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Topological and symmetry-enriched random quantum critical points

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We study how symmetry can enrich strong-randomness quantum critical points and phases, and lead to robust topological edge modes coexisting with critical bulk fluctuations. These are the disordered analogs of gapless topological phases. Using real-space and density matrix renormalization group approaches, we analyze the boundary and bulk critical behavior of such symmetry-enriched random quantum spin chains. We uncover a new class of symmetry-enriched infinite randomness fixed points: while local bulk properties are indistinguishable from conventional random singlet phases, nonlocal observables, and boundary critical behavior are controlled by a different renormalization group fixed point. We also illustrate how such new quantum critical points emerge naturally in Floquet systems.

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I. INTRODUCTION

Topological phases form a cornerstone of modern condensed matter physics, extending beyond the Landau-Ginzburg paradigm of symmetry-breaking order. An especially important class of topological states are symmetry-protected topological (SPT) phases [1–11], which are gapped systems characterized by nonlocal order parameters and symmetry-protected topological edge modes. Prominent examples of SPT phases include fermionic topological insulators [12–19], protected by time-reversal and charge conservation symmetry, or the Haldane phase in quantum spin chains [20–23], protected by spin-rotation symmetry.

Recently, the concept of SPT order was extended to gapless systems [24–56]: surprisingly, many of the key features of SPT physics carry over to the gapless case, despite the nontrivial coupling between topological edge modes and bulk critical fluctuations. It is also helpful to think of gapless SPT (gSPT) states [45] as symmetry-enriched quantum critical points (SEQCP) [54], where global symmetries can enrich the critical behavior of critical systems. This led to the discovery of new critical points and phases with unusual nonlocal scaling operators, which imply an anomalous surface critical behavior, and symmetry-protected topological edge modes. In certain cases, such SEQCPs are naturally realized as phase transitions separating SPT and symmetry-broken phases: while the bulk universality class is locally dictated by the Landau-Ginzburg theory of spontaneous symmetry-breaking, the nonlocal operators, and the surface critical behavior are affected by the neighboring SPT phase.

In this work, we show that the mechanism protecting gapless SPT phases persists upon adding disorder. We focus on one-dimensional systems, where the bulk criticality flows to infinite-randomness fixed points [57–62]. We first discuss the paradigmatic infinite-randomness Ising criticality, where we find that—similar to the clean case [54]—there are topologically distinct versions in the presence of an additional $\mathbb{Z}_2^T$ symmetry. We find that one of these classes has topologically protected edge states. While this is a fine-tuned critical point, our second example is a stable random singlet phase of matter. Moreover, in the latter case, there are additional gapped degrees of freedom, which are able to make the edge mode exponentially localized. We also illustrate how this topological random quantum criticality can emerge naturally in periodically driven (Floquet) systems.

II. ISING* TRANSITION

We consider the spin-1/2 chain

$$H = -\sum_i J_i Z_i Z_{i+1} - \sum_i h_i X_i - \sum_i g_i Z_{i-1} X_i Z_{i+1}, \qquad (1)$$

where X, Y, Z denote the Pauli matrices. The model has a $\mathbb{Z}_2$ spin-flip symmetry (generated by $P = \prod_i X_i$) and a time-reversal symmetry $\mathbb{Z}_2^T$ (acting as the complex conjugation $T = K$). Let us first consider the clean case, where the coefficients $J_i \equiv J$, $h_i \equiv h$ and $g_i \equiv g$ are site independent. In this case, the $J, h, g \geqslant 0$ terms, respectively, drive the system towards ferromagnetic (FM), trivial paramagnetic (PM), and $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry protected topological (SPT) [63–67] phases, the latter sometimes being called the cluster or Haldane SPT phase. The phase diagram is shown in Fig. 1(a), with the gray solid lines indicating Ising criticalities.

