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Letter

Quantum quench in a driven Ising chain

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We consider the Ising chain driven by oscillatory transverse magnetic fields. For certain parameter regimes, we reveal a hidden integrable structure in the problem, which allows access to the *exact time evolution* in this driven quantum system. We compute time-evolved one- and two-point functions following a quench that activates the driving. It is shown that this model does not heat up to infinite temperature, despite the absence of energy conservation, and we further discuss the generalization to a family of driven Hamiltonians that do not appear to suffer heating to infinite temperature, despite the absence of integrability and disorder. The particular model studied in detail also presents a route for realizing exotic physics (in this case, signatures of the E_8 structure associated with perturbing a critical quantum Ising chain with a small longitudinal magnetic field) by suitably tuning the driving frequency. In particular, we numerically confirm that the ratio of the masses of the two lowest meson excitations is given by the golden ratio.

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Introduction. Over the last decade, the nonequilibrium dynamics of quantum systems has attracted a great deal of attention [1–12], motivated by the desire to address fundamental questions: When and how do quantum systems relax to equilibrium? How does one describe this equilibrium? What influences the dynamics and equilibration? Understanding these issues is important when developing descriptions of a growing number of experiments that examine nonequilibrium dynamics, both in cold atomic gases [13,14] and in the solid state [15,16]. The insights gained may play an important role in the development of quantum computing resources, especially when considering how to protect quantum information from the scrambling associated with thermalization.

Recently, attention has turned to understanding driven quantum systems, partially due to the realization that such systems can host interesting topological phases (see, e.g., Refs. [17–23]) and other exotic behavior (such as time crystal phases [24–30]). These studies have generated much discussion of how to extend and apply the concepts of equilibrium statistical mechanics in the presence of driving. A particular issue is that, generically, driven quantum systems do not conserve energy. As a result, in the long-time limit entropy maximization leads them to heat up to infinite temperature, leading to trivial ergodic behavior. As a result, quantum information is completely scrambled [31–33]. Routes to avoid this behavior include introducing disorder to induce a many body localization transition (see, e.g., Refs. [24,34–36]), or to consider models that are, in some sense, integrable [37].

In this Letter, we consider a driven model that, at each point in time, is nonintegrable but nonetheless possesses the dynamics which is governed by a hidden integrability. Using this, we compute the nonequilibrium dynamics of equal-time correlation functions following a quench in which the driving is initiated. The method for attacking this problem can be generalized to a (infinite) family of Hamiltonians, opening the door for future nonperturbative, exact studies. We will see that this whole family of driven quantum systems, each of which is generically nonintegrable, does not undergo heating to infinite temperature. We will also see that breaking the special structure of this family leads to thermalization to infinite temperature.

The driven Ising chain. We consider a one-dimensional spin-$\frac{1}{2}$ Ising magnet, driven by oscillatory transverse fields. The Hamiltonian reads

$$H(t) = -J\sum_{l=1}^{L}\sigma_l^z\sigma_{l+1}^z + h^z\sum_{l=1}^{L}\sigma_l^z - g\sum_{l=1}^{L}(e^{-i\Omega t}\sigma_l^+ + e^{i\Omega t}\sigma_l^-), \qquad (1)$$

with $J > 0$ being the Ising exchange parameter, h^z a static longitudinal field, g the strength of the transverse fields, which oscillate at frequency Ω, and L the system size. The spin operators σ_l^α act at the lth site of the lattice, $\sigma_l^\pm = (\sigma_l^x \pm i\sigma_l^y)/2$, and we impose periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$. The Hamiltonian (1) is periodic in time $H(t) = H(t+T)$ with period $T = 2\pi/\Omega$ and could be realized in the quasi-one-dimensional (quasi-1D) ferromagnet $CoNb_2O_6$ [38,39] by application of oscillating transverse fields.

At a generic time, the Hamiltonian consists of an Ising interaction term and fields in all (x, y, z) directions. Thus, instantaneously, the Hamiltonian $H(t)$ is *nonintegrable*, and

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the exact computation of quantities seems unlikely. In the following we will see that this is in fact not the case—there exists a hidden integrable line within this model where exact results can be obtained. Furthermore, away from this integrability we will draw general insights.

