Functional-integral approach to Gaussian fluctuations in Eliashberg theory

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The Eliashberg theory of superconductivity is based on a dynamical electron-phonon interaction as opposed to a static interaction present in BCS theory. The standard derivation of Eliashberg theory is based on an equation of motion approach, which incorporates certain assumptions such as Migdal's approximation for the pairing vertex. In this paper we provide a functional-integral-based derivation of Eliashberg theory and we also consider its Gaussian-fluctuation extension. The functional approach enables a self-consistent method of computing the mean-field equations, which arise as saddle-point conditions, and here we observe that the conventional Eliashberg self-energy and pairing function both appear as Hubbard-Stratonovich auxiliary fields. An important consequence of this fact is that it provides a systematic derivation of the Cooper and density-channel interactions in the Gaussian-fluctuation response. We also investigate the fluctuation contribution to the diamagnetic susceptibility near the critical temperature.

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I. INTRODUCTION

In the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity [1,2], the ordered phase originates from a static effective attraction between electrons near the Fermi surface, which leads to electron pairing and the formation of Cooper pairs [3]. Eliashberg theory [4,5], however, is based on a dynamical electron-phonon interaction, and this results in a frequency-dependent pairing function. Early reviews of Eliashberg theory were given in Refs. [6–8], followed by discussions of the critical temperature [9] and thermodynamics [10], and more recent reviews on the subject matter can be found in Refs. [11,12]. One of the standard approaches to deriving the self-consistent Eliashberg equations involves the equation of motion method [12,13]. As outlined in Ref. [14], there are a number of assumptions in the standard Eliashberg theory, such as Migdal's approximation [15] and ignoring particle-hole interactions [16]. While these assumptions can be tested *a posteriori*, it is beneficial to have a robust theoretical treatment where the approximations made at each stage of the derivation are clearly articulated.

There has recently [17,18] been a new approach to studying electron-phonon interactions. The method is again based on an equation of motion approach, but now it is cast in the form of the functional Schwinger-Dyson equations [19]. One advantage of this formulation of the problem is that it allows the tools from quantum field theory to be utilized, and in particular the Ward identities for the electron-phonon system can be derived in the same manner as the Ward identities for quantum electrodynamics. Other functional-integral applications to electron-phonon systems include topological superfluids [20] and generalizations of the Sachdev-Ye-Kitaev model [21,22]. With this renewed interest in functional formulations of Eliashberg theory, here we aim to understand electron-phonon systems by using a functional-integral approach, and in addition we aim to bring the theoretical understanding of the Eliashberg fluctuation theory to the same level as the well-understood BCS counterpart [23–26].

In this paper, we develop a model describing electron-phonon interactions and we derive the mean-field Eliashberg equations using a functional-integral approach, which achieves what has been done [23–26] for the counterpart BCS theory. We show that when the effective electron interaction is decoupled in the density and Cooper channels, the saddle-point solutions of this description correspond to conventional Eliashberg theory. Deriving Eliashberg theory from the functional-integral formalism has the benefit of providing an alternative way to understand the theory compared to the canonical derivation, making it accessible to a wider range of physicists and also enabling those who are familiar with the canonical derivations to see the theory in a new light.

A natural extension of the functional formalism is to consider fluctuations of the Hubbard-Stratonovich (HS) fields about their mean-field values. In the case of Gaussian-fluctuation theory for a static electron-electron interaction, normal-state fluctuations have been studied in the diagrammatic [23,27] and functional-integral [28,29] approaches. The extension to fluctuations in the superconducting phase has also been derived using the functional approach [25,26,28,29] and the advantage of this method is that gauge invariance and thermodynamic sum rules are manifestly satisfied [26]. Normal-state fluctuations in Eliashberg theory have not been extensively investigated, and the main studies have been on the fluctuation contributions to the electrical conductivity [30,31], the specific heat [32], and the Hall and Nernst effects [33]. The functional-integral formalism we develop enables us to systematically derive the fluctuation action and response at

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the Gaussian level, and moreover, since this method clearly identifies the apposite fields which can fluctuate, we will rigorously derive the physics investigated in Refs. [30–32]. Our paper thus provides the theoretical framework for the fluctuation theory of Eliashberg superconductors, and in principle it should be possible to extend it to the superconducting phase and also to incorporate other effects such as Coulomb interactions. An experimental quantity of interest [34] is the diamagnetic susceptibility, and here we calculate the fluctuation contribution in the strong-coupling limit.

The structure of the paper is as follows. In Sec. II we present the electron-phonon model and integrate out the phonons to obtain an attractive interaction. In Sec. III we perform the HS analysis and then in Sec. IV we obtain the mean-field Eliashberg equations. The Gaussian-fluctuation response is studied in Sec. V and in particular we compute the Aslamazov-Larkin (AL) contribution to the diamagnetic susceptibility near the critical temperature. The conclusion is presented in Sec. VI.

II. THE ELECTRON-PHONON INTERACTION

A. The model

We consider the following Hamiltonian for an interacting electron-phonon system [13,24]:

$$\hat{H} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \Omega_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{q}}\hat{b}_{\mathbf{q}} + \sum_{\mathbf{k},\mathbf{q},\sigma} g_{\mathbf{q}}(\hat{b}_{\mathbf{q}} + \hat{b}^\dagger_{-\mathbf{q}})(\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}+\mathbf{q}\uparrow} + \hat{c}^\dagger_{-\mathbf{k}-\mathbf{q}\downarrow}\hat{c}_{-\mathbf{k}\downarrow}). \tag{2.1}$$

Here, $\hat{c}^\dagger$ and $\hat{b}^\dagger$ are the respective electron and phonon creation operators, whereas $\hat{c}$ and $\hat{b}$ are the corresponding annihilation operators. The electron and phonon dispersion relations are, respectively, denoted by $\epsilon_{\mathbf{k}}$ and $\Omega_{\mathbf{q}}$. The electron-phonon interaction is $g_{\mathbf{q}}$, which satisfies $g^*_{\mathbf{q}} = g_{-\mathbf{q}}$. In the case of a multibranch phonon dispersion, we take $\mathbf{q}$ to also include branch indices.

As in BCS theory, the energy can be minimized by the pairing of electrons with equal and opposite momentum and spin [13]. To ensure this type of scattering is predominant, a macroscopic number of phonons with equal momentum is introduced. The interaction physically describes the scattering of an electron from $\mathbf{k}+\mathbf{q}\uparrow$ to $\mathbf{k}\uparrow$ due to the emission of a phonon of momentum $\mathbf{q}$, which can then be absorbed by scattering an electron from $-\mathbf{k}-\mathbf{q}\downarrow$ to $-\mathbf{k}\downarrow$.

To examine the model in Eq. (2.1) using the finite-temperature functional-integral formalism [24], we construct an action functional through the Legendre transform:

$$S[\bar{c}, c, \bar{b}, b] = \int_0^\beta d\tau \left[\sum_{\mathbf{k},\sigma} \bar{c}_{\mathbf{k}\sigma}(\partial_\tau + \mu)c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \bar{b}_{\mathbf{q}}\partial_\tau b_{\mathbf{q}} + H(\bar{c}, c, \bar{b}, b)\right]. \tag{2.2}$$

Here, τ denotes imaginary time and $\beta = 1/T$. In this paper we use Natural units $c = \hbar = k_B = 1$. The operators $\hat{c}^\dagger$, $\hat{c}$, $\hat{b}^\dagger$, $\hat{b}$ from the Hamiltonian formalism have been replaced by the Grassmann-valued functions $\bar{c}(\tau)$, $c(\tau)$, and the complex-valued functions $\bar{b}(\tau)$, $b(\tau)$, respectively. The fermionic chemical potential is denoted by μ, and we assume there is no chemical potential imbalance: $\mu_\uparrow = \mu_\downarrow \equiv \mu$. Since the number of phonons is not fixed, the phonons have zero chemical potential [35].