Although the FM-PM and FM-SPT transition are both described by the Ising conformal field theory (CFT), the time-reversal symmetry acts differently on the disorder operator, leading to different symmetry enriched CFTs (or

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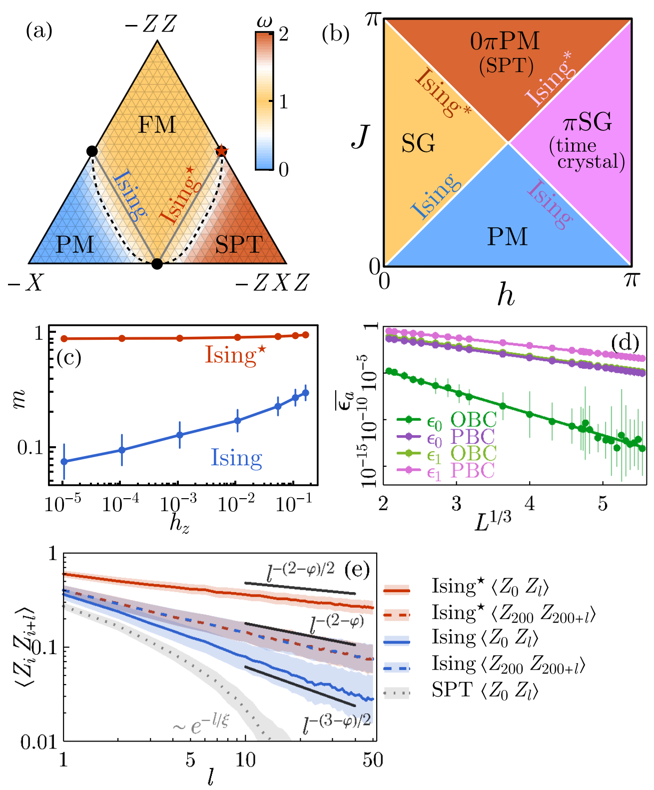

FIG. 1. Random Ising* transition. (a) Phase diagram of the random Ising Hamiltonian (1) for clean (solid lines) and disordered (dotted lines, averaged over 2000 realizations), showing the topological winding number ω for the dual fermionic description (see main text). (b) Floquet phase diagram of Eq. (4), which shows *two* topologically nontrivial Ising* transitions. (c) Boundary magnetization under small Zeeman field, showing spontaneous magnetization at the Ising* transition [red star in (a)]. (d) Finite-size energy splitting of boundary spins at the Ising* transition. (e) Spin-spin correlations involving bulk and boundary spins (averaged over 1.5×10^5 realizations), compared to theory predictions (solid black lines), where φ is the golden ratio. Calculations are performed using the SBRG method on a 512-site spin chain.

gapless SPTs) [45,48,51,54,56]. To briefly review this, note that an Ising CFT has a unique local and a unique nonlocal scaling operator with scaling dimension $\Delta = 1/8$, commonly denoted by σ and μ, respectively. These are the order parameters of the nearby phases, i.e., $\sigma(n) \sim Z_n$ is the Ising order parameter, whereas the disorder operator $\mu(n)$ is the Kramers-Wannier-dual string order parameter of the symmetry-preserving phase. In the trivial PM, $\mu(n) \sim \prod_{j=-\infty}^{n} X_j$, whereas in the SPT phase, $\mu(n) \sim \prod_{j=-\infty}^{n} Z_{j-1}X_jZ_{j+1} = \cdots X_{n-2}X_{n-1}Y_nZ_{n+1}$ [53,54,68,69]. We see that the two Ising critical lines are distinguished by the discrete invariant $T\mu T = \pm\mu$ [54]. This means they must be separated by a phase transition. Indeed, in Fig. 1(a) they meet at a multicritical point where the central charge is $c = 1$.

We refer to the nontrivial case, where the nonlocal bulk operator is charged $T\mu T = -\mu$, as Ising*. This supports a localized zero-energy edge state [54]. Intuitively, the edge of the Ising* criticality spontaneously breaks the Ising $\mathbb{Z}_2$ symmetry. This unusual degenerate boundary fixed point is stable since μ is charged and hence cannot be used to disorder the boundary. The finite-size splitting of this edge mode is parametrically faster than the finite-size bulk gap $\sim 1/L$. In particular, if the model is dual to free-fermions [such as Eq. (1)] then the edge mode is exponentially localized [48] whereas with interactions, the splitting becomes $\sim 1/L^{14}$ [54].

III. RANDOM ISING* TRANSITION

We now study the fate of Ising* upon disordering the system. The coefficients J_i, h_i, and g_i in Eq. (1) are now independently drawn from power-law distributions $P(J) = (J/J_0)^{1/\Gamma}/(\Gamma J)$ for $J \in [0, J_0]$ [similarly for $P(h)$ and $P(g)$], where Γ controls the width of the distribution in logarithmic scale. The limit $\Gamma \to 0$ would recover the clean case. We will take $\Gamma = 1$, i.e., the uniform distribution.