Time evolution of observables. We will now consider how a state $|\Psi_0\rangle$ evolves under the Hamiltonian (1) at times $t > 0$. The time-evolved state $|\Psi(t)\rangle$ will be a solution of the time-dependent Schrödinger equation

$$[i\hbar\partial_t - H(t)]|\Psi(t)\rangle = 0, \tag{2}$$

subject to the initial condition $|\Psi(t=0)\rangle = |\Psi_0\rangle$. Herein we set $\hbar = 1$, which defines our units. The formal solution of Eq. (2) is well known:

$$|\Psi(t)\rangle = \mathbb{T}\exp\left(-i\int_0^t \mathrm{d}t'\, H(t')\right)|\Psi_0\rangle, \tag{3}$$

however, using this to compute time-evolution is a challenge due to the explicit time-ordering ($\mathbb{T}$) of the exponential. To make some headway on this problem we apply a time-dependent unitary transformation $U(t)$,[1] multiplying both sides of Eq. (2) from the left by $U(t)$ and inserting a factor of $\mathbb{1} = U(t)^\dagger U(t)$ between the wave function and the operators:

$$U(t)[i\partial_t - H(t)]U^\dagger(t)U(t)|\Psi(t)\rangle = 0. \tag{4}$$

The problem can become much simpler if there is a choice of $U(t)$ such that this reduces to an effective *time-independent* Schrödinger equation. Choosing [43]

$$U(t) = \exp\left(\frac{i\Omega t}{2}\sum_l \sigma_l^z\right) \equiv e^{\frac{i\Omega t}{2}\sigma^z_{\rm tot}}, \tag{5}$$

we map Eq. (2) to a time-independent Schrödinger equation $(i\partial_t - H_{\rm st})|\Phi(t)\rangle = 0$ with an effective static Hamiltonian

$$H_{\rm st} = \sum_{l=1}^{L}\left[-J\sigma_l^z\sigma_{l+1}^z + \left(h^z - \frac{\Omega}{2}\right)\sigma_l^z - g\sigma_l^x\right]. \tag{6}$$

The wave function transforms as $|\Phi(t)\rangle = U(t)|\Psi(t)\rangle$. This reduction to a static problem is not evident in the Magnus expansion [53].

Diagonalizing (6) to obtain eigenstates $|E_n\rangle$ with energies E_n, the time-evolved state can be written as

$$|\Psi(t)\rangle = \sum_n \exp\left[-i\left(E_n + \frac{\Omega}{2}\sigma^z_{\rm tot}\right)t\right]|E_n\rangle\langle E_n|\Psi_0\rangle\,. \tag{7}$$

The states $|E_n\rangle$ are *not eigenstates* of $\sigma^z_{\rm tot}$ and thus each term in Eq. (7) undergoes nontrivial dynamics. While Eq. (7) is highly nontrivial, there is no need to despair. Our problem reduces to a tractable one if we focus on equal-time correlation functions, as one can use that the operator $U(t)$ acts in a simple manner on the spin operators:

$$\begin{aligned} U(t)\sigma_l^{x,y}U(t)^\dagger &= \cos(\Omega t)\sigma_l^{x,y} \mp \sin(\Omega t)\sigma_l^{y,x},\\ U(t)\sigma_l^z U(t)^\dagger &= \sigma_l^z. \end{aligned} \tag{8}$$

[1]As discussed in Refs. [40–42], there is a nice geometric interpretation of unitary transformations that depends on a continuous parameter (e.g., time t) in terms of gauge potentials.

Mapping to a "sudden quench." Let us now consider the time evolution of one-point functions $s^\alpha(t) = \langle\Psi(t)|\sigma_l^\alpha|\Psi(t)\rangle$, where the result is independent of l by translational invariance. Using Eq. (8) these become

$$\begin{aligned} s^z(t) &= s^z_{\rm st}(t),\\ s^x(t) &= \cos(\Omega t)s^x_{\rm st}(t) - \sin(\Omega t)s^y_{\rm st}(t),\\ s^y(t) &= \cos(\Omega t)s^y_{\rm st}(t) + \sin(\Omega t)s^x_{\rm st}(t). \end{aligned} \tag{9}$$

Here each time-dependent expectation value on the right-hand side describes time-evolution induced by a sudden quench to the static Hamiltonian (6) when starting from the initial state $|\Psi_0\rangle$:

$$s^\alpha_{\rm st}(t) = \sum_{n,m} e^{i(E_n - E_m)t}\langle\Psi_0|E_n\rangle\langle E_n|\sigma_l^\alpha|E_m\rangle\langle E_m|\Psi_0\rangle. \tag{10}$$

Equations similar to Eq. (9) can be written for the two-point functions, $s^{\alpha\beta}(\ell;t) = \langle\Psi(t)|\sigma_j^\alpha\sigma_{j+\ell}^\beta|\Psi(t)\rangle$. These are tractable but a little unwieldy and so are given in the Supplemental Material [53]. All time-evolved correlation functions are reduced to oscillatory factors multiplying "sudden quench" correlation functions. Thus, for this driven problem, we can apply the techniques developed for sudden quantum quenches to compute the time-evolution of observables.