We can express the field operators in terms of Matsubara frequencies by taking the Fourier transform of the imaginary-time expressions:

$$c_{\mathbf{k}\sigma}(\tau) = \frac{1}{\beta}\sum_{\omega_n} c_{k\sigma}e^{-i\omega_n\tau}; \quad b_{\mathbf{q}}(\tau) = \frac{1}{\beta}\sum_{\Omega_m} b_q e^{-i\Omega_m\tau}. \tag{2.3}$$

Here, $\omega_n \equiv (2n+1)\pi T$, where $n \in \mathbb{Z}$ are the fermionic Matsubara frequencies and $\Omega_m \equiv 2m\pi T$, where $m \in \mathbb{Z}$ are the bosonic Matsubara frequencies. Defining the four-vector momenta by $k \equiv (i\omega_n, \mathbf{k})$ and $q \equiv (i\Omega_m, \mathbf{q})$, the action in Eq. (2.2) becomes

$$S[\bar{c}, c, \bar{b}, b] = -\sum_{k,\sigma} \bar{c}_{k\sigma}G_0^{-1}(k)c_{k\sigma} - \sum_q \bar{b}_q D_0^{-1}(q) b_q + \sum_{k,q,\sigma} g_{\mathbf{q}}(b_q + \bar{b}_{-q})\bar{c}_{k\sigma}c_{k+q\sigma}. \tag{2.4}$$

Here, $\sum_k \equiv T\sum_{i\omega_n}\sum_{\mathbf{k}}$ and $\sum_q \equiv T\sum_{i\Omega_m}\sum_{\mathbf{q}}$. The free-particle fermionic (inverse) Green's function is defined by $G_0^{-1}(k) \equiv i\omega_n - \xi_{\mathbf{k}}$, where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$, and $D_0^{-1}(q) \equiv i\Omega_m - \Omega_{\mathbf{q}}$ is the (inverse) scalar propagator for the b_q field. Given this action functional, the finite-temperature partition function is then $Z = \int \mathfrak{D}[\bar{c}, c, \bar{b}, b]e^{-S[\bar{c}, c, \bar{b}, b]}$.

B. Integrating out the phonons

The fermionic density function is defined by $\rho_q \equiv \sum_{k,\sigma}\bar{c}_{k\sigma}c_{k+q\sigma}$ and it satisfies $\rho_q = \bar{\rho}_{-q}$. Using this definition, we can factor out the part of the partition function depending on the phonon variables and write

$$Z_{\rm ph}[\bar{c}, c] = \int \mathfrak{D}[\bar{b}, b] \exp\left[\sum_q \left(\bar{b}_q D_0^{-1}(q) b_q - g_{\mathbf{q}}(b_q + \bar{b}_{-q})\rho_q\right)\right]. \tag{2.5}$$

The functional integral over the fields $\bar{b}$ and b is Gaussian, and thus it can be computed exactly by performing shifts in the fields given by $\bar{b}_q \to \bar{b}_q + g_{\mathbf{q}}D_0(q)\bar{\rho}_{-q}$ and $b_q \to b_q + g_{-\mathbf{q}}D_0(q)\rho_{-q}$. As in the previous section, $\bar{b}_q \neq b_{-q}$ and thus we can perform independent transformations of these fields. On performing these transformations, the action is then written explicitly as a quadratic function of $\bar{b}$ and b, and thus the functional integral may be computed exactly using the standard formula [24]. The result is

$$Z_{\rm ph}[\bar{c}, c] = \int \mathfrak{D}[\bar{b}, b] \exp\left[\sum_q \left(\bar{b}_q D_0^{-1}(q) b_q - |g_{\mathbf{q}}|^2 D_0(q)\rho_q\bar{\rho}_q\right)\right] = \mathcal{N}\exp\left[-\sum_q |g_{\mathbf{q}}|^2 D_0(q)\rho_q\bar{\rho}_q\right]. \tag{2.6}$$

The prefactor $\mathcal{N}$ is an unimportant constant independent of ρ_q. In obtaining the result above, we have used the properties $g^*_{\mathbf{q}} = g_{-\mathbf{q}}$ and $\bar{\rho}_q = \rho_{-q}$. Notice that in the effective interaction term the propagator $D_0(q)$ is coupled to a quantity that is manifestly even in q. Therefore, only the even part of $D_0(q)$ contributes to the sum, i.e., only the propagator for the sum of scalar field operators, $b_q + \bar{b}_{-q}$, is important. Hence, we write the phonon part of the partition function as

$$Z_{\text{ph}}[\bar{c}, c] \sim \exp\left[-\frac{1}{2}\sum_q |g_{\mathbf{q}}|^2 \mathcal{D}_0(q)\rho_q\bar{\rho}_q\right], \tag{2.7}$$

where the phonon propagator $\mathcal{D}_0$ appearing above is [35]:

$$\mathcal{D}_0(q) \equiv D_0(q) + D_0(-q) = \frac{2\Omega_{\mathbf{q}}}{(i\Omega_m)^2 - \Omega^2_{\mathbf{q}}}. \tag{2.8}$$

We emphasize that D_0 is the propagator for the field b_q, whereas $\mathcal{D}_0$ is the propagator for $b_q + \bar{b}_{-q}$. The full partition function thus reduces to

$$Z = \int \mathfrak{D}[\bar{c}, c] \exp\left[\sum_{k,\sigma} \bar{c}_{k\sigma} G_0^{-1}(k) c_{k\sigma} + \frac{1}{2}\sum_q \lambda(q)\rho_q\bar{\rho}_q\right], \tag{2.9}$$

where $\lambda(q) \equiv -|g_{\mathbf{q}}|^2\mathcal{D}_0(q)$ and we have ignored the prefactor term as it does not affect the electronic quantities of interest. At this stage of the development, integrating out the phonons has resulted in a dynamical density-density interaction term for the fermion fields, and the full partition function is of the form $\int \mathfrak{D}[\bar{c}, c]e^{-S_F[\bar{c},c]}$. The dynamical properties of this interaction term give rise to a frequency-dependent pairing function, in distinction to the static pairing function present in the BCS theory of superconductivity. Formulated in real space, the fermionic action reads

$$\begin{aligned} S_F[\bar{c}, c] = &-\int_{x,y}\sum_\sigma \bar{c}_{x\sigma}\, G_0^{-1}(x-y)\, c_{y\sigma} \\ &- \frac{1}{2}\int_{x,y}\sum_{\sigma,\sigma'} \lambda(x-y)\bar{c}_{x\sigma}c_{x\sigma}\bar{c}_{y\sigma'}c_{y\sigma'}. \end{aligned} \tag{2.10}$$

Here, $\int_x \equiv \int d^3r \int d\tau$ denotes integration over both space and imaginary time. Sums over repeated spin labels are taken to be implicit from now on.