In the presence of randomness, the Ising CFT flows towards the infinite-randomness fixed point ($\Gamma \to \infty$) [59,61]. We will explore the symmetry enriched infinite-randomness fixed point as the many-body localized counterpart of gapless SPT states. The disordered phase diagram is shown in Fig. 1(a), which is qualitatively unchanged from the clean case. This was obtained by mapping Eq. (1) to free fermions (using a Jordan-Wigner transformation) and using the transfer matrix method to determine the topological winding number ω [70]; in this case the PM, FM, and SPT phases map to the trivial ($\omega = 0$), Kitaev chain ($\omega = 1$) and two Kitaev chains ($\omega = 2$). In the original spin chain language, one can interpret ω as encoding the ground-state degeneracy 2^ω with open boundary conditions, which is 0, 2, and 4, respectively.

Similar to the Ising CFT, the infinite-randomness Ising fixed point also has a local σ and nonlocal μ scaling operator. While their scaling dimensions have changed ($\Delta^{\text{bulk}} = 1 - \varphi/2 \approx 0.191$ where $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio [61]), their lattice expressions are as before—indeed, the nearby gapped phases are still characterized by the same order parameters. We thus still have the bulk topological invariant $T\mu T = \pm\mu$, distinguishing two distinct symmetry-enriched infinite-randomness Ising fixed points, which we refer to as the Ising and Ising*. For the same reasons as before, we expect that the disordered Ising* criticality has spontaneously fixed boundary conditions. This would come with at least three physical fingerprints: (i) a nonzero spontaneous magnetization at the boundary, (ii) a degenerate edge mode whose finite-size splitting is parametrically smaller than the bulk gap, and (iii) spin-spin correlations near the boundary should have a boundary scaling dimension [71,72] $\Delta_\sigma^{\text{bdy}} = 1/2$ (or 0) for free (or spontaneously fixed) boundary condition, characterizing the Ising (Ising*) case.

We now test these predictions numerically. Because we will be interested in including interactions, we use the spectrum bifurcation renormalization group (SBRG) method [73–76], which is a numerical real-space renormalization group approach that progressively transforms the original Hamiltonian H to its diagonal form $H_{\text{MBL}} = \sum_a \epsilon_a \tau_a + \sum_{ab} \epsilon_{ab} \tau_a \tau_b + \cdots$ as a many-body localization (MBL) effective Hamiltonian [77–79], and constructs the (approximate) local integrals of motion τ_a of the MBL system in the form of Pauli strings. The approximation is asymptotically exact in the strong-disorder limit. The rescaled parameters

$(\tilde{J}, \tilde{h}, \tilde{g}) \equiv (J_0, h_0, g_0)^{1/\Gamma}$ are invariant under the renormalization group (RG) flow, and should be considered as tuning parameters. SBRG can be thought of as an implementation of the strong disorder real-space renormalization group (RSRG) [57–62] and its generalization to excited states (RSRG-X) [80–83] in operator space. While SBRG can be used to study MBL physics and excited states, in the following we focus on $T = 0$ ground-state properties.

IV. SBRG RESULTS

We focus on the Ising* transition at $(\tilde{J}, \tilde{h}, \tilde{g}) = (1, 0, 1)$ [red star in Fig. 1(a)]. We have verified [76] that in the bulk, the Ising* transition flows to an infinite-randomness fixed point with dynamical scaling $l \sim (\log t)^2 \sim (-\log \epsilon)^2$ that relates the length scale l and the energy scale ϵ [61], and logarithmic scaling of the entanglement entropy [84,85]. This is not surprising because with periodic boundary conditions, Ising, and Ising* are unitarily equivalent.

We now probe the boundary properties. To include the effect of interactions, we follow Ref. [54] and add a generic $\mathbb{Z}_2 \times \mathbb{Z}_2^T$-symmetric boundary perturbation $H_V = -V(X_0Z_1Z_2 + Z_{L-2}Z_{L-1}X_L)$, with V a random variable ten times smaller than the bulk couplings. Microscopically, this perturbation can flip the boundary Ising spin. Nevertheless, if we study the boundary magnetization $m = \langle Z_0 \rangle$ in response to a small Zeeman field h_z applied along the z axis, we find that it tends to a nonzero limit as $h_z \to 0$ (with h_z smaller than the finite-size bulk gap, but larger than the ground states splitting, see below), shown in Fig. 1(d). This is in contrast to the trivial Ising fixed point, where the boundary magnetization is known to vanish as $m(h_z) \sim 1/|\log h_z|$ [71].

