0$, Eq. (16) expectedly reduces to the well-known invariant for a plasma with a growing density of free particles [22,30,39],

$$W\omega = \text{const.} \tag{17}$$

In the opposite limit $\omega \ll \Omega_0$, Eq. (16) coincides with the invariant for a nondispersive dielectric [20–22],

$$W/\omega = \text{const.} \tag{18}$$

Equation (16), including its asymptotic forms Eqs. (17) and (18), predicts that the frequency shift is accompanied by energy loss. Indeed, for the initial wave on the upper dispersion curve in Fig. 1(a), i.e., with $\omega > \Omega_0$, the wave frequency ω increases [upward arrow in Fig. 1(a) and upper curves in Fig. 1(b)] as the oscillator density grows. According to Eq. (16), it leads to a decrease of W. For the wave on the lower dispersion curve in Fig. 1(a), i.e., with $\omega < \Omega_0$, the wave frequency ω decreases [downward arrow in Fig. 1(a) and lower curves in Fig. 1(b)] as the oscillator density grows, and according to Eq. (16), W decreases as well. The relation of the wave energy loss to the wave frequency shift is illustrated in Fig. 2(a).

The mechanism of the energy loss can be attributed to the excitation of natural oscillations in the medium at the intrinsic frequency Ω_0. The excitation of natural oscillations was demonstrated for a steplike increase in the oscillator density [35]. It was shown that the polarization components of the initial and newly created oscillators move at Ω_0 out of phase, and therefore, the total polarization equals zero. Although the oscillations at Ω_0 do not manifest themselves through any field, they can consume a substantial part of the wave energy. Since a gradual growth of the oscillator density

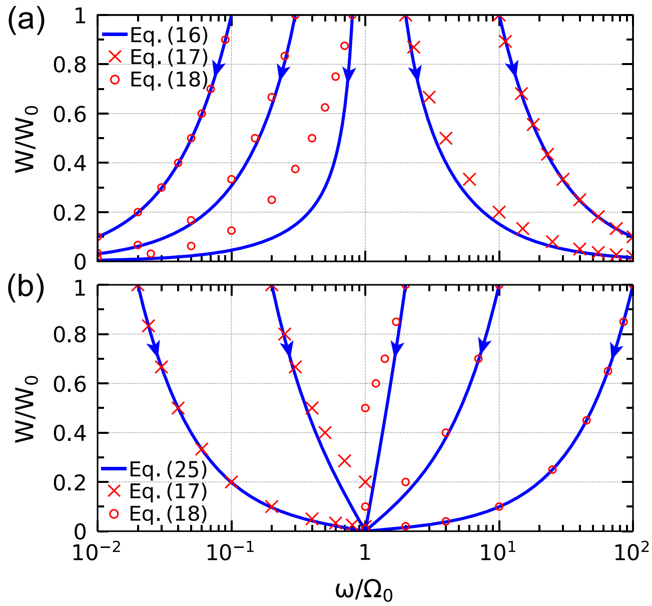

FIG. 2. Diagrams illustrating the relation of the wave energy with the wave frequency according to the adiabatic invariants, (a) Eq. (16) (increasing N) and (b) Eq. (25) (decreasing N). The crosses and circles show the limiting relations given by Eqs. (17) and (18).

may be represented as a series of small density steps, the loss mechanism revealed in Ref. [35] can be extended to the case under consideration.

III. LORENTZ MEDIUM WITH A DECREASING OSCILLATOR DENSITY

Let us consider now the electromagnetic properties of a Lorentz medium with an oscillator density decreasing in time.

A. Constitutive relations

We assume that the oscillator density decreases in time according to an arbitrary function $N(t)$. Contrary to the above-studied case of a growing oscillator density, in a medium with a decreasing oscillator density there is no multistream motion of the oscillators. Therefore, the current density can simply be written as

$$\mathbf{J} = N(t)\frac{\partial \mathbf{p}}{\partial t}, \tag{19}$$

with the dipole moment of an oscillator $\mathbf{p}$ determined by Eq. (1). Thus, Eqs. (1) and (19) comprise the constitutive relations for a Lorentz medium with a decreasing oscillator density.

It is interesting to note that the commonly accepted relation $\mathbf{J} = \partial\mathbf{P}/\partial t$ [22,33] is not valid for a Lorentz medium with a decreasing oscillator density, i.e.,

$$\mathbf{J} \neq \frac{\partial \mathbf{P}}{\partial t}, \tag{20}$$

with $\mathbf{P}(t) = N(t)\mathbf{p}(t)$. The reason can be explained as follows. In the case of decreasing the oscillator density, the polarization $\mathbf{P}(t)$ changes not only due to a change in $\mathbf{p}(t)$, which is, indeed, related to a current flow but also due to the withdrawal of a part of the oscillating dipoles from creating the current.