III. SUPERCONDUCTIVITY IN THE COOPER CHANNEL

A. Hubbard-Stratonovich transformation

The interaction term in the fermionic action in Eq. (2.10) is quartic in the fermionic fields $\bar{c}$ and c. At present there are no known techniques for exactly computing the functional integral of such an interaction, but we can make progress by reformulating the calculation in terms of auxiliary fields using the Hubbard-Stratonovich (HS) transformation. This procedure eliminates the interaction term at the cost of introducing a functional integral over the auxiliary fields. We first expand the interaction term in the action into the two parts where $\sigma = \sigma'$ and $\sigma \neq \sigma'$, which results in

$$\begin{aligned} S_{\text{Int}}[\bar{c}, c] &\equiv -\frac{1}{2}\int_{x,y}\lambda(x-y)\bar{c}_{x\sigma}c_{x\sigma}\bar{c}_{y\sigma'}c_{y\sigma'} \\ &= -\int_{x,y}\lambda(x-y)\left(\frac{1}{2}\bar{c}_{x\sigma}c_{x\sigma}\bar{c}_{y\sigma}c_{y\sigma} + \bar{c}_{x\uparrow}\bar{c}_{y\downarrow}c_{y\downarrow}c_{x\uparrow}\right). \end{aligned} \tag{3.1}$$

We now introduce the bosonic auxiliary fields $\bar{\phi}$, ϕ, Σ and a measure $\mathfrak{D}[\bar{\phi}, \phi, \Sigma]$ chosen such that

$$1 = \int \mathfrak{D}[\bar{\phi}, \phi, \Sigma] \exp\left[-\int_{x,y}\frac{\bar{\phi}_{xy}\phi_{xy} + \frac{1}{2}\Sigma^\sigma_{xy}\Sigma^\sigma_{yx}}{\lambda(x-y)}\right], \tag{3.2}$$

which is then inserted into the partition function, giving

$$\begin{aligned} Z = &\int \mathfrak{D}[\bar{c}, c, \bar{\phi}, \phi, \Sigma] \\ &\times \exp\left[-S_F[\bar{c}, c] - \int_{x,y}\frac{\bar{\phi}_{xy}\phi_{xy} + \frac{1}{2}\Sigma^\sigma_{xy}\Sigma^\sigma_{yx}}{\lambda(x-y)}\right]. \end{aligned} \tag{3.3}$$

In principle, the interaction term in Eq. (3.1) can be decoupled in three possible channels [24]—Cooper, Density, and Exchange—which capture different physical phenomena. The Cooper channel is apposite for describing superconductivity, the density channel encapsulates density fluctuations, and the exchange channel describes electron-hole interactions. Weighting these channels is nontrivial, and while there are examples of multichannel HS decompositions in the literature [36–38], we use physical arguments to proceed. Since we are interested in describing singlet superconductivity, it is natural to use the Cooper channel to decouple the part of the interaction in Eq. (3.1) with opposite spins. Hence, we can identify ϕ as being related to fermion pairing and superconductivity. The term with the same spins is decoupled in the density channel, since the spins appearing on the fermions are not pertinent, and thus Σ acts as a collective density fluctuation. This decomposition of a single interaction into two different channels, which are treated on equal footing, is one of the main tenets of Eliashberg theory [14]. Our neglect of decoupling the interaction in the exchange channel will preclude this analysis from obtaining Kohn-Luttinger corrections, which have recently been considered in the diagrammatic framework [14,16].

We now shift the fields according to the transformations:

$$\bar{\phi}_{xy} \to \bar{\phi}_{xy} - \lambda(x-y)\bar{c}_{x\uparrow}\bar{c}_{y\downarrow}, \tag{3.4}$$

$$\phi_{xy} \to \phi_{xy} - \lambda(x-y)c_{y\downarrow}c_{x\uparrow}, \tag{3.5}$$

$$\Sigma^\sigma_{xy} \to \Sigma^\sigma_{xy} + i\lambda(x-y)\bar{c}_{y\sigma}c_{x\sigma}. \tag{3.6}$$

The resulting action now has no quartic interaction term, albeit at the expense of coupling the fermionic fields $\bar{c}$ and c to the HS fields $\bar{\phi}$, ϕ, and Σ. This transformation results in a new action given by

$$S[\bar{c}, c, \bar{\phi}, \phi, \Sigma] = \int_{x,y} \left[\frac{\bar{\phi}_{xy}\phi_{xy} + \frac{1}{2}\Sigma^{\sigma}_{xy}\Sigma^{\sigma}_{yx}}{\lambda(x-y)} - \bar{c}_{x\sigma} G_0^{-1}(x-y) c_{y\sigma} - \phi_{xy}\bar{c}_{x\uparrow}\bar{c}_{y\downarrow} - \bar{\phi}_{xy} c_{y\downarrow} c_{x\uparrow} + i\Sigma^{\sigma}_{xy}\bar{c}_{x\sigma} c_{y\sigma} \right]. \tag{3.7}$$

We consider translation-invariant systems, and thus the HS fields $\bar{\phi}, \phi$, and Σ depend only on a relative coordinate. After performing the Fourier transform of the action, the momentum-space form is expressed as

$$S[\bar{c}, c, \bar{\phi}, \phi, \Sigma] = \sum_{k,k'} \frac{\bar{\phi}(k)\phi(k') + \frac{1}{2}\Sigma^{\sigma}(k)\Sigma^{\sigma}(k')}{\lambda(k-k')} - \sum_{k} \left\{ \bar{c}_{k\sigma}\left[G_0^{-1}(k) - i\Sigma^{\sigma}(k)\right] c_{k\sigma} + \phi(k)\bar{c}_{k\uparrow}\bar{c}_{-k\downarrow} + \bar{\phi}(k) c_{-k\downarrow} c_{k\uparrow} \right\}. \tag{3.8}$$

B. Integrating out the fermions

We now introduce the Nambu fields ψ_k and $\bar{\psi}_k$, defined by $\psi_k = (c_{k\uparrow}, \bar{c}_{-k\downarrow})^T$ and $\bar{\psi}_k = (\bar{c}_{k\uparrow}, c_{-k\downarrow})$, which facilitate writing the action as

$$S[\bar{\psi}, \psi, \bar{\phi}, \phi, \Sigma] = -\sum_{k} \bar{\psi}_k \mathcal{G}^{-1}(k)\psi_k + \sum_{k,k'} \frac{\bar{\phi}(k)\phi(k') + \frac{1}{2}\Sigma^{\sigma}(k)\Sigma^{\sigma}(k')}{\lambda(k-k')}, \tag{3.9}$$

where we define $G_{n,\sigma}^{-1}(k) \equiv G_0^{-1}(k) - i\Sigma^{\sigma}(k)$ and the inverse Nambu Green's function is

$$\mathcal{G}_k^{-1} \equiv \begin{pmatrix} G_{n,\uparrow}^{-1}(k) & \phi(k) \\ \bar{\phi}(k) & -G_{n,\downarrow}^{-1}(-k) \end{pmatrix}. \tag{3.10}$$

The action is quadratic in the Nambu fields $\bar{\psi}$ and ψ, and hence the functional integral over these fields can be performed exactly to produce

$$\int \mathfrak{D}[\bar{\psi}, \psi] \exp\left(-S[\bar{\psi}, \psi, \bar{\phi}, \phi, \Sigma]\right) = \exp\left[\mathrm{Tr}\ln(-\beta\mathcal{G}^{-1}) - \sum_{k,k'} \frac{\bar{\phi}(k)\phi(k') + \frac{1}{2}\Sigma^{\sigma}(k)\Sigma^{\sigma}(k')}{\lambda(k-k')} \right] \equiv \exp\left(-S_{\mathrm{HS}}[\bar{\phi}, \phi, \Sigma]\right). \tag{3.11}$$

The trace operation Tr represents a matrix trace over the Nambu indices and an integration over the spatial degrees of freedom. The original, microscopic model of electrons interacting with phonons has been reformulated as a description in terms of an HS action S_{HS} which depends on the fields $\bar{\phi}, \phi$, and Σ. Excitations in the field ϕ are coupled to the annihilation of two fermions, whereas the Σ field is coupled to fermion density fluctuations.