Having reduced the problem from one with driving to an effective sudden quench, let us return to the static Hamiltonian (6). This describes a quantum Ising chain with both transverse g and longitudinal $h = h^z - \Omega/2$ fields. The two fields can be independently controlled via the amplitude g and frequency Ω of the driving, see Eq. (1). Two interesting cases are immediately apparent. First, if the frequency of the driving is tuned to a $\Omega = 2h^z$, the longitudinal field is removed from the static Hamiltonian, which then describes the integrable quantum Ising chain [54]. Second, one can consider tuning both the amplitude and the frequency such that $g = J$ and $|h^z - \Omega/2| \ll g$, where one realizes a lattice limit of the exotic critical Ising field theory perturbed by the spin operator [55] (which has recently received renewed attention thanks to its nonthermal properties [56–61], despite an absence of integrability). In this work, we focus on the first scenario and describe the full time evolution of one- and two-point functions in this driven problem. We touch upon the second case toward the end.

When $\Omega = 2h^z$, the static Hamiltonian reads

$$H^0_{\rm st} = -J\sum_{l=1}^{L}\sigma_l^z\sigma_{l+1}^z - g\sum_{l=1}^{L}\sigma_l^x\,. \tag{11}$$

This is the quantum Ising chain, which can be mapped to free fermions and so is exactly solvable [54]. This reveals that, along the line $\Omega = 2h^z$, there is a hidden integrability in the problem [despite, instantaneously, the Hamiltonian $H(t)$ being nonintegrable]. Sudden quenches in the transverse field Ising model have been extensively studied, with many exact results being known; see in particular the works of Calabrese, Essler, and Fagotti [62–64]. We will exploit some of these results, alongside some new ones, to *analytically* compute the dynamics of observables starting from an initial state $|\Psi_0\rangle$ that is then time evolved with the driven Hamiltonian (1). The