Thus the boundary is spontaneously magnetized in the Ising* case despite the Hamiltonian (1) being symmetric. Schematically, on a finite system we have two spontaneously fixed ferromagnetic (FM) ground states $|\uparrow_L \uparrow_R\rangle$ and $|\downarrow_L \downarrow_R\rangle$, where L and R denote the configurations of the left and right edge modes (note that these are split from $|\uparrow_L \downarrow_R\rangle$ and $|\downarrow_L \uparrow_R\rangle$ by the critical bulk penalizing antiferromagnetic states) [45,54]. The above perturbation H_V can couple these FM states at second order in V, which should lead to a finite-size splitting. The claim that we have a ground-state degeneracy is only meaningful if this splitting is smaller than the bulk finite-size gap. To confirm this, we arrange the energy coefficients ϵ_a obtained from SBRG in the ascending order $\epsilon_0 < \epsilon_1 < \cdots$, and focus on the lowest two. For the Ising* transition with open boundary condition (OBC), ϵ_0 characterizes the smallest energy splitting between $|\uparrow_L \uparrow_R\rangle \pm |\downarrow_L \downarrow_R\rangle$ whereas ϵ_1 characterizes the bulk excitation gap. As shown in Fig. 1(e), both splittings $\overline{\epsilon_0}$ and $\overline{\epsilon_1}$ follow $\overline{\epsilon_a} \sim \exp(-\alpha_a L^{1/3})$ but with different exponents $\alpha_0 = 5.4 \pm 0.6$ and $\alpha_1 = 2.51 \pm 0.02$, i.e., $\overline{\epsilon_0} \approx \overline{\epsilon_1}^2$. The finite-size splitting $\overline{\epsilon_0}$ of the symmetry-protected edge modes decays significantly faster with the system size L compared to $\overline{\epsilon_1}$. This provides a quantitative distinction between the topological edge modes and the bulk excitations. To further verify this interpretation, we switch to the periodic boundary condition (PBC), the fast-decaying topological splitting disappears and the smallest splitting decays with the bulk exponent as $\alpha_0 = 2.45 \pm 0.02$.

The Ising and Ising* states can be further distinguished by their average boundary-bulk spin-spin correlation functions $\overline{\langle Z_0 Z_l \rangle}$, which decay as $\sim 1/l^{\Delta_\sigma^{\text{bdy}} + \Delta^{\text{bulk}}}$, where $\Delta_\sigma^{\text{bdy}}$ (Δ^{bulk}) is the boundary (bulk) scaling dimension of the Ising order parameter mentioned before. We thus predict

$$\overline{\langle Z_0 Z_l \rangle} \sim \begin{cases} l^{-(3-\varphi)/2} \approx l^{-0.69} & \text{Ising}, \\ l^{-(2-\varphi)/2} \approx l^{-0.19} & \text{Ising}^\star. \end{cases} \tag{2}$$

In Fig. 1(f), we find that the boundary-bulk correlation follows $\overline{\langle Z_0 Z_l \rangle} \sim l^{-(0.67 \pm 0.08)}$ for Ising and $l^{-(0.20 \pm 0.02)}$ for Ising*, which matches Eq. (2) within error bars. We also checked that the bulk-bulk correlation $\overline{\langle Z_i Z_{i+l} \rangle} \sim l^{-(0.42 \pm 0.05)}$ decays with the expected exponent $2\Delta^{\text{bulk}} = 2 - \varphi \approx 0.38$ for both Ising and Ising* transitions.

V. SYMMETRY-ENRICHED RANDOM SINGLET PHASE

The Ising* transition provides a clear example of symmetry-enriched random quantum critical point, with stretched-exponentially localized edge modes. It is natural to ask whether this notion can be extended to random critical phases, and whether the topological edge modes can be made exponentially localized despite the absence of a bulk gap. Here, we answer both questions in the positive, by introducing a symmetry-enriched random singlet phase.