IV. HS ACTION ANALYSIS

A. Saddle-point conditions

We expect large contributions to the partition function from the field configurations where the effective action is slowly varying about its extremum. Enforcing that the effective action is stationary with respect to the HS fields Σ and ϕ results in two saddle-point equations, the solutions of which are the saddle-point or mean-field values Σ_{mf} and ϕ_{mf}. For the HS action in Eq. (3.11), the saddle-point condition for Σ is

$$\begin{aligned} 0 &= \left.\frac{\delta S_{\mathrm{HS}}[\bar{\phi}, \phi, \Sigma]}{\delta\Sigma_k^{\sigma}}\right|_{\Sigma=\Sigma_{\mathrm{mf}}, \phi=\phi_{\mathrm{mf}}} \\ &= \sum_{k'} \left\{ \lambda^{-1}(k-k')\Sigma^{\sigma}_{\mathrm{mf}}(k') + i\mathrm{Tr}\left[\mathcal{G}_{k'} \begin{pmatrix} \delta_{k,k'}\delta_{\sigma\uparrow} & 0 \\ 0 & -\delta_{k,-k'}\delta_{\sigma\downarrow} \end{pmatrix} \right] \right\} \\ &= \sum_{k'} \lambda^{-1}(k-k')\Sigma^{\sigma}_{\mathrm{mf}}(k') + \frac{iG_{n\bar{\sigma}}^{-1}(-k)}{G_{n,\sigma}^{-1}(k)G_{n,\bar{\sigma}}^{-1}(-k) + |\phi_{\mathrm{mf}}(k)|^2}. \end{aligned} \tag{4.1}$$

Using the identity $\delta(k-k') = \sum_{k''}\lambda(k-k'')\lambda^{-1}(k''-k')$, we can simplify the above equation to solve for Σ^{σ}. By taking advantage of the spin symmetry of the system, i.e., $\mu_{\uparrow} = \mu_{\downarrow}$, we can define $\Sigma \equiv i\Sigma^{\uparrow} = i\Sigma^{\downarrow}$ and $G_n \equiv G_{n,\uparrow} = G_{n,\downarrow}$, to obtain

$$\Sigma_{\mathrm{mf}}(k) = \sum_{k'} \lambda(k-k') \frac{G_n^{-1}(-k')}{G_n^{-1}(k')G_n^{-1}(-k') + |\phi_{\mathrm{mf}}(k')|^2}. \tag{4.2}$$

This equation motivates defining the full Green's function as

$$G(k) \equiv \frac{G_n^{-1}(-k)}{G_n^{-1}(k)G_n^{-1}(-k) + |\phi_{\mathrm{mf}}(k)|^2}. \tag{4.3}$$

The self-consistent equation for Σ_{mf} in Eq. (4.2) now becomes

$$\Sigma_{\mathrm{mf}}(k) = \sum_{k'} \lambda(k-k') G(k'). \tag{4.4}$$

The saddle-point analysis of the pairing field ϕ can also be performed, leading to

$$\begin{aligned} 0 &= \left.\frac{\delta S_{\mathrm{HS}}[\bar{\phi}, \phi, \Sigma]}{\delta\bar{\phi}_k}\right|_{\Sigma=\Sigma_{\mathrm{mf}}, \phi=\phi_{\mathrm{mf}}} \\ &= \sum_{k'} \lambda^{-1}(k-k')\phi_{\mathrm{mf}}(k') - \mathrm{Tr}\left[\mathcal{G}_k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right]. \end{aligned} \tag{4.5}$$

Solving this equation for ϕ gives

$$\phi_{\mathrm{mf}}(k) = \sum_{k'} \lambda(k-k') \frac{\phi_{\mathrm{mf}}(k')}{G_n^{-1}(k')G_n^{-1}(-k') + |\phi_{\mathrm{mf}}(k')|^2}. \tag{4.6}$$

This homogeneous equation has the familiar form of a gap equation for the superconducting order parameter ϕ_{mf}. One possible solution to the above equation is $\phi = 0$, which represents a normal-state system.

Based on these mean-field equations, we can now gain some physical intuition about the parameters Σ_{mf} and ϕ_{mf}.

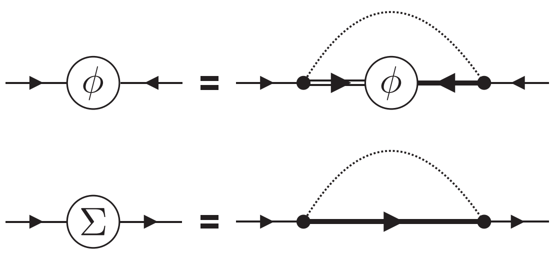

FIG. 1. A diagrammatic representation of the mean-field Eliashberg equations (4.7) and (4.8). The dotted lines terminated by circles denote λ, i.e., the phonon propagator and the coupling constants. The double-struck solid lines denote G_n and the bold solid lines are the full fermion propagator G. The lines at the ends of the diagram are external legs.

The simplest case is the normal state, where $\phi_{\rm mf} = 0$. The full Green's function then reduces to $G = G_n$, where $G_n^{-1} = G_0^{-1} - \Sigma_{\rm mf}$. Thus, when $\phi_{\rm mf} = 0$, G_n is a normal-state Green's function dressed by $\Sigma_{\rm mf}$, with $\Sigma_{\rm mf}$ playing the role of the normal-state self-energy. While Σ is an HS field whose physical interpretation is unknown *a priori*, based on the mean-field equation we find that $\Sigma_{\rm mf}$ can be interpreted as a self-energy (hence the choice in notation for this particular HS field). For nonzero ϕ, however, G_n is modified by superconducting interactions. For Eliashberg theory, we can define the self-energy $\Sigma_{\rm sc}(k) = -|\phi_{\rm mf}(k)|^2 G_n(-k)$, which is nonzero only in the superconducting phase, and then the inverse Green's function can be written as $G^{-1}(k) = G_n^{-1}(k) - \Sigma_{\rm sc}(k)$, and depicted in Fig. 2.

In the conventional [11–13] formulation of Eliashberg theory, based on the equation of motion technique, it is not apparent that the self-energy $\Sigma_{\rm mf}$ arises from the saddle-point value of an HS field. The importance of this result, which naturally appears in the functional-integral approach, is that it enables one to consider fluctuations beyond the saddle-point condition and possible corrections to the mean-field Eliashberg equations. This will be explored further in the next subsection. For convenience, from now on we drop the subscripts in $\Sigma_{\rm mf}$ and $\phi_{\rm mf}$.