B. Adiabatic invariant

Now let us derive an adiabatic invariant for an electromagnetic wave of the quasimonochromatic form (9) propagating in a medium with a decreasing oscillator density. For this purpose, we reduce Eqs. (10), (19), and (1) to a single equation for the only component of the dipole moment p_x,

$$\frac{\partial^4 p_x}{\partial t^4} + \left(\Omega_0^2 + \Omega_p^2 + c^2k^2\right)\frac{\partial^2 p_x}{\partial t^2} + \frac{d\Omega_p^2}{dt}\frac{\partial p_x}{\partial t} + \Omega_0^2 c^2 k^2 p_x = 0. \tag{21}$$

Then, we substitute p_x in a quasimonochromatic form

$$p_x(z,t) = p(t)e^{i\varphi(t) - ikz} \tag{22}$$

with a slowly varying amplitude $p(t)$ into Eq. (21) and apply the WKB method.

In the zeroth-order approximation, we arrive at the same Eq. (12) for the wave frequency $\omega(t)$ as in the case of growing oscillator density. In the first-order approximation, we obtain

$$2\omega\left(\omega^4 - \Omega_0^2 c^2 k^2\right)\frac{dp}{dt} + \left(3\omega^4 + \Omega_0^2 c^2 k^2\right)p\frac{d\omega}{dt} = 0. \tag{23}$$

By using Eq. (1), the electric field amplitude $\mathcal{E}(t)$ can be related to the dipole moment amplitude $p(t)$ as $\mathcal{E} = p(\Omega_0^2 - \omega^2)m/e^2$. This allows expressing the wave energy density (15) through $|p|^2$ as

$$W = \frac{m^2|p|^2}{4\pi e^4}\left[\left(1 + \frac{c^2k^2}{\omega^2}\right)\left(\omega^2 - \Omega_0^2\right)^2 + \Omega_p^2\left(\omega^2 + \Omega_0^2\right)\right]. \tag{24}$$

Multiplying Eq. (23) by p^* and using Eq. (24), we obtain the desired adiabatic invariant in the form

$$W\frac{\omega}{\omega^2 - \Omega_0^2} = \text{const}. \tag{25}$$

The invariant (25) differs from Eq. (16) derived in Sec. II. In the limit $\omega \gg \Omega_0$, Eq. (25) reduces to Eq. (18), which is the known invariant for a decaying plasma [30,39]. In the opposite limit $\omega \ll \Omega_0$, Eq. (25) reduces to Eq. (17). By using Eq. (25), Fig. 1(a) (with reversed directions of the arrows), and Fig. 1(b), one can conclude that frequency shift is again accompanied by energy loss [Fig. 2(b)]. The loss is due to removing part of the oscillators, together with their energy, from the oscillatory motion.

IV. LORENTZ MEDIUM WITH A TIME-VARYING INTRINSIC FREQUENCY

Let us consider now a dynamic Lorentz medium with a constant density of the oscillators ($N = \text{const}$) but with a time-varying intrinsic frequency $\Omega_0(t)$ of an oscillator.

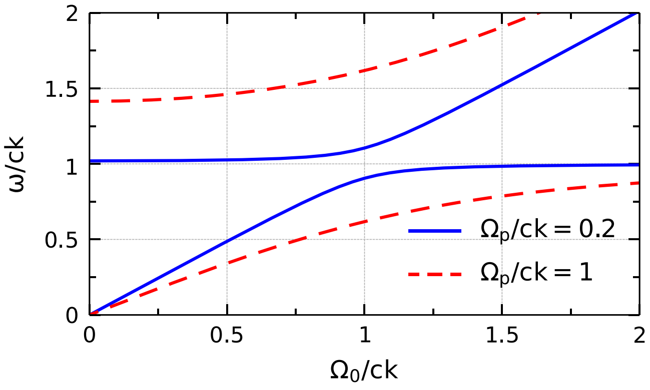

FIG. 3. The frequencies of the higher- and lower-frequency modes as functions of Ω_0 for two values of Ω_p.