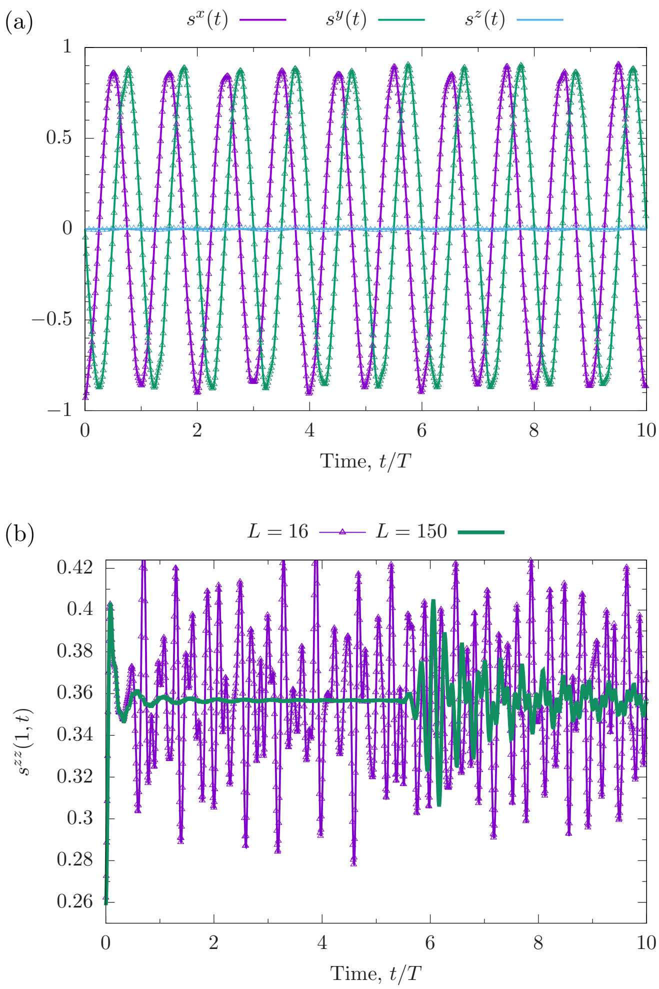

FIG. 1. (a) Time-evolution of one-point functions starting from the ground state of the quantum Ising chain, $H(t=0)$ with $J=1$, $h^z=0$, $g=2$ and time-evolved with the driven Hamiltonian $H(t)$ (1) with $J=1$, $\Omega=1$, $h^z=0.5$, $g=1.5$ for a system with $L=16$ sites. The behavior remains the same for bigger system sizes and larger times. Lines represent analytical results, while points show numerically exact time-evolution. (b) Time evolution of the two-point function $s^{zz}(1,t)$ of two adjacent sites for the same quench, for different system sizes L. We see that $s^{zz}(1,t)$ converges to a nonzero value. The revival of the fluctuations is a finite-size effect, as can be seen by increasing the system size.

derivation of these results is rather technical, so we provide the details in the Supplemental Material [53].

Time evolution in the driven model. Let us now present the time evolution of correlation functions in the driven model (1) governed by the effective static Hamiltonian (11). We compare our analytical results to numerical results obtained on small finite lattices (our numerical algorithm is explained in the Supplemental Material. [53]).

In Fig. 1 we present the results for one- and two-point functions for a particular quench. We see that the one-point functions synchronize to the driving frequency Ω and no heating to infinite temperature occurs, not even if we restrict the study of the system to stroboscopic times. For the two-point functions, we see that $s^{zz}(1,t)$ converges to a nonzero stationary value, confirming the absence of infinite heating. Although not included in the figure, we mention that the remaining two-point functions synchronize to the period Ω like the one-point functions, except for $s^{yz}(1,t)$ and $s^{xz}(1,t)$ that converge to zero.

A particularly simple and solvable in closed form scenario is realized when $H_{\rm st}$ coincides with the initial Hamiltonian $H(t<0)$. In this case the "sudden quench" correlation functions in expressions such as Eq. (9) reduce to *equilibrium correlation functions*, known since the seminal works of Barouch *et al.* [65] and Barouch and McCoy [66,67] in the 1970s. Detailed results in this case are presented in the Supplemental Material [53] and are, to our knowledge, some of the few closed form exact results known for correlation functions in models with driving.

Absence of heating to infinite temperature. With observables mapping in a simple manner to those from a sudden quench, it is expected that the system cannot undergo heating to infinite temperature, as is usually assumed to occur in driven systems [31–33]. This is easily seen for observables that feature only σ_l^z operators, which map exactly to "sudden quench" observables [see, e.g., the first line of Eq. (9)]. The long-time limit of observables after a sudden quench will be described via the relevant statistical ensemble; for the case detailed above this is the generalized Gibbs ensemble [12,68,69]. Generically, when $H_{\rm st}$ is nonintegrable, this will be a finite-temperature Gibbs ensemble [70]. We conjecture that the absence of heating to infinite temperature is not a result of integrability but instead is due to the structure of the driving term. In Fig. S1 of the Supplemental Material [53], we show an explicit example of a nonintegrable system with absence of heating to infinite temperature by working outside the integrable line $\Omega = 2h^z$.

We can then ask, what happens if this structure is broken such that we do not map to an effective sudden-quench problem? We then expect that, in the long-time limit, the system thermalizes to infinite temperature, due to the absence of both energy conservation and the mapping to a sudden quench problem, combined with entropy maximization. We can examine this numerically by adding terms to our Hamiltonian (1), for example:

$$H_X(t) = H(t) + J_X \sum_{l=1}^{L} \sigma_l^x \sigma_{l+1}^x. \qquad (12)$$

The added term breaks σ^z conservation, and thus evolves nontrivially under the transformation $U(t)$. This breaks the mapping to a static Hamiltonian, and hence we expect heating to infinite temperature. It is worth noting that the thermalization timescale in Floquet systems can be very large, see, e.g., Refs. [71–74]. [The Floquet model studied in Ref. [74] bears some similarity to Eq. (12).]

In Fig. 2 we present the time-evolution of one-point functions in the driven model (12). With the addition of the J_X term, we see that the system evolves towards a state with $\lim_{t\to\infty}\langle\sigma_j^\alpha(t)\rangle = 0$, corresponding to infinite temperature, at least at the level of one-point subsystems.