Together, these mean-field equations produce the standard [11–13] Eliashberg equations depicted in Fig. 1:

$$\phi(i\omega_n, \mathbf{k}) = T \sum_{\mathbf{k}', i\omega_m} \lambda_{\mathbf{k}-\mathbf{k}'}(i\omega_n - i\omega_m) F(i\omega_m, \mathbf{k}'). \tag{4.7}$$

$$\Sigma(i\omega_n, \mathbf{k}) = T \sum_{\mathbf{k}', i\omega_m} \lambda_{\mathbf{k}-\mathbf{k}'}(i\omega_n - i\omega_m) G(i\omega_m, \mathbf{k}'). \tag{4.8}$$

$$G_n(i\omega_n, \mathbf{k}) = \left[G_0^{-1}(i\omega_n, \mathbf{k}) - \Sigma(i\omega_n, \mathbf{k})\right]^{-1}. \tag{4.9}$$

$$G(i\omega_n, \mathbf{k}) = \frac{G_n^{-1}(-i\omega_n, -\mathbf{k})}{G_n^{-1}(i\omega_n, \mathbf{k}) G_n^{-1}(-i\omega_n, -\mathbf{k}) + |\phi(i\omega_n, \mathbf{k})|^2}. \tag{4.10}$$

$$F(i\omega_n, \mathbf{k}) = \phi(i\omega_n, \mathbf{k}) G_n(-i\omega_n, -\mathbf{k}) G(i\omega_n, \mathbf{k}). \tag{4.11}$$

A diagrammatic representation of the propagators is shown in Fig. 2. The superconducting phase corresponds to a nontrivial solution for $\phi(i\omega_n, \mathbf{k})$. The critical temperature T_c, computed at the mean-field level, is the highest temperature for which a nonzero $\phi(i\omega_n, \mathbf{k})$ exists. Extensive discussion on methods to determine the mean-field T_c can be found in Ref. [9].

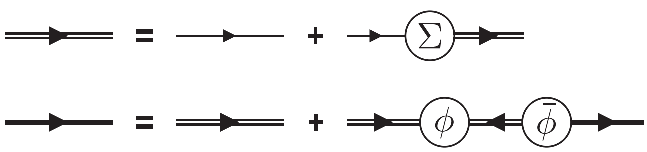

FIG. 2. A diagrammatic representation of the dressed propagators appearing in the Eliashberg equations (4.9) and (4.10).

B. Gaussian fluctuations in the normal state

In this section we shall focus on normal-state fluctuations, that is, fluctuations about $\phi_{\rm mf} = 0$. We allow ϕ and Σ to depend on both a relative and a center-of-mass coordinate:

$$\phi_{xy} = \phi\left(x - y; \frac{x+y}{2}\right); \Sigma^\sigma_{xy} = \Sigma^\sigma\left(x - y; \frac{x+y}{2}\right). \tag{4.12}$$

The Fourier transforms of ϕ_{xy} and Σ^σ_{xy} are denoted by $\phi(k;q)$ and $\Sigma^\sigma(k;q)$, respectively, and those of $\bar{c}_{x\uparrow}\bar{c}_{y\downarrow}$ and $\bar{c}_{x\sigma}c_{y\sigma}$ are $\bar{c}_{k+q/2\uparrow}\bar{c}_{-k+q/2\downarrow}$ and $\bar{c}_{k+q/2\sigma}c_{k-q/2\sigma}$, respectively. We restrict our attention to small fluctuations away from the saddle-point condition, where we enforce that $q = 0$, and consider

$$\phi(k;q) \equiv \phi(k)\delta_{q,0} + \eta(k;q) \ , \tag{4.13}$$

$$\Sigma^\sigma(k;q) \equiv \Sigma^\sigma(k)\delta_{q,0} + \zeta^\sigma(k;q). \tag{4.14}$$

Here, $\phi(k)$ and $\Sigma(k)$ are given by the mean-field solutions to the Eliashberg equations (4.7)–(4.11), and $\eta(k,q)$ and $\zeta(k,q)$ are small perturbations about these solutions. We emphasize that the four-momentum k in $\eta(k,q)$ is not transferred but rather acts merely as a label, whereas the four-momentum q is transferred. This is required on the grounds that the Matsubara frequency associated with k, i.e., $k_0 = i\omega_n$, is fermionic, whereas ϕ (and hence η) are bosonic fields.

We now expand the inverse Green's function about its mean-field value via $\mathcal{G}^{-1} = \mathcal{G}^{-1}_{\rm mf}(1 - \mathcal{G}_{\rm mf}\Lambda_\eta - \mathcal{G}_{\rm mf}\Lambda_\zeta)$, where

$$\Lambda_\eta(k;q) \equiv \begin{pmatrix} 0 & -\eta(k;q) \\ -\bar{\eta}(k;-q) & 0 \end{pmatrix}, \tag{4.15}$$

$$\Lambda_\zeta(k;q) \equiv \begin{pmatrix} i\zeta^\uparrow(k;q) & 0 \\ 0 & -i\zeta^\downarrow(-k;q) \end{pmatrix}. \tag{4.16}$$

Hence, the action may be written as

$$\begin{aligned} S_{\rm HS}[\bar{\phi}, \phi, \bar{\eta}, \eta, \Sigma, \zeta] &= \int_{x,y} \frac{(\bar{\phi}_{xy} + \bar{\eta}_{xy})(\phi_{xy} + \eta_{xy}) + \frac{1}{2}\left(\Sigma^\sigma_{xy} + \zeta^\sigma_{xy}\right)\left(\Sigma^\sigma_{yx} + \zeta^\sigma_{yx}\right)}{\lambda(x-y)} \\ &\quad - \mathrm{Tr}\ln\left[-\beta\mathcal{G}^{-1}_{\rm mf}(1 - \mathcal{G}_{\rm mf}\Lambda_\eta - \mathcal{G}_{\rm mf}\Lambda_\zeta)\right]. \end{aligned} \tag{4.17}$$

Using the fact that the perturbations η and ζ^σ are small, we can expand the logarithm to quadratic order, noting that the saddle-point conditions ensure that the terms linear in ϕ and Σ vanish identically. The HS action is thus

$$\begin{aligned} S_{\rm HS}[\bar{\phi}, \phi, \bar{\eta}, \eta, \Sigma, \zeta] &= S_{\rm mf}[\bar{\phi}, \phi, \Sigma] + \int_{x,y} \frac{\bar{\eta}_{xy}\eta_{xy} + \frac{1}{2}\zeta^\sigma_{xy}\zeta^\sigma_{yx}}{\lambda(x-y)} \\ &\quad + \frac{1}{2}\mathrm{Tr}[(\mathcal{G}_{\rm mf}\Lambda_\eta + \mathcal{G}_{\rm mf}\Lambda_\zeta)^2]. \end{aligned} \tag{4.18}$$

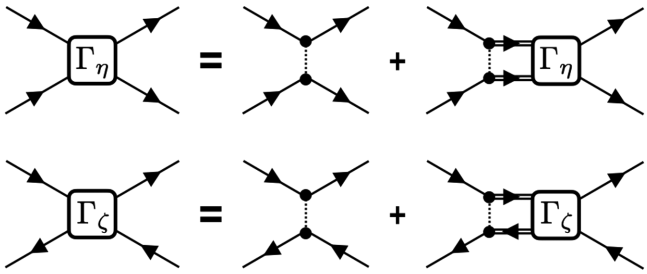

FIG. 3. A diagrammatic representation of the Dyson equation for the normal-state Eliashberg fluctuation propagators.