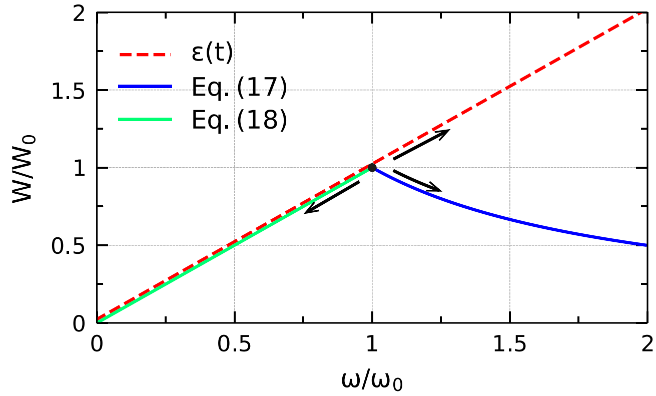

FIG. 4. The wave energy vs the wave frequency for a nondispersive Lorentz medium with a time-varying dipole density $N(t)$ (solid lines) in comparison to the $\varepsilon(t)$ model (dashed line). The arrows show the directions of motion along the curves from the initial wave point with frequency ω_0 and energy W_0.

A. Constitutive relation

The dipole moment of an oscillator is now determined by the equation

$$\frac{d^2\mathbf{p}}{dt^2} + \Omega_0^2(t)\mathbf{p} = \frac{e^2}{m}\mathbf{E}. \tag{26}$$

Multiplying Eq. (26) by a constant density of the oscillators N yields the constitutive relation

$$\frac{\partial^2\mathbf{P}}{\partial t^2} + \Omega_0^2(t)\mathbf{P} = \frac{\Omega_p^2}{4\pi}\mathbf{E}. \tag{27}$$

Equation (27) is valid for an arbitrary law of the intrinsic frequency variation $\Omega_0(t)$. Further, we apply it to study an adiabatic wave evolution in the medium with a slowly varying $\Omega_0(t)$.

B. Adiabatic invariant

We reduce the system of Eqs. (10) and (27) to a single equation for the polarization P_x,

$$\frac{\partial^4 P_x}{\partial t^4} + \left(\Omega_0^2 + \Omega_p^2 + c^2k^2\right)\frac{\partial^2 P_x}{\partial t^2} + 2\frac{d\Omega_0^2}{dt}\frac{\partial P_x}{\partial t} + \left(\frac{d^2\Omega_0^2}{dt^2} + \Omega_0^2c^2k^2\right)P_x = 0, \tag{28}$$

and apply the same WKB approach as in Secs. II B and III B.

In the zeroth-order approximation, the wave frequency ω is again determined by Eq. (13) but with a time-varying frequency Ω_0, rather than Ω_p. From Fig. 3, the frequencies of both higher- and lower-frequency modes grow with Ω_0.

In the first-order approximation, we obtain an equation for a slowly varying amplitude $\mathcal{P}(t)$ of P_x,

$$2\omega(\omega^2 - c^2k^2)\left[(\omega^2 - c^2k^2)^2 + \Omega_p^2c^2k^2\right]\frac{d\mathcal{P}}{dt} + \left[(\omega^2 - c^2k^2)^3 - \Omega_p^2c^2k^2(3\omega^2 + c^2k^2)\right]\mathcal{P}\frac{d\omega}{dt} = 0, \tag{29}$$

that, by also using Eq. (24) with $p = \mathcal{P}/N$, leads to the adiabatic invariant (18). Here, this invariant is universal, i.e., applicable to both an increase and decrease of $\Omega_0(t)$ for any ratio of ω and Ω_0, contrary to the case of varying $\Omega_p(t)$ (Secs. II and III).

For a decreasing intrinsic frequency $\Omega_0(t)$, the wave frequency also decreases (Fig. 3), and according to the invariant (18), the wave energy decreases as well. The energy is, evidently, lost due to the negative work done by the external force when changing the oscillator parameters to decrease Ω_0.

For growing $\Omega_0(t)$, however, the growth of the wave frequency (Fig. 3) is accompanied by an increase in the wave energy, which can be attributed to positive work of the external force. Earlier [11,21,22,24,25], the wave amplification in a dynamic medium was predicted in the model of a nondispersive dielectric. Here, we extend the result to a dispersive dynamic medium, such as a Lorentz medium with a varying intrinsic frequency. An example of such a medium is the structures with time-modulated distributed capacitance, such as an array of subwavelength spaced varactors modulated by an rf bias [4,5,43,44].