Realizing a perturbed critical model. Let us finish with an illustration of the second interesting case discussed above. We consider tuning the driving such that the static Hamiltonian

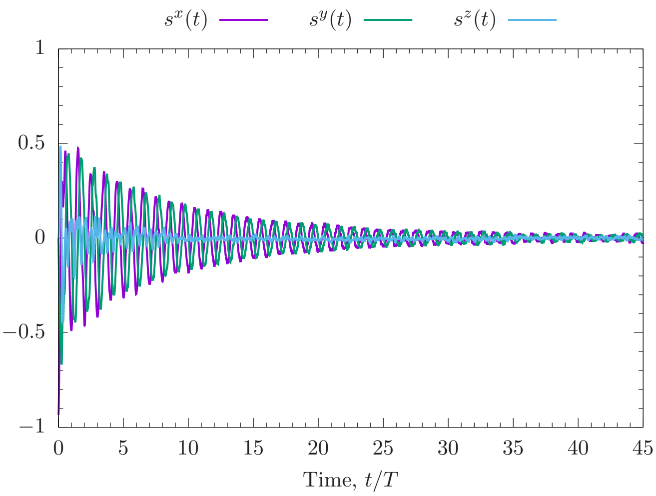

FIG. 2. Numerically exact time evolution of one-point functions with the driven Hamiltonian $H_X(t)$, Eq. (12). This shows that, at the level of one-point functions, breaking the structure of the drive leads to thermalization to infinite temperature, as all the expectation values converge to zero. The parameters considered were those of Fig. (1) with $J_x = 0.5$, and a Chebyshev expansion of order 64 with a time step $\Delta t = 0.001$ (see the Supplemental Material [53] for the details of the numerical algorithm employed).

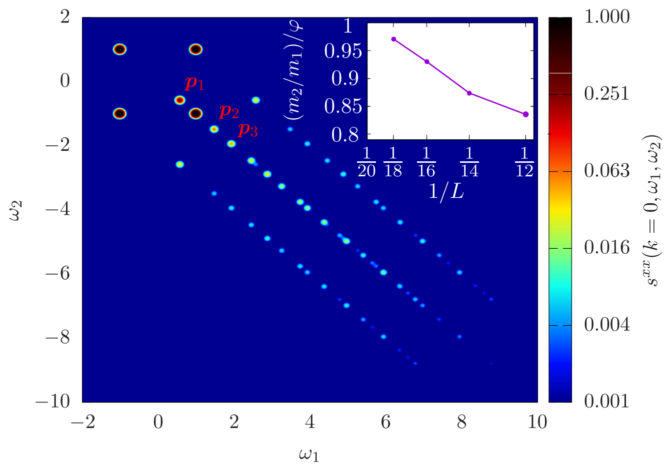

FIG. 3. Dynamical correlation function $s^{xx}(k=0,\omega_1,\omega_2)$ (14) for the perturbed critical model with $J = g = 1$, $\Omega = 1$, $h_z = \Omega/2 + 0.1$, and $L = 16$. The data were normalized so that the maximum value of the plot is one. The four dominant peaks at $(\pm\Omega, \pm\Omega)$ and $(\mp\Omega, \pm\Omega)$ come from the driving frequency Ω, while the next three at $\boldsymbol{p}_i = (m_i - \Omega, \Omega - m_i)$ come from the masses of the mesons excitations m_i. Although m_2/m_1 is not equal to the golden ratio φ, we verify in the inset that, as the size of the system is increased, m_2/m_1 gets closer to φ.

describes the perturbed critical Ising chain:

$$H_{\rm st}^1 = -J\sum_{l=1}^{L}\sigma_l^z\sigma_{l+1}^z - J\sum_{l=1}^{L}\sigma_l^x + h\sum_{l=1}^{L}\sigma_l^z. \qquad (13)$$

When $h = 0$, $H_{\rm st}^1$ realizes the critical point of the Ising chain. For $h \neq 0$, $H_{\rm st}^1$ is no longer integrable, but its low-energy physics is well understood thanks to Zamolodchikov [55]. Pairs of fermions (corresponding to domain walls in the ordered phase) are confined by the presence of the longitudinal field h and form "meson" excitations. In the scaling limit, the algebraic structure of the theory allows the prediction of these meson masses, including the beautiful result that the ratio of masses of the first and second meson states realizes the golden ratio, $m_2/m_1 = \varphi$. With integrability absent, we are limit to performing small system numerics, such as in Fig. 3.