We assume that the system under study is above the superconducting critical temperature T_c and set $\phi = \bar{\phi} = 0$. After performing some matrix algebra, the squared trace term simplifies to the following result expressed in momentum space

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}[(\mathcal{G}_{\mathrm{mf}}\Lambda_\eta + \mathcal{G}_{\mathrm{mf}}\Lambda_\zeta)^2] \\
&\quad = \frac{1}{2}\mathrm{Tr}(\mathcal{G}_n\Lambda_\eta\mathcal{G}_n\Lambda_\eta) + \frac{1}{2}\mathrm{Tr}(\mathcal{G}_n\Lambda_\zeta\mathcal{G}_n\Lambda_\zeta) \\
&\quad = -\sum_{k,q}\Big[\bar{\eta}(k-q/2;q)G_n(k)G_n(q-k)\eta(k-q/2;q) \\
&\qquad + \frac{1}{2}\zeta^\sigma(k-q/2;q)G_n(k)G_n(k-q)\zeta^\sigma(k-q/2;-q)\Big].
\end{aligned} \tag{4.19}$$

Thus, the HS action is now of the form $S_{\mathrm{HS}}[\bar{\phi},\phi,\Sigma,\bar{\eta},\eta,\zeta] = S_{\mathrm{mf}}[\bar{\phi},\phi,\Sigma] + S[\bar{\eta},\eta,\zeta]$, where

$$\begin{aligned}
&S[\bar{\eta},\eta,\zeta] \\
&\quad = -\sum_{k,k',q}\Big[\bar{\eta}(k-q/2;q)\Gamma_\eta^{-1}(k,k';q)\eta(k'-q/2;q) \\
&\qquad + \frac{1}{2}\zeta^\sigma(k-q/2;q)\Gamma_\zeta^{-1}(k,k';q)\zeta^\sigma(k'-q/2;-q)\Big].
\end{aligned} \tag{4.20}$$

We have introduced $\Gamma_\eta(k,k';q)$ and $\Gamma_\zeta(k,k';q)$ as fluctuation propagators for the fields η and ζ^σ respectively, and they are defined by the following Dyson equations

$$\Gamma_\eta^{-1}(k,k';q) = -\lambda^{-1}(k-k') + \delta(k-k')G_n(k)G_n(q-k), \tag{4.21}$$

$$\Gamma_\zeta^{-1}(k,k';q) = -\lambda^{-1}(k-k') + \delta(k-k')G_n(k)G_n(k-q). \tag{4.22}$$

The Dyson equations for the fluctuation propagators are shown in Fig. 3. Notice that Γ_η has a particle-particle loop, whereas Γ_ζ has a particle-hole loop; this is due to the fact that the η field corresponds to fluctuations in the Cooper channel, whereas ζ^σ represents fluctuations in the density channel. After performing the functional integral over the fields $\bar{\eta}$, η, and ζ we obtain the fluctuation action S_{fluc}, defined by $\int \mathfrak{D}[\bar{\eta},\eta,\zeta]\exp(-S[\bar{\eta},\eta,\zeta]) = \exp(-S_{\mathrm{fluc}})$, where

$$S_{\mathrm{fluc}} = \sum_p \ln\left[-\Gamma_\eta^{-1}(p)\right] + \frac{1}{2}\sum_{p,\sigma}\ln\left[-\Gamma_\zeta^{-1}(p)\right]. \tag{4.23}$$

To invert Eqs. (4.21) and (4.22), we multiply Eq. (4.21) by $\Gamma_\eta(k',p';q)\lambda(p-k)$ and integrate over k' and k to obtain

$$\begin{aligned}
&\int dk'dk\,[-\lambda^{-1}(k-k')\Gamma_\eta(k',p';q)\lambda(p-k) \\
&\qquad + \delta(k-k')G_n(k)G_n(q-k)\Gamma_\eta(k',p';q)\lambda(p-k)] \\
&\quad = -\Gamma_\eta(p,p';q) + \int dk' G_n(k')G_n(q-k') \\
&\qquad \times \Gamma_\eta(k',p';q)\lambda(p-k') \\
&\quad = \int dk'dk\,\Gamma_\eta^{-1}(k,k';q)\Gamma_\eta(k',p';q)\lambda(p-k) \\
&\quad = \lambda(p-p').
\end{aligned} \tag{4.24}$$

The same analysis can be done for Γ_ζ. On rearranging terms and relabeling variables, we obtain the following two integral equations for Γ_η and Γ_ζ:

$$\begin{aligned}
\Gamma_\eta(p,p';q) &= -\lambda(p-p') + \sum_{p''}\lambda(p-p'') \\
&\qquad \times G_n(p'')G_n(q-p'')\Gamma_\eta(p'',p';q).
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\Gamma_\zeta(p,p';q) &= -\lambda(p-p') + \sum_{p''}\lambda(p-p'') \\
&\qquad \times G_n(p'')G_n(p''-q)\Gamma_\zeta(p'',p';q).
\end{aligned} \tag{4.26}$$

The fluctuation propagators satisfy a Bethe-Salpeter-like integral equation [39], where a symmetric combination of Green's functions appears in the integral kernel. Other approaches, like FLEX [40], go beyond Gaussian fluctuation theory and are not considered here.

Our result for Γ_η in Eq. (4.25) agrees with that found in Eq. (1) of Refs. [30–32], after restoring the coupling constant in their expressions. The function $\Gamma_\eta(k,k';q)$ physically represents the propagator for two electrons above T_c, with relative momentum k and center-of-mass momentum q, to form a short-lived Cooper pair before separating into two electrons with relative momentum k'. The condition that $\Gamma_\eta^{-1}(k,k';0) \to 0$ would imply that these fluctuations proliferate at $q = 0$, signaling a phase transition to the superconducting state. In Ref. [32], the fluctuation contribution to the electron self-energy was ignored; here, however, we have incorporated this effect. In particular, the function $\Gamma_\zeta(k,k';q)$ parameterizes fluctuations in the Σ field and physically accounts for density interactions. The Dyson equation for Γ_ζ has a similar form to the phonon propagator in Migdal-Eliashberg theory [11,41–43], where the bare phonon propagator is corrected by a phonon self-energy involving a particle-hole loop of two fermionic Green's functions.

Using the results from Eqs. (4.18) and (4.23), the full normal-state action is given by

$$S(T > T_c) = -\sum_{p,p'} \Sigma(p)\lambda^{-1}(p-p')\Sigma(p') - 2\sum_{p} \ln\left[-G_n^{-1}(p)\right] + \sum_{p} \ln\left[-\Gamma_\eta^{-1}(p)\right] + \frac{1}{2}\sum_{p,\sigma} \ln\left[-\Gamma_\zeta^{-1}(p)\right]. \tag{4.27}$$

The normal-state action was also considered in Ref. [32], by adding Cooper-pair fluctuations to the normal-state result of Ref. [41]. Here we have incorporated fluctuations in the Σ field and rigorously derived the Copper-pair fluctuations using the functional integral. We emphasize that the result above has fixed Σ at its mean-field value, but also incorporated fluctuations about this quantity. As a result, this action is not extremal with respect to variations in Σ.