In order to obtain a critical phase in one dimension, we consider a system with charge conservation and particle-hole symmetry. For concreteness, we will focus on the random antiferromagnetic spin-1/2 XXZ spin chain $H_{\text{A}} = \sum_i J_i (X_i^A X_{i+1}^A + Y_i^A Y_{i+1}^A + \Delta_i Z_i^A Z_{i+1}^A)$, with $J_i > 0$ and $0 < \Delta_i < 1$ random couplings specified later. It has a symmetry group $G_A = U(1) \rtimes \mathbb{Z}_2^A$ with the $\mathbb{Z}_2^A$ spin flip generated by $\prod_i X_i^A$, while the $U(1)$ part corresponds to $\sum_i Z_i^A$ conservation. For uniform couplings, this spin chain is in a Luttinger liquid phase; while for random couplings, its low-energy properties can be captured by a real-space renormalization group (RSRG) procedure very similar to the SBRG approach above (but restricted to the ground state). The random XXZ spin chain forms a random singlet phase [60], where the ground state is asymptotically made of non-crossing pairs of singlets of all ranges, with quantum critical properties similar to the random Ising transition (which itself can be thought of as a random singlet state of Majorana fermions). In particular, the entanglement entropy grows logarithmically with effective central charge $c_{\text{eff}} = \log 2$ [85,86], and the gap closes stretched exponentially with system size (dynamical exponent $z = \infty$).

To obtain a topological random singlet phase, we use the decorated domain walls construction [87] to twist the random XXZ chain. To that effect, we introduce another spin species B, with Ising symmetry $G_B = \mathbb{Z}_2^B$, with Hamiltonian $H_B = -\sum_i X_i^B + g_B Z_i^B Z_{i+1}^B$. We take $g_B \ll 1$ so that the B spins are disordered, deep into a quantum paramagnetic phase. We then couple the two models by attaching charges of the $G_B = \mathbb{Z}_2^B$ symmetry to the domain walls of the A spins. This is achieved by the unitary transformation $U = \prod_{\text{DW}(A)} (-1)^{(1-Z_i^B)/2}$, where the product runs over all the domain walls of the A spins in the Z basis, with $U^2 = 1$. After unitary rotation (twist) of $H_A + H_B + V$, we find

$$H = \sum_i J_i\left[Z_{i-1}^B\left(X_i^A X_{i+1}^A + Y_i^A Y_{i+1}^A\right)Z_{i+1}^B + \Delta_i Z_i^A Z_{i+1}^A\right] - \sum_i Z_i^A X_i^B Z_{i+1}^A + g_B Z_i^B Z_{i+1}^B + V', \tag{3}$$

where $V' = UVU$ represents arbitrary small perturbations that preserves the $G_A \times G_B$ symmetry. Following the terminology of Refs. [45,51], we refer to Eq. (3) and $H_A + H_B + V = UHU$ as the gSPT and gTrivial (gapless, topologically trivial) Hamiltonians, respectively.

For periodic boundary conditions, H is unitarily related to $H_A + H_B$ plus perturbations, and thus corresponds to random singlet A spins coupled to the gapped paramagnetic B spins. Nevertheless, the two models are topologically distinct. Like Ising and Ising* above, they can be distinguished by the charges of nonlocal scaling operators. In fact, since now there are additional gapped degrees of freedom, one can consider a string order parameter with long-range order: in the trivial case $H_A + H_B$ this is $\cdots X_{j-2}^B X_{j-1}^B X_j^B$ whereas in the topological case H it is $\cdots X_{j-2}^B X_{j-1}^B X_j^B Z_{j+1}^A$. In the latter case, this string order parameter for the gapped B variables is charged under G_A. This discrete invariant shows that we have two distinct symmetry-enriched versions of the same underlying infinite-randomness fixed point. Relatedly, for open boundary conditions, we have $H = J_0\Delta_0 Z_0^A Z_1^A + Z_0^A X_0^B Z_1^B + Z_0^B Z_1^B + \dots$, and in the absence of additional perturbations ($V = 0$), we see that $[Z_0^A, H] = 0$, providing an exact edge mode.

Going away from this special limit, we expect exponentially localized topological edge modes to be protected by the finite gap of the B spins, as in the clean case [45,51]. We confirmed numerically the presence of exponentially localized edge modes coexisting with bulk random singlet criticality using density-matrix renormalization group (DMRG) [88,89] techniques (Fig. 2), including generic symmetry-preserving perturbations [76].