In Fig. 3 we plot the dynamical correlation function

$$s^{xx}(k=0,\omega_1,\omega_2) = \sum_{\ell}\int dt_1 dt_2 e^{i(\omega_1 t_1 + \omega_2 t_2)} \times \langle\Psi_0|\sigma_{j+\ell}^x(t_1)\sigma_j^x(t_2)|\Psi_0\rangle, \qquad (14)$$

where $\sigma_n^x(t)$ denotes the time evolution of σ_n^x in the Heisenberg picture and, for simplicity, we assume the initial state of the system was prepared to be the ground state of the static Hamiltonian (13). Note that, because of the driving, energy is not conserved, and $\langle\Psi_0|\sigma_{j+\ell}^x(t_1)\sigma_j^x(t_2)|\Psi_0\rangle$ is no longer a function of the time difference $(t_1 - t_2)$, so we considered the Fourier transform of both times.

Note that there are four dominant peaks in Fig. 3, these correspond to the driving frequency Ω and are located at $(\omega_1, \omega_2) = (\pm\Omega, \pm\Omega), (\mp\Omega, \pm\Omega)$. The remaining dominant peaks (marked $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ in the figure) correspond to the first excitations or masses of the static system, and their coordinates are $\boldsymbol{p}_i = (m_i - \Omega, \Omega - m_i)$, where m_i denotes the masses of the meson excitations. Although these masses do not satisfy the equality $m_2/m_1 = \varphi$, we verify that this is a finite-size effect in the inset of the figure, as m_2/m_1 gets closer to φ when the size of the system is increased.

Our results reiterate the fact that driven systems, such as (1), can be used to realize exotic physics that is far from accessible with equilibrium probes in its undriven state. This complements existing studies of such systems where exotic physics may instead be accessed via sudden quantum quenches [56–61].

Discussion. In this Letter, we have explored an example of a driven system that is instantaneously nonintegrable but can nonetheless be solved exactly. This is due to a hidden integrability in the problem that is not apparent from the time-dependent Schrödinger equation: the instantaneous Hamiltonian $H(t)$ is nonintegrable, but dynamics of observables are nonetheless controlled by an effective static, integrable Hamiltonian. This may provide a route to protecting quantum information from the scrambling associated with thermalization through the addition of driving; this is an interesting direction for future studies.

The methods applied within this Letter can be used to tackle the dynamics of an infinite family of Hamiltonians (not necessarily integrable). Based on our results, we conjecture that this family of driven systems does not undergo heating to infinite temperature, even though they are lacking disorder and (generically) integrability. For example, consider the Hamiltonian $\widetilde{H}$ of any spin-$\frac{1}{2}$ chain that conserves total $\sigma_{\rm tot}^z$ magnetization (this need not be translationally invariant), which is driven as in Eq. (1):

$$\widetilde{H}(t) = \widetilde{H} - \widetilde{g}\sum_l (e^{-i\widetilde{\Omega}t}\sigma_l^+ + \text{H.c.}). \qquad (15)$$

The transformation (5) still maps Eq. (2) to a time-independent Schrödinger equation with the new effective static Hamiltonian $\widetilde{H}_{\rm st} = \widetilde{H} - (\widetilde{\Omega}/2)\sum_l \sigma_l^z - \widetilde{g}\sum_l \sigma_l^x$. The time evolution of observables in a driven system has once again been mapped to a sudden quench problem. It would be interesting to explore this idea further in interacting models, such as when $\tilde{H}_{\rm st}$ describes the Heisenberg or XXZ model, where potentially integrability can be harnessed to perform exact calculations.

Another scenario worthy of attention is to consider a problem in which the parameters of the static Hamiltonian describe a different phase to the initial Hamiltonian. One may then expect to see signatures of dynamical phase transitions in the nonequilibrium dynamics, such as kinks in the Lochsmidt echo [75,76]. Further exploring the lattice limit of Zamolodchikov's perturbed Ising field theory [55], which features interesting collective excitations related to an exotic hidden E_8 algebraic structure, is interesting. Such studies would require detailed numerical analysis (perhaps in the scaling limit [77]), an avenue left to future works.

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