V. DISCUSSION OF THE NONDISPERSIVE LIMIT

The commonly accepted model of a nondispersive dynamic dielectric is based on the constitutive relation [11,20–23]

$$\mathbf{D} = \varepsilon(t)\mathbf{E}, \tag{30}$$

which relates the electric displacement $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ to the electric field $\mathbf{E}$ by means of a time-varying dielectric permittivity $\varepsilon(t)$. Equation (30) is widely used in the current literature [1,5,26,45,46]. For an electromagnetic wave in a medium with a slowly varying function $\varepsilon(t)$, both increasing and decreasing, using Eq. (30) leads to the adiabatic invariant in the form of Eq. (18) with $W = \varepsilon|\mathcal{E}|^2/(2\pi)$ [20–22]. Importantly, for a decreasing $\varepsilon(t)$ the wave frequency $\omega(t)$ increases according to the dispersion equation

$$c^2k^2 - \omega^2(t)\varepsilon(t) = 0 \tag{31}$$

($k = \text{const}$), and Eq. (18) indicates that the wave energy $W(t)$ increases as well (Fig. 4).

The model of a Lorentz medium can be reduced to that of a nondispersive dielectric by omitting the oscillator inertia term $d^2\mathbf{p}/dt^2$ in Eqs. (1) and (26). For a growing oscillator density

$N(t)$, this simplifies the constitutive relation (7) to the form

$$\mathbf{P} = \frac{\Omega_p^2(t)}{4\pi\Omega_0^2}\mathbf{E} \tag{32}$$

and leads to Eq. (30) with

$$\varepsilon(t) = 1 + \Omega_p^2(t)/\Omega_0^2. \tag{33}$$

The general adiabatic invariant (16) reduces to Eq. (18), which predicts energy loss with the growth of $\varepsilon(t)$ and decrease of $\omega(t)$ (Fig. 4).

In the case of a decreasing oscillator density $N(t)$, however, Eq. (32), and therefore Eq. (33), cannot be used, as we discussed in Sec. III A for the general case. Instead, one should rather use Eq. (19), which, neglecting the inertia, simplifies to the form

$$\mathbf{J} = \frac{\Omega_p^2(t)}{4\pi\Omega_0^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{34}$$

Equation (34) cannot be reduced to Eq. (30). Therefore, even a nondispersive medium with a density of inertialess dipoles decreasing in time cannot be described by dielectric permittivity $\varepsilon(t)$. For such a medium, the general adiabatic invariant (25) reduces to Eq. (17), which predicts, taking into account the growth of $\omega(t)$, a decrease of the wave energy $W(t)$, contrary to the prediction of the $\varepsilon(t)$ model (Fig. 4).

Thus, in accord with the general theory (Secs. II B and III B), there can be no wave amplification in a nondispersive medium with nonstationarity due to a time-varying dipole density $N(t)$, both increasing and decreasing.

For a Lorentz medium with $N = \text{const}$ and $\Omega_0(t)$, the nondispersive approximation $[\omega(t) \ll \Omega_0(t)]$ reduces the constitutive relation (27) to Eq. (30) with

$$\varepsilon(t) = 1 + \Omega_p^2/\Omega_0^2(t), \tag{35}$$

which is valid for both increasing and decreasing function $\Omega_0(t)$. The adiabatic invariant remains in the form of Eq. (18). Thus, one can conclude that nonstationarity due to a time-varying intrinsic frequency $\Omega_0(t)$ is the most adequate mechanism for the constitutive relation (30).

VI. CONCLUSION

To conclude, we have derived constitutive relations for a Lorentz medium with a time-varying oscillator density and for a medium with a time-varying intrinsic frequency of the oscillators. By using these relations, we have obtained adiabatic invariants for an electromagnetic wave in a dynamic Lorentz medium. The adiabatic invariants for a medium with a time-varying oscillator density differ from the invariants known for nondispersive dielectrics and plasmas. These adiabatic invariants have a different form in the cases of increasing and decreasing oscillator density, but in both cases they predict a decrease of the wave energy. For a Lorentz medium with a time-varying intrinsic frequency of the oscillators, the invariant coincides with the one for a nondispersive dielectric and predicts the wave amplification and attenuation with the increase and decrease of the frequency, respectively.

It was also shown that even for a nondispersive medium consisting of inertialess dipoles the applicability of the commonly used constitutive relation $\mathbf{D} = \varepsilon(t)\mathbf{E}$ depends on the mechanism of the medium nonstationarity. In particular, this relation is inapplicable to the medium with a time-decreasing density of the dipoles. In this case, Eq. (34) should be used instead.

The results obtained clarify the energetics of electromagnetic wave transformation in natural and artificial slowly time-variant media with intrinsic resonances and put limits on the efficiency of the transformation. The considered case of slow (on the scale of the wave period) time variations of the medium parameters, such as the density of its structural elements (molecules, meta-atoms) or the intrinsic frequency of the element, is characteristic of the nonstationarity produced by chemical reactions [32] or electrical modulation of meta-atoms [4,5,43,44,47].

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