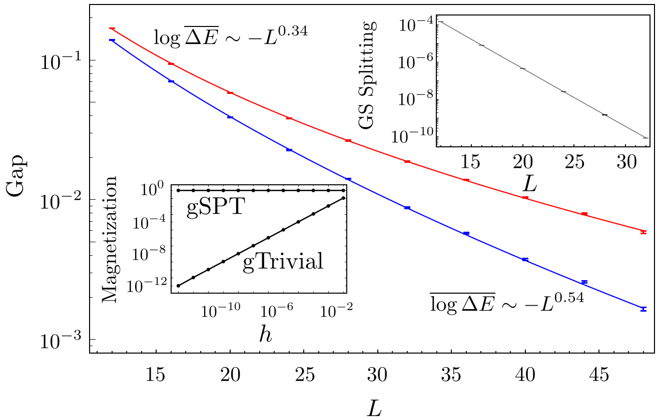

FIG. 2. Symmetry-enriched random singlet phase. DMRG results on Eq. (3) including various perturbations [76]. Fits of the typical and average finite size gaps, showing a scaling compatible with the random-singlet $z = \infty$ scalings $\Delta E_{\text{typical}} \equiv e^{\overline{\log \Delta E}} \sim e^{-\sqrt{L}}$ and $\overline{\Delta E} \sim e^{-L^{1/3}}$. Top-right inset: the splitting between the two ground states vanishes exponentially with system size, indicating exponentially localized edge modes. Bottom-left inset: spontaneous boundary magnetization in the presence of a small symmetry-breaking magnetic field h.

VI. FLOQUET ISING* CRITICALITY

To close this paper, we illustrate how such novel universality classes emerge naturally in the context of periodically driven (Floquet) systems. We focus on the driven quantum Ising chain characterized by the single-period evolution (Floquet) operator [90]

$$F = e^{-\frac{i}{2}\sum_i J_i Z_i Z_{i+1} + \dots} e^{-\frac{i}{2}\sum_i h_i X_i + \dots} \tag{4}$$

where the dots represent small but arbitrary interactions preserving the $\mathbb{Z}_2$ symmetry $G = \prod_i X_i$. For strong enough disorder, this system admits four dynamical phases protected by MBL [90]. In addition to the familiar paramagnetic (PM) and spin glass (SG) Ising phases, there are two more phases called π-SG (also known as time crystal [90–94]) and 0π PM (a nontrivial SPT phase); see Fig. 1(b). This phase structure is due to an emergent $\mathbb{Z}_2$ symmetry inherited from time translation symmetry. The transitions between those phases have been argued to be in the random Ising universality class [95,96] (ignoring potential instabilities towards thermalization in the presence of interactions [97–99]). Here we note that the transitions out of the 0π PM are actually in the random Ising* universality class described above, protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (one of the $\mathbb{Z}_2$'s being emergent). This is because the 0π PM is closely related to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ equilibrium SPT [90,100–103]. We find exponentially localized edge modes at the transitions separating the 0π PM to either the SG or π-SG, which are protected due to the disorder operator μ for the critical $\mathbb{Z}_2$ symmetry again being charged with respect to the second $\mathbb{Z}_2$ symmetry, as detailed in the Supplemental Material [76]. (The edge mode localization is exponential as in the random singlet phase above, as the protecting symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $\mathbb{Z}_2 \times \mathbb{Z}_2^T$.)

VII. DISCUSSION

We have demonstrated the existence of symmetry-enriched infinite-randomness fixed points with robust topological edge modes coexisting with all the characteristics of strong disorder quantum criticality. In particular, we have shown that the paradigmatic random Ising critical point and XXZ random singlet phase come in topologically distinct versions in the presence of an additional $\mathbb{Z}_2^T$ or $\mathbb{Z}_2$ symmetry. The topological edge modes couple nontrivially to gapless bulk fluctuations, leading to anomalous boundary critical behavior. We expect our findings to extend to essentially all known strong- and infinite-randomness critical points: finding examples of symmetry-enriched random critical points in 2+1d [62,104] and 3+1d represents an interesting direction for future works. It would also be interesting to investigate the consequences of our results for dynamical properties [96,105–107].

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