As discussed in Sec. IV A, T_c is the maximum temperature at which ϕ is nonzero. We have not investigated how Gaussian fluctuations affect T_c; that is, we still use the mean-field equations to define T_c. The importance of how fluctuations affect the number equation and superfluid density has been studied in the BCS case [25,44]. The Gaussian-fluctuation analysis can also be performed about a nonzero value of ϕ, which will result in a fluctuation action that depends on ϕ. In principle, one could then examine the solutions to $\delta S_{\rm fluc}/\delta\phi = 0$, which would yield Gaussian-fluctuation corrections to the gap. Such an analysis has been done for the fluctuation extension of BCS theory [45]. Our primary focus, however, is on the normal-state response, as in Refs. [30–32], and in the next section we study the diamagnetic susceptibility.

V. FLUCTUATION RESPONSE

In this section we derive the normal-state fluctuation response based on the propagators derived in the previous section. The temperature region over which fluctuations are important is characterized in terms of the Ginzburg-Levanyuk number (Gi), which is defined as the value of the reduced temperature ($\epsilon \equiv T/T_c - 1$) at which the fluctuation correction of the heat capacity (in zero magnetic field) equals the value of the discontinuity in the heat capacity [23]. For ultraclean three-dimensional systems, Ginzburg-Landau (GL) fluctuation theory predicts $Gi \sim 80(T_c/E_F)^4 \ll 1$, where E_F is the Fermi energy. Thus, fluctuations are important only in a very narrow range of temperatures near T_c. To discern the effect of electron-phonon interactions on Gi, we let λ denote the electron-phonon coupling (not to be confused with the bare phonon propagator); for an Einstein phonon spectrum, λ is related to $g_{\mathbf{q}}$ by $\lambda = 2N|g_{\mathbf{q}}|^2/\omega_E$, where N is the single-spin density of states (DOS) [11]. From Eq. (10) of Ref. [32], we can infer the value of Gi to be $Gi \sim \lambda^9(T_c/E_F)^4$, as $\lambda \to \infty$. Thus, very strong electron-phonon coupling would increase the range of temperatures over which fluctuations are important; however, as noted in Ref. [32], since real metals have $\lambda \sim 1$, the effect on the prefactor may be roughly an order of magnitude.

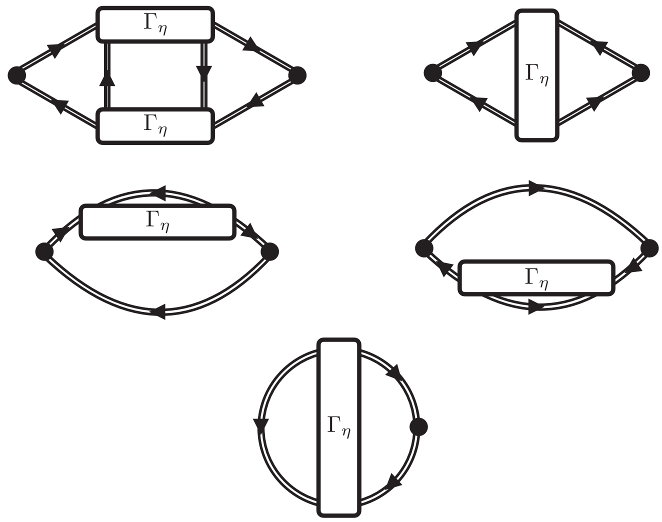

FIG. 4. Cooper-channel fluctuation response diagrams in the normal state. The top left diagram is the AL diagram, next is the MT diagram, followed by the two DOS diagrams, and finally the diamagnetic diagram.

To derive the electromagnetic response, the standard method [24] is to introduce an external gauge field A_μ and apply a minimal-coupling procedure. To simplify the derivation, we assume that $p, p' \approx p_F$ in the propagators $\Gamma_\eta(p, p'; q)$ and $\Gamma_\zeta(p, p'; q)$, where p_F is the Fermi momentum. Thus, if we let q denote the momentum of the external gauge field A_μ, then we can derive the response in the same manner as the conventional fluctuation response [29,46]. The fluctuation action, in the presence of the external field, has the form $S_{\rm fluc}[A] \sim \ln[-\Gamma_\eta^{-1}[A]] + \frac{1}{2}\ln[-\Gamma_\zeta^{-1}[A]]$. The response is then obtained using the definition $K^{\mu\nu} = \delta^2 S[A]/\delta A_\mu \delta A_\nu|_{A=0}$. Performing these derivatives, we obtain

$$K^{\mu\nu}_{\rm fluc} \sim \sum_{a=\eta,\zeta}\left[-\Gamma_a\frac{\delta\Gamma_a^{-1}}{\delta A_\mu}\Gamma_a\frac{\delta\Gamma_a^{-1}}{\delta A_\mu} + \Gamma_a\frac{\delta^2\Gamma_a^{-1}}{\delta A_\mu\delta A_\nu}\right]. \tag{5.1}$$

The first terms are the AL diagrams, one with the fluctuation propagator Γ_η describing the transport of Cooper pairs and the other with the fluctuation propagator Γ_ζ describing the density fluctuations. The second term comprises the Maki-Thompson (MT), DOS, and diamagnetic contributions [23,29,46]. These diagrams are shown in Figs. 4 and 5.

Here we compute the singular contribution to the diamagnetic susceptibility for the Eliashberg fluctuation response. This was initially investigated in Ref. [47], albeit the conclusions reached were disputed [32].The Kubo formula for diamagnetic susceptibility is [48]:

$$\chi = -\frac{1}{q^2}K^{xx}(i\Omega_m = 0, q^y)|_{q^y\to 0, q^x=q^z=0}. \tag{5.2}$$

The absence of a normal-state Meissner effect requires [13] that $K^{xx}(i\Omega_m = 0, \mathbf{q} \to 0) = 0$, thus the above Kubo formula is well defined. Near the transition temperature the fluctuation propagator becomes singular, and since the AL diagram has one more propagator than the other fluctuation diagrams, we may expect this term to provide the sole contribution to the diamagnetic susceptibility [49]. Since χ is a thermodynamic

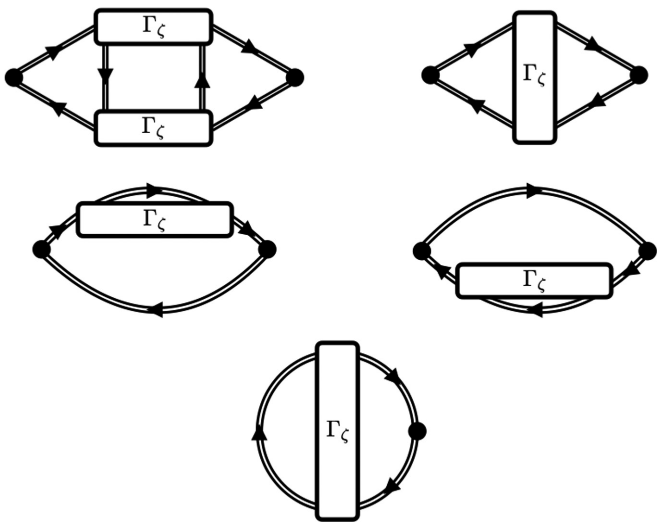

FIG. 5. Density-channel fluctuation response diagrams in the normal state.

quantity, no analytic continuation needs to be performed, and thus there is no anomalous MT contribution to consider [23]. As a result, the AL term does indeed provide the dominant singular contribution near T_c.

The vertex function Γ_η can be written as [30]: $\Gamma_\eta(p,p';q)=\phi(p)L(q)\bar{\phi}(p')$, where ϕ is the solution to the linearized version of Eq. (4.7) and L is the fluctuation propagator, which depends only on the center-of-mass momentum q. Following Refs. [23,30], the AL diagram for the fluctuation Cooper pairs, i.e., η fluctuations, in the top left diagram in Fig. 4, is then

$$K^{xx}_{\mathrm{AL},\eta}(0,q^y)=-\frac{4e^2}{m^2}T\sum_{\mathbf{p},\epsilon_m}L(i\epsilon_m,\mathbf{p}_+)C^x(p_+,p_-)\times L(i\epsilon_m,\mathbf{p}_-)C^x(p_-,p_+). \tag{5.3}$$

Here, $\mathbf{p}_\pm=\mathbf{p}\pm\mathbf{q}/2$. The vertex C^x is defined in position space by $C^x(x,y,z)\sim\delta L^{-1}[A](x,y)/\delta A^x(z)$; for the momentum-space form, see Ref. [30]. The critical regime is characterized by $L^{-1}(0,\mathbf{q}\to 0)\to 0$, thus the Matsubara frequency equal to zero provides the dominant contribution and so we set $\epsilon_m=0$. Performing the Taylor expansion of the fluctuation propagators and using the Kubo formula, we obtain

$$\chi_{\mathrm{fluc}}=\frac{e^2}{m^2}T\sum_{\mathbf{p}}[C^x(p,p)]^2[LL''-(L')^2]. \tag{5.4}$$

Here the prime denotes differentiation with respect to p^y. The vertex $C^x(p,p)$ has the form $C^x(p,p)=p^xC(0)$. Performing the derivatives and using integration by parts, we obtain

$$\chi_{\mathrm{fluc}}=-\frac{2}{3}\frac{e^2}{m^2}T\sum_{\mathbf{p}}[C^x(p,p)]^2L^3(0,\mathbf{p})[L^{-1}(0,\mathbf{p})]''. \tag{5.5}$$

The inverse retarded fluctuation propagator has the form [30]: $L^{-1}_{\mathrm{R}}(\Omega,\mathbf{p})=N(\frac{i\Omega}{T_c}b-Dp^2-\epsilon)$, where $\epsilon=T/T_c-1$. Here, N is the single-spin DOS at the Fermi surface, b is a constant, and D is related to the square of the coherence length [50]. Inserting the propagator into Eq. (5.5) and performing the integration, we find

$$\begin{aligned}\chi_{\mathrm{fluc}}&=\frac{4}{3}ND\frac{e^2}{m^2}T\sum_{\mathbf{p}}[p^xC(0)]^2L^3(0,\mathbf{p})\\&=\frac{2}{9\pi^2}ND[C(0)]^2\frac{e^2}{m^2}T\int_0^\infty dp\,p^4L^3\\&=-\frac{e^2T}{24\pi}\left[\frac{C(0)}{NDm}\right]^2\sqrt{\frac{D}{\epsilon}}.\end{aligned} \tag{5.6}$$

The critical exponent for the fluctuation contribution to diamagnetic susceptibility agrees with the GL result [23,49]. Indeed, independent of the particular pairing mechanism, whether it be electron-phonon based or BCS based, the critical exponent is dependent only on the dimensionality of the system. This observation was also noted in Refs. [30,31], in the context of the fluctuation contribution to electrical conductivity. The result we have obtained here is for ultraclean systems. However, the exponent is still expected to be the same for systems with impurities. In Ref. [51], the diamagnetic susceptibility was studied for dirty superconductors near the Anderson localization transition, and the critical exponent was found to be the same as in the ultraclean case—the only differences were certain prefactors and the definition of the coherence length. This phenomenon is also discussed in Ref. [52].

Note that the factor in Eq. (5.6) in square brackets is dimensionless, although it does depend on the strength of the coupling constant. In the conventional fluctuation theory, $[C(0)/(NDm)]^2=4$, and in this case we recover Aslamazov and Larkin's result [49]:

$$\chi_{\mathrm{AL}}=-\frac{e^2T}{6\pi}\sqrt{\frac{D}{\epsilon}}. \tag{5.7}$$

The presence of the electron-phonon interaction can significantly modify the prefactor appearing above. Indeed, in the strong-coupling limit, Ref. [30] finds that

$$N\sim mp_F,\ b\sim\frac{1}{\lambda},\ D\sim\frac{1}{\lambda^3}\frac{v_F^2}{T_c^2},\ C(0)\sim\frac{1}{\lambda^2}\frac{p_F^3}{T_c^2}. \tag{5.8}$$

Thus, the effect of strong coupling is to modify the diamagnetic susceptibility by

$$\frac{\chi_{\mathrm{fluc}}}{\chi_{\mathrm{AL}}}\sim\lambda^{\frac{1}{2}},\quad \lambda\to\infty. \tag{5.9}$$

In the limit of large electron-phonon coupling, the diamagnetic susceptibility is increasingly large due to the prefactor $\lambda^{\frac{1}{2}}$. Thus, in comparison to the seminal results of Ref. [49], when the electron-phonon coupling becomes increasingly large the fluctuation regime for large diamagnetic susceptibility is broadened.

In regard to future directions for functional applications of Eliashberg theory, we note that the functional-integral approach is a natural method to incorporate many fluctuating degrees of freedom. The Coulomb interaction, for example, can be naturally incorporated by extending the HS analysis to include another HS field [53,54]. The resulting EM response, with Coulomb and phase fluctuations, then has a matrix structure for the phase, amplitude, and Coulomb degrees of freedom. In principle, this can be extended to the case of the ordered phase of Eliashberg theory, where the interaction constant would be replaced by the phonon propagator. One could also study the EM response at the Gaussian level within Eliashberg theory, following similar lines to Ref. [26] for the BCS case.

VI. CONCLUSION

The Eliashberg theory of superconductivity incorporates the dynamical nature of the electron-phonon interaction, and conventionally it is derived within an equation of motion approach. Here we have shown how to derive the mean-field Eliashberg equations using a functional-integral method. Importantly, this analysis illustrates that the electron self-energy appears as an HS field, which enables its fluctuations about the mean-field solution to be considered. We reproduced the standard mean-field equations as saddle-point conditions of the HS action, without recourse to intricate diagrammatic arguments. In addition, we considered the Gaussian-fluctuation response and obtained both Cooper- and density-channel excitations. The former had been studied previously, but our systematic analysis naturally obtains both fluctuations at the same time. As a result, we obtained the fluctuation diagrams for Eliashberg theory and have provided the complementary account for what has been done for BCS theory. Furthermore, we computed the fluctuation diamagnetic susceptibility near the critical temperature and determined its strong-coupling form. Our functional approach enables clear pathways for going beyond the traditional Eliashberg theory framework by considering alternative HS decouplings